CHAPTER II

PRELIMINARIES

In this chapter we give precise definitions, notations and basic results which will be used in thesis.

2.1 Basic results.

We start by giving the definitions of Γ-semigroups as follows:

Definition 2.1.1. [4] Let S and Γ be two non-empty sets. S is called a Γ-*semigroup* if there exists a mapping $S \times \Gamma \times S \to S$, written as $(a, \gamma, b) \longmapsto a\gamma b$, satisfies the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. We see that Γ-semigroups generalize semigroups. For examples, see [9], [10] and [11].

Let S be a Γ-semigroup. For any nonempty subsets A and B of S, let

$$A\Gamma B := \{a\alpha b \mid a \in A, b \in B \text{ and } \alpha \in \Gamma\}[9].$$

Definition 2.1.2. [17] Let S be a Γ-semigroup. An element $a \in S$ is called *regular* if $a \in a\Gamma S\Gamma a$, where

$$a\Gamma S\Gamma a := \{a\alpha b\beta a \mid b \in S \text{ and } \alpha, \beta \in \Gamma\}.$$

Definition 2.1.3. [17] A Γ-semigroup S is called a *regular* Γ-semigroup if every element of S is regular.

Definition 2.1.4. [8] Let S be a Γ-semigroup and $a \in S$ and $\alpha, \beta \in \Gamma$. An element x of S is said to be an (α, β)-*inverse of* a if $a = a\alpha x\beta a$ and $x = x\beta a\alpha x$.

The set of all (α, β)-inverses of a is denoted by $V_\alpha^\beta(a)$. Thus

$$V_\alpha^\beta(a) := \{x \in S \mid x = x\beta a\alpha x \textit{ and } a = a\alpha x\beta a\}.$$

Definition 2.1.5. [8] Let S be a Γ-semigroup. An element $e \in S$ is said to be an *α-idempotent* of S, where $\alpha \in \Gamma$ if $e\alpha e = e$.

We denote the set of all α-idempotents of S by $E_\alpha(S)$, that is

$$E_\alpha(S) := \{e \in S \mid e\alpha e = e\}.$$

We denote $\bigcup_{\alpha\in\Gamma} E_\alpha(S)$ by $E(S)$.

Definition 2.1.6. [8] The Green's equivalence relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ on a Γ-semigroup S are defined by the following rules:

(1) $a \ \mathcal{L} \ b$ if and only if $S\Gamma a \cup \{a\} = S\Gamma b \cup \{b\}$.

(2) $a \ \mathcal{R} \ b$ if and only if $a\Gamma S \cup \{a\} = b\Gamma S \cup \{b\}$.

(3) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

The $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{H}$-class containing the element a will be denoted by L_a, R_a and H_a, respectively.

Remark 2.1.7. [8, p. 188] *Let S be a Γ-semigroup. Then for all $a, b \in S$, we have*

(1) $a \ \mathcal{L} \ b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$.

(2) $a \ \mathcal{R} \ b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = b\alpha x$ and $b = a\beta y$.

(3) $a \ \mathcal{H} \ b$ if and only if $a \ \mathcal{L} \ b$ and $a \ \mathcal{R} \ b$.

Theorem 2.1.8. [8, p. 189] *Let S be a Γ-semigroup, $\alpha \in \Gamma$ and e be an α-idempotent. Then the following statements hold:*

(1) $a\alpha e = a$ for all $a \in L_e$.

(2) $e\alpha a = a$ for all $a \in R_e$.

(3) $a\alpha e = a = e\alpha a$ for all $a \in H_e$.

(4) For all $a \in S$, $|H_a \cap E_\alpha| \leq 1$.