CHAPTER VI

EXISTENCE AND ITERATIVE APPROXIMATION FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN BANACH SPACES

6.1 Existence theorems and iterative approximation methods for generalized mixed equilibrium problems for a countable family of nonexpansive mappings

In this section, we introduce the new generalized mixed equilibrium problem basing on hemicontinuous and relaxed monotonic mapping.

For solving the mixed equilibrium problem, let us assume the following conditions for a bifunction f:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e. $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $y \in C$, $f(\cdot, y)$ is weakly upper semicontinuous;

(A4) for all $x \in C$, $f(x, \cdot)$ is convex.

Let $f : C \times C \to \mathbb{R}$ be a bifunction and $\varphi : C \to \mathbb{R}$ a real-valued function. We consider the following new generalized mixed equilibrium problem :

$$\text{Find } x \in C \text{ such that } f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{6.1.1}$$

The set of such $x \in C$ is denoted by $EP(f,T)$, i.e.,

$$EP(f,T) = \{x \in C : f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) \geq \varphi(x), \ \forall y \in C\}.$$

Using the KKM technique, we obtain the existence of solutions for the generalized mixed equilibrium problem in a Banach space. Furthermore, we also introduce a hybrid projection algorithm for finding a common element in the solution set of a generalized mixed equilibrium problem and the common fixed point set of a countable family of nonexpansive mappings.

Lemma 6.1.1. *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed $\eta - \xi$ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) and (A4) and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and $z \in C$. Assume that*

(i) $\eta(x,x) = 0, \quad \forall x \in C.$

(ii) for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x,u)\rangle$ is convex.

Then the following problems (6.1.2) and (6.1.3) are equivalent:
Find $x \in C$ such that

$$f(x,y) + \varphi(y) + \langle Tx, \eta(y,x)\rangle + \frac{1}{r}\langle y - x, J(x-z)\rangle \geq \varphi(x),\ \forall y \in C; \quad (6.1.2)$$

Find $x \in C$ such that

$$f(x,y) + \langle Ty, \eta(y,x)\rangle + \varphi(y) + \frac{1}{r}\langle y - x, J(x-z)\rangle \geq \varphi(x) + \xi(y-x),\ \forall y \in C. \quad (6.1.3)$$

Proof. Let $x \in C$ be a solution of the problem (6.1.2). Since T is relaxed $\eta - \xi$ monotone, we have

$$\begin{aligned} & f(x,y) + \langle Ty, \eta(y,x)\rangle + \varphi(y) + \frac{1}{r}\langle y - x, J(x-z)\rangle \\ \geq\ & f(x,y) + \xi(y-x) + \varphi(y) + \frac{1}{r}\langle y - x, J(x-z)\rangle + \langle Tx, \eta(y,x)\rangle \\ \geq\ & \varphi(x) + \xi(y-x), \quad \forall y \in C. \end{aligned} \quad (6.1.4)$$

Thus $x \in C$ is a solution of the problem (6.1.3).

Conversely, let $x \in C$ be a solution of the problem (6.1.3). Letting

$$y_t = (1-t)x + ty, \quad \forall t \in (0,1), \tag{6.1.5}$$

then $y_t \in C$. Since $x \in C$ is a solution of the problem (6.1.3), it follows that

$$\begin{aligned} & f(x, y_t) + \langle Ty_t, \eta(y_t, x)\rangle + \varphi(y_t) + \frac{1}{r}\langle y_t - x, J(x-z)\rangle \\ \geq\ & \varphi(x) + \xi(y_t - x) \\ =\ & \varphi(x) + t^p \xi(y - x). \end{aligned} \tag{6.1.6}$$

Using (i), (ii), (A1) and (A4), we have

$$\begin{aligned} \langle Ty_t, \eta(y_t, x)\rangle &\leq (1-t)\langle Ty_t, \eta(x,x)\rangle + t\langle Ty_t, \eta(y,x)\rangle \\ &= t\langle T(x + t(y-x)), \eta(y,x)\rangle \end{aligned} \tag{6.1.7}$$

and

$$f(x, y_t) \leq (1-t)f(x,x) + tf(x,y) = tf(x,y). \tag{6.1.8}$$

The convexity of the function φ implies that

$$\varphi(y_t) = \varphi((1-t)\alpha + ty) \leq (1-t)\varphi(x) + t\varphi(y). \tag{6.1.9}$$

It follows from (6.1.6)-(6.1.9) that

$$\begin{aligned} tf(x,y) + t\langle T(x + t(y-x)), \eta(y,x)\rangle + t\varphi(y) \geq\ & f(x, y_t) + \langle Ty_t, \eta(y_t, x)\rangle + \varphi(y_t) \\ & - (1-t)\varphi(x). \end{aligned} \tag{6.1.10}$$

Moreover, we observe that

$$\begin{aligned} \frac{1}{r}\langle y_t - x, J(x-z)\rangle &= \frac{1}{r}\langle (1-t)x + ty - x, J(x-z)\rangle \\ &= \frac{1}{r}\langle t(y-x), J(x-z)\rangle \end{aligned}$$

$$= \frac{t}{r}\langle y - x, J(x - z)\rangle. \tag{6.1.11}$$

From (6.1.6) and (6.1.10), we obtain

$$\begin{aligned} & tf(x,y) + t\langle T(x + t(y-x)), \eta(y,x)\rangle + t\varphi(y) + \frac{t}{r}\langle y - x, J(x-z)\rangle \\ \geq\ & f(x, y_t) + \langle Ty_t, \eta(y_t, x)\rangle + \varphi(y_t) \\ & -(1-t)\varphi(x) + \frac{1}{r}\langle y_t - x, J(x-z)\rangle \\ \geq\ & \varphi(x) + t^p\xi(y-x) - (1-t)\varphi(x) \\ =\ & t\varphi(x) + t^p\xi(y-x). \end{aligned} \tag{6.1.12}$$

Hence

$$\begin{aligned} & f(x,y) + \langle T(x + t(y-x)), \eta(y,x)\rangle + \varphi(y) + \frac{1}{r}\langle y - x, J(x-z)\rangle \\ \geq\ & \varphi(x) + t^{p-1}\xi(y-x). \end{aligned} \tag{6.1.13}$$

Since T is η-hemicontinuous and $p > 1$, taking $t \to 0$ in (6.1.13), we get

$$f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) + \frac{1}{r}\langle y - x, J(x-z)\rangle \geq \varphi(x), \quad \forall y \in C.$$

Therefore, x is also a solution of the problem (6.1.2). This completes the proof. □

Next we use the concept of KKM mapping to prove the following two lemmas for our main result.

Lemma 6.1.2. *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed $\eta - \xi$ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1), (A3) and (A4) and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and $z \in C$. Assume that*

(i) $\eta(x,y) + \eta(y,x) = 0$ *for all* $x, y \in C$;

(ii) for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x,u)\rangle$ is convex and lower semicontinuous;

(iii) $\xi : E \to \mathbb{R}$ is weakly lower semicontinuous; that is, for any net $\{x_\beta\}, x_\beta$ converges to x in $\sigma(E, E^*)$ which implies that $\xi(x) \leq \liminf \xi(x_\beta)$.

Then, the solution set of the problem (6.1.2) is nonempty; that is, there exists $x_0 \in C$ such that

$$\begin{aligned} & f(x_0, y) + \langle Tx_0, \eta(y, x_0)\rangle + \varphi(y) + \frac{1}{r}\langle y - x_0, J(x_0 - z)\rangle \\ \geq\ & \varphi(x_0), \ \forall y \in C. \end{aligned} \tag{6.1.14}$$

Proof. Let $z \in C$. Define two set-valued mappings $F_z, G_z : C \to 2^E$ as follows:

$$F_z(y) = \left\{x \in C : f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) + \frac{1}{r}\langle y - x, J(x-z)\rangle \geq \varphi(x)\right\}$$

and

$$G_z(y) = \left\{x \in C : f(x,y) + \langle Ty, \eta(y,x)\rangle + \varphi(y) + \frac{1}{r}\langle y - x, J(x-z)\rangle \geq \varphi(x) + \xi(y-x)\right\},$$

for every $y \in C$. It is easily to seen that $y \in F_z(y)$ and $y \in G_z(y)$, and hence $F_z(y) \neq \emptyset$ and $G_z(y) \neq \emptyset$.

(a) We claim that F_z is a KKM mapping. If F_z is not a KKM mapping, then there exist $\{y_1, \ldots, y_n\} \subset C$ and $t_i > 0, i = 1, \ldots, n$, such that

$$\sum_{i=1}^{n} t_i = 1, \quad y = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} F_z(y_i). \tag{6.1.15}$$

By the definition of F_z, we have

$$f(y, y_i) + \langle Ty, \eta(y_i, y)\rangle + \varphi(y_i) - \varphi(y) + \frac{1}{r}\langle y_i - y, J(y-z)\rangle < 0, \quad \forall i = 1, \ldots, n. \tag{6.1.16}$$

It follows from (A1), (A4), (ii) and the convexity of φ that

$$\begin{aligned} 0 &= f(y,y) + \langle Ty, \eta(y,y)\rangle + \varphi(y) - \varphi(y) + \frac{1}{r}\langle y - y, J(y-z)\rangle \\ &\leq \sum_{i=1}^{n} t_i \left(f(y,y_i) + \langle Ty, \eta(y_i,y)\rangle + \varphi(y_i) - \varphi(y) + \frac{1}{r}\langle y_i - y, J(y-z)\rangle \right) \\ &< 0, \end{aligned} \tag{6.1.17}$$

which is a contradiction. This implies that F_z is a KKM mapping.

(b) We claim that G_z is a KKM mapping. It is sufficient to show that

$$F_z(y) \subset G_z(y), \quad \forall y \in C. \tag{6.1.18}$$

For any given $y \in C$, taking $x \in F_z(y)$, then

$$f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) - \varphi(x) + \frac{1}{r}\langle y - x, J(x-z)\rangle \geq 0. \tag{6.1.19}$$

It follows from (6.1.19) and the relaxed $\eta - \xi$ monotonicity of T that

$$\begin{aligned} & f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) - \varphi(x) + \frac{1}{r}\langle y - x, J(x-z)\rangle \\ \geq\ & f(x,y) + \langle Tx, \eta(y,x)\rangle + \xi(y-x) + \varphi(y) - \varphi(x) + \frac{1}{r}\langle y - x, J(x-z)\rangle \\ \geq\ & \xi(y-x). \end{aligned} \tag{6.1.20}$$

It follows that $x \in G_z(y)$ and so

$$F_z(y) \subset G_z(y), \quad \forall y \in C. \tag{6.1.21}$$

This implies that G_z is also a KKM mapping.

(c) Next, we show that $G_z(y)$ is closed for all $y \in C$. Let $\{\varrho_n\}$ be a sequence in $G_z(y)$ such that $\varrho_n \to \varrho$ as $n \to \infty$. It then follows from $\varrho_n \in G_z(y)$ that,

$$f(\varrho_n, y) + \langle Ty, \eta(y, \varrho_n)\rangle + \varphi(y) + \frac{1}{r}\langle y - z_n, J(\varrho_n - z)\rangle$$

$$\geq \varphi(\varrho_n) + \xi(y - \varrho_n). \tag{6.1.22}$$

By (A3), (ii), the continuity of J, and the lower semicontinuity of φ, ξ and $\|\cdot\|^2$, we obtain from (6.1.22) that

$$\begin{aligned}
& \varphi(\varrho) + \xi(y - \varrho) + \langle Ty, \eta(\varrho, y)\rangle \\
\leq\ & \liminf_{n\to\infty} \varphi(\varrho_n) + \liminf_{n\to\infty} \xi(y - \varrho_n) + \liminf_{n\to\infty} \langle Ty, \eta(\varrho_n, y)\rangle \\
\leq\ & \liminf_{n\to\infty} (\varphi(\varrho_n) + \xi(y - \varrho_n) + \langle Ty, \eta(\varrho_n, y)\rangle) \\
\leq\ & \limsup_{n\to\infty} (\varphi(\varrho_n) + \xi(y - \varrho_n) + \langle Ty, \eta(\varrho_n, y)\rangle) \\
=\ & \limsup_{n\to\infty} (\varphi(\varrho_n) + \xi(y - \varrho_n) - \langle Ty, \eta(y, \varrho_n)\rangle) \\
\leq\ & \limsup_{n\to\infty} (f(\varrho_n, y) + \varphi(y) + \frac{1}{r}\langle y - \varrho_n, J(\varrho_n - z)\rangle) \\
=\ & \limsup_{n\to\infty} (f(\varrho_n, y) + \varphi(y) + \frac{1}{r}\langle y - z, J(\varrho_n - z)\rangle + \frac{1}{r}\langle z - \varrho_n, J(\varrho_n - z)\rangle) \\
=\ & \limsup_{n\to\infty} (f(\varrho_n, y) + \varphi(y) + \frac{1}{r}\langle y - z, J(\varrho_n - z)\rangle - \frac{1}{r}\|z - \varrho_n\|^2) \\
\leq\ & \limsup_{n\to\infty} (f(\varrho_n, y) + \varphi(y)) + \limsup_{n\to\infty} \frac{1}{r}\langle y - z, J(\varrho_n - z)\rangle - \frac{1}{r}\liminf_{n\to\infty} \|z - \varrho_n\|^2 \\
\leq\ & f(\varrho, y) + \varphi(y) + \frac{1}{r}\langle y - z, J(\varrho - z)\rangle - \frac{1}{r}\|\varrho - z\|^2 \\
\leq\ & f(\varrho, y) + \varphi(y) + \frac{1}{r}\langle y - z, J(\varrho - z)\rangle - \frac{1}{r}\langle \varrho - z, J(\varrho - z)\rangle \\
=\ & f(\varrho, y) + \varphi(y) + \frac{1}{r}\langle y - \varrho, J(\varrho - z)\rangle.
\end{aligned} \tag{6.1.23}$$

Thus,

$$f(\varrho, y) + \langle Ty, \eta(y, \varrho) + \varphi(y) + \frac{1}{r}\langle y - \varrho, J(\varrho - z)\rangle \geq \varphi(\varrho) + \xi(y - \varrho).$$

This shows that $\varrho \in G_z(y)$ and hence $G_z(y)$ is closed for all $y \in C$.

(d) We prove that $G_z(y)$ is weakly compact. Indeed, we equip E with the weak topology. Then C, as a closed bounded convex subset in a reflexive space, is weakly compact. Hence $G_z(y)$ is also weakly compact.

By using (a)-(d) and Lemma 6.1.1 and Lemma 2.1.76 that

$$\bigcap_{y\in C} F_z(y) = \bigcap_{y\in C} G_z(y) \neq \emptyset.$$

Hence there exists $x_0 \in C$ satisfying the inequality (6.1.14). This completes the proof. □

Motivated by Takahashi and Zembayashi [63], and Ceng and Yao [69], we next prove the following crucial lemma concerning the generalized mixed equilibrium problem in a strictly convex, reflexive and smooth Banach space.

Lemma 6.1.3. *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed $\eta - \xi$ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and and define a mapping $\Phi_r : E \to C$ as follows:*

$$\begin{aligned}\Phi_r(x) = \Big\{ z \in C : f(z,y) + \langle Tz, \eta(y,z)\rangle + \varphi(y) + \frac{1}{r}\langle y - z, J(z - x)\rangle \\ \geq \varphi(z), \forall y \in C \Big\} \qquad (6.1.24)\end{aligned}$$

for all $x \in E$. Assume that

(i) $\eta(x,y) + \eta(y,x) = 0$, for all $x, y \in C$;

(ii) for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x,u)\rangle$ is convex and lower semicontinuous and the mapping $x \mapsto \langle Tu, \eta(v,x)\rangle$ is lower semicontinuous;

(iii) $\xi : E \to \mathbb{R}$ is weakly lower semicontinuous;

(iv) for any $x, y \in C$, $\xi(x - y) + \xi(y - x) \geq 0$.

Then, the following holds:

(1) Φ_r *is single-valued;*

(2) $\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle$ *for all* $x, y \in E$*;*

(3) $F(\Phi_r) = EP(f, T)$*;*

(4) $EP(f, T)$ *is closed and convex.*

Proof. For each $x \in E$, by Lemma 6.1.2, we conclude that $\Phi_r(x)$ is nonempty.

(1) We prove that Φ_r is single-valued. Indeed, for $x \in E$ and $r > 0$, let z_1, $z_2 \in \Phi_r x$. Then,

$$f(z_1, z_2) + \langle Tz_1, \eta(z_2, z_1)\rangle + \varphi(z_2) + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x)\rangle \geq \varphi(z_1)$$

and

$$f(z_2, z_1) + \langle Tz_2, \eta(z_1, z_2)\rangle + \varphi(z_1) + \frac{1}{r}\langle z_1 - z_2, J(z_2 - x)\rangle \geq \varphi(z_2).$$

Adding the two inequalities, from (i) we have

$$f(z_2, z_1) + f(z_1, z_2) + \langle Tz_1 - Tz_2, \eta(z_2, z_1)\rangle + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq 0. \tag{6.1.25}$$

From (A2), we have

$$\langle Tz_1 - Tz_2, \eta(z_2, z_1)\rangle + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq 0, \tag{6.1.26}$$

That is,

$$\frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq \langle Tz_2 - Tz_1, \eta(z_2, z_1)\rangle. \tag{6.1.27}$$

Since T is relaxed $\eta - \xi$ monotone and $r > 0$, one has

$$\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq r\xi(z_2 - z_1) \tag{6.1.28}$$

In (6.1.27) exchanging the position of z_1 and z_2, we get

$$\frac{1}{r}\langle z_1 - z_2, J(z_2 - x) - J(z_1 - x)\rangle \geq \langle Tz_1 - Tz_2, \eta(z_1, z_2)\rangle, \tag{6.1.29}$$

that is,

$$\langle z_1 - z_2, J(z_2 - x) - J(z_1 - x)\rangle \geq r\xi(z_1 - z_2) \tag{6.1.30}$$

Now, adding the inequalities (6.1.28) and (6.1.30), by using (iv) we have

$$2\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq r(\xi(z_2 - z_1) + \xi(z_1 - z_2)) \geq 0. \tag{6.1.31}$$

Hence,

$$0 \leq \langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle = \langle (z_2 - x) - (z_1 - x), J(z_1 - x) - J(z_2 - x)\rangle.$$

Since J is monotone and E is strictly convex, we obtain that $z_1 - x = z_2 - x$ and hence $z_1 = z_2$. Therefore S_r is single-valued.

(2) For $x, y \in C$, we have

$$f(\Phi_r x, \Phi_r y) + \langle T\Phi_r x, \eta(\Phi_r y, \Phi_r x)\rangle + \varphi(\Phi_r y) - \varphi(\Phi_r x) + \frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x)\rangle \geq 0$$

and

$$f(\Phi_r y, \Phi_r x) + \langle T\Phi_r y, \eta(\Phi_r x, \Phi_r y)\rangle + \varphi(\Phi_r x) - \varphi(\Phi_r y) + \frac{1}{r}\langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle \geq 0$$

Adding the above two inequalities and by (i) and (A2), we get

$$\langle T\Phi_r x - T\Phi_r y, \eta(\Phi_r y, \Phi_r x)\rangle + \frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq 0 \tag{6.1.32}$$

that is,

$$\frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \;\geq\; \langle T\Phi_r y - T\Phi_r x, \eta(\Phi_r y, \Phi_r x)\rangle$$

$$\geq \xi(\Phi_r y - \Phi_r x). \tag{6.1.33}$$

In (6.1.33) exchanging the position of $\Phi_r x$ and $\Phi_r y$, we get

$$\frac{1}{r}\langle \Phi_r x - \Phi_r y, J(\Phi_r y - y) - J(\Phi_r x - x)\rangle \geq \xi(\Phi_r x - \Phi_r y). \tag{6.1.34}$$

Adding the inequalities (6.1.33) and (6.1.34), we have

$$2\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq r(\xi(\Phi_r x - \Phi_r y) + \xi(\Phi_r y - \Phi_r x)). \tag{6.1.35}$$

It follows from (iv) that

$$\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq 0. \tag{6.1.36}$$

Hence

$$\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle. \tag{6.1.37}$$

(3) Next, we show that $F(\Phi_r) = EP(f, T)$. Indeed, we have the following:

$$\begin{aligned} u \in F(\Phi_r) \quad &\Leftrightarrow \quad u = \Phi_r u \\ &\Leftrightarrow \quad f(u,y) + \langle Tu, \eta(y,u)\rangle + \varphi(y) + \frac{1}{r}\langle y - u, J(u-u)\rangle \geq \varphi(u), \quad \forall y \in C \\ &\Leftrightarrow \quad f(u,y) + \langle Tu, \eta(y,u)\rangle + \varphi(y) \geq \varphi(u), \quad \forall y \in C \\ &\Leftrightarrow \quad u \in EP(f,T). \end{aligned} \tag{6.1.38}$$

Hence, $F(\Phi_r) = EP(f, T)$.

(4) Finally, we prove that $EP(f,T)$ is closed and convex. For each $y \in C$, we define the mapping $G : C \to 2^E$ by

$$G(y) = \{x \in C : f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) \geq \varphi(x)\}.$$

Since $y \in G(y)$, we have $G(y) \neq \emptyset$. We prove that G is a KKM mapping on C. Suppose that there exists a finite subset $\{z_1, z_2, ..., z_m\}$ of C and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\hat{z} = \sum_{j=1}^m \alpha_j z_j \notin G(z_i)$ for all $i = 1, 2, ..., m$. Then

$$f(\hat{z}, z_i) + \langle T\hat{z}, \eta(z_i, \hat{z})\rangle + \varphi(z_i) - \varphi(\hat{z}) < 0, \quad i = 1, 2, \ldots, m.$$

From (A1), (A4), (ii) and the convexity of φ, we have

$$\begin{aligned}
0 &= f(\hat{z}, \hat{z}) + \langle T\hat{z}, \eta(\hat{z}, \hat{z})\rangle + \varphi(\hat{z}) - \varphi(\hat{z}) \\
&= f\left(\hat{z}, \sum_{j=1}^m \alpha_j z_j\right) + \langle T\hat{z}, \eta\left(\sum_{j=1}^m \alpha_j z_j, \hat{z}\right)\rangle + \varphi\left(\sum_{j=1}^m \alpha_j z_j\right) - \varphi(\hat{z}) \\
&\leq \sum_{j=1}^m \alpha_j (f(\hat{z}, z_j) + \langle T\hat{z}, \eta(z_j, \hat{z}) + \varphi(z_j) - \varphi(\hat{z})) \\
&< 0.
\end{aligned}$$

which is a contradiction. Thus G is a KKM mapping on C.

Next, we prove that $G(y)$ is closed for each $y \in C$. For any $y \in C$, let $\{x_n\}$ be any sequence in $G(y)$ such that $x_n \to x_0$. We claim that $x_0 \in G(y)$. Then, for each $y \in C$, we have

$$f(x_n, y) + \langle Tx_n, \eta(y, x_n)\rangle + \varphi(y) \geq \varphi(x_n).$$

By (A3), (i), (ii) and the lower semicontinuity of of φ, we obtain the following

$$\begin{aligned}
\varphi(x_0) + \langle Tx_0, \eta(x_0, y)\rangle &\leq \liminf_{n\to\infty} \varphi(x_n) + \liminf_{n\to\infty} \langle Tx_n, \eta(x_n, y)\rangle \\
&\leq \liminf_{n\to\infty} (\varphi(x_n) + \langle Tx_n, \eta(x_n, y)\rangle) \\
&= \liminf_{n\to\infty} (\varphi(x_n) - \langle Tx_n, \eta(y, x_n)\rangle) \\
&\leq \limsup_{n\to\infty} (\varphi(x_n) - \langle Tx_n, \eta(y, x_n)\rangle) \\
&\leq \limsup_{n\to\infty} (f(x_n, y) + \varphi(y)) \\
&\leq f(x_0, y) + \varphi(y).
\end{aligned}$$

Hence,

$$f(x_0, y) + \langle Tx_0, \eta(y, x_0)\rangle + \varphi(y) \geq \varphi(x_0).$$

This shows that $x_0 \in G(y)$ and hence $G(y)$ is closed for each $y \in C$. Thus $EP(f,T) = \bigcap_{y \in C} G(y)$ is also closed.

Next, we observe that $G(y)$ is weakly compact. In fact, since C is bounded, closed and convex, we also have $G(y)$ is weakly compact in the weak topology. By Lemma 2.1.76, we can conclude that $\bigcap_{y \in C} G(y) = EP(f,T) \neq \emptyset$.

Finally, we prove that $EP(f,T)$ is convex. In fact, let $u, v \in F(\Phi_r)$ and $z_t = tu + (1-t)v$ for $t \in (0,1)$. From (2), we know that

$$\langle \Phi_r u - \Phi_r z_t, J(\Phi_r z_t - z_t) - J(\Phi_r u - u) \rangle \geq 0.$$

This yields that

$$\langle u - \Phi_r z_t, J(\Phi_r z_t - z_t) \rangle \geq 0. \tag{6.1.39}$$

Similarly, we also have

$$\langle v - \Phi_r z_t, J(\Phi_r z_t - z_t) \rangle \geq 0. \tag{6.1.40}$$

It follows from (6.1.39) and (6.1.40) that

$$\begin{aligned} \|z_t - \Phi_r z_t\|^2 &= \langle z_t - \Phi_r z_t, J(z_t - \Phi_r z_t) \rangle \\ &= t\langle u - \Phi_r z_t, J(z_t - \Phi_r z_t) \rangle + (1-t)\langle v - \Phi_r z_t, J(z_t - \Phi_r z_t) \rangle \\ &\leq 0. \end{aligned}$$

Hence $z_t \in F(\Phi_r) = EP(f,T)$ and hence $EP(f,T)$ is convex. This completes the proof. □

Next, we prove a strong convergence theorem by using a hybrid projection algorithm in a uniformly convex and smooth Banach space.

Theorem 6.1.4. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from*

$C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T : C \to E^*$ be an η-hemicontinuous and relaxed $\eta - \xi$ monotone mapping and let $\{S_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(S_n) \cap EP(f,T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C generated by

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - S_n z\| \leq t_n \|x_n - S_n x_n\|\}, \quad n \geq 0, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + \langle Tu_n, \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases} \tag{6.1.41}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0,1)$ and $\liminf_{n\to\infty} r_n > 0$.

Assume that $\{S_n\}_{n=0}^{\infty}$ satisfy the NST-condition. Then, the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. Firstly, we rewrite the algorithm (6.1.41) as the following :

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - S_n z\| \leq t_n \|x_n - S_n x_n\|\}, \quad n \geq 1, \\ D_n = \{z \in D_{n-1} : \langle \Phi_{r_n} x_n - z, J(x_n - \Phi_{r_n} x_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases} \tag{6.1.42}$$

where Φ_r is the mapping defined by (6.1.24) for all $r > 0$. We first show that the sequence $\{x_n\}$ is well-defined. It is easy to verify that $C_n \cap D_n$ is closed and convex and $\Omega \subset C_n$ for all $n \geq 0$. Next, we prove that $\Omega \subset C_n \cap D_n$. Since $D_0 = C$, we also have $\Omega \subset C_0 \cap D_0$. Suppose that $\Omega \subset C_{k-1} \cap D_{k-1}$ for $k \geq 2$. It follows from Lemma 6.1.3 (2) that

$$\langle \Phi_{r_k} x_k - \Phi_{r_k} u, J(\Phi_{r_k} u - u) - J(\Phi_{r_k} x_k - x_k) \rangle \geq 0,$$

for all $u \in \Omega$. This implies that

$$\langle \Phi_{r_k} x_k - u, J(x_k - \Phi_{r_k} x_k) \rangle \geq 0,$$

for all $u \in \Omega$. Hence $\Omega \subset D_k$. By the mathematical induction, we get that $\Omega \subset C_n \cap D_n$ for each $n \geq 0$ and hence $\{x_n\}$ is well-defined. Put $w = P_\Omega x_0$. Since $\Omega \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|w - x_0\|, n \geq 0. \qquad (6.1.43)$$

Since $x_{n+2} \in D_{n+1} \subset D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|x_{n+2} - x_0\|.$$

This implies that $\{\|x_n - x_0\|\}$ is nondecreasing and hence $\lim_{n\to\infty} \|x_n - x_0\| = d$ for some a constant d. Moreover, by the convexity of D_n, we also have $\frac{1}{2}(x_{n+1} + x_{n+2}) \in D_n$ and hence

$$\|x_0 - x_{n+1}\| \leq \left\| x_0 - \frac{x_{n+1} + x_{n+2}}{2} \right\| \leq \frac{1}{2}\Big(\|x_0 - x_{n+1}\| + \|x_0 - x_{n+2}\| \Big).$$

This implies that

$$\lim_{n\to\infty} \left\| \frac{1}{2}(x_0 - x_{n+1}) + \frac{1}{2}(x_0 - x_{n+2}) \right\| = \lim_{n\to\infty} \left\| x_0 - \frac{x_{n+1} + x_{n+2}}{2} \right\| = d.$$

By Lemma 2.1.45, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \qquad (6.1.44)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \rightharpoonup \tilde{x} \in C \quad \text{as } k \to \infty. \tag{6.1.45}$$

Next, we show that $\tilde{x} \in \bigcap_{n=0}^{\infty} F(S_n)$. Since $x_{n+1} \in C_n$ and $t_n > 0$, there exists $m \in \mathbb{N}$, $\{\lambda_0, \lambda_1, \ldots, \lambda_m\} \subset [0,1]$ and $\{y_0, y_1, \ldots, y_m\} \subset C$ such that

$$\sum_{i=0}^{m} \lambda_i = 1, \quad \Big\|x_{n+1} - \sum_{i=0}^{m} \lambda_i y_i\Big\| < t_n, \quad \text{and } \|y_i - S_n y_i\| \le t_n \|x_n - S_n x_n\|$$

for each $i = 0, 1, \ldots, m$. Since C is bounded, by Lemma 2.1.46, we have

$$\begin{aligned}
\|x_n - S_n x_n\| &\le \|x_n - x_{n+1}\| + \Big\|x_{n+1} - \sum_{i=0}^{m} \lambda_i y_i\Big\| + \Big\|\sum_{i=0}^{m} \lambda_i y_i - \sum_{i=0}^{m} \lambda_i S_n y_i\Big\| \\
&\quad + \Big\|\sum_{i=0}^{m} \lambda_i S_n y_i - S_n(\sum_{i=0}^{m} \lambda_i y_i)\Big\| + \Big\|S_n(\sum_{i=0}^{m} \lambda_i y_i) - S_n x_n\Big\| \\
&\le \|x_n - x_{n+1}\| + t_n + \sum_{i=0}^{m} \lambda_i \|y_i - S_n y_i\| \\
&\quad + \gamma^{-1}\Big(\max_{0 \le j \le k \le m} (\|y_j - y_k\| - \|S_n y_j - S_n y_k\|)\Big) + \Big\|\sum_{i=0}^{m} \lambda_i y_i - x_n\Big\| \\
&\le \|x_n - x_{n+1}\| + t_n + t_n M + \|x_{n+1} - x_n\| \\
&\quad + \gamma^{-1}\Big(\max_{0 \le j \le k \le m} (\|y_j - S_n y_j\| + \|y_k - S_n y_k\|)\Big) + \Big\|\sum_{i=0}^{m} \lambda_i y_i - x_n\Big\| \\
&\le 2\|x_n - x_{n+1}\| + (2 + M)t_n + \gamma^{-1}(2M t_n),
\end{aligned}$$

where $M = \sup_{n \ge 0} \|x_n - S_n x_n\|$. Applying (C1) and (6.1.44) into the above inequality, we obtain

$$\lim_{n \to \infty} \|x_n - S_n x_n\| = 0. \tag{6.1.46}$$

In particular, we obtain

$$\lim_{k \to \infty} \|x_{n_k} - S_{n_k} x_{n_k}\| = 0. \tag{6.1.47}$$

Using (6.1.45), (6.1.47) and the NST-condition, we have $\tilde{x} \in \bigcap_{n=0}^{\infty} F(S_n)$.

Next, we show that $\tilde{x} \in EP(f,T)$. By the construction of D_n, we see from Theorem 2.1.44 that $\Phi_{r_n}x_n = P_{D_n}x_n$. Since $x_{n+1} \in D_n$, we obtain

$$\|x_n - \Phi_{r_n}x_n\| \le \|x_n - x_{n+1}\| \to 0,$$

as $n \to \infty$. From (C2), we also have

$$\frac{1}{r_n}\Big\|J(x_n - \Phi_{r_n}x_n)\Big\| = \frac{1}{r_n}\|x_n - \Phi_{r_n}x_n\| \to 0, \tag{6.1.48}$$

as $n \to \infty$. By (6.1.48), we also have $\Phi_{r_{n_k}}x_{n_k} \rightharpoonup \tilde{x}$. By the definition of $\Phi_{r_{n_k}}$, for each $y \in C$, we obtain

$$f(\Phi_{r_{n_k}}x_{n_k}, y) + \langle T\Phi_{r_{n_k}}x_{n_k}, \eta(y, \Phi_{r_{n_k}}x_{n_k})\rangle + \varphi(y) + \frac{1}{r_{n_k}}\langle y - \Phi_{r_{n_k}}x_{n_k}, J(\Phi_{r_{n_k}}x_{n_k} - x_{n_k})\rangle$$
$$\ge \varphi(\Phi_{r_{n_k}}x_{n_k}).$$

By (A3), (6.1.48), (ii), the weakly lower semicontinuity of φ and η-hemicontinuity of T we have

$$\begin{aligned}
\varphi(\tilde{x}) &\le \liminf_{k\to\infty} \varphi(\Phi_{r_{n_k}}x_{n_k}) \\
&\le \liminf_{k\to\infty} f(\Phi_{r_{n_k}}x_{n_k}, y) + \liminf_{k\to\infty}\langle T\Phi_{r_{n_k}}x_{n_k}, \eta(y, \Phi_{r_{n_k}}x_{n_k})\rangle \\
&\quad + \varphi(y) + \liminf_{k\to\infty} \frac{1}{r_{n_k}}\langle y - \Phi_{r_{n_k}}x_{n_k}, J(\Phi_{r_{n_k}}x_{n_k} - x_{n_k})\rangle \\
&\le f(\tilde{x}, y) + \varphi(y) + \langle T\tilde{x}, \eta(y, \tilde{x})\rangle.
\end{aligned}$$

Hence,

$$f(\tilde{x}, y) + \varphi(y) + \langle T\tilde{x}, \eta(y, \tilde{x})\rangle \ge \varphi(\tilde{x}).$$

This shows that $\tilde{x} \in EP(f,T)$ and hence $\tilde{x} \in \Omega := \bigcap_{n=0}^{\infty} F(S_n) \cap EP(f,T)$.

Finally, we show that $x_n \to w$ as $n \to \infty$, where $w := P_\Omega x_0$. By the weakly lower semicontinuity of the norm, it follows from (6.1.43) that

$$\|x_0 - w\| \le \|x_0 - \tilde{x}\| \le \liminf_{k\to\infty}\|x_0 - x_{n_k}\| \le \limsup_{k\to\infty}\|x_0 - x_{n_k}\| \le \|x_0 - w\|.$$

This shows that

$$\lim_{k\to\infty} \|x_0 - x_{n_k}\| = \|x_0 - w\| = \|x_0 - \tilde{x}\|$$

and $\tilde{x} = w$. Since E is uniformly convex, we obtain that $x_0 - x_{n_k} \to x_0 - w$. It follows that $x_{n_k} \to w$. So we have $x_n \to w$ as $n \to \infty$. This completes the proof. □

Applying Lemma 2.1.48 and Theorem 6.1.4, we obtain the following results immediately.

Corollary 6.1.5. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T : C \to E^*$ be an η-hemicontinuous and relaxed $\eta-\xi$ monotone mapping and let $\{S_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(S_n) \cap EP(f,T) \neq \emptyset$. Let $\{\beta_n^k\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that*

(i) $\sum_{k=1}^{n} \beta_n^k = 1$ *for every* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \beta_n^k > 0$ *for every* $k \in \mathbb{N}$

and let $G_n = \alpha_n I + (1-\alpha_n)\sum_{k=1}^{n} \beta_n^k S_k$ for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$ with $a \leq b$. Let $\{x_n\}$ be the sequence in C generated by

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - G_n z\| \leq t_n \|x_n - G_n x_n\|\}, \quad n \geq 0, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + \langle Tu_n, \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0,1)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

If we take $S_n \equiv S$, a nonexpansive mapping on C, for all $n \geq 0$ in Theorem 6.1.4, then we obtain the following result.

Theorem 6.1.6. *Let* E *be a uniformly convex and smooth Banach space and let* C *be a nonempty, bounded, closed and convex subset of* E. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4). Let* $T : C \to E^*$ *be an* η*-hemicontinuous and relaxed* $\eta - \xi$ *monotone mapping and let* S *be a nonexpansive mapping of* C *into itself such that* $\Omega := F(S) \cap EP(f,T) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence in* C *generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - Sz\| \leq t_n\|x_n - Sx_n\|\}, \quad n \geq 0, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + \langle Tu_n, \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases}$$

where $\{t_n\}$ *and* $\{r_n\}$ *are sequences which satisfy the conditions:*

(C1) $\{t_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0,1)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

If we take $T \equiv 0$ and $\varphi \equiv 0$ in Theorem 6.1.4, then we obtain the following result concerning an equilibrium problem in a Banach space setting.

Theorem 6.1.7. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\{S_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(S_n) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - S_n z\| \leq t_n \|x_n - S_n x_n\|\}, \quad n \geq 0, \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq 0, \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases} \tag{6.1.49}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0,1)$ and $\liminf_{n\to\infty} r_n > 0$.

Assume that $\{S_n\}_{n=0}^{\infty}$ satisfy the NST-condition, the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

If we take $f \equiv 0$ and $T \equiv 0$ and $\varphi \equiv 0$ in Theorem 6.1.4, then we obtain the following result.

Theorem 6.1.8. *Let E be a uniformly convex and smooth Banach space, C a nonempty, bounded, closed and convex subset of E and $\{S_n\}_{n=0}^{\infty}$ a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(S_n) \neq \emptyset$ and suppose that $\{S_n\}_{n=0}^{\infty}$ satisfies the NST-condition. Let $\{x_n\}$ be the sequence in C generated* *by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - S_n z\| \le t_n \|x_n - S_n x_n\|\}, \quad n \ge 0, \\ x_{n+1} = P_{C_n} x_0, \quad n \ge 0. \end{cases} \tag{6.1.50}$$

If $\{t_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} t_n = 0$, *then* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Remark 6.1.9. By Lemma 2.1.48, if we define $T_n = \alpha_n I + (1-\alpha_n)\sum_{k=1}^{n} \beta_n^k S_k$ for all $n \ge 0$ in Theorem 6.1.7 and Theorem 6.1.8, then the theorems also hold.

6.2 Existence and iterative approximation for generalized equilibrium problems for a countable family of nonexpansive mappings in Banach spaces

The purpose of this section we first prove the existence of a solution of the generalized equilibrium problem (GEP) by using the KKM mapping in a Banach space setting. We construct a hybrid algorithm for finding a common element in the solution set of a generalized equilibrium problem and the fixed point set of a countable family of nonexpansive mappings in Banach spaces.

Let $f : C \times C \to \mathbb{R}$ be a bifunction and $A : C \to E^*$ be a nonlinear mapping. We consider the following generalize equilibrium problem (GEP):

$$\text{Find } u \in C \text{ such that } f(u,y) + \langle Au, y-u\rangle \ge 0, \quad \forall y \in C. \tag{6.2.1}$$

The set of such $u \in C$ is denoted by $GEP(f)$, i.e.,

$$GEP(f) = \{u \in C : f(u,y) + \langle Au, y-u\rangle \ge 0, \ \forall y \in C\}.$$

Theorem 6.2.1. *Let* C *be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space* E, *let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4), where*

(A1) $f(x,x) = 0$ *for all* $x \in C$;

(A2) f is monotone, i.e. $f(x,y)+f(y,x)\leq 0$ for all $x,y\in C$;

(A3) for all $y\in C$, $f(.,y)$ is weakly upper semicontinuous;

(A4) for all $x\in C$, $f(x,.)$ is convex.

Let A be α-inverse strongly monotone of C into E^*. For all $r>0$ and $x\in E$, define the mapping $S_r: E\longrightarrow 2^C$ as follows:

$$S_r(x)=\{z\in C: f(z,y)+\langle Az,y-z\rangle+\frac{1}{r}\langle y-z,J(z-x)\rangle\geq 0,\quad \forall y\in C\}. \quad (6.2.2)$$

Then the following statements hold:

(1) for each $x\in E$, $S_r(x)\neq\emptyset$;

(2) S_r is single-valued;

(3) $\langle S_r(x)-S_r(y),J(S_rx-x)\rangle\leq\langle S_r(x)-S_r(y),J(S_ry-y)\rangle$ for all $x,y\in E$;

(4) $F(S_r)=GEP(f)$;

(5) GEP(f) is nonempty, closed and convex.

Proof. (1) Let x_0 be any given point in E. For each $y\in C$, we define the mapping $G: C\to 2^E$ by

$$G(y)=\{z\in C: f(z,y)+\langle Az,y-z\rangle+\frac{1}{r}\langle y-z,J(z-x_0)\rangle\geq 0\} \text{ for all } y\in C.$$

It is easily to seen that $y\in G(y)$, and hence $G(y)\neq\emptyset$.

(a) First, we will show that G is a KKM mapping. Suppose that there exists a finite subset $\{y_1,y_2,...,y_m\}$ of C and $\alpha_i>0$ with $\sum_{i=1}^m\alpha_i=1$ such that $\hat{x}=\sum_{i=1}^m\alpha_i y_i\notin G(y_i)$ for all $i=1,2,\ldots,m$. It follows that

$$f(\hat{x},y_i)+\langle A\hat{x},y_i-\hat{x}\rangle+\frac{1}{r}\langle y_i-\hat{x},J(\hat{x}-x_0)\rangle<0, \text{ for all } i=1,2,\ldots,m.$$

By (A1) and (A4), we have

$$\begin{aligned} 0 &= f(\hat{x},\hat{x}) + \langle A\hat{x}, \hat{x}-\hat{x}\rangle + \frac{1}{r}\langle \hat{x}-\hat{x}, J(\hat{x}-x_0)\rangle \\ &\leq \sum_{i=1}^{m}\Big(f(\hat{x},y_i) + \langle A\hat{x}, y_i-\hat{x}\rangle + \frac{1}{r}\langle y_i-\hat{x}, J(\hat{x}-x_0)\rangle\Big) < 0, \end{aligned}$$

which is a contradiction. Thus G is a KKM mapping on C.

(b) Next, we show that $G(y)$ is closed for all $y \in C$. Let $\{z_n\}$ be a sequence in $G(y)$ such that $z_n \to z$ as $n \to \infty$. It then follows from $z_n \in G(y)$ that,

$$f(z_n, y) + \langle Az_n, y - z_n\rangle + \frac{1}{r}\langle y - z_n, J(z_n - x)\rangle \geq 0. \tag{6.2.3}$$

By (A3), the continuity of J, and the lower semicontinuity of $\|\cdot\|^2$, we obtain from (6.2.3) that

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty}[f(z_n,y) + \langle Az_n, y-z_n\rangle + \frac{1}{r}\langle y-z_n, J(z_n-x_0)\rangle] \\ &\leq \limsup_{n\to\infty}[f(z_n,y) + \langle Az_n, y-z_n\rangle + \frac{1}{r}\langle y-x_0, J(z_n-x_0)\rangle + \frac{1}{r}\langle x_0-z_n, J(z_n-x_0)\rangle] \\ &= \limsup_{n\to\infty}[f(z_n,y) + \langle Az_n, y-z_n\rangle + \frac{1}{r}\langle y-x_0, J(z_n-x_0)\rangle - \frac{1}{r}\|z_n-x_0\|^2] \\ &\leq \limsup_{n\to\infty} f(z_n,y) + \limsup_{n\to\infty}\langle Az_n, y-z_n\rangle + \frac{1}{r}\limsup_{n\to\infty}\langle y-x_0, J(z_n-x_0)\rangle \\ &\quad - \frac{1}{r}\liminf_{n\to\infty}\|z_n-x_0\|^2 \\ &\leq f(z,y) + \langle Az, y-z\rangle + \frac{1}{r}\langle y-x_0, J(z-x_0)\rangle - \frac{1}{r}\|z-x_0\|^2 \\ &= f(z,y) + \langle Az, y-z\rangle + \frac{1}{r}\langle y-x_0, J(z-x_0)\rangle - \frac{1}{r}\langle z-x_0, J(z-x_0)\rangle \\ &= f(z,y) + \langle Az, y-z\rangle + \frac{1}{r}\langle y-z, J(z-x_0)\rangle. \end{aligned}$$

This shows that $z \in G(y)$ and hence $G(y)$ is closed for all $y \in C$.

(c) We prove that $G(y)$ is weakly compact. We now equip E with the weak topology. Then C, as a closed bounded convex subset in a reflexive space, is weakly compact. Hence $G(y)$ is also weakly compact.

Using (a), (b) and (c) and Lemma 2.1.76, we have $\bigcap_{y\in C} G(y) \neq \emptyset$. It is easily to seen that

$$S_r(x_0) = \bigcap_{y\in C} G(y)$$

Hence $S_r(x_0) \neq \emptyset$. Since x_0 is arbitrary, we can conclude that $S_r(x) \neq \emptyset$ for all $x \in E$.

(2) We prove that S_r is single-valued. In fact, for $x \in C$ and $r > 0$, let $z_1, z_2 \in S_r(x)$. Then,

$$f(z_1, z_2) + \langle Az_1, z_2 - z_1\rangle + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x)\rangle \geq 0.$$

and

$$f(z_2, z_1) + \langle Az_2, z_1 - z_2\rangle + \frac{1}{r}\langle z_1 - z_2, J(z_2 - x)\rangle \geq 0.$$

Adding the two inequalities. From the condition (A2) and monotonicity of A, we have

$$\begin{aligned} 0 &\leq f(z_1, z_2) + f(z_2, z_1) + \langle Az_1, z_2 - z_1\rangle + \langle Az_2, z_1 - z_2\rangle \\ &\quad + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \\ &\leq \langle Az_1 - Az_2, z_2 - z_1\rangle + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \\ &\leq -\alpha\|Az_1 - Az_2\|^2 + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \\ &\leq \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle, \end{aligned} \tag{6.2.4}$$

and hence

$$\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq 0.$$

Hence

$$0 \leq \langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle = \langle (z_2 - x) - (z_1 - x), J(z_1 - x) - J(z_2 - x)\rangle.$$

Since J is monotone and E is strictly convex, we obtain that $z_1 - x = z_2 - x$ and hence $z_1 = z_2$. Therefore S_r is single-valued.

(3) For $x, y \in C$, we have

$$f(S_r x, S_r y) + \langle AS_r x, S_r y - S_r x\rangle + \frac{1}{r}\langle S_r y - S_r x, J(S_r x - x)\rangle \geq 0$$

and

$$f(S_r y, S_r x) + \langle AS_r y, S_r x - S_r y\rangle + \frac{1}{r}\langle S_r x - S_r y, J(S_r y - y)\rangle \geq 0.$$

Again, adding the two inequalities, we also have

$$\langle AS_r x - AS_r y, S_r y - S_r x\rangle + \langle S_r y - S_r x, J(S_r x - x) - J(S_r y - y)\rangle \geq 0.$$

It follows from monotonicity of A that

$$\langle S_r y - S_r x, J(S_r x - x)\rangle \leq \langle S_r y - S_r x, J(S_r y - y)\rangle.$$

(4) It is easy to see that

$$\begin{aligned} z \in F(S_r) &\Leftrightarrow z = S_r z \\ &\Leftrightarrow f(z, y) + \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, J(z - z)\rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow f(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow z \in GEP(f). \end{aligned}$$

Hence $F(S_r) = GEP(f)$.

(5) Finally, we claim that $GEP(f)$ is nonempty, closed and convex. For each $y \in C$, we define the mapping $\Theta : C \to 2^E$ by

$$\Theta(y) = \{x \in C : f(x, y) + \langle Ax, y - x\rangle \geq 0\}.$$

Since $y \in \Theta(y)$, we have $\Theta(y) \neq \emptyset$. We prove that Θ is a KKM mapping on C. Suppose that there exists a finite subset $\{z_1, z_2, ..., z_m\}$ of C and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\hat{z} = \sum_{i=1}^m \alpha_i z_i \notin \Theta(z_i)$ for all $i = 1, 2, ..., m$. Then

$$f(\hat{z}, z_i) + \langle A\hat{z}, z_i - \hat{z}\rangle < 0, \quad i = 1, 2, \ldots, m.$$

From (A1) and (A4), we have

$$0 = f(\hat{z}, \hat{z}) + \langle A\hat{z}, \hat{z} - \hat{z} \rangle \leq \sum_{i=1}^{m} \alpha_i \Big(f(\hat{z}, z_i) + \langle A\hat{z}, z_i - \hat{z} \rangle \Big) < 0,$$

which is a contradiction. Thus Θ is a KKM mapping on C.

Next, we prove that $\Theta(y)$ is closed for each $y \in C$. For any $y \in C$, let $\{x_n\}$ be any sequence in $\Theta(y)$ such that $x_n \to x_0$. We claim that $x_0 \in \Theta(y)$. Then, for each $y \in C$, we have

$$f(x_n, y) + \langle Ax_n, y - x_n \rangle \geq 0.$$

By (A3), we see that

$$f(x_0, y) + \langle Ax_0, y - x_0 \rangle \geq \limsup_{n\to\infty} f(x_n, y) + \lim_{n\to\infty} \langle Ax_n, y - x_n \rangle \geq 0.$$

This shows that $x_0 \in \Theta(y)$ and $\Theta(y)$ is closed for each $y \in C$. Thus $\bigcap_{y \in C} \Theta(y) = GEP(f)$ is also closed.

We observe that $\Theta(y)$ is weakly compact. In fact, since C is bounded, closed and convex, we also have $\Theta(y)$ is weakly compact in the weak topology. By Lemma 2.1.76, we can conclude that $\bigcap_{y \in C} \Theta(y) = GEP(f) \neq \emptyset$.

Finally, we prove that $GEP(f)$ is convex. In fact, let $u, v \in F(S_r)$ and $z_t = tu + (1-t)v$ for $t \in (0,1)$. From (3), we know that

$$\langle S_r u - S_r z_t, J(S_r z_t - z_t) - J(S_r u - u) \rangle \geq 0.$$

This yields that

$$\langle u - S_r z_t, J(S_r z_t - z_t) \rangle \ \geq \ 0. \tag{6.2.5}$$

Similarly, we also have

$$\langle v - S_r z_t, J(S_r z_t - z_t) \rangle \ \geq \ 0. \tag{6.2.6}$$

It follows from (6.2.5) and (6.2.6) that

$$\begin{aligned}\|z_t - S_r z_t\|^2 &= \langle z_t - S_r z_t, J(z_t - S_r z_t)\rangle \\ &= t\langle u - S_r z_t, J(z_t - S_r z_t)\rangle + (1-t)\langle v - S_r z_t, J(z_t - S_r z_t)\rangle \\ &\leq 0.\end{aligned}$$

Hence $z_t \in F(S_r) = GEP(f)$ and hence $GEP(f)$ is convex. This completes the proof. □

6.2.1 Strong convergence theorems

In this section, we prove a strong convergence theorem by using a hybrid projection algorithm in a uniformly convex and smooth Banach space.

Theorem 6.2.2. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), A an α-inverse strongly monotone mapping of C into E^* and $\{T_n\}_{n=0}^{\infty}$ a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap GEP(f) \neq \emptyset$ and suppose that $\{T_n\}_{n=0}^{\infty}$ satisfies the NST-condition. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - T_n z\| \leq t_n \|x_n - T_n x_n\|\}, \quad n \geq 1, \\ u_n \in C \ \text{ such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq 0, \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases} \tag{6.2.7}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0,1)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Proof. Firstly, we rewrite the algorithm (6.2.7) as the following :

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - T_n z\| \le t_n\|x_n - T_n x_n\|\}, \quad n \ge 1, \\ D_n = \{z \in D_{n-1} : \langle S_{r_n}x_n - z, J(x_n - S_{r_n}x_n)\rangle \ge 0\}, \quad n \ge 1, \\ x_{n+1} = P_{C_n \cap D_n}x_0, \quad n \ge 0, \end{cases} \tag{6.2.8}$$

where S_r is the mapping defined by (6.2.2) for all $r > 0$. We first show that the sequence $\{x_n\}$ is well-defined. It is easy to verify that $C_n \cap D_n$ is closed and convex and $\Omega \subset C_n$ for all $n \ge 0$. Next, we prove that $\Omega \subset C_n \cap D_n$. Since $D_0 = C$, we also have $\Omega \subset C_0 \cap D_0$. Suppose that $\Omega \subset C_{k-1} \cap D_{k-1}$ for $k \ge 2$. It follows from Lemma 6.2.1 (3) that

$$\langle S_{r_k}x_k - S_{r_k}u, J(S_{r_k}u - u) - J(S_{r_k}x_k - x_k)\rangle \ge 0,$$

for all $u \in \Omega$. This implies that

$$\langle S_{r_k}x_k - u, J(x_k - S_{r_k}x_k)\rangle \ge 0,$$

for all $u \in \Omega$. Hence $\Omega \subset D_k$. By the mathematical induction, we get that $\Omega \subset C_n \cap D_n$ for each $n \ge 0$ and hence $\{x_n\}$ is well-defined. Put $w = P_F x_0$. Since $\Omega \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n}x_0$, we have

$$\|x_{n+1} - x_0\| \le \|w - x_0\|, n \ge 0. \tag{6.2.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. Since $x_{n+2} \in D_{n+1} \subset D_n$ and $x_{n+1} = P_{C_n \cap D_n}x_0$, we have

$$\|x_{n+1} - x_0\| \le \|x_{n+2} - x_0\|.$$

Since $\{x_n - x_0\}$ is bounded, we have $\lim_{n\to\infty}\|x_n - x_0\| = d$ for some a constant d. Moreover, by the convexity of D_n, we also have $\frac{1}{2}(x_{n+1}+x_{n+2}) \in D_n$ and hence

$$\|x_0 - x_{n+1}\| \le \left\|x_0 - \frac{x_{n+1}+x_{n+2}}{2}\right\| \le \frac{1}{2}\Big(\|x_0 - x_{n+1}\| + \|x_0 - x_{n+2}\|\Big).$$

This implies that

$$\lim_{n\to\infty}\left\|\frac{1}{2}(x_0 - x_{n+1}) + \frac{1}{2}(x_0 - x_{n+2})\right\| = \lim_{n\to\infty}\left\|x_0 - \frac{x_{n+1}+x_{n+2}}{2}\right\| = d.$$

By Lemma 2.1.45, we have $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$.

Next, we show that $v \in \bigcap_{n=0}^{\infty} F(T_n)$. Since $x_{n+1} \in C_n$ and $t_n > 0$, there exists $m \in \mathbb{N}$, $\{\lambda_0, \lambda_1, \dots, \lambda_m\} \subset [0,1]$ and $\{y_0, y_1, \dots, y_m\} \subset C$ such that

$$\sum_{i=1}^{m}\lambda_i = 1, \quad \left\|x_{n+1} - \sum_{i=0}^{m}\lambda_i y_i\right\| < t_n, \quad \text{and } \|y_i - T_n y_i\| \le t_n\|x_n - T_n x_n\|$$

for each $i = 0, 1, \dots, m$. Since C is bounded, by Lemma 2.1.46, we have

$$\begin{aligned}
\|x_n - T_n x_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - \sum_{i=0}^{m}\lambda_i y_i\right\| + \left\|\sum_{i=0}^{m}\lambda_i y_i - \sum_{i=0}^{m}\lambda_i T_n y_i\right\| \\
&\quad + \left\|\sum_{i=0}^{m}\lambda_i T_n y_i - T_n\Big(\sum_{i=0}^{m}\lambda_i y_i\Big)\right\| + \left\|T_n\Big(\sum_{i=0}^{m}\lambda_i y_i\Big) - T_n x_n\right\| \\
&\le 2\|x_n - x_{n+1}\| + (2+2M)t_n \\
&\quad + \gamma^{-1}\Big(\max_{0\le i\le j\le m}(\|y_i - y_j\| - \|T_n y_i - T_n y_j\|)\Big) \\
&\le 2\|x_n - x_{n+1}\| + (2+2M)t_n \\
&\quad + \gamma^{-1}\Big(\max_{0\le i\le j\le m}(\|y_i - T_n y_i\| - \|y_j - T_n y_j\|)\Big) \\
&\le 2\|x_n - x_{n+1}\| + (2+2M)t_n + \gamma^{-1}(4Mt_n),
\end{aligned}$$

where $M = \sup_{n\ge 0}\|x_n - w\|$. It follows from (C1) that $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. Since $\{T_n\}$ satisfies the NST-condition, we have $v \in \bigcap_{n=0}^{\infty} F(T_n)$.

Next, we show that $v \in GEP(f)$. By the construction of D_n, we see from

(2.1.44) that $S_{r_n}x_n = P_{D_n}x_n$. Since $x_{n+1} \in D_n$, we obtain

$$\|x_n - S_{r_n}x_n\| \leq \|x_n - x_{n+1}\| \to 0,$$

as $n \to \infty$. From (C2), we also have

$$\frac{1}{r_n}\Big\|J(x_n - S_{r_n}x_n)\Big\| = \frac{1}{r_n}\|x_n - S_{r_n}x_n\| \to 0, \tag{6.2.10}$$

as $n \to \infty$. Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_i}\}$ which weakly converges to some $v \in E$. By (6.2.10), we also have $S_{r_{n_i}} \rightharpoonup v$. By the definition of $S_{r_{n_j}}$, for each $y \in C$, we obtain

$$f(S_{r_{n_i}}x_{n_i}, y) + \langle AS_{r_{n_i}}x_{n_i}, y - S_{r_{n_i}}x_{n_i}\rangle + \frac{1}{r_{n_i}}\langle y - S_{r_{n_i}}x_{n_i}, J(S_{r_{n_i}}x_{n_i} - x_{n_i})\rangle \geq 0.$$

By (A3) and (6.2.10), we have

$$f(v, y) + \langle Av, y - v\rangle \geq 0, \quad \forall y \in C.$$

This shows that $v \in GEP(f)$ and hence $v \in \Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap GEP(f)$.

Note that $w = P_\Omega x_0$. Finally, we show that $x_n \to w$ as $n \to \infty$. By the weakly lower semicontinuity of the norm, it follows from (6.2.9) that

$$\|x_0 - w\| \leq \|x_0 - v\| \leq \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \leq \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \leq \|x_0 - w\|.$$

This shows that

$$\lim_{i\to\infty} \|x_0 - x_{n_i}\| = \|x_0 - w\| = \|x_0 - v\|$$

and $v = w$. Since E is uniformly convex, we obtain that $x_0 - x_{n_i} \to x_0 - w$. It follows that $x_{n_i} \to w$. So we have $x_n \to w$ as $n \to \infty$. This completes the proof. □

If we take $f \equiv 0$ and $A \equiv 0$ in Theorem 6.2.2, then we obtain the following result.

Theorem 6.2.3. *Let E be a uniformly convex and smooth Banach space, C a nonempty, bounded, closed and convex subset of E and $\{T_n\}_{n=0}^{\infty}$ a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$ and suppose that $\{T_n\}_{n=0}^{\infty}$ satisfies the NST-condition. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - T_n z\| \leq t_n \|x_n - T_n x_n\|\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n} x_0, \quad n \geq 0. \end{cases} \tag{6.2.11}$$

If $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$, then $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Remark 6.2.4. By Lemma 2.1.48, if we define $T_n = \alpha_n I + (1-\alpha_n)\sum_{k=1}^{n} \beta_n^k S_k$ for all $n \geq 0$ in Theorem 6.2.1 and Theorem 6.2.3, then the theorems also hold.

If we take $T_n \equiv I$, the identity mapping on C, for all $n \geq 0$ in Theorem 6.2.2, then we obtain the following result.

Theorem 6.2.5. *Let E be a uniformly convex and smooth Banach space, C a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and A an α-inverse strongly monotone mapping of C into E^*. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq 0, \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{D_n} x_0, \quad n \geq 0. \end{cases} \tag{6.2.12}$$

If $\{r_n\} \subset (0,1)$ and $\liminf_{n\to\infty} r_n > 0$, then $\{x_n\}$ converges strongly to $P_{GEP(f)} x_0$.

If we take $A \equiv 0$ in Theorem 6.2.2, then we obtain the following result concerning an equilibrium problem in a Banach space setting.

Theorem 6.2.6. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap EP(f) \neq \emptyset$ and suppose that $\{T_n\}_{n=0}^{\infty}$ satisfy the NST-condition. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - T_n z\| \leq t_n \|x_n - T_n x_n\|\}, \quad n \geq 1, \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq 0, \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases} \tag{6.2.13}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0,1)$ and $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.