CHAPTER VII

CONCLUSION

The following results are all main theorems of this dissertation:

1. Let $\mathcal{S} = (T(s))_{s\geq 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \to H$ be an α-contraction, let $A : H \to H$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$ and let γ be a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $G : H \times H \to \mathbb{R}$ be a mapping satisfying hypotheses (E1)-(E4) and $\Psi : H \to H$ an inverse-strongly monotone mappings with coefficients δ such that $F(\mathcal{S}) \cap GEP(G, \Psi) \neq \emptyset$. Let the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_1 \in H \text{ chosen arbitrary}, \\ G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ y_n = \beta_n x_n + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \forall n \geq 1, \end{cases}$$

where the real sequences $\{r_n\} \subset (0, 2\delta)$, $\{s_n\} \subset (0, \infty)$ and $\{\alpha_n\}$, $\{\beta_n\}$ in $(0, 1)$ satisfy the following conditions:

(D1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$,

(D2) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(D3) $\lim_{n\to\infty} s_n = +\infty$, $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$, and

(D4) $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0$.

Then the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to z which is a unique solution in $F(\mathcal{S}) \cap GEP(G, \Psi)$ of the variational inequality

$$\langle(\gamma f - A)z, p - z\rangle \leq 0, \quad \forall p \in F(\mathcal{S}) \cap GEP(G, \Psi).$$

2. Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $S_1, S_2, \ldots, S_N : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{1 + k_{p(n)}^{i(n)}\}$ respectively, such that $k_{p(n)}^{i(n)} \to 0$ as $n \to \infty$, $h_n := \max_{1 \leq i(n) \leq N}\{k_{p(n)}^{i(n)}\}$ and $\Gamma := \cap_{i=1}^{N} F(S_i)$,

$$\Gamma = F(S_N S_{N-1} S_{N-2} \ldots S_1) = F(S_1 S_N \ldots S_2) = \cdots = F(S_{N-1} S_{N-2} \cdots S_1 S_N).$$

Let $A : C \to H$ be an α-inverse strongly monotone mapping. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4) such that $\Omega := EP \cap \Gamma$ is nonempty. Let $F : C \to H$ be δ-strongly monotone and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, $f : C \to H$ a ρ-contraction, γ a positive real number such that $\gamma < (1 - \sqrt{(1-\delta)/\lambda})/\rho$ and r a constant such that $r \in (0, 2\alpha)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by (3.2.2). Suppose that $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{h_n}{\alpha_n} = 0$;

(C2) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$ and $\lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\alpha_{n+1}} = 0$.

Assume that $\sum_{n=1}^{\infty} \sup_{z \in B} \|S_{i(n+1)}^{p(n+1)} z - S_{i(n)}^{p(n)} z\| < \infty$, for each bounded subset B of C. Then, the sequence $\{x_n\}$ converges strongly to x^* of the following variational inequality

$$\langle(F - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in \Omega$$

or equivalently $\tilde{x} = P_\Omega(I - F + \gamma f)\tilde{x}$, where P_Ω is the metric projection of H onto Ω.

3. Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in F$. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from C into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap VI(A, C)$. Assume that T_i and S_i are uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algolithm:

$$\begin{cases} x_0 = x \in C, \text{ chosen arbitrary,} \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT_n x_n + \gamma_n JS_n w_n), \\ y_n = J^{-1}(\delta_n Jx_1 + (1 - \delta_n)Jz_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1 - \delta_n)[\alpha_n \phi(u, x_{n-1}) + (1 - \alpha_n)\phi(u, x_n)]\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where $T_n = T_{n(\text{mod } N)}, S_n = S_{n(\text{mod } N)}$, and J is the normalized duality mapping on E. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0, 1]$ satisfying the restrictions:

(C1) $\lim_{n\to\infty} \delta_n = 0$;

(C2) $r_n \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ and if one of the following conditions is satisfied

(a) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$ and $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$ and

(b) $\lim_{n\to\infty} \alpha_n = 0$ and $\liminf_{n\to\infty} \beta_n \gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from C onto F.

4. Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. Assume that A is a continuous operator of C into E^* satisfying conditions (2.3.4) and (2.3.5) and $S, T : C \to C$ are relatively weak nonexpansive mappings with $F := F(S) \cap F(T) \cap VI(A,C) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 = x \in C \text{chosen arbitrary}, C_0 = C, \\ z_n = \Pi_C(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)J\Pi_C(Jz_n - \beta Az_n)), \\ u_n \in C \ \ \textit{such that}\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} Jx \quad \forall n \geq 0. \end{cases}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;

(C2) $0 \leq \delta_n < 1, \limsup_{n\to\infty} \delta_n < 1$;

(C3) $\{r_n\} \subset [a, \infty)$ for some $a > 0$; and

(C4) $\liminf_{n\to\infty} \alpha_n\beta_n > 0, \liminf_{n\to\infty} \alpha_n\gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x$.

5. Let E be a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to$

E a L_i-Lipchitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, \frac{d_i - c_i L_i^2}{K^2 L_i^2})$, respectively for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of uniformly ε-strict pseudo-contractions. Define a mapping $S_n : E \to E$ by

$$S_n x = (1 - \frac{\varepsilon}{K^2})x + \frac{\varepsilon}{K^2} T_n x \text{ for all } x \in C \text{ and } n \geq 1.$$

Assume that $\Omega := \cap_{n=1}^{\infty} F(T_n) \cap F(Q) \neq \emptyset$, where Q is defined as Lemma 5.1.3. Let $f : E \to E$ be an α-contraction, let $A : E \to E$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$ with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $x_1 = u \in E$ and $\{x_n\}$ a sequence generated by

$$\begin{cases} z_n = J_{(M_2,\rho_2)}(x_n - \rho_2 \Psi_2 x_n), \\ y_n = J_{(M_1,\rho_1)}(z_n - \rho_1 \Psi_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)[\mu S_n x_n + (1 - \mu) y_n], \quad \forall n \geq 1, \end{cases}$$

where $\mu \in (0, 1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Suppose that $\{S_n\}$ satisfies AKTT-condition. Let $S : E \to E$ be the mapping defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in E$ and suppose that $F(S) = \cap_{n=1}^{\infty} F(S_n)$. If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z) \rangle \leq 0, z \in \Omega,$$

and $(\tilde{x}, \tilde{y})$ is a solution of general system of variational inequality problem (5.1.2) such that $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2 \Psi_2 \tilde{x})$.

6. Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_{φ}, and C a nonempty closed convex subset of E. Let $\{T_n : C \to C\}$ be a family of uniformly λ-strict pseudo-contractions with respect to

p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0, 1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy the following conditions:

1. $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$;
2. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$;
3. $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \xi$, where $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ be the sequence generated by the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \geq 1. \end{cases}$$

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality :

$$\langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T).$$

7. Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed $\eta - \xi$ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1), (A3) and (A4) and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and $z \in C$. Assume that

(i) $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$;

(ii) for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x, u)\rangle$ is convex and lower semicontinuous;

(iii) $\xi : E \to \mathbb{R}$ is weakly lower semicontinuous; that is, for any net $\{x_\beta\}, x_\beta$ converges to x in $\sigma(E, E^*)$ which implies that $\xi(x) \leq \liminf \xi(x_\beta)$.

Then, the solution set of the problem (6.1.2) is nonempty; that is, there exists $x_0 \in C$ such that

$$f(x_0, y) + \langle Tx_0, \eta(y, x_0)\rangle + \varphi(y) + \frac{1}{r}\langle y - x_0, J(x_0 - z)\rangle \geq \varphi(x_0), \quad \forall y \in C.$$

8. Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T : C \to E^*$ be an η-hemicontinuous and relaxed $\eta - \xi$ monotone mapping and let $\{S_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(S_n) \cap EP(f, T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C generated by

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - S_n z\| \leq t_n \|x_n - S_n x_n\|\}, \quad n \geq 0, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + \langle Tu_n, \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0, 1)$ and $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0, 1)$ and $\liminf_{n\to\infty} r_n > 0$.

Assume that $\{S_n\}_{n=0}^{\infty}$ satisfy the NST-condition. Then, the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

9. Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), A an α-inverse strongly monotone mapping of C into E^* and $\{T_n\}_{n=0}^{\infty}$ a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap GEP(f) \neq \emptyset$ and suppose that $\{T_n\}_{n=0}^{\infty}$ satisfies the NST-condition. Let $\{x_n\}$ be the sequence in C generated by

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - T_n z\| \leq t_n \|x_n - T_n x_n\|\}, \quad n \geq 1, \\ u_n \in C \ \ \textit{such that} \ f(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq 0, \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0,1)$ and $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.