CHAPTER III

COMPLETELY REGULAR ELEMENTS IN Γ-SEMIGROUPS

In this chapter we divide in three sections. In the first section we give a definition of a completely regular Γ-semigroup, and we study the relationship between $\mathcal{H}$-class and completely regular elements. In the second section we establish the set of union of all subgroups of a Γ-semigroup. We also study and extend the properties of completely regular elements which is related to Green's relations and subgroups on semigroups to Γ-semigroups. In the third section we introduce the concept of an inverse Γ-semigroup satisfying the structure of Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, and we study the relationship between such inverse and completely regular elements.

3.1 $\mathcal{H}$-class and Completely Regular Elements in a Γ-semigroup

For this section, we introduce the definition of a completely regular Γ-semigroup, which is more general definition of [15], and we shall use repeatedly. We consider and study the relationship between $\mathcal{H}$-class and a completely regular element in a Γ-semigroup, which will be used in the main body for this research.

Definition 3.1.1. Let S be a Γ-semigroup and $\gamma \in \Gamma$. Define $*$ on S by, for all $a, b \in S$, $a * b = a\gamma b$. Then $(S, *)$ is a semigroup. Such semigroup is denoted by $(S_\gamma, *)$.

A nonempty subset T of S is called a *subgroup of* S_γ if T is a group under the operation $*$.

Definition 3.1.2. Let S be a Γ-semigroup and $a \in S$. If $a = a\alpha x\beta a$ and $a\alpha x = x\beta a$ for some $x \in S, \alpha, \beta \in \Gamma$, then an element a is called *an (α, β)-completely regular element of S.*

A Γ-semigroup S will be called *completely regular* if every element of S is an (α, β)-completely regular element for some $\alpha, \beta \in \Gamma$.

We shall give two examples of completely regular Γ-semigroups as follows.

Example 3.1.3. Let $S = \{\begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R}\}$ and $\Gamma = \{\begin{bmatrix} c & 0 \\ d & 0 \\ 0 & 0 \end{bmatrix} \mid c, d \in \mathbb{R}\}$.
We shall show that S is completely regular. It is easy to show that S is a Γ-semigroup under the usual multiplication of matrices. Let $A = \begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \in S$.

Case $a = b = 0$. Choose $\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Case $a \neq 0$ and $b = 0$. We choose $\alpha = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} \frac{1}{a} & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Case $a = 0$ and $b \neq 0$. Set $\alpha = \begin{bmatrix} 0 & 0 \\ \frac{1}{b} & 0 \\ 0 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & 0 \\ \frac{1}{b} & 0 \\ 0 & 0 \end{bmatrix}$ where $d \in \mathbb{R}$.

Case $a \neq 0$ and $b \neq 0$. Choose $\alpha = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & 0 \\ \frac{1}{b} & 0 \\ 0 & 0 \end{bmatrix}$.
Then $A = A\alpha A\beta A$ and $A\alpha A = A\beta A$. Thus A is (α, β)-completely regular. Hence S is completely regular.

Example 3.1.4. Let $S = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\end{bmatrix}\}$ and $\Gamma = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$. We shall show that S is completely regular. Clearly, S is a Γ-semigroup. Set $\alpha = \begin{bmatrix}1\\0\end{bmatrix}$ and $\beta = \begin{bmatrix}0\\1\end{bmatrix}$. Consider

$$\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} \text{ and } \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix},$$

so $\begin{bmatrix}1 & 0\end{bmatrix}$ is (α, α)-completely regular. Consider

$$\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} \text{ and } \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix},$$

we get $\begin{bmatrix}0 & 1\end{bmatrix}$ is (β, β)-completely regular.

$$\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} \text{ and } \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix},$$

we obtain $\begin{bmatrix}1 & 1\end{bmatrix}$ is (β, α)-completely regular. Clearly, $\begin{bmatrix}0 & 0\end{bmatrix}$ is (α, β)-completely regular. Thus S is completely regular under the usual multiplication of matrices.

Theorem 3.1.5. *Let S be a Γ-semigroup and $e \in E_\gamma(S)$. Then H_e is a subgroup of S_γ.*

Proof. Let $a, b \in H_e$. By Theorem 2.1.8 (3), we have

$$b\gamma e = b = e\gamma b \text{ and } a\gamma e = a = e\gamma a. \tag{3.1.1}$$

If $a = e$ or $b = e$, then, by (3.1.1), $a\gamma b = e\gamma b = b$ or $a\gamma b = a\gamma e = a$. This implies that $a\gamma b \in H_e$. Suppose that $a \neq e$ and $b \neq e$. Since $a \,\mathcal{H}\, e$ and $b \,\mathcal{H}\, e$, there exist $w, x, y, z \in S, \delta, \zeta, \eta, \theta \in \Gamma$ such that

$$e = w\delta a,\ e = x\zeta b,\ e = a\eta y \text{ and } e = b\theta z.$$

Since $a\gamma b = e\gamma a\gamma b$ and $e = a\eta y = a\gamma e\eta y = a\gamma b\theta z\eta y$, we have $a\gamma b \in R_e$. Also, from $a\gamma b = a\gamma b\gamma e$ and $e = x\zeta b = x\zeta e\gamma b = x\zeta w\delta a\gamma b$, so $a\gamma b \in L_e$. This implies that $a\gamma b \in H_e$.

Clearly, e is the identity of H_e.

Let $c \in H_e$. By Theorem 2.1.8 (3), we have $c\gamma e = c = e\gamma c$. If $c = e$, then $c\gamma c = e\gamma e = e$, so c is an inverse of c in H_e. Suppose that $c \neq e$. Since $c \mathcal{H} e$, there exist $u, v \in S, \vartheta, \mu \in \Gamma$ such that $e = u\vartheta c$ and $e = c\mu v$. Then $e = u\vartheta c = u\vartheta e\gamma c$ and $e = c\mu v = c\gamma e\mu v$. We claim that $u\vartheta e = e\mu v$. Now $u\vartheta e = u\vartheta c\mu v = e\mu v$. Hence $u\vartheta e$ is the inverse of c in H_e. It follows that H_e is a subgroup of S_γ. □

Corollary 3.1.6. *Let S be a Γ-semigroup and $e \in E(S)$. Then H_e is a subgroup of S_γ for some $\gamma \in \Gamma$.*

Proof. It follows directly from Theorem 3.1.5. □

Theorem 3.1.7. *Let S be a Γ-semigroup. Then the following statements are equivalent:*

(1) *S is completely regular.*

(2) *Each element of S lies in a subgroup of S_γ for some $\gamma \in \Gamma$.*

(3) *Every $\mathcal{H}$-class is a subgroup of S_δ for some $\delta \in \Gamma$.*

Proof. (1)⇒(2) Let $a \in S$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that

$$a = a\alpha x\beta a \text{ and } a\alpha x = x\beta a.$$

Set $e = a\alpha x = x\beta a$. Then $a \in L_e \cap R_e = H_e$ and $e \in E_\alpha(S)$. It follows that H_e is a subgroup of S_α by Theorem 3.1.5.

(2)⇒(3) Let $a \in S$. By assumption, there exists $\gamma \in \Gamma$ such that a is contained in some subgroup T of S_γ. Then we have

$$a\gamma e = a = e\gamma a \text{ and } a\gamma x = e = x\gamma a \text{ for some } x \in T,$$

where e is the identity of T. It follows that $a \mathcal{H} e$, and hence $H_a = H_e$. Note that $e \in E_\gamma(S)$. By Theorem 3.1.5, H_e is a subgroup of S_γ. This implies that H_a is a subgroup of S_γ.

(3)⇒(1) Let $a \in S$. By assumption, H_a is a subgroup of S_γ for some $\gamma \in \Gamma$. Then we have

$$a\gamma e = a = e\gamma a \text{ and } a\gamma x = e = x\gamma a \text{ for some } x \in T,$$

where e is the identity of H_a. Thus we have $a = a\gamma e = a\gamma x\gamma a$. It follows that S is completely regular. □

Theorem 3.1.8. *Let S be a Γ-semigroup and $a \in S$, $\alpha, \beta \in \Gamma$. Then the following statements hold:*

(1) If a is an (α, β)-completely regular element of S, then H_a is a subgroup of S_α and S_β with the same identity and the same inverse of a.

(2) If H_a is a subgroup of S_α and S_β with the same identity, then a is an (α, β)-completely regular element of S.

(3) a is an (α, β)-completely regular element of S if and only if H_a is a subgroup of S_α and S_β with the same identity.

Proof. (1) By assumption, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that

$$a = a\alpha x\beta a \text{ and } a\alpha x = x\beta a.$$

Set $e = a\alpha x = x\beta a$. Then $a \in L_e \cap R_e = H_e$ and $e \in E_\alpha(S) \cap E_\beta(S)$. Thus $H_e = H_a$. But H_e is a subgroup of S_α and S_β by Theorem 3.1.5, it follows that H_a is a subgroup of S_α and S_β containing the identity element e. Then we have

$$a\alpha y = e = y\alpha a \text{ and } a\beta z = e = z\beta a \text{ for some } y, z \in H_a.$$

Now,

$$y = y\alpha e = y\alpha a\beta z = e\beta z = z.$$

Therefore H_a is a subgroup of S_α and S_β with the same identity and the same inverse of a.

The proof of (2) is trivial, and (3) follows directly from (1) and (2). □

3.2 The Relationship between Completely Regular Elements, Green's Relations and Subgroups in Γ-semigroups

In this section, we start by study and extend the properties of completely regular elements which is related to Green's relations on semigroups to Γ-semigroups as follows:

Proposition 3.2.1. *Let S be a Γ-semigroup and $a \in S$. If a is a regular element of S, then we have*

(1) $R_a = a\Gamma S \cap R_a\Gamma S\Gamma R_a \cap R_a$,

(2) $L_a = S\Gamma a \cap L_a\Gamma S\Gamma L_a \cap L_a$,

(3) $H_a = a\Gamma S\Gamma a \cap L_a\Gamma S\Gamma R_a \cap R_a\Gamma S\Gamma L_a \cap H_a$.

Proof. Assume that a is a regular element of S. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

(1) Let $b \in R_a$. Then $a = b$ or there exist $m, n \in S$ and $\gamma, \delta \in \Gamma$ such that $a = b\gamma m$ and $b = a\delta n$. Thus we have the following two cases:

Case 1: $a = b$. Then

$$b = a\alpha x\beta a = b\alpha x\beta a \in R_a\Gamma S\Gamma R_a \cap a\Gamma S.$$

Hence $R_a \subseteq a\Gamma S \cap R_a\Gamma S\Gamma R_a \cap R_a$.

Case 2: $a = b\gamma m$ and $b = a\delta n$. Consider,

$$b = a\delta n = a\alpha x\beta a\delta n = b\gamma m\alpha x\beta b.$$

It follows that $b \in a\Gamma S \cap R_a\Gamma S\Gamma R_a$. Hence we have (1).

(2) It is similar to the proof of (1).

(3) Let $c \in H_a$. Then $c \in L_a \cap R_a$. We consider the following two cases:

Case 1: $a = c$. Then

$$c = a = a\alpha x\beta a = c\alpha x\beta c,$$

which implies that $c \in a\Gamma S\Gamma a \cap L_a\Gamma S\Gamma R_a \cap R_a\Gamma S\Gamma L_a$.

Case 2: There exist $s, t, u, v \in S$ and $\zeta, \eta, \theta, \vartheta \in \Gamma$ such that $a = s\zeta c$, $c = t\eta a$, $a = c\theta u$ and $c = a\vartheta v$. Now

$$c = a\vartheta v = a\alpha x\beta a\vartheta v = a\alpha x\beta c = a\alpha x\beta t\eta a \in a\Gamma S\Gamma a$$

and

$$c = t\eta a = t\eta a\alpha x\beta a = c\alpha x\beta s\zeta c \in L_a\Gamma S\Gamma R_a \cap R_a\Gamma S\Gamma L_a.$$

Hence we have (3). □

Proposition 3.2.2. *Let S be a Γ-semigroup and $a \in S$. Then a is a regular element of S if and only if $H_a = R_a\Gamma S\Gamma L_a \cap H_a$.*

Proof. Necessity. Assume that a is a regular element of S. Then by Proposition 3.2.1, we have $H_a \subseteq R_a\Gamma S\Gamma L_a \cap H_a$. Obviously, $R_a\Gamma S\Gamma L_a \cap H_a \subseteq H_a$. This implies that $H_a = R_a\Gamma S\Gamma L_a \cap H_a$.

Sufficiency. Notice that $a \in H_a$, we get $a \in R_a\Gamma S\Gamma L_a \cap H_a$. Then there exist $x \in R_a$, $y \in S$, $\alpha, \beta \in \Gamma$, $z \in L_a$ such that

$$a = x\alpha y\beta z. \tag{3.2.1}$$

Since $x \in R_a$, we obtain $a = x$ or there exist $q \in S$ and $\gamma \in \Gamma$ such that $x = a\gamma q$. Since $z \in L_a$, we have $a = z$ or there exist $r \in S$ and $\delta \in \Gamma$ such that $z = r\delta a$. Now, we shall show that a is a regular element of S. We consider the following:

Case 1: $a = x = z$. Then we substitute in (3.2.1), we get $a = a\alpha y\beta a$. Hence a is a regular element of S.

Case 2: $a = x$ and $z = r\delta a$. Then by (3.2.1), we have $a = a\alpha(y\beta r)\delta a$. So a is a regular element of S.

Case 3: $x = a\gamma q$ and $a = z$. Then by (3.2.1), we obtain $a = a\gamma(q\alpha y)\beta a$. Thus a is a regular element of S.

Case 4: $x = a\gamma q$ and $z = r\delta a$. Then by (3.2.1), we have $a = a\gamma(q\alpha y\beta r)\delta a$. Therefore a is a regular element of S. This completes the proof. □

Remark 3.2.3. Recall that an element a is an (α, β)-completely regular element of a Γ-semigroup S if $a = a\alpha x\beta a$ and $a\alpha x = x\beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$. Then

$$a\alpha a = a\alpha a\alpha x\beta a = a\alpha x\beta a\beta a = a\beta a.$$

Proposition 3.2.4. *Let S be a Γ-semigroup and $a \in S$, $\alpha, \beta \in \Gamma$. If a is an (α, β)-completely regular element of S, then $a\alpha a \ \mathcal{L}\ a \ \mathcal{R}\ a\beta a$ and $a\alpha a \ \mathcal{R}\ a \ \mathcal{L}\ a\beta a$.*

Proof. Assume that a is an (α, β)-completely regular element of S. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$ and $a\alpha x = x\beta a$. From

$$a = a\alpha x\beta a = a\alpha a\alpha x \text{ and } a = a\alpha x\beta a = x\beta a\beta a,$$

we deduce that $a\alpha a \ \mathcal{R}\ a$ and $a \ \mathcal{L}\ a\beta a$, and by Remark 3.2.3, we see that $a\alpha a \ \mathcal{L}\ a$ and $a \ \mathcal{R}\ a\beta a$. □

Proposition 3.2.5. *Let S be a Γ-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. If a is an (α, β)-completely regular element of S, then we have*

(1) $R_a = a\zeta a\Gamma S \cap R_{a\zeta a}\Gamma S\Gamma R_{a\zeta a} \cap R_{a\zeta a}$,

(2) $L_a = S\Gamma a\zeta a \cap L_{a\zeta a}\Gamma S\Gamma L_{a\zeta a} \cap L_{a\zeta a}$,

(3) $H_a = L_{a\zeta a}\Gamma S\Gamma R_{a\zeta a} \cap R_{a\zeta a}\Gamma S\Gamma L_{a\zeta a} \cap H_{a\zeta a}$,

where $\zeta \in \{\alpha, \beta\}$.

Proof. Let $a \in S$ be such that $a = a\alpha x\beta a$ and $a\alpha x = x\beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$.

(1) Let $b \in R_a$. Then $a = b$ or there exist $q \in S$ and $\gamma, \delta \in \Gamma$ such that $a = b\gamma q$ and $b = a\delta r$. Thus we have the following two cases:

Case 1: $a = b$. By Remark 3.2.3,

$$b = a = a\alpha x\beta a = a\alpha a\alpha x \in a\alpha a\Gamma S = a\beta a\Gamma S.$$

Hence $R_a \subseteq a\zeta a\Gamma S$, where $\zeta \in \{\alpha, \beta\}$.

Case 2: $a = b\gamma q$ and $b = a\delta r$. By Remark 3.2.3,

$$b = a\delta r = a\alpha x\beta a\delta r = a\alpha a\alpha x\delta r \in a\alpha a\Gamma S\Gamma S \subseteq a\alpha a\Gamma S = a\beta a\Gamma S.$$

Thus $R_a \subseteq a\zeta a\Gamma S$, where $\zeta \in \{\alpha, \beta\}$.

By Proposition 3.2.1, 3.2.4, and two above cases, we conclude that $R_a \subseteq R_{a\gamma a}\Gamma S\Gamma R_{a\zeta a} \cap R_{a\zeta a}$, where $\zeta \in \{\alpha, \beta\}$. Again, by Proposition 3.2.4, we obtain that the converse holds.

Similarly, we can prove that (2) hold, and by using Proposition 3.2.1 and 3.2.4, we obtain (3) holds. □

Proposition 3.2.6. *Let S be a Γ-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. Then a is an (α, β)-completely regular element of S if and only if $H_a = R_{a\alpha a}\Gamma L_{a\beta a} \cap H_{a\zeta a}$ and $a\alpha a = a\beta a$, where $\zeta \in \{\alpha, \beta\}$.*

Proof. Necessity. Let $a \in S$ be such that $a = a\alpha x\beta a$ and $a\alpha x = x\beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$. Let $y \in H_a$. Then $a = y$ or there exist $q, r, s \in S$ and $\gamma, \delta, \zeta \in \Gamma$ such that $y = a\gamma q$, $y = r\delta a$ and $a = s\zeta y$.

Case 1: $a = y$. Then $y = a\alpha x\beta a$. Obviously, $a\alpha x \in R_a$, so we have $y \in R_a\Gamma L_a$. By Proposition 3.2.4, $y \in R_{a\alpha a}\Gamma L_{a\beta a}$.

Case 2: $y = a\gamma q$, $y = r\delta a$ and $a = s\zeta y$. Then

$$y = a\gamma q = a\alpha x\beta a\gamma q = a\alpha x\beta y = a\alpha x\beta r\delta a. \tag{3.2.2}$$

Since $a = s\zeta y = s\zeta r\delta a$, we get $r\delta a \in L_a$, and obvious that $a\alpha x \in R_a$. Thus by (3.2.2), we obtain $y \in R_a\Gamma L_a$. By Proposition 3.2.4, we have $y \in R_{a\alpha a}\Gamma L_{a\beta a}$. Again, we using Proposition 3.2.4, we obtain the converse hold.

Sufficiency. Note that $a \in H_a$. Then $a \in R_{a\alpha a}\Gamma L_{a\beta a}$. Thus there exist $b \in R_{a\alpha a}$, $\gamma \in \Gamma$, $c \in L_{a\beta a}$ such that

$$a = b\gamma c. \tag{3.2.3}$$

Since $b\ \mathcal{R}\ a\alpha a$, we get $b = a\alpha a$ or there exist $q \in S$ and $\zeta \in \Gamma$ such that $b = a\alpha a\zeta q$. Since $c\ \mathcal{L}\ a\beta a$, we have $c = a\beta a$ or there exist $r \in S$ and $\theta \in \Gamma$ such that $c = r\theta a\beta a$. We shall show that a is an (α, β)-completely regular element of S. Now

Case 1: $b = a\alpha a$ and $c = a\beta a$. Then by (3.2.3), we obtain $a = a\alpha a\gamma a\beta a$. Set

$x = a\gamma a$. Then $a = a\alpha x\beta a$. By assumption, we have

$$a\alpha x\alpha a\alpha x = a\alpha a\gamma a\alpha a\alpha a\gamma a = a\alpha a\gamma a\beta a\alpha a\gamma a = a\alpha a\gamma a = a\alpha x.$$

Thus we have $a\alpha x \in E_\alpha(S)$. Now

$$\begin{aligned} a\alpha x &= a\alpha x\alpha a\alpha x \\ &= a\alpha a\gamma a\alpha a\alpha a\gamma a \\ &= a\alpha a\gamma a\beta a\alpha a\gamma a \\ &= a\alpha a\gamma a \\ &= a\alpha a\gamma a\alpha a\gamma a\beta a \\ &= a\alpha a\gamma a\beta a\gamma a\beta a \\ &= a\gamma a\beta a \\ &= x\beta a. \end{aligned}$$

Hence a is an (α, β)-completely regular element of S.

Case 2: $b = a\alpha a$ and $c = r\theta a\beta a$. Then by (3.2.3), we have $a = a\alpha a\gamma r\theta a\beta a$. Set $x = a\gamma r\theta a$. Then $a = a\alpha x\beta a$. By assumption, we obtain

$$\begin{aligned} a\alpha x\alpha a\alpha x &= a\alpha a\gamma r\theta a\alpha a\alpha a\gamma r\theta a \\ &= a\alpha a\gamma r\theta a\beta a\alpha a\gamma r\theta a \\ &= a\alpha a\gamma r\theta a \\ &= a\alpha x. \end{aligned}$$

It follows that $a\alpha x \in E_\alpha(S)$. Consider

$$\begin{aligned} a\alpha x &= a\alpha x\alpha a\alpha x \\ &= a\alpha a\gamma r\theta a\alpha a\alpha a\gamma r\theta a \\ &= a\alpha a\gamma r\theta a\beta a\alpha a\gamma r\theta a \\ &= a\alpha a\gamma r\theta a \\ &= a\alpha a\gamma r\theta a\alpha a\gamma r\theta a\beta a \end{aligned}$$

$$= a\alpha a\gamma r\theta a\beta a\gamma r\theta a\beta a$$

$$= a\gamma r\theta a\beta a$$

$$= x\beta a.$$

This implies that a is an (α, β)-completely regular element of S.

Case 3: $b = a\alpha a\zeta q$ and $c = a\beta a$. Then by (3.2.3), we have $a = a\alpha a\zeta q\gamma a\beta a$. Set $x = a\zeta q\gamma a$. Then $a = a\alpha x\beta a$. By assumption, we have

$$\begin{aligned} a\alpha x\alpha a\alpha x &= a\alpha a\zeta q\gamma a\alpha a\alpha a\zeta q\gamma a \\ &= a\alpha a\zeta q\gamma a\beta a\alpha a\zeta q\gamma a \\ &= a\alpha a\zeta q\gamma a = a\alpha x. \end{aligned}$$

Hence $a\alpha x \in E_\alpha(S)$. Now we consider the following:

$$\begin{aligned} a\alpha x &= a\alpha x\alpha a\alpha x \\ &= a\alpha a\zeta q\gamma a\alpha a\alpha a\zeta q\gamma a \\ &= a\alpha a\zeta q\gamma a\beta a\alpha a\zeta q\gamma a \\ &= a\alpha a\zeta q\gamma a \\ &= a\alpha a\zeta q\gamma a\alpha a\zeta q\gamma a\beta a \\ &= a\alpha a\zeta q\gamma a\beta a\zeta q\gamma a\beta a \\ &= a\zeta q\gamma a\beta a \\ &= x\beta a. \end{aligned}$$

Therefore a is an (α, β)-completely regular element of S.

Case 4: $b = a\alpha a\zeta q$ and $c = r\theta a\beta a$. Then by (3.2.3), we have $a = a\alpha a\zeta q\gamma r\theta a\beta a$. Set $x = a\zeta q\gamma r\theta a$. Then $a = a\alpha x\beta a$. By assumption, we obtain

$$\begin{aligned} a\alpha x\alpha a\alpha x &= a\alpha a\zeta q\gamma r\theta a\alpha a\alpha a\zeta q\gamma r\theta a \\ &= a\alpha a\zeta q\gamma r\theta a\beta a\alpha a\zeta q\gamma r\theta a \\ &= a\alpha a\zeta q\gamma r\theta a = a\alpha x. \end{aligned}$$

Thus $a\alpha x \in E_\alpha(S)$. Now

$$\begin{aligned} a\alpha x &= a\alpha x\alpha a\alpha x \\ &= a\alpha a\zeta q\gamma r\theta a\alpha a\alpha a\zeta q\gamma r\theta a \\ &= a\alpha a\zeta q\gamma r\theta a\beta a\alpha a\zeta q\gamma r\theta a \\ &= a\alpha a\zeta q\gamma r\theta a \\ &= a\alpha a\zeta q\gamma r\theta a\alpha a\zeta q\gamma r\theta a\beta a \\ &= a\alpha a\zeta q\gamma r\theta a\beta a\zeta q\gamma r\theta a\beta a \\ &= a\zeta q\gamma r\theta a\beta a \\ &= x\beta a. \end{aligned}$$

This implies that a is an (α, β)-completely regular element of S, so we can conclude that satisfy the requisite conditions. □

Proposition 3.2.7. *Let S be a Γ-semigroup, $a, b, c \in S$ and $\gamma \in \Gamma$. If $a\ \mathcal{R}\ b\ \mathcal{L}\ c$ and $a\gamma c\ \mathcal{H}\ b$, then $a\ \mathcal{L}\ h\ \mathcal{R}\ c$ for some $h \in E_\gamma(S)$.*

Proof. Assume that $a\ \mathcal{R}\ b\ \mathcal{L}\ c$ and $a\gamma c\ \mathcal{H}\ b$. Then $a\gamma c \in R_a \cap L_c$. Since $a\gamma c\ \mathcal{R}\ a$, we have $a\gamma c = a$ or there exist $m \in S$ and $\alpha \in \Gamma$ such that $a = a\gamma c\alpha m$. Since $a\gamma c\ \mathcal{L}\ c$, we get $a\gamma c = c$ or there exist $n \in S$ and $\beta \in \Gamma$ such that $c = n\beta a\gamma c$. Thus we consider the following cases:

Case 1: $a = a\gamma c = c$. Set $h = a$. Then we have done.

Case 2: $a\gamma c = a$ and $c = n\beta a\gamma c$. Set $h = n\beta a$. Then we get

$$a = a\gamma c = a\gamma n\beta a\gamma c = a\gamma n\beta a = a\gamma h,$$

so $a\ \mathcal{L}\ h$. Since $c = n\beta a\gamma c = n\beta a = h$, we obtain

$$h = n\beta a = n\beta a\gamma c = h\gamma c = c\gamma c.$$

Therefore $h\ \mathcal{R}\ c$. From

$$h = n\beta a = n\beta a\gamma c = n\beta a\gamma n\beta a\gamma c = n\beta a\gamma n\beta a = h\gamma h,$$

it follows that $h \in E_\gamma(S)$.

Case 3: $a = a\gamma c\alpha m$ and $a\gamma c = c$. Set $h = c\alpha m$. Then

$$c = a\gamma c = a\gamma c\alpha m\gamma c = c\alpha m\gamma c = h\gamma c,$$

so we obtain $h\ \mathcal{R}\ c$. Since $a = a\gamma c\alpha m = c\alpha m = h$, we have

$$h = c\alpha m = a\gamma c\alpha m = a\gamma h = a\gamma a.$$

Hence $a\ \mathcal{L}\ h$. Now

$$h = c\alpha m = a\gamma c\alpha m = a\gamma c\alpha m\gamma c\alpha m = c\alpha m\gamma c\alpha m = h\gamma h,$$

which implies that $h \in E_\gamma(S)$.

Case 4: $a = a\gamma c\alpha m$ and $c = n\beta a\gamma c$. Set $h = c\alpha m$. Then we have

$$h = c\alpha m = n\beta a\gamma c\alpha m = n\beta a.$$

Thus $c = n\beta a\gamma c = h\gamma c$ and $a = a\gamma c\alpha m = a\gamma h$, and hence $a\ \mathcal{L}\ h\ \mathcal{R}\ c$. From

$$h\gamma h = n\beta a\gamma c\alpha m = c\alpha m = h,$$

It follows that $h \in E_\gamma(S)$. This completes the proof. □

We establish and consider the set of an union of all subgroups of Γ-semigroup as follows:

Definition 3.2.8. Let S be a Γ-semigroup. Define the set $G_\Gamma(S)$ to be the set of union of all subgroups of S_γ, for each $\gamma \in \Gamma$; that is

$$G_\Gamma(S) = \cup_{\gamma\in\Gamma} \cup_{T\in\mathcal{S}(\gamma)} T,$$

where $\mathcal{S}(\gamma) = \{T \mid T\ is\ a\ subgroup\ of\ S_\gamma\}$.

Theorem 3.2.9. *Let S be a Γ-semigroup and $a \in S$. Then the following statements are equivalent:*

(1) $a \in G_\Gamma(S)$.

(2) *a is an (α,β)-completely regular element of S for some $\alpha,\beta \in \Gamma$.*

(3) *a has a (γ,δ)-inverse x with $a\gamma x = x\delta a$ for some $x \in S$, $\gamma,\delta \in \Gamma$.*

(4) $a \in \bigcup_{e\in E_\Gamma(S)} H_e$.

Proof. (1)⇒(2) Assume that $a \in G_\Gamma(S)$. Then there exists a subgroup T of S_γ for some $\gamma \in \Gamma$ such that $a \in T$. Thus a is an (α,β)-completely regular element of S for some $\alpha,\beta \in \Gamma$ by Theorem 3.1.7.

(2)⇒(3) Assume that a is an (α,β)-completely regular element of S for some $\alpha,\beta \in \Gamma$, we have H_a is a subgroup of S_α and S_β with the same identity and the same inverse of a by Theorem 3.1.8 (1). Thus

$$a\alpha x = x\alpha a = a\beta x = x\beta a = e \text{ for some } x \in H_a,$$

where e is the identity of H_a. From

$$a = a\alpha e = a\alpha x\beta a \text{ and } x = x\beta e = x\beta a\alpha x,$$

so we have $x \in V_\alpha^\beta(a)$.

(3)⇒(4) Let $x \in S, \gamma$ and $\delta \in \Gamma$ be such that $a = a\gamma x\delta a$, $x = x\delta a\gamma x$ and $a\gamma x = x\delta a$. Set $e = a\gamma x$. Then $e \in E_\delta(S)$, and we also have $a \in H_e$.

(4)⇒(1) It follows directly from Theorem 3.1.5 and 3.1.7, respectively. □

Theorem 3.2.10. *Let S be a Γ-semigroup. Then the following two statements hold:*

(1) $G_\Gamma(S)$ is the set of all (α,β)-completely regular elements of S for some $\alpha,\beta \in \Gamma$.

(2) $G_\Gamma(S) = \bigcup_{e \in E_\Gamma(S)} H_e$.

Proof. It follows directly from Theorem 3.1.5 and 3.1.7. □

3.3 $(\mathcal{L},\mathcal{R},\mathcal{H})$-inverses in Γ-semigroups

The following three lemmas will be used for the main body of this section.

Lemma 3.3.1. *Let S be a Γ-semigroup and $a,b \in S$, $\alpha,\beta \in \Gamma$. Then H_b contains (α,β)-inverse of a if and only if there exist $e \in E_\alpha(S), f \in E_\beta(S)$ such that*

$$a \;\mathcal{L}\; e \;\mathcal{R}\; b \;\mathcal{L}\; f \;\mathcal{R}\; a.$$

Moreover, an (α,β)-inverse of a in H_b is unique.

Proof. Necessity. Let $x \in V_\alpha^\beta(a) \cap H_b$. Then $a = a\alpha x\beta a$, $x = x\beta a\alpha x$, $x \in H_b$. Set $e = x\beta a$ and $f = a\alpha x$. Then $e \in E_\alpha(S)$ and $f \in E_\beta(S)$. Since $a = a\alpha x\beta a = a\alpha e$ and $e = x\beta a$, we have $a\ \mathcal{L}\ e$. Since $a = a\alpha x\beta a = f\beta a$ and $f = a\alpha x$. Hence $f\ \mathcal{R}\ a$. Since $x \in H_b$, we have $x\ \mathcal{H}\ b$. Then $a = x$ or there exist $u, v, w, z \in S, \delta, \gamma, \theta, \lambda \in \Gamma$ such that $x = w\gamma b$, $b = z\delta x$, $x = b\theta u$ and $b = x\lambda v$. Now, we shall show that $e\ \mathcal{R}\ b\ \mathcal{L}\ f$. If $x = b$, then $f = a\alpha x = a\alpha b$ and $b = x = x\beta a\alpha x = x\beta f$. Thus $b\ \mathcal{L}\ f$. Also, $e = x\beta a = b\beta a$ and $b = x = x\beta a\alpha x = e\alpha x$. Therefore $e\ \mathcal{R}\ b$. Suppose that $x = w\gamma b$, $b = z\delta x$, $x = b\theta u$ and $b = x\lambda v$. Since $f = a\alpha x = a\alpha w\gamma b$ and $b = z\delta x = z\delta x\beta a\alpha x = z\delta x\beta f$, we have $b\ \mathcal{L}\ f$. Since $e = x\beta a = b\theta u\beta a$ and $b = x\lambda v = x\beta a\alpha x\lambda v = e\alpha x\lambda v$, we have $e\ \mathcal{R}\ b$. This implies that $a\ \mathcal{L}\ e\ \mathcal{R}\ b\ \mathcal{L}\ f\ \mathcal{R}\ a$.

Sufficiency. Let $e \in E_\alpha(S)$ and $f \in E_\beta(S)$ be such that

$$a\ \mathcal{L}\ e\ \mathcal{R}\ b\ \mathcal{L}\ f\ \mathcal{R}\ a.$$

Thus $a\ \mathcal{L}\ e$ and $f\ \mathcal{R}\ a$. By Theorem 2.1.8 (1), (2), we have $a = a\alpha e$ and $a = f\beta a$. Since $a\ \mathcal{L}\ e$, we have $a = e$ or there exist $s, t \in S, \gamma, \delta \in \Gamma$ such that $a = s\gamma e$ and $e = t\delta a$. Since $f\ \mathcal{R}\ a$, we have $a = f$ or there exist $u, v \in S$ and $\zeta, \eta \in \Gamma$ such that $a = f\zeta u$ and $f = a\eta v$. We shall show that there exists a unique (α, β)-inverse of a in H_b. To show that (α, β)-inverse of a exists in H_b, we consider the following cases:

Case 1: $a = e = f$. Set $x = f$. Then $a = a\alpha e = a\alpha a = a\alpha f\beta a$ and $f = f\beta f = f\beta a = f\beta a\alpha e = f\beta a\alpha f$. So we have $x \in V_\alpha^\beta(a)$. By assumption, we have $x\ \mathcal{R}\ b\ \mathcal{L}\ x$, so $x \in H_b$.

Case 2: $a = e$ and $a \neq f$. Set $x = f$. Then $a = a\alpha e = a\alpha a = a\alpha f\beta a$ and $f = f\beta f = f\beta a\eta v = f\beta a\alpha e\eta v = f\beta a\alpha a\eta v = f\beta a\alpha f$. Thus $x \in V_\alpha^\beta(a)$. From $e = a = f\zeta u = x\zeta u$ and $x = f = a\eta v = e\eta v$, we deduce that $x\ \mathcal{R}\ e$. By assumption, $x\ \mathcal{R}\ e\ \mathcal{R}\ b\ \mathcal{L}\ x$. This implies that $x \in H_b$.

Case 3: $a \neq e$ and $a = f$. Set $x = e$. Then

$$a = a\alpha e = a\alpha t\delta a = a\alpha t\delta f\beta a = a\alpha t\delta a\beta a = a\alpha e\beta a$$

and

$$e = t\delta a = t\delta f\beta a = t\delta f\beta a\alpha e = t\delta a\beta a\alpha e = e\beta a\alpha e.$$

Thus $x \in V_\alpha^\beta(a)$. Since $f = a = s\gamma e = s\gamma x$ and $x = e = t\delta a = t\delta f$, we have $f\ \mathcal{L}\ x$. By assumption, $x\ \mathcal{R}\ b\ \mathcal{L}\ f \mathcal{L}\ x$. It follows that $x \in H_b$.

Case 4: $a \neq e$ and $a \neq f$. Set $x = e\eta v\beta a\alpha t\delta f$. Then

$$a = a\alpha e = f\beta a\alpha e = a\eta v\beta a\alpha t\delta a = a\alpha e\eta v\beta a\alpha t\delta f\beta a = a\alpha x\beta a$$

and

$$\begin{aligned} x\beta a\alpha x &= e\eta v\beta a\alpha t\delta f\beta a\alpha e\eta v\beta a\alpha t\delta f \\ &= e\eta v\beta a\alpha t\delta f\beta a\eta v\beta a\alpha t\delta f \\ &= e\eta v\beta a\alpha t\delta a\eta v\beta a\alpha t\delta f \\ &= e\eta v\beta a\alpha e\eta v\beta a\alpha t\delta f \\ &= e\eta v\beta a\eta v\beta a\alpha t\delta f \\ &= e\eta v\beta f\beta a\alpha t\delta f \\ &= e\eta v\beta a\alpha t\delta f = x. \end{aligned}$$

Hence $x \in V_\alpha^\beta(a)$. We claim that $x\ \mathcal{L}\ f$ and $x\ \mathcal{R}\ e$. Now

$$\begin{aligned} f &= a\eta v \\ &= a\alpha x\beta a\eta v \\ &= a\alpha e\eta v\beta a\alpha t\delta f\beta a\eta v \\ &= a\alpha e\eta v\beta a\alpha t\delta f\beta f \\ &= a\alpha e\eta v\beta a\alpha t\delta f \\ &= a\alpha x \end{aligned}$$

and we get $x = e\delta w\beta a\alpha u\gamma f$, so we have $x\ \mathcal{L}\ f$. Since

$$\begin{aligned} e &= t\delta a \\ &= t\delta a\alpha x\beta a \\ &= t\delta a\alpha e\eta v\beta a\alpha t\delta f\beta a \\ &= e\alpha e\eta v\beta a\alpha t\delta f\beta a \\ &= e\eta v\beta a\alpha t\delta f\beta a \\ &= x\beta a \end{aligned}$$

and $x = e\delta w\beta a\alpha u\gamma f$, we have $x\ \mathcal{R}\ e$. By assumption, we have $x\ \mathcal{R}\ e\ \mathcal{R}\ b\ \mathcal{L}\ f\ \mathcal{L}\ x$. This implies that $x \in H_b$.

Finally, we prove the uniqueness. Let $m, n \in V_\alpha^\beta(a) \cap H_b$. Then $a = a\alpha m\beta a$, $m = m\beta a\alpha m$, $a = a\alpha n\beta a$, $n = n\beta a\alpha n$, and $m, n \in H_b$. Thus $m\ \mathcal{H}\ b\ \mathcal{H}\ n$ and so $m\ \mathcal{H}\ n$. Suppose that $m = p\theta n$ and $m = n\vartheta q$ for some $p, q \in S, \theta, \vartheta \in \Gamma$. Then $m = p\theta n = p\theta n\beta a\alpha n = m\beta a\alpha n$ and $m = n\vartheta q = n\beta a\alpha n\vartheta q = n\beta a\alpha m$. Thus

$$\begin{aligned} m &= m\beta a\alpha m \\ &= n\beta a\alpha m\beta a\alpha m\beta a\alpha n \\ &= n\beta a\alpha m\beta a\alpha n \\ &= n\beta a\alpha n \\ &= n. \end{aligned}$$

Hence the (α, β)-inverse of a in H_b is unique. □

Lemma 3.3.2. *Let S be a Γ-semigroup, $a, b \in S$ and $\gamma \in \Gamma$. Then $a\gamma b \in R_a \cap L_b$ if and only if $L_a \cap R_b$ is a subgroup of S_γ.*

Proof. Necessity. Let $a\gamma b \in R_a \cap L_b$. Then $a\gamma b\ \mathcal{R}\ a$ and $a\gamma b\ \mathcal{L}\ b$. Since $a\gamma b\ \mathcal{R}\ a$, we have $a\gamma b = a$ or there exist $m, n \in S$ and $\zeta, \eta \in \Gamma$ such that

$$a = a\gamma b\zeta m \text{ and } a\gamma b = a\eta n. \tag{3.3.1}$$

Since $a\gamma b\ \mathcal{L}\ b$, we have $a\gamma b = b$ or there exist $o, p \in S$ and $\theta, \vartheta \in \Gamma$ such that

$$b = o\theta a\gamma b \text{ and } a\gamma b = p\vartheta b. \tag{3.3.2}$$

We shall now show that $L_a \cap R_b$ is a subgroup of S_γ. Consider

Case 1: $a = a\gamma b = b$. Then $a = a\gamma b = a\gamma a$, so $a \in E_\gamma(S)$. By Theorem 3.1.5, H_a is a subgroup of S_γ and also have $H_a = L_a \cap R_b$. It follows that $L_a \cap R_b$ is a subgroup of S_γ.

Case 2: $a\gamma b = a$ and (3.3.2). Then $b = o\theta a\gamma b = o\theta a$. Thus $b = o\theta a\gamma b = b\gamma b$, so $b \in E_\gamma(S)$ and also have $a\ \mathcal{L}\ b$. It follows that H_b is a subgroup of S_γ by Theorem 3.1.5. Since $a\ \mathcal{L}\ b$, we have $L_a = L_b$. Hence $H_b = L_b \cap R_b = L_a \cap R_b$. Therefore $L_a \cap R_b$ is a subgroup of S_γ.

Case 3: (3.3.2) and $a\gamma b = b$. Then $a = a\gamma b\zeta m = b\zeta m$. Thus $a = a\gamma b\zeta m = a\gamma a$, so $a \in E_\gamma(S)$ and also have $a\ \mathcal{R}\ b$, it follows that H_a is a subgroup of S_γ by Theorem 3.1.5. Since $a\ \mathcal{R}\ b$, we have $R_a = R_b$. Hence $H_a = L_a \cap R_a = L_a \cap R_b$. So $L_a \cap R_b$ is a subgroup of S_γ.

Case 4: (3.3.1) and (3.3.2). Set $e = b\zeta m$. Then $e = b\zeta m = o\theta a\gamma b\zeta m = o\theta a$. We see that $H_e = L_a \cap R_b$. From $e\gamma e = o\theta a\gamma b\zeta m = b\zeta m = e$, it follows that $e \in E_\gamma(S)$. By Theorem 3.1.5, we have H_e is a subgroup of S_γ and so $L_a \cap R_b$ is a subgroup of S_γ.

Sufficiency. Suppose now that $L_a \cap R_b$ is a subgroup of S_γ. Let e be an identity of $L_a \cap R_b$. Then $e \in L_a \cap R_b$. It follows that $e\ \mathcal{L}\ a$ and $e\ \mathcal{R}\ b$. Since $e\ \mathcal{L}\ a$, we have $e = a$ or there exist $i \in S, \zeta \in \Gamma$ such that $e = i\zeta a$. Since $e\ \mathcal{R}\ b$, we get $e = b$ or there exist $j \in S, \eta \in \Gamma$ such that $e = b\eta j$. Note that $e \in E_\gamma(S)$. Again from $e \in L_a \cap R_b$ it follows that $a = a\gamma e$ and $b = e\gamma b$ by Theorem 2.1.8 (1), (2). We consider the following cases:

Case 1: $a = e = b$. Since $a = a\gamma e = a\gamma e\gamma e = a\gamma b\gamma e$ and $a\gamma b = a\gamma b$, we have $a\gamma b\ \mathcal{R}\ a$. Since $a\gamma b = a\gamma b$ and $b = e\gamma b = e\gamma e\gamma b = e\gamma a\gamma b$, we botain $a\gamma b\ \mathcal{L}\ b$. Therefore $a\gamma b \in R_a \cap L_b$.

Case 2: $e = a$ and $e = b\eta j$. Since $a = a\gamma e = a\gamma b\eta j$ and $a\gamma b = a\gamma b$, we get

$a\gamma b \mathcal{R} a$. Since $a\gamma b = a\gamma b$ and $b = e\gamma b = e\gamma e\gamma b = e\gamma a\gamma b$, we have $a\gamma b \mathcal{L} b$. Hence $a\gamma b \in R_a \cap L_b$.

Case 3: $e = i\zeta a$ and $e = b$. Since $a = a\gamma e = a\gamma e\gamma e = a\gamma b\gamma e$ and $a\gamma b = a\gamma b$, we obtain $a\gamma b \mathcal{R} a$. Since $a\gamma b = a\gamma b$ and $b = e\gamma b = i\zeta a\gamma b$, we have $a\gamma b \mathcal{L} b$. Thus $a\gamma b \in R_a \cap L_b$.

Case 4: $e = i\zeta a$ and $e = b\eta j$. Since $a = a\gamma e = a\gamma b\eta j$ and $a\gamma b = a\gamma b$, we get $a\gamma b \mathcal{R} a$. Since $a\gamma b = a\gamma b$ and $b = e\gamma b = i\zeta a\gamma b$, we have $a\gamma b \mathcal{L} b$. It follows that $a\gamma b \in R_a \cap L_b$. This completes the proof. □

Lemma 3.3.3. *Let S be a Γ-semigroup and $\alpha, \beta, \gamma \in \Gamma$. If $e \in E_\alpha(S)$, $f \in E_\beta(S)$, $x \in V_\alpha^\beta(e\gamma f)$, $g = e\gamma f\alpha x\beta e$, and $h = f\alpha x\beta e\gamma f$, then the following two statements hold:*

(1) *$g \in E_\alpha(S)$ and $h \in E_\beta(S)$.*

(2) *$g \mathcal{R} e\gamma f = g\gamma h \mathcal{L} h$.*

Proof. (1) From hypothesis, we get

$$\begin{aligned} g\alpha g &= e\gamma f\alpha x\beta e\alpha e\gamma f\alpha x\beta e \\ &= e\gamma f\alpha x\beta e\gamma f\alpha x\beta e \\ &= e\gamma f\alpha x\beta e \\ &= g, \end{aligned}$$

so we have $g \in E_\alpha(S)$.

From

$$\begin{aligned} h\beta h &= f\alpha x\beta e\gamma f\beta f\alpha x\beta e\gamma f \\ &= f\alpha x\beta e\gamma f\alpha x\beta e\gamma f \\ &= f\alpha x\beta e\gamma f \\ &= h, \end{aligned}$$

it follows that $h \in E_\beta(S)$.

(2) We claim that $g\gamma h = e\gamma f$. Consider

$$e\gamma f = e\gamma f\alpha x\beta e\gamma f = e\gamma f\alpha x\beta e\gamma f\alpha x\beta e\gamma f = g\gamma h. \tag{3.3.3}$$

We get $g = e\gamma f\alpha x\beta e$ and (3.3.3), so we have $g\ \mathcal{R}\ e\gamma f$. Consider

$$h = f\alpha x\beta e\gamma f = f\alpha x\beta g\gamma h. \tag{3.3.4}$$

Note that $g\gamma h = g\gamma h$, and we get that (3.3.4), hence $g\gamma h\ \mathcal{L}\ h$. This implies that $g\ \mathcal{R}\ e\gamma f = g\gamma h\ \mathcal{L}\ h$. □

We shall give definitions of $(\mathcal{L}, \alpha, \beta)$-inverse of a, $(\mathcal{R}, \alpha, \beta)$-inverse of a and $(\mathcal{H}, \alpha, \beta)$-inverse of a, respectively.

Definition 3.3.4. Let S be a Γ-semigroup, $a, b \in S$ and $\alpha, \beta \in \Gamma$. Then we say that

(1) b is an $(\mathcal{L}, \alpha, \beta)$-inverse of a if $b \in V_L^{(\alpha,\beta)}(a) = V_\alpha^\beta(a) \cap L_a$,

(2) b is an $(\mathcal{R}, \alpha, \beta)$-inverse of a if $b \in V_R^{(\alpha,\beta)}(a) = V_\alpha^\beta(a) \cap R_a$,

(3) b is an $(\mathcal{H}, \alpha, \beta)$-inverse of a if $b \in V_H^{(\alpha,\beta)}(a) = V_\alpha^\beta(a) \cap H_a$.

The following an our main results in this section.

Theorem 3.3.5. *Let S be a Γ-semigroup, $a, b \in S$ and $\alpha, \beta \in \Gamma$. Then the following statements hold:*

(1) $V_L^{(\alpha,\beta)}(a) \cap V_L^{(\alpha,\beta)}(b) = \emptyset$ *or* $V_L^{(\alpha,\beta)}(a) = V_L^{(\alpha,\beta)}(b)$.

(2) $V_R^{(\alpha,\beta)}(a) \cap V_R^{(\alpha,\beta)}(b) = \emptyset$ *or* $V_R^{(\alpha,\beta)}(a) = V_R^{(\alpha,\beta)}(b)$.

(3) $V_H^{(\alpha,\beta)}(a) \cap V_H^{(\alpha,\beta)}(b) = \emptyset$ *or* $V_H^{(\alpha,\beta)}(a) = V_H^{(\alpha,\beta)}(b)$.

Proof. (1) Suppose that $V_L^{(\alpha,\beta)}(a) \cap V_L^{(\alpha,\beta)}(b) \neq \emptyset$, and let $c \in V_L^{(\alpha,\beta)}(a) \cap V_L^{(\alpha,\beta)}(b)$, and $d \in V_L^{(\alpha,\beta)}(a)$. Then $c = c\beta a\alpha c$, $a = a\alpha c\beta a$, $c = c\beta b\alpha c$, $b = b\alpha c\beta b$, $d = d\beta a\alpha d$, $a = a\alpha d\beta a$, and $a\ \mathcal{L}\ b\ \mathcal{L}\ c\ \mathcal{L}\ d$. It follows that $a\ \mathcal{L}\ b$ and $c\ \mathcal{L}\ d$. Thus we consider the following:

If $a = b$ or $c = d$, then we have

$$d = d\beta a\alpha d = d\beta b\alpha d \text{ and } b = a = a\alpha d\beta a = b\alpha d\beta b \tag{3.3.5}$$

or

$$d = c = c\beta b\alpha c = d\beta b\alpha d \text{ and } b = b\alpha c\beta b = b\alpha d\beta b. \tag{3.3.6}$$

Thus by (3.3.5) or (3.3.6), we obtain that $d \in V_\alpha^\beta(b)$.

Suppose that $a \neq b$ and $c \neq d$. Then there exist $s, t, u, v \in S, \gamma, \delta, \zeta, \eta \in \Gamma$ such that $a = s\gamma b$, $b = v\eta a$, $c = u\zeta d$ and $d = t\delta c$. Now

$$\begin{aligned} d &= d\beta a\alpha d \\ &= d\beta s\gamma b\alpha d \\ &= d\beta s\gamma b\alpha c\beta b\alpha d \\ &= d\beta a\alpha c\beta b\alpha d \\ &= t\delta c\beta a\alpha c\beta b\alpha d \\ &= t\delta c\beta b\alpha d \\ &= d\beta b\alpha d \end{aligned}$$

and

$$\begin{aligned} b &= b\alpha c\beta b \\ &= b\alpha u\zeta d\beta b \\ &= b\alpha u\zeta d\beta a\alpha d\beta b \\ &= b\alpha c\beta a\alpha d\beta b \\ &= v\eta a\alpha c\beta a\alpha d\beta b \\ &= v\eta a\alpha d\beta b \\ &= b\alpha d\beta b. \end{aligned}$$

Hence $d \in V_\alpha^\beta(b)$. Therefore $V_L^{(\alpha,\beta)}(a) \subseteq V_L^{(\alpha,\beta)}(b)$. In the same way, we can prove that $V_L^{(\alpha,\beta)}(b) \subseteq V_L^{(\alpha,\beta)}(a)$. It follows that $V_L^{(\alpha,\beta)}(a) = V_L^{(\alpha,\beta)}(b)$.

Similarly, we can prove that (2) and (3) hold. □

The next theorem provides the existence of any kind of these inverses for an element of S.

Theorem 3.3.6. *Let S be a Γ-semigroup and $a \in S$. Then the following statements are equivalent:*

(1) *a is an (α, β)-completely regular element of S for some $\alpha, \beta \in \Gamma$.*

(2) *a has an $(\mathcal{H}, \gamma, \delta)$-inverse for some $\gamma, \delta \in \Gamma$.*

(3) *a has an $(\mathcal{L}, \zeta, \eta)$-inverse for some $\zeta, \eta \in \Gamma$.*

(4) *a has an $(\mathcal{R}, \theta, \vartheta)$-inverse for some $\theta, \vartheta \in \Gamma$.*

Proof. Obviously, (2) implies (3) and (2) implies (4).

(3)$\Rightarrow$(1) Let $x \in S$, $\zeta, \eta \in \Gamma$ be such that $x \in V_L^{(\zeta,\eta)}(a)$. Then $a = a\zeta x\eta a$, $x = x\eta a\zeta x$ and $x \in L_a$. Set $e = a\zeta x$. Then we have $e \in E_\eta(S)$. By Theorem 3.1.5, we have H_e is a subgroup of S_η. Since $a \ \mathcal{L} \ x$, we have $a = x$ or there exist $u, v \in S$ and $\lambda, \mu \in \Gamma$ such that $a = u\lambda x$ and $x = v\mu a$. Thus we consider the following two cases:

Case 1: $a = x$. Then it is easy to see that $a \in H_e$. By Theorem 3.1.7, we have a is an (α, β)-completely regular element of S for some $\alpha, \beta \in \Gamma$.

Case 2: $a \neq x$. Then $a = u\lambda x = u\lambda x\eta a\zeta x = u\lambda x\eta e$ and $e = a\zeta x = a\zeta v\mu a$. It follows that $a \ \mathcal{L} \ e$. Clearly, $a \ \mathcal{R} \ e$. Therefore $e \ \mathcal{H} \ a$, and so $a \in H_e$. By Theorem 3.1.7, we obtain that a is an (α, β)-completely regular element of S for some $\alpha, \beta \in \Gamma$. Similarly, we can prove that (4) implies (1) hold.

(1)$\Rightarrow$(2) Assume that a is an (α, β)-completely regular element of S for some $\alpha, \beta \in \Gamma$. Then by Theorem 3.1.8 (1), we obtain H_a is a subgroup of S_α and S_β with the same identity and the same inverse of a. Thus

$$a\alpha x = x\alpha a = a\beta x = x\beta a = e \quad \text{for some} \quad x \in H_a,$$

where e is the identity of H_a. From

$$a = a\alpha e = a\alpha x\beta a \text{ and } x = x\beta e = x\beta a\alpha x,$$

so we have $x \in V_H^{(\alpha,\beta)}(a) \cap H_a$. $\square$

Corollary 3.3.7. *Let S be a Γ-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. If a is an (α, β)-completely regular element of S, then the following statements hold:*

(1) *a has an $(\mathcal{H}, \alpha, \beta)$-inverse.*

(2) *a has an $(\mathcal{L}, \alpha, \beta)$-inverse.*

(3) *a has an $(\mathcal{R}, \alpha, \beta)$-inverse.*

Proof. In the proof of Theorem 3.3.6 ((1) $\Rightarrow$ (2)), we have (1). As a consequence, we obtain (2) and (3) hold. □

As a consequence fix elements α, β in Γ, we have the existence of any kind of these (α, β)-inverses for an element a of S if a is an (α, β)-completely regular element of S. For an element of S, the next theorem provides the sets of all such inverses.

Theorem 3.3.8. *Let S be a Γ-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. If a is an (α, β)-completely regular element of S, then the following statements hold:*

(1) $V_L^{(\alpha,\beta)}(a) = \{f\gamma x \mid f \in E_\alpha(L_a), \gamma \in \{\alpha, \beta\}\}$

$$= \{q \in S \mid a = a\alpha(q\delta a) = (a\delta a)\alpha q,\ q = (q\delta a)\alpha q = (q\alpha q)\delta a, \text{ where } \delta \in \{\alpha, \beta\}\},$$

(2) $V_R^{(\alpha,\beta)}(a) = \{x\gamma f \mid f \in E_\beta(R_a), \gamma \in \{\alpha, \beta\}\}$

$$= \{r \in S \mid a = (a\zeta r)\beta a = r\beta(a\zeta a),\ r = r\beta(a\zeta r) = a\zeta(r\beta r), \text{ where } \zeta \in \{\alpha, \beta\}\},$$

(3) $V_H^{(\alpha,\beta)}(a) = \{x\}$

$$= \{s \in S \mid a = a\eta s\eta a,\ s = s\eta a\eta s,\ a\eta s = s\eta a, \text{ where } \eta \in \{\alpha, \beta\}\},$$

where x is both an inverse of a in H_a of S_α and S_β.

Proof. Let a be an (α, β)-completely regular element. Then by Theorem 3.1.8 (1), we have H_a is a subgroup of S_α and S_β with the same identity e and the same inverse x of a. Thus we obtain

$$a\alpha x = x\alpha a = a\beta x = x\beta a = e.$$

(1) Let $y \in V_L^{(\alpha,\beta)}(a)$. Then $a = a\alpha y\beta a$, $y = y\beta a\alpha a$ and $y \in L_a$. Set $f = y\beta a$. Then $a\ \mathcal{L}\ f$ and $y\ \mathcal{R}\ f$. It follows that $L_a = L_f$ and $R_y = R_f$, respectively. Note that $f \in E_\alpha(S)$. We claim that $H_a = L_f \cap R_x$. Since $a\ \mathcal{R}\ x$, we have $R_a = R_x$. Hence $L_f \cap R_x = L_a \cap R_a = H_a$ is a subgroup of S_α and S_β. By Lemma 3.3.2, $f\alpha x,\ f\beta x \in R_f \cap L_x$. We claim that $H_y = R_f \cap L_x$. Since $y\ \mathcal{L}\ a\ \mathcal{L}\ x$, we have $y\ \mathcal{L}\ x$ and so $L_y = L_x$. So we get $H_y = R_y \cap L_y = R_f \cap L_x$. Therefore $f\alpha x, f\beta x \in H_y$. Now

$$f\beta x\beta a\alpha f\beta x = f\beta x\beta a\beta x = f\beta e\beta x = f\beta x$$

and

$$a\alpha f\beta x\beta a = a\beta x\beta a = e\beta a = a.$$

Hence $f\beta x \in V_\alpha^\beta(a) \cap H_y$. In the same way we have $f\alpha x \in V_\alpha^\beta(a) \cap H_y$. It follows that $y, f\gamma x \in V_\alpha^\beta(a) \cap H_y$, where $\gamma \in \{\alpha, \beta\}$. Since $a\ \mathcal{L}\ f\ \mathcal{R}\ y\ \mathcal{L}\ e\ \mathcal{R}\ a$, we get $|V_\alpha^\beta(a)| = 1$ by Lemma 3.3.1. Thus $y = f\gamma x$.

Let $q = f\gamma x$, where $f \in E_\alpha(L_a)$, $\gamma \in \{\alpha, \beta\}$, and x is both an inverse of a in H_a of S_α and S_β. We consider the following:

$$q\beta a\alpha q = f\gamma x\beta a\alpha f\gamma x = f\gamma x\beta a\gamma x = f\gamma e\gamma x = f\gamma x = q$$

and

$$a\alpha q\beta a = a\alpha f\gamma x\beta a = a\gamma x\beta a = a\gamma e = a.$$

Hence $q \in V_\alpha^\beta(a)$. And then

$$a\gamma a\alpha q = a\gamma a\alpha f\gamma x = a\gamma a\gamma x = a\gamma e = a$$

and

$$q\alpha q\gamma a = f\gamma x\alpha f\gamma x\gamma a = f\gamma x\alpha f\gamma e = f\gamma x\gamma e = f\gamma x = q.$$

Obviously, $\{q \in S \mid a = a\alpha(q\delta a) = (a\delta a)\alpha q,\ q = (q\delta a)\alpha q = (q\alpha q)\delta a$, where $\delta \in \{\alpha, \beta\}\} \subseteq V_L^{(\alpha,\beta)}(a)$, so we can conclude that it satisfies the requisite conditions.

(2) Let $y \in V_R^{(\alpha,\beta)}(a)$. Then $a = a\alpha y\beta a$, $y = y\beta a\alpha a$ and $y \in R_a$. Set $f = a\alpha y$. Then $a\ \mathcal{R}\ f$ and $y\ \mathcal{L}\ f$. It follows that $R_a = R_f$ and $L_y = L_f$, respectively. Note that $f \in E_\beta(S)$. We claim that $H_a = L_x \cap R_f$. Since $a\ \mathcal{L}\ x$, we have $L_a = L_x$. Hence $L_x \cap R_f = L_a \cap R_a = H_a$ is a subgroup of S_α and S_β. By Lemma (3.3.2), $x\alpha f, x\beta f \in R_x \cap L_f$. We claim that $H_y = R_x \cap L_f$. Since $y\ \mathcal{R}\ a\ \mathcal{R}\ x$, we have $y\ \mathcal{R}\ x$ and so $R_y = R_x$. So we have $H_y = R_y \cap L_y = R_x \cap L_f$. Therefore $x\alpha f, x\beta f \in H_y$. Now

$$x\beta f\beta a\alpha x\beta f = x\beta a\alpha x\beta f = e\alpha x\beta f = x\beta f$$

and

$$a\alpha x\beta f\beta a = a\alpha x\beta a = a\alpha e = a.$$

Hence $x\beta f \in V_\alpha^\beta(a) \cap H_y$. In the same way we have $x\alpha f \in V_\alpha^\beta(a) \cap H_y$. It follows that $y, x\gamma f \in V_\alpha^\beta(a) \cap H_y$, where $\gamma \in \{\alpha, \beta\}$. Since $a\ \mathcal{L}\ e\ \mathcal{R}\ y\ \mathcal{L}\ f\ \mathcal{R}\ a$, we have $|V_\alpha^\beta(a)| = 1$ by Lemma 3.3.1. Thus $y = x\gamma f$.

Let $r = x\gamma f$, where $f \in E_\beta(R_a)$, $\gamma \in \{\alpha, \beta\}$, and x is both an inverse of a in H_a of S_α and S_β. We consider the following:

$$r\beta a\alpha r = x\gamma f\beta a\alpha x\gamma f = x\gamma a\alpha x\gamma f = x\gamma e\gamma f = x\gamma f = r$$

and

$$a\alpha r\beta a = a\alpha x\gamma f\beta a = a\alpha x\gamma a = a\alpha e = a.$$

Hence $r \in V_\alpha^\beta(a)$. And

$$r\beta a\gamma a = x\gamma f\beta a\gamma a = x\gamma a\gamma a = e\gamma a = a$$

and

$$a\gamma r\beta r = a\gamma x\gamma f\beta x\gamma f = a\gamma x\gamma x\gamma f = e\gamma x\gamma f = x\gamma f = r.$$

Obviously, $\{r \in S \mid a = (a\zeta r)\beta a = r\beta(a\zeta a),\ r = r\beta(a\zeta r) = a\zeta(r\beta r), where\ \zeta \in \{\alpha,\beta\}\} \subseteq V_R^{(\alpha,\beta)}(a)$, so we can conclude that it satisfies the requisite conditions.

It is easy to see that the final condition holds. □

Corollary 3.3.9. *Let S be a Γ-semigroup and $\alpha,\beta \in \Gamma$. If $e \in E_\alpha(S) \cap E_\beta(S)$, then $V_L^{(\alpha,\beta)}(e) = E_\alpha(L_e)$ and $V_R^{(\alpha,\beta)}(e) = E_\beta(R_e)$.*

Proof. Let $e \in E_\alpha(S) \cap E_\beta(S)$. Consider

Let $q \in V_L^{(\alpha,\beta)}(e)$. Then $e = e\alpha q\beta e$, $q = q\beta e\alpha q$ and $e\ \mathcal{L}\ q$. By Theorem 2.1.8 (1), $q = q\beta e$. Thus we have $q = q\beta e\alpha q = q\alpha q$ and so $q \in E_\alpha(S)$. Therefore $q \in E_\alpha(L_a)$. Conversely, let $r \in E_\alpha(L_e)$. By Theorem 2.1.8 (1), we have

$$r = r\alpha r,\ r = r\beta e \text{ and } e = e\alpha r.$$

From $r = r\beta e = r\beta e\alpha r$ and $e = e\alpha r = e\alpha r\beta e$, so we have $r \in V_\alpha^\beta(e)$. By hypothesis, $r \in L_e$, and hence $r \in V_L^{(\alpha,\beta)}(e)$. Therefore $V_L^{(\alpha,\beta)}(e) = E_\alpha(L_e)$.

Finally, let $s \in V_R^{(\alpha,\beta)}(e)$. Then $e = e\alpha s\beta e$, $s = s\beta e\alpha s$, and $e\ \mathcal{R}\ s$. By Theorem 2.1.8(2), $s = e\alpha s$. Thus $s = s\beta e\alpha s = s\beta s$. It follows that $s \in E_\beta(S)$. Therefore $s \in E_\beta(R_e)$. Conversely, let $t \in E_\beta(R_e)$. By Theorem 2.1.8 (2), we have

$$t = t\beta t,\ t = e\alpha t \text{ and } e = t\beta e.$$

Since $t = e\alpha t = t\beta e\alpha t$ and $e = t\beta e = e\alpha t\beta e$, we have $t \in V_\alpha^\beta(e)$. By hypothesis, $t \in R_e$, so we have $t \in V_R^{(\alpha,\beta)}(e)$. This implies that $V_e^{(\alpha,\beta)}(e) = E_\beta(R_e)$. □