CHAPTER V

STRONG CONVERGENCE THEOREMS FOR STRICT PSEUDO-CONTRACTIONS IN BANACH SPACES

5.1 A general iterative process for solving a system of variational inclusions in Banach spaces

In this section, we introduce a general iterative method for finding solutions of a general system of variational inclusions with Lipchitzian relaxed cocoercive mappings. Strong convergence theorems are established in strictly convex and 2-uniformly smooth Banach spaces. Moreover, we apply our result to the problem of finding a common fixed point of a countable family of strict pseudo-contraction mappings.

In a smooth Banach space, a mapping $A : C \to E$ is called *strongly positive* [27] if there exists a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, J(x)\rangle \geq \bar{\gamma}\|x\|^2,\ \|aI-bA\| = \sup_{\|x\|\leq 1} |\langle (aI-bA)x,\ J(x)\rangle|,\ a\in[0,1],\ b\in[-1,1], \tag{5.1.1}$$

where I is the identity mapping and J is the normalized duality mapping.

Next, we consider a system of quasi-variational inclusions:
Find $(x^*, y^*) \in E \times E$ such that

$$\begin{aligned} &0 \in x^* - y^* + \rho_1(\Psi_1 y^* + M_1 x^*),\\ &0 \in y^* - x^* + \rho_2(\Psi_2 x^* + M_2 y^*), \end{aligned} \tag{5.1.2}$$

where $\Psi_i : E \to E$ and $M_i : E \to 2^E$ are nonlinear mappings for each $i = 1, 2$.

As special cases of the problem (5.1.2), we have the following:

(1) If $\Psi_1 = \Psi_2 = \Psi$ and $M_1 = M_2 = M$, then the problem (5.1.2) is reduced to the following:

Find $(x^*, y^*) \in E \times E$ such that

$$0 \in x^* - y^* + \rho_1(\Psi y^* + Mx^*),$$
$$0 \in y^* - x^* + \rho_2(\Psi x^* + My^*). \tag{5.1.3}$$

(2) Further, if $x^* = y^*$ in the problem (5.1.3), then the problem (5.1.3) is reduced to the following:

Find $x^* \in E$ such that

$$0 \in \Psi x^* + Mx^*. \tag{5.1.4}$$

Definition 5.1.1. [67] Let $M : E \to 2^E$ be a multi-valued maximal accretive mapping. The single-valued mapping $J_{(M,\rho)} : E \to E$ defined by

$$J_{(M,\rho)}(u) = (I + \rho M)^{-1}(u), \quad \forall u \in E$$

is called the resolvent operator associated with M, where ρ is any positive number and I is the identity mapping.

Lemma 5.1.2. [67] *$u \in E$ is a solution of variational inclusion (5.1.4) if and only if $u = J_{(M,\rho)}(u - \rho\Psi u)$, $\forall \rho > 0$, that is,*

$$VI(E, \Psi, M) = F(J_{(M,\rho)}(I - \rho\Psi)), \; \forall \rho > 0,$$

where $VI(E, \Psi, M)$ denotes the set of solutions to the problem (5.1.4) .

Lemma 5.1.3. [68] *For any $(x^*, y^*) \in E \times E$, where $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2\Psi_2 x^*)$, (x^*, y^*) is a solution of the problem (5.1.2) if and only if x^* is a fixed point of the mapping Q defined by*

$$Q(x) = J_{(M_1,\rho_1)}[J_{(M_2,\rho_2)}(x - \rho_2\Psi_2 x) - \rho_1\Psi_1 J_{(M_2,\rho_2)}(x - \rho_2\Psi_2 x)].$$

Theorem 5.1.4. [27, Lemma 1.9] *Let C be a nonempty closed convex subset of a reflexive, smooth Banach space E which admits a weakly sequentially continuous duality mapping J from E to E^*. Let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty, $f : C \to C$ a contraction with coefficient $\alpha \in (0,1)$, and let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Then the net $\{x_t\}$ defined by*

$$x_t = t\gamma f(x_t) + (1 - tA)Tx_t \tag{5.1.5}$$

converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z)\rangle \leq 0, \ \forall z \in F(T). \tag{5.1.6}$$

Lemma 5.1.5. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the smoothness constant K. Let $\Psi : C \to E$ be a L_Ψ-Lipchitzian and relaxed (c,d)-cocoercive mapping with $d > cL_\Psi^2$. Then*

$$\|(I - \lambda\Psi)x - (I - \lambda\Psi)y\|^2 \leq (1 + 2\lambda c L_\Psi^2 - 2\lambda d + 2\lambda^2 K^2 L_\Psi^2)\|x - y\|^2. \tag{5.1.7}$$

If $0 < \lambda \leq \frac{d - cL_\Psi^2}{K^2 L_\Psi^2}$, then $I - \lambda\Psi$ is nonexpansive.

Proof. Using Lemma 2.1.52 and the cocoercivity of the mapping Ψ, we have, for all $x, y \in C$,

$$\begin{aligned}
\|(I - \lambda\Psi)x - (I - \lambda\Psi)y\|^2 &= \|(x - y) - (\lambda\Psi x - \lambda\Psi y)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle \Psi x - \Psi y, J(x - y)\rangle \\
&\quad + 2\lambda^2 K^2 \|\Psi x - \Psi y\|^2 \\
&\leq \|x - y\|^2 - 2\lambda[-c\|\Psi x - \Psi y\|^2 + d\|x - y\|^2] \\
&\quad + 2\lambda^2 K^2 \|\Psi x - \Psi y\|^2 \\
&= \|x - y\|^2 - 2\lambda d\|x - y\|^2 + 2\lambda c\|\Psi x - \Psi y\|^2 \\
&\quad + 2\lambda^2 K^2 \|\Psi x - \Psi y\|^2
\end{aligned}$$

$$\leq (1 + 2\lambda c L_{\Psi}^2 - 2\lambda d + 2\lambda^2 K^2 L_{\Psi}^2)\|x - y\|^2.$$

Hence (5.1.7) is proved. Assume that $\lambda \leq \frac{d - cL_{\Psi}^2}{K^2 L_{\Psi}^2}$. Then, we have $(1+2\lambda c L_{\Psi}^2 - 2\lambda d + 2\lambda^2 K^2 L_{\Psi}^2) \leq 1$. This together with (5.1.7) implies that $I - \lambda\Psi$ is nonexpansive. □

Lemma 5.1.6. *Let E be a strictly convex and 2-uniformly smooth Banach space admits a weakly sequentially continuous duality mapping with the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipchitzian and relaxed (c_i, d_i)-cocoercive mapping with $d_i > c_i L_i^2$. Let $\rho_i \in (0, \frac{d_i - c_i L_i^2}{K^2 L_i^2})$, respectively for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of uniformly ε-strict pseudo-contractions. Define a mapping $S_n : E \to E$ and $G_n : E \to E$ by*

$$S_n x = (1 - \frac{\varepsilon}{K^2})x + \frac{\varepsilon}{K^2} T_n x, \ \forall x \in C \text{ and } n \geq 1.$$

and

$$G_n = \mu S_n + (1 - \mu) Q,$$

where Q is defined as in Lemma 5.1.3. Assume that $\Omega := \cap_{n=1}^{\infty} F(T_n) \cap F(Q) \neq \emptyset$. Let $f : E \to E$ be an α-contraction, let $A : E \to E$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$ with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Then the following hold :

1. *For each $n \in \mathbb{N}$, G_n is nonexpansive such that*

$$F(G_n) = F(S_n) \cap F(Q) = F(T_n) \cap F(Q).$$

2. *Suppose that $\{G_n\}$ satisfies AKTT-condition. Let $G : E \to E$ be the mapping defined by $Gy = \lim_{n\to\infty} G_n y$ for all $y \in E$ and suppose that $F(G) = \cap_{n=1}^{\infty} F(G_n)$. The net $\{x_t\}$ defined by $x_t = t\gamma f(x_t) + (I - tA)Gx_t$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of G which solves the variational inequality:*

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z)\rangle \leq 0, \ \forall z \in F(G), \tag{5.1.8}$$

and $(\tilde{x}, \tilde{y})$ is a solution of general system of variational inequality problem (5.1.2) such that $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2\Psi_2\tilde{x})$.

Proof. It follows from Lemma 2.1.50 that S_n is nonexpansive such that $F(S_n) = F(T_n)$ for each $n \in \mathbb{N}$. Next, we prove that Q is nonexpansive. Indeed, we observe that

$$\begin{aligned} Q(x) &= J_{(M_1,\rho_1)}[J_{(M_2,\rho_2)}(x - \rho_2\Psi_2 x) - \rho_1\Psi_1 J_{(M_2,\rho_2)}(x - \rho_2\Psi_2 x)]. \\ &= J_{(M_1,\rho_1)}(I - \rho_1\Psi_1)J_{(M_2,\rho_2)}(I - \rho_2\Psi_2)x. \end{aligned}$$

The nonexpansivity of $J_{(M_1,\rho_1)}$, $J_{(M_2,\rho_2)}$, $I - \rho_1\Psi_1$ and $I - \rho_2\Psi_2$ implies that Q is nonexpansive. By Lemma 2.1.58, we have G_n is nonexpansive such that

$$F(G_n) = F(S_n) \cap F(Q) = F(T_n) \cap F(Q) \neq \emptyset, \forall n \in \mathbb{N}.$$

Hence (i) is proved. It is well-known that if E is uniformly smooth, then E is reflexive. Hence Theorem 2.1.53 implies that $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of G which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z)\rangle \leq 0, \ \forall z \in F(G), \tag{5.1.9}$$

and $(\tilde{x}, \tilde{y})$ is a solution of the problem (5.1.2), where $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2\Psi_2\tilde{x})$. This completes the proof of (ii). □

Theorem 5.1.7. *Let E be a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipchitzian and relaxed (c_i, d_i)-cocoercive mapping with $d_i > c_i L_i^2$. Let $\rho_i \in (0, \frac{d_i - c_i L_i^2}{K^2 L_i^2})$, respectively for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of uniformly ε-strict pseudo-contractions. Define a mapping $S_n : E \to E$ by*

$$S_n x = (1 - \frac{\varepsilon}{K^2})x + \frac{\varepsilon}{K^2}T_n x \text{ for all } x \in C \text{ and } n \geq 1.$$

Assume that $\Omega := \cap_{n=1}^{\infty} F(T_n) \cap F(Q) \neq \emptyset$, *where* Q *is defined as Lemma 5.1.3. Let* $f : E \to E$ *be an* α*-contraction, let* $A : E \to E$ *be a strongly positive linear bounded self adjoint operator with coefficient* $\bar{\gamma}$ *with* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. *Let* $x_1 = u \in E$ *and* $\{x_n\}$ *a sequence generated by*

$$\begin{cases} z_n = J_{(M_2,\rho_2)}(x_n - \rho_2 \Psi_2 x_n), \\ y_n = J_{(M_1,\rho_1)}(z_n - \rho_1 \Psi_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)[\mu S_n x_n + (1-\mu) y_n], \quad \forall n \geq 1, \end{cases} \tag{5.1.10}$$

where $\mu \in (0,1)$, *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$. *Suppose that* $\{S_n\}$ *satisfies AKTT-condition. Let* $S : E \to E$ *be the mapping defined by* $Sy = \lim_{n\to\infty} S_n y$ *for all* $y \in E$ *and suppose that* $F(S) = \cap_{n=1}^{\infty} F(S_n)$. *If the control consequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following restrictions*

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then $\{x_n\}$ *converges strongly to* $\tilde{x}$ *which solves the variational inequality:*

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z) \rangle \leq 0, z \in \Omega,$$

and $(\tilde{x}, \tilde{y})$ *is a solution of general system of variational inequality problem (5.1.2) such that* $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2 \Psi_2 \tilde{x})$.

Proof. First, we show that sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.
By the control condition (C2), we may assume, with no loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$.
Since A is a linear bounded operator on E, by (5.1.1), we have

$$\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in E, \|u\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I-\alpha_n A)u, J(u)\rangle &= 1-\beta_n-\alpha_n\langle Au, J(u)\rangle\\ &\geq 1-\beta_n-\alpha_n\|A\|\\ &\geq 0.\end{aligned}$$

It follows that

$$\begin{aligned}\|(1-\beta_n)I-\alpha_n A\| &= \sup\{\langle((1-\beta_n)I-\alpha_n A)u, J(u)\rangle : u\in E, \|u\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Au, J(u)\rangle : u\in E, \|u\|=1\}\\ &\leq 1-\beta_n-\alpha_n\bar{\gamma}.\end{aligned}$$

Therefore, taking $\bar{x}\in\Omega$, one has

$$\bar{x}=J_{(M_1,\rho_1)}[J_{(M_2,\rho_2)}(\bar{x}-\rho_2\Psi_2\bar{x})-\rho_1\Psi_1 J_{(M_2,\rho_2)}(\bar{x}-\rho_2\Psi_2\bar{x})]. \tag{5.1.11}$$

Putting $\bar{y}=J_{(M_2,\rho_2)}(\bar{x}-\rho_2\Psi_2\bar{x})$, one sees that

$$\bar{x}=J_{(M_1,\rho_1)}(\bar{y}-\rho_1\Psi_1\bar{y}). \tag{5.1.12}$$

It follows from Lemma 2.1.61 and Lemma 5.1.5 that

$$\begin{aligned}\|z_n-\bar{y}\| &= \|J_{(M_2,\rho_2)}(x_n-\rho_2\Psi_2 x_n)-J_{(M_2,\rho_2)}(\bar{x}-\rho_2\Psi_2\bar{x})\|\\ &\leq \|(x_n-\rho_2\Psi_2 x_n)-(\bar{x}-\rho_2\Psi_2\bar{x})\|\\ &\leq \|x_n-\bar{x}\|. \end{aligned}\tag{5.1.13}$$

This implies that

$$\begin{aligned}\|y_n-\bar{x}\| &= \|J_{(M_1,\rho_1)}(z_n-\rho_1\Psi_1 z_n)-J_{(M_1,\rho_1)}(\bar{y}-\rho_1\Psi_1\bar{y})\|\\ &\leq \|(z_n-\rho_1\Psi_1 z_n)-(\bar{y}-\rho_1\Psi_1\bar{y})\|\\ &\leq \|z_n-\bar{y}\|\end{aligned}$$

$$\leq \|x_n - \bar{x}\|. \tag{5.1.14}$$

Setting $t_n = \mu S_n x_n + (1-\mu)y_n$ and applying Lemma 2.1.50, we have S_n is a nonexpansive mapping such that $F(S_n) = F(T_n)$ for all $n \geq 1$ and hence $\cap_{n=1}^{\infty} F(S_n) = \cap_{n=1}^{\infty} F(T_n)$. Then

$$\begin{aligned} \|t_n - \bar{x}\| &= \|\mu S_n x_n + (1-\mu)y_n - \bar{x}\| \\ &\leq \mu\|S_n x_n - \bar{x}\| + (1-\mu)\|y_n - \bar{x}\| \\ &\leq \|x_n - \bar{x}\|. \end{aligned} \tag{5.1.15}$$

It follows from the last inequality that

$$\begin{aligned} \|x_{n+1} - \bar{x}\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)t_n - \bar{x}\| \\ &= \|\alpha_n(\gamma f(x_n) - A\bar{x}) + \beta_n(x_n - \bar{x}) + ((1-\beta_n)I - \alpha_n A)(t_n - \bar{x})\| \\ &\leq (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - \bar{x}\| + \beta_n\|x_n - \bar{x}\| + \alpha_n\|\gamma f(x_n) - A\bar{x}\| \\ &\leq (1 - \alpha_n\bar{\gamma})\|x_n - \bar{x}\| + \alpha_n\gamma\alpha\|x_n - \bar{x}\| + \alpha_n\|\gamma f(\bar{x}) - A\bar{x}\| \\ &= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - \bar{x}\| + \alpha_n\|\gamma f(\bar{x}) - A\bar{x}\|. \end{aligned}$$

By induction, we have

$$\|x_n - \bar{x}\| \leq \max\left\{\|x_1 - \bar{x}\|, \frac{\|\gamma f(\bar{x}) - A\bar{x}\|}{\bar{\gamma} - \gamma\alpha}\right\}, \quad n \geq 1.$$

This shows that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$.

On the other hand, from the nonexpansivity of the mappings $J_{(M_2,\rho_2)}$, one sees that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|J_{(M_1,\rho_1)}(z_{n+1} - \rho_1\Psi_1 z_{n+1}) - J_{(M_1,\rho_1)}(z_n - \rho_1\Psi_1 z_n)\| \\ &\leq \|(z_{n+1} - \rho_1\Psi_1 z_{n+1}) - (z_n - \rho_1\Psi_1 z_n)\| \\ &\leq \|z_{n+1} - z_n\|. \end{aligned} \tag{5.1.16}$$

In a similar way, one can obtain that

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\|. \tag{5.1.17}$$

It follows that

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|. \tag{5.1.18}$$

This implies that

$$\begin{aligned}
\|t_{n+1} - t_n\| &= \|\mu S_{n+1}x_{n+1} + (1-\mu)y_{n+1} - (\mu S_n x_n + (1-\mu)y_n)\| \\
&= \|\mu S_{n+1}x_{n+1} - \mu S_{n+1}x_n + (1-\mu)y_{n+1} + \mu S_{n+1}x_n - \mu S_n x_n - (1-\mu)y_n\| \\
&\leq \mu\|S_{n+1}x_{n+1} - S_{n+1}x_n\| + (1-\mu)\|y_{n+1} - y_n\| + \mu\|S_{n+1}x_n - S_n x_n\| \\
&\leq \mu\|x_{n+1} - x_n\| + (1-\mu)\|x_{n+1} - x_n\| + \mu \sup_{z\in\{x_n\}} \|S_{n+1}z - S_n z\| \\
&= \|x_{n+1} - x_n\| + \mu \sup_{z\in\{x_n\}} \|S_{n+1}z - S_n z\|. \qquad (5.1.19)
\end{aligned}$$

Setting

$$x_{n+1} = (1-\beta_n)e_n + \beta_n x_n, \quad \forall n \geq 1, \tag{5.1.20}$$

one sees that

$$\begin{aligned}
& e_{n+1} - e_n \\
= & \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)t_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)t_n}{1-\beta_n} \\
= & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - At_{n+1}) + t_{n+1} - \frac{\alpha_n}{1-\beta_n}(\gamma f(x_n) - At_n) - t_n
\end{aligned}$$

and so it follows that

$$\|e_{n+1} - e_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - At_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - At_n\| + \|t_{n+1} - t_n\|,$$

which combines with (5.1.19) yields that

$$\begin{aligned}\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\| \leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-At_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-At_n\| \\ & +\mu \sup_{z\in\{x_n\}}\|S_{n+1}z-S_nz\|. \end{aligned} \tag{5.1.21}$$

Using the conditions (C1) and (C2) and AKTT-condition of $\{S_n\}$, we have

$$\limsup_{n\to\infty}(\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Hence, from Lemma 2.1.56, it follows that

$$\lim_{n\to\infty}\|e_n-x_n\|=0. \tag{5.1.22}$$

From (5.1.20), it follows that

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|e_n-x_n\|.$$

By (5.1.22), one sees that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{5.1.23}$$

On the other hand, one has

$$x_{n+1}-x_n = \alpha_n(\gamma f(x_n)-Ax_n)+((1-\beta_n)I-\alpha_nA)(t_n-x_n). \tag{5.1.24}$$

It follows that

$$(1-\beta_n-\alpha_n\bar{\gamma})\|t_n-x_n\| \leq \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-Ax_n\|. \tag{5.1.25}$$

From the conditions (C1), (C2) and (5.1.23), one sees that

$$\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{5.1.26}$$

Define the mapping G_n by

$$G_n = \mu S_n + (1-\mu)Q,$$

where Q is defined as in Lemma 5.1.3. From Lemma 6.2.1 (i), we see that G_n is nonexpansive such that

$$F(G_n) = F(T_n) \cap F(Q) = F(S_n) \cap F(Q). \tag{5.1.27}$$

From (5.1.26), it follows that

$$\lim_{n\to\infty} \|G_n x_n - x_n\| = 0. \tag{5.1.28}$$

Since $\{S_n\}$ satisfies AKTT-condition and $S : E \to E$ is the mapping defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in E$, we have $\{G_n\}$ satisfies AKTT-condition. Let the mapping $G : E \to E$ is the mapping defined by $Gy = \lim_{n\to\infty} G_n y$ for all $y \in E$. It follows from the nonexpansivity of S and

$$Gy = \mu S y + (1-\mu)Q$$

that G is nonexpansive such that

$$F(G) = F(S) \cap F(Q) = \cap_{n=1}^{\infty} F(S_n) \cap F(Q) = \cap_{n=1}^{\infty} F(T_n) \cap F(Q) = \cap_{n=1}^{\infty} F(G_n).$$

Next, we prove that

$$\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_n - \tilde{x})\rangle \le 0, \tag{5.1.29}$$

where $\tilde{x} = \lim_{t\to 0} x_t$ with x_t be the fixed point of the contraction

$$x \mapsto t\gamma f(x) + (I - tA)Gx.$$

Then x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)Gx_t$. It follows from Lemma 5.1.6 (ii) that $\tilde{x} \in F(G) = \Omega$, which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z)\rangle \le 0, \ \forall z \in F(G),$$

and $(\tilde{x}, \tilde{y})$ is a solution of general system of variational inequality problem (5.1.2) such that $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2\Psi_2\tilde{x})$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n_k} - \tilde{x})\rangle = \limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_n - \tilde{x})\rangle. \tag{5.1.30}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in C$ as $i \to \infty$. It follows from (5.1.28) and the nonexpansivity of G, we have $w \in F(G)$ by Lemma 2.1.59. Since the duality map J is single-valued and weakly sequentially continuous from E to E^*, we get that

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_n - \tilde{x})\rangle &= \lim_{k\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n_k} - \tilde{x})\rangle \\ &= \lim_{i\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n_{k_i}} - \tilde{x})\rangle \\ &= \langle (A - \gamma f)\tilde{x}, J(\tilde{x} - w)\rangle \leq 0\end{aligned}$$

as required. Now from Lemma 2.1.52, we have

$$\begin{aligned}&\|x_{n+1} - \tilde{x}\|^2 \\ = \;& \|\alpha_n\gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n A]t_n - \tilde{x}\|^2 \\ = \;& \|[(1-\beta_n)I - \alpha_n A](t_n - \tilde{x}) + \alpha_n(\gamma f(x_n) - A\tilde{x}) + \beta_n(x_n - \tilde{x})\|^2 \\ \leq \;& (1-\beta_n-\alpha_n\bar{\gamma})^2\|t_n - \tilde{x}\|^2 + 2\langle \alpha_n(\gamma f(x_n) - A\tilde{x}) + \beta_n(x_n - \tilde{x}), J(x_{n+1} - \tilde{x})\rangle \\ = \;& (1-\beta_n-\alpha_n\bar{\gamma})^2\|t_n - \tilde{x}\|^2 + 2\beta_n\langle x_n - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\ & + 2\alpha_n\langle \gamma f(x_n) - A\tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\ = \;& (1-\beta_n-\alpha_n\bar{\gamma})^2\|t_n - \tilde{x}\|^2 + 2\beta_n\langle x_n - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\ & + 2\alpha_n\langle \gamma f(x_n) - \gamma f(\tilde{x}), J(x_{n+1} - \tilde{x})\rangle + 2\alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\ \leq \;& (1-\beta_n-\alpha_n\bar{\gamma})^2\|t_n - \tilde{x}\|^2 + 2\beta_n\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| \\ & + 2\alpha_n\|\gamma f(x_n) - \gamma f(\tilde{x})\|\|x_{n+1} - \tilde{x}\| + 2\alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\ \leq \;& (1-\beta_n-\alpha_n\bar{\gamma})^2\|x_n - \tilde{x}\|^2 + \beta_n(\|x_{n+1} - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2) \\ & + \alpha_n\gamma\alpha(\|x_{n+1} - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2) + 2\alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n+1} - \tilde{x})\rangle\end{aligned}$$

$$= [(1-\beta_n-\alpha_n\bar{\gamma})^2+\beta_n+\alpha_n\gamma\alpha]\|x_n-\tilde{x}\|^2+(\beta_n+\alpha_n\gamma\alpha)\|x_{n+1}-\tilde{x}\|^2$$
$$+2\alpha_n\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle, \qquad (5.1.31)$$

which implies that

$$\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2\\
\leq\ & \frac{(1-\beta_n-\alpha_n\bar{\gamma})^2+\beta_n+\alpha_n\gamma\alpha}{1-\beta_n-\alpha_n\gamma\alpha}\|x_n-\tilde{x}\|^2+\frac{2\alpha_n}{1-\beta_n-\alpha_n\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle\\
=\ & \left[1-\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\beta_n-\alpha_n\gamma\alpha}\right]\|x_n-\tilde{x}\|^2+\frac{\beta_n^2+2\beta_n\alpha_n\bar{\gamma}+\alpha_n^2\bar{\gamma}^2}{1-\beta_n-\alpha_n\gamma\alpha}\|x_n-\tilde{x}\|^2\\
&+\frac{2\alpha_n}{1-\beta_n-\alpha_n\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle\\
=\ & \left[1-\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\beta_n-\alpha_n\gamma\alpha}\right]\|x_n-\tilde{x}\|^2+\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\beta_n-\alpha_n\gamma\alpha}\left[\frac{\beta_n^2+2\beta_n\alpha_n\bar{\gamma}+\alpha_n^2\bar{\gamma}^2}{2\alpha_n(\bar{\gamma}-\gamma\alpha)}M_3\right.\\
&\left.+\frac{1}{\bar{\gamma}-\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle\right], \qquad (5.1.32)
\end{aligned}$$

where M_3 is an appropriate constant such that $M_3 \geq \sup_{n\geq 0}\|x_n-\tilde{x}\|^2$. Put

$$j_n=\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\beta_n-\alpha_n\gamma\alpha} \text{ and } k_n=\frac{\beta_n^2+2\beta_n\alpha_n\bar{\gamma}+\alpha_n^2\bar{\gamma}^2}{2\alpha_n(\bar{\gamma}-\gamma\alpha)}M_3+\frac{1}{\bar{\gamma}-\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle,$$

that is,

$$\|x_{n+1}-\tilde{x}\|^2\leq(1-j_n)\|x_n-\tilde{x}\|^2+j_nk_n. \qquad (5.1.33)$$

It follows that from conditions (C1), (C2) and (5.1.29) that

$$\lim_{n\to\infty} j_n=0,\quad \sum_{n=1}^{\infty} j_n=\infty \text{ and } \limsup_{n\to\infty} k_n\leq 0.$$

Apply Lemma 2.2.10 to (5.1.33) to conclude $x_n\to\tilde{x}$ as $n\to\infty$. This completes the proof. □

Setting $A\equiv I, \gamma=1,\ f:=u$, we have the following result.

Theorem 5.1.8. *Let E be a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i: E\to 2^E$ be a maximal monotone mapping and $\Psi_i: E\to$*

E a L_i-Lipchitzian and relaxed (c_i, d_i)-cocoercive mapping with $d_i > c_iL_i^2$. Let $\rho_i \in (0, \frac{d_i - c_iL_i^2}{K^2L_i^2})$, respectively for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of uniformly ε-strict pseudo-contractions. Define a mapping $S_n : E \to E$ by

$$S_nx = (1 - \frac{\varepsilon}{K^2})x + \frac{\varepsilon}{K^2}T_nx \text{ for all } x \in C \text{ and } n \geq 1.$$

Assume that $\Omega := \cap_{n=1}^{\infty}F(T_n) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 5.1.3. Let $x_1 = u \in E$ and $\{x_n\}$ a sequence generated by

$$\begin{cases} z_n = J_{(M_2,\rho_2)}(x_n - \rho_2\Psi_2x_n), \\ y_n = J_{(M_1,\rho_1)}(z_n - \rho_1\Psi_1z_n), \\ x_{n+1} = \alpha_nu + \beta_nx_n + (1 - \beta_n - \alpha_n)[\mu S_nx_n + (1-\mu)y_n], \quad \forall n \geq 1, \end{cases} \tag{5.1.34}$$

where $\mu \in (0,1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that $\{S_n\}$ satisfies AKTT-condition. Let $S : E \to E$ be the mapping defined by $Sy = \lim_{n\to\infty} S_ny$ for all $y \in E$ and suppose that $F(S) = \cap_{n=1}^{\infty}F(S_n)$. If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality:

$$\langle (I - f)\tilde{x}, J(\tilde{x} - z)\rangle \leq 0, z \in \Omega,$$

and $(\tilde{x}, \tilde{y})$ is a solution of general system of variational inequality problem (5.1.2) such that $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2\Psi_2\tilde{x})$.

Remark 5.1.9. Theorem 5.1.7 mainly improves Theorem 2.1 of Qin et al. [68], in the following respects:

(a) From the class of inverse-strongly accretive mappings to the class of Lipchitzian and relaxed cocoercive mappings.

(b) From a ε-strict pseudo-contraction to the countable family of uniformly ε-strict pseudo-contractions.

(c) From a uniformly convex and 2-uniformly smooth Banach space to a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping.

Further, if $\{T_n : E \to E\}$ be a countable family of nonexpansive mappings, then Theorem 5.1.7 is reduced to the following result.

Theorem 5.1.10. *Let E be a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipchitzian and relaxed (c_i, d_i)-cocoercive mapping with $d_i > c_i L_i^2$. Let $\rho_i \in (0, \frac{d_i - c_i L_i^2}{K^2 L_i^2})$, respectively for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings. Assume that $\Omega := \cap_{n=1}^{\infty} F(T_n) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 5.1.3. Let $f : E \to E$ be an α-contraction, let $A : E \to E$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$ with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $x_1 = u \in E$ and $\{x_n\}$ a sequence generated by*

$$\begin{cases} z_n = J_{(M_2,\rho_2)}(x_n - \rho_2 \Psi_2 x_n), \\ y_n = J_{(M_1,\rho_1)}(z_n - \rho_1 \Psi_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)[\mu T_n x_n + (1-\mu) y_n], \quad \forall n \geq 1, \end{cases} \tag{5.1.35}$$

where $\mu \in (0,1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that $\{T_n\}$ satisfies AKTT-condition. Let $T : E \to E$ be the mapping defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in E$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

then $\{x_n\}$ *converges strongly to* $\tilde{x}$ *which solves the variational inequality:*

$$\langle (A-\gamma f)\tilde{x}, J(\tilde{x}-z)\rangle \leq 0, z \in \Omega,$$

and $(\tilde{x},\tilde{y})$ *is a solution of general system of variational inequality problem (5.1.2) such that* $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x}-\rho_2\Psi_2\tilde{x})$.

Remark 5.1.11. As in [33, Theorem 4.1], we can generate a sequence $\{T_n\}$ of nonexpansive mappings satisfying AKTT-condition i.e. $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_nz\| : z \in B\} < \infty$ for any bounded subset B of E by using convex combination of a general sequence $\{S_k\}$ of nonexpansive mappings with a common fixed point. To be more precise, they obtained the following lemma.

Lemma 5.1.12. [33] *Let C be a closed convex subset of a smooth Banach space E. Suppose that $\{S_k\}$ is a sequence of nonexpansive mappings of E into itself with a common fixed point. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by*

$$T_nx = \sum_{k=1}^{n}\beta_n^kS_kx, \quad \forall x \in E, \tag{5.1.36}$$

where $\{\beta_n^k\}$ is a family of nonnegative numbers with indices $n,k \in \mathbb{N}$ with $k \leq n$ such that

(i) $\sum_{k=1}^{n}\beta_n^k = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty}\beta_n^k > 0$ *for every* $k \in \mathbb{N}$;

(iii) $\sum_{n=1}^{\infty}\sum_{k=1}^{n}|\beta_{n+1}^k - \beta_n^k| < \infty$.

Then

(1) Each T_n is a nonexpansive mapping.

(2) $\{T_n\}$ satisfies AKTT-condition.

(3) If $T : C \to C$ is defined by

$$Tx = \sum_{k=1}^{\infty}\beta_n^kS_kx, \quad \forall x \in C,$$

then $Tx = \lim_{n\to\infty}T_nx$ *and* $F(T) = \cap_{n=1}^{\infty}F(T_n) = \cap_{k=1}^{\infty}F(S_k)$.

Theorem 5.1.13. *Let E be a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipchitzian and relaxed (c_i, d_i)-cocoercive mapping with $d_i > c_i L_i^2$. Let $\rho_i \in (0, \frac{d_i - c_i L_i^2}{K^2 L_i^2})$, respectively for each $i = 1, 2$. Let $\{S_k : E \to E\}_{k=1}^{\infty}$ be a countable family of nonexpansive mappings. Assume that $\Omega := \cap_{k=1}^{\infty} F(S_k) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 5.1.3. Let $f : E \to E$ be an α-contraction, let $A : E \to E$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$ with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $x_1 = u \in E$ and $\{x_n\}$ a sequence generated by*

$$\begin{cases} z_n = J_{(M_2,\rho_2)}(x_n - \rho_2 \Psi_2 x_n), \\ y_n = J_{(M_1,\rho_1)}(z_n - \rho_1 \Psi_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)[\mu \sum_{k=1}^{n} \beta_n^k S_k x_n + (1-\mu) y_n], \quad \forall n \geq 1, \end{cases} \tag{5.1.37}$$

where $\{\beta_n^k\}$ satisfies conditions (i)-(iii) of Lemma 5.1.12, $\mu \in (0,1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that $\{T_n\}$ satisfies AKTT-condition. Let $T : E \to E$ be the mapping defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in E$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z) \rangle \leq 0, z \in \Omega,$$

and $(\tilde{x}, \tilde{y})$ is a solution of general system of variational inequality problem (5.1.2) such that $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2 \Psi_2 \tilde{x})$.

Proof. We write the iteration (5.1.37) as

$$\begin{cases} z_n = J_{(M_2,\rho_2)}(x_n - \rho_2\Psi_2 x_n), \\ y_n = J_{(M_1,\rho_1)}(z_n - \rho_1\Psi_1 z_n), \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)[\mu T_n x_n + (1-\mu)y_n], \quad \forall n \geq 1, \end{cases}$$

where T_n is defined by (5.1.36). It is clear that each mapping T_n is nonexpansive. By Theorem 5.1.10 and Lemma 5.1.12, the conclusion follows. □

The following example appears in [33] shows that there exists $\{\beta_n^k\}$ satisfying the conditions of Lemma 5.1.12.

Example 5.1.14. *Let $\{\beta_n^k\}$ be defined by*

$$\beta_n^k = \begin{cases} 2^{-k} & (k < n) \\ 2^{1-k} & (k = n), \end{cases}$$

for all $n, k \in \mathbb{N}$ with $k \leq n$. In this case, the sequence $\{T_n\}$ of mappings generated by $\{S_k\}$ is defined as follows: For $x \in C$,

$$T_1 x = S_1 x,$$

$$T_2 x = \tfrac{1}{2}S_1 x + \tfrac{1}{2}S_2 x,$$

$$T_3 x = \tfrac{1}{2}S_1 x + \tfrac{1}{4}S_2 x + \tfrac{1}{4}S_3 x,$$

$$T_4 x = \tfrac{1}{2}S_1 x + \tfrac{1}{4}S_2 x + \tfrac{1}{8}S_3 x + \tfrac{1}{8}S_4 x,$$

$$\vdots$$

$$T_n x = \tfrac{1}{2}S_1 x + \tfrac{1}{4}S_2 x + \tfrac{1}{8}S_3 x + \tfrac{1}{16}S_4 x + \cdots + \tfrac{1}{2^{n-1}}S_{n-1}x + \tfrac{1}{2^{n-1}}S_n x.$$

5.2 Strong convergence theorems of viscosity iterative methods for a countable family of strict pseudo-contractions in Banach spaces

In this section, we consider a countable family $\{T_n\}_{n=1}^{\infty}$ of strictly pseudo-contractions, a strong convergence of viscosity iteration is shown in order to find a common fixed point of $\{T_n\}_{n=1}^{\infty}$ in either p-uniformly convex Banach space which admits a weakly continuous duality mapping or p-uniformly convex Banach space with uniformly Gâteaux differentiable norm.

Definition 5.2.1. A countable family of mapping $\{T_n : C \to C\}_{i=1}^{\infty}$ is called a *family of uniformly λ-strict pseudo-contractions with respect to p*, if there exists a constant $\lambda \in [0,1)$ such that

$$\|T_n x - T_n y\|^p \leq \|x-y\|^p + \lambda\|(I-T_n)x - (I-T_n)y\|^p, \quad \forall x,y \in C, \quad \forall n \geq 1.$$

For $T : C \to C$ a nonexpansive mapping, $t \in (0,1)$ and $f \in \Pi_C$, $tf+(1-t)T : C \to C$ defines a contraction mapping. Thus, by the Banach contraction mapping principle, there exists a unique fixed point x_t^f satisfying

$$x_t^f = tf(x_t) + (1-t)Tx_t^f. \tag{5.2.1}$$

For simplicity we will write x_t for x_t^f provided no confusion occurs. Next, we will prove the following lemma.

Lemma 5.2.2. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ. Let C be a nonempty closed convex subset of E, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Then the net $\{x_t\}$ defined by (5.2.1) converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality :*

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x}-z)\rangle \leq 0, z \in F(T). \tag{5.2.2}$$

Proof. We first show that the uniqueness of a solution of the variational inequality (5.2.2). Suppose both $\tilde{x} \in F(T)$ and $x^* \in F(T)$ are solutions to (5.2.2), then

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x}-x^*)\rangle \le 0 \tag{5.2.3}$$

and

$$\langle (I-f)x^*, J_\varphi(x^*-\tilde{x})\rangle \le 0. \tag{5.2.4}$$

Adding (5.2.3) and (5.2.4), we obtain

$$\langle (I-f)\tilde{x}-(I-f)x^*, J_\varphi(\tilde{x}-x^*)\rangle \le 0. \tag{5.2.5}$$

Noticing that for any $x, y \in E$,

$$\begin{aligned}\langle (I-f)x-(I-f)y, J_\varphi(x-y)\rangle &= \langle x-y, J_\varphi(x-y)\rangle - \langle f(x)-f(y), J_\varphi(x-y)\rangle \\ &\ge \|x-y\|\varphi(\|x-y\|) - \|f(x)-f(y)\|\varphi(\|x-y\|) \\ &\ge \Phi(\|x-y\|) - \alpha\Phi(\|x-y\|) \\ &= (1-\alpha)\Phi(\|x-y\|) \ge 0. \end{aligned} \tag{5.2.6}$$

From (5.2.5), we conclude that $\Phi(\|\tilde{x}-x^*\|)=0$. This implies that $\tilde{x}=x^*$ and the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (5.2.2). Next, we will prove that $\{x_t\}$ is bounded. Take a $p \in F(T)$, then we have

$$\begin{aligned}\|x_t-p\| &= \|tf(x_t)+(1-t)Tx_t-p\| \\ &= \|(1-t)Tx_t-(1-t)p+t(f(x_t)-p)\| \\ &\le (1-t)\|x_t-p\|+t(\alpha\|x_t-p\|+\|f(p)-p\|).\end{aligned}$$

It follows that

$$\|x_t-p\| \le \frac{1}{1-\alpha}\|f(p)-p\|.$$

Hence $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{T(x_t)\}$. The definition of $\{x_t\}$ implies that

$$\|x_t-Tx_t\| = t\|f(x_t)-Tx_t\| \to 0 \text{ as } t \to 0. \tag{5.2.7}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_t\}$ that there exists $\{x_{t_n}\}$ which is a subsequence of $\{x_t\}$ converging weakly to $w \in C$ as $n \to \infty$. Since J_φ is weakly sequentially continuous, we have by Lemma 2.1.43 that

$$\limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - w\|) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

Let

$$H(x) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|), \text{ for all } x \in E.$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

Since

$$\|x_{t_n} - Tx_{t_n}\| = t_n\|f(x_{t_n}) - Tx_{t_n}\| \to 0 \text{ as } n \to \infty.$$

We obtain

$$\begin{aligned} H(Tw) &= \limsup_{n\to\infty} \Phi(\|x_{t_n} - Tw\|) = \limsup_{n\to\infty} \Phi(\|Tx_{t_n} - Tw\|) \\ &\leq \limsup_{n\to\infty} \Phi(\|x_{t_n} - w\|) = H(w). \end{aligned} \tag{5.2.8}$$

On the other hand, however,

$$H(Tw) = H(w) + \Phi(\|T(w) - w\|). \tag{5.2.9}$$

It follows from (5.2.8) and (5.2.9) that

$$\Phi(\|T(w) - w\|) = H(Tw) - H(w) \leq 0.$$

This implies that $Tw = w$. Next we show that $x_{t_n} \to w$ as $n \to \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau, \forall t \geq 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t).$$

Following Lemma 2.1.43, we have

$$\begin{aligned}
\Phi(\|x_{t_n} - w\|) &= \Phi(\|(1-t_n)Tx_{t_n} - (1-t_n)w + t_n(f(x_{t_n}) - w)\|) \\
&= \Phi(\|(1-t_n)Tx_{t_n} - (1-t_n)w\|) + t_n\langle f(x_{t_n}) - w, J(x_{t_n} - w)\rangle \\
&\leq \Phi((1-t_n)\|x_{t_n} - w\|) + t_n\langle f(x_{t_n}) - f(w), J(x_{t_n} - w)\rangle \\
&\quad + t_n\langle f(w) - w, J(x_{t_n} - w)\rangle \\
&\leq (1-t_n)\Phi(\|x_{t_n} - w\|) + t_n\|f(x_{t_n}) - f(w)\|\|J(x_{t_n} - w)\| \\
&\quad + t_n\langle f(w) - w, J(x_{t_n} - w)\rangle \\
&\leq (1-t_n)\Phi(\|x_{t_n} - w\|) + t_n\alpha\|x_{t_n} - w\|\|J_\varphi(x_{t_n} - w)\| \\
&\quad + t_n\langle f(w) - w, J(x_{t_n} - w)\rangle \\
&= (1-t_n)\Phi(\|x_{t_n} - w\|) + t_n\alpha\Phi(\|x_{t_n} - w\|) \\
&\quad + t_n\langle f(w) - w, J(x_{t_n} - w)\rangle \\
&= (1-t_n(1-\alpha))\Phi(\|x_{t_n} - w\|) \\
&\quad + t_n\langle f(w) - w, J(x_{t_n} - w)\rangle. \qquad (5.2.10)
\end{aligned}$$

This implies that

$$\Phi(\|x_{t_n} - w\|) \leq \frac{1}{1-\alpha}\langle f(w) - w, J(x_{t_n} - w)\rangle.$$

Now observing that $x_{t_n} \rightharpoonup w$ implies $J_\varphi(x_{t_n} - w) \rightharpoonup 0$, we conclude from the last inequality that

$$\Phi(\|x_{t_n} - w\|) \to 0 \text{ as } n \to \infty.$$

Hence $x_{t_n} \to w$ as $n \to \infty$. Next we prove that w solves the variational inequality (5.2.2). For any $z \in F(T)$, we observe that

$$\begin{aligned}
\langle (I-T)x_t - (I-T)z, J_\varphi(x_t - z)\rangle &= \langle x_t - z, J_\varphi(x_t - z)\rangle + \langle Tx_t - Tz, J_\varphi(x_t - z)\rangle \\
&= \Phi(\|x_t - z\|) - \langle Tz - Tx_t, J_\varphi(x_t - z)\rangle \\
&\geq \Phi(\|x_t - z\|) - \|Tz - Tx_t\|\|J_\varphi(x_t - z)\|
\end{aligned}$$

$$\begin{aligned} &\geq \Phi(\|x_t - z\|) - \|z - x_t\|\|J_\varphi(x_t - z)\| \\ &= \Phi(\|x_t - z\|) - \Phi(\|x_t - z\|) = 0. \qquad (5.2.11) \end{aligned}$$

Since

$$x_t = tf(x_t) + (1-t)Tx_t,$$

we can derive that

$$(I - f)(x_t) = -\frac{1}{t}(I - T)x_t + (I - T)x_t.$$

Thus

$$\begin{aligned} \langle (I - f)(x_t), J_\varphi(x_t - z)\rangle &= -\frac{1}{t}\langle (I - T)x_t - (I - T)z, J_\varphi(x_t - z)\rangle \\ &\quad + \langle (I - T)x_t, J_\varphi(x_t - z)\rangle \\ &\leq \langle (I - T)x_t, J_\varphi(x_t - z)\rangle. \qquad (5.2.12) \end{aligned}$$

Noticing that

$$x_{t_n} - Tx_{t_n} \to w - T(w) = w - w = 0.$$

Now replacing t in (5.2.12) with t_n and letting $n \to \infty$, we have

$$\langle (I - f)w, J_\varphi(w - z)\rangle \leq 0.$$

So, $w \in F(T)$ is a solution of the variational inequality (5.2.2), and hence $w = \tilde{x}$ by the uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$(at $t \to 0$) equals $\tilde{x}$. Therefore, $x_t \to \tilde{x}$ as $t \to 0$. This completes the proof. □

Theorem 5.2.3. *Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_φ, and C a nonempty closed convex subset of E. Let $\{T_n : C \to C\}$ be a family of uniformly λ-strict pseudo-contractions with respect to p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:*

1. $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

2. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$;

3. $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \xi$, *where* $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ *be the sequence generated by the following*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \geq 1. \end{cases} \tag{5.2.13}$$

Suppose that $\{T_n\}$ *satisfies the AKTT-condition. Let* T *be a mapping of* C *into itself defined by* $Tz = \lim_{n\to\infty} T_n z$ *for all* $z \in C$ *and suppose that* $F(T) = \cap_{n=1}^{\infty} F(T_n)$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\tilde{x}$ *which solves the variational inequality :*

$$\langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T). \tag{5.2.14}$$

Proof. Rewrite the iterative sequence (5.2.13) as follows:

$$x_{n+1} = \alpha_n f(x_n) + \beta'_n x_n + \gamma'_n S_n x_n, \quad n \geq 1, \tag{5.2.15}$$

where $\beta'_n = \beta_n - \frac{\gamma_n}{\xi}(1 - \xi)$, $\gamma'_n = \frac{\gamma_n}{\xi}$ and $S_n := (1 - \xi)I + \xi T_n$, I is the identity mapping. By Lemma 2.1.54, S_n is nonexpansive such that $F(S_n) = F(T_n)$ for all $n \in \mathbb{N}$. Taking any $q \in \cap_{n=1}^{\infty} F(T_n)$. From (5.2.15), it implies that

$$\begin{aligned} \|x_{n+1} - q\| &\leq \alpha_n \|f(x_n) - q\| + \beta'_n \|x_n - q\| + \gamma'_n \|S_n x_n - q\| \\ &\leq \alpha_n k \|x_n - q\| + \alpha_n \|f(q) - q\| + (1 - \alpha_n)\|x_n - q\| \\ &= \alpha_n (1 - k) \frac{1}{1-k} \|f(q) - q\| + (1 - \alpha_n(1 - k))\|x_n - q\| \\ &\leq \max\left\{ \|x_1 - q\|, \frac{1}{1-k}\|f(q) - q\| \right\}. \end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded, and so are the sequences $\{f(x_n)\}$, $\{S_n x_n\}$. Since $S_n x_n = (1 - \xi_n)x_n + \xi_n T_n x_n$ and $\liminf \xi_n > 0$, we know that $\{T_n x_n\}$ is bounded. We note that for any bounded subset B of C,

$$\begin{aligned}
&\sup_{z\in B}\|S_{n+1}z - S_n z\| \\
=\ & \sup_{z\in B}[\|((1-\xi_{n+1})z + \xi_{n+1}T_{n+1}z) - ((1-\xi_n)z + \xi_n T_n z)\|] \\
\le\ & |\xi_{n+1} - \xi_n| \sup_{z\in B}\|z\| + \xi_{n+1}\sup_{z\in B}\|T_{n+1}z - T_n z\| + |\xi_{n+1} - \xi_n| \sup_{z\in B}\|T_n z\| \\
=\ & |\xi_{n+1} - \xi_n| \sup_{z\in B}(\|z\| + \|Tz\|) + \xi_{n+1}\sup_{z\in B}\|T_{n+1}z - T_n z\|.
\end{aligned}$$

From $\sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty$ and $\{T_n\}$ satisfies AKTT-condition, we obtain that

$$\sum_{n=1}^{\infty} \sup_{z\in B}\|S_{n+1}z - S_n z\| < \infty,$$

that is the sequence $\{S_n\}$ satisfies AKTT-condition. Applying Lemma 2.2.9, we can take the mapping $S : C \to C$ defined by

$$Sz = \lim_{n\to\infty} S_n z, \quad \forall z \in C. \tag{5.2.16}$$

Moreover, we have S is nonexpansive and

$$Sz = \lim_{n\to\infty} S_n z = \lim_{n\to\infty}((1-\xi_n)z + \xi_n T_n z) = (1-\xi)z + \xi Tz.$$

It is easy to see that $F(S) = F(T)$. Hence $F(S) = \cap_{n=1}^{\infty} F(T_n) = \cap_{n=1}^{\infty} F(S_n)$. The iterative sequence (5.2.15) can be expressed as follows:

$$x_{n+1} = \beta'_n x_n + (1-\beta'_n) y_n,$$

where

$$y_n = \frac{\alpha_n}{1-\beta'_n} f(x_n) + \frac{\gamma'_n}{1-\beta'_n} S_n x_n. \tag{5.2.17}$$

We estimate from (5.2.17)

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
=\ & \left\| \frac{\alpha_{n+1}}{1-\beta'_{n+1}} f(x_{n+1}) + \frac{\gamma'_{n+1}}{1-\beta'_{n+1}} S_{n+1}x_{n+1} - \frac{\alpha_n}{1-\beta'_n} f(x_n) + \frac{\gamma'_n}{1-\beta'_n} S_n x_n \right\| \\
\le\ & \frac{\alpha_{n+1}}{1-\beta'_{n+1}} k\|x_{n+1} - x_n\| + \frac{\gamma'_{n+1}}{1-\beta'_{n+1}} \|S_{n+1}x_{n+1} - S_n x_n\|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}-\frac{\alpha_n}{1-\beta'_n}\right|\|f(x_n)-S_nx_n\| \\
\leq\ & \frac{\alpha_{n+1}}{1-\beta'_{n+1}}k\|x_{n+1}-x_n\|+\frac{\gamma'_{n+1}}{1-\beta'_{n+1}}[\|S_{n+1}x_{n+1}-S_{n+1}x_n\|+\|S_{n+1}x_n-S_nx_n\|] \\
&+\left|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}-\frac{\alpha_n}{1-\beta'_n}\right|\|f(x_n)-S_nx_n\| \\
\leq\ & \frac{\alpha_{n+1}}{1-\beta'_{n+1}}k\|x_{n+1}-x_n\|+\frac{\gamma'_{n+1}}{1-\beta'_{n+1}}[\|x_{n+1}-x_n\|+\sup_{z\in\{x_n\}}\|S_{n+1}z-S_nz\|] \\
&+\left|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}-\frac{\alpha_n}{1-\beta'_n}\right|\|f(x_n)-S_nx_n\|. \qquad (5.2.18)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| \ \leq\ & \frac{\alpha_{n+1}}{1-\beta'_{n+1}}k\|x_{n+1}-x_n\|+\frac{\gamma'_{n+1}}{1-\beta'_{n+1}}\sup_{z\in\{x_n\}}\|S_{n+1}z-S_nz\| \\
&+\left|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}-\frac{\alpha_n}{1-\beta'_n}\right|\|f(x_n)-S_nx_n\|. \qquad (5.2.19)
\end{aligned}
$$

Since $\lim_{n\to\infty}\alpha_n = 0$, and $\lim_{n\to\infty}\sup_{z\in\{x_n\}}\|S_{n+1}z - S_nz\| = 0$, we have from (5.2.19) that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Hence, by Lemma 2.1.56, we obtain

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \qquad (5.2.20)$$

From (5.2.17), we get

$$\lim_{n\to\infty}\|y_n-S_nx_n\|=\lim_{n\to\infty}\frac{\alpha_n}{1-\beta'_n}\|f(x_n)-S_nx_n\|=0, \qquad (5.2.21)$$

and so it follows from (5.2.20) and (5.2.21) that

$$\lim_{n\to\infty}\|x_n-S_nx_n\|=0. \qquad (5.2.22)$$

It follows from Lemma 2.2.9 and (5.2.22), we have

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - S_nx_n\| + \|S_nx_n - Sx_n\| \\ &\leq \|x_n - S_nx_n\| + \sup\{\|S_nz - Sz\| : z \in \{x_n\}\} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{5.2.23}$$

Since S is a nonexpansive mapping, we have from Lemma 2.1.54 that the net $\{x_t\}$ generated by

$$x_t = tf(x_t) + (1-t)Sx$$

converges strongly to $\tilde{x} \in F(S)$, as $t \to 0^+$. Next, we prove that

$$\limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \leq 0, \tag{5.2.24}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle = \limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_n - \tilde{x})\rangle. \tag{5.2.25}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in C$ as $i \to \infty$. Since J_φ is weakly continuous, we have by Lemma 2.1.43 that

$$\limsup_{i\to\infty}\Phi(\|x_{n_{k_i}} - x\|) = \limsup_{i\to\infty}\Phi(\|x_{n_{k_i}} - w\|) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

Let

$$H(x) = \limsup_{i\to\infty}\Phi(\|x_{n_{k_i}} - x\|), \text{ for all } x \in E.$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

From (5.2.23), we obtain

$$H(Sw) = \limsup_{i\to\infty}\Phi(\|x_{n_{k_i}} - Sw\|) = \limsup_{i\to\infty}\Phi(\|Sx_{n_{k_i}} - Sw\|)$$

$$\leq \limsup_{i\to\infty} \Phi(\|x_{n_{k_i}} - w\|) = H(w) \tag{5.2.26}$$

On the other hand, however,

$$H(Sw) = H(w) + \Phi(\|S(w) - w\|) \tag{5.2.27}$$

It follows from (5.2.26) and (5.2.27) that

$$\Phi(\|S(w) - w\|) = H(Sw) - H(w) \leq 0.$$

This implies that $Sw = w$ that is $w \in F(S) = F(T)$. Since the duality map J_φ is single-valued and weakly continuous, we get that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_n - \tilde{x})\rangle &= \lim_{k\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle \\
&= \lim_{i\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n_{k_i}} - \tilde{x})\rangle \\
&= \langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - w)\rangle \leq 0
\end{aligned}$$

as required. Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$.

$$\begin{aligned}
\Phi(\|x_{n+1} - \tilde{x}\|) &= \Phi(\|\alpha_n(f(x_n) - f(\tilde{x})) + \beta_n'(x_n - \tilde{x}) + \gamma_n'(S_n x_n - \tilde{x}) + \alpha_n(f(\tilde{x}) - \tilde{x})\|) \\
&\leq \Phi(\|\alpha_n(f(x_n) - f(\tilde{x})) + \beta_n'(x_n - \tilde{x}) + \gamma_n'(S_n x_n - \tilde{x})\|) \\
&\quad + \alpha_n\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\leq \Phi(\|\alpha_n k\|x_n - \tilde{x}\| + \beta_n'\|x_n - \tilde{x}\| + \gamma_n'\|x_n - \tilde{x}\|) \\
&\quad + \alpha_n\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&= \Phi((1 - \alpha_n(1-k))\|x_n - \tilde{x}\|) + \alpha_n\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\leq (1 - \alpha_n(1-k))\Phi(\|x_n - \tilde{x}\|) + \alpha_n\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle. \qquad (5.2.28)
\end{aligned}$$

It follows that from condition (i) and (5.2.24) that

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty \text{ and } \limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \leq 0.$$

Apply Lemma 2.2.10 to (5.2.28) to conclude $\Phi(\|x_{n+1} - \tilde{x}\|) \to 0$ as $n \to \infty$; that is, $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

If $\{T_n : C \to C\}$ is a family of nonexpansive mappings, then we obtain the following results:

Corollary 5.2.4. *Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_φ, and C a nonempty closed convex subset of E. Let $\{T_n : C \to C\}$ be a family of nonexpansive mappings such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:*

1. $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;
2. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$;
3. $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Let $\{x_n\}$ be the sequence generated by the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \geq 1. \end{cases}$$

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ converges strongly $\tilde{x}$ which solves the variational inequality :

$$\langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T).$$

Corollary 5.2.5. *Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_φ, and C a nonempty closed convex subset of E. Let $T : C \to C$ be a λ-strict pseudo-contraction with respect to p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $F(T) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:*

1. $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

2. $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = +\infty$;

3. $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < \xi$, *where* $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ *be the sequence generated by the following*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n, \quad n \ge 1. \end{cases}$$

Then the sequence $\{x_n\}$ *converges strongly to* $\tilde{x}$ *which solves the variational inequality :*

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x}-z)\rangle \le 0, z \in F(T).$$

Theorem 5.2.6. *Let* E *be a real* p*-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and* C *a nonempty closed convex subset of* E *which has the fixed point property for nonexpansive mappings. Let* $\{T_n : C \to C\}$ *be a family of uniformly* λ*-strict pseudo-contractions with respect to* p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ *and* $\cap_{n=1}^{\infty}F(T_n) \ne \emptyset$. *Let* $f : C \to C$ *be a* k*-contraction with* $k \in (0,1)$. *Assume that real sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy the following conditions:*

1. $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

2. $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = +\infty$;

3. $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < \xi$, *where* $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ *be the sequence generated by the following*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \ge 1. \end{cases} \tag{5.2.29}$$

Suppose that $\{T_n\}$ *satisfies the AKTT-condition. Let* T *be a mapping of* C *into itself defined by* $Tz = \lim_{n\to\infty}T_n z$ *for all* $z \in C$ *and suppose that* $F(T) = \cap_{n=1}^{\infty}F(T_n)$. *Then the sequence* $\{x_n\}$ *converges strongly to a common fixed point* $\tilde{x}$ *of* $\{T_n\}$.

Proof. It follows from the same argumentation as Theorem 5.2.3 that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$, where S is a nonexpansive mapping defined by (5.2.16). From Lemma 2.1.55 that the net $\{x_t\}$ generated by $x_t = tf(x_t) + (1-t)Sx_t$ converges strongly to $\tilde{x} \in F(S) = F(T)$, as $t \to 0^+$. Obviously,

$$x_t - x_n = (1-t)(Sx_t - x_n) + t(f(x_t) - x_n).$$

In view of Lemma 2.1.43, we calculate

$$\begin{aligned}\|x_t - x_n\|^2 &\leq (1-t)^2\|Sx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\ &\leq (1-2t+t^2)(\|x_t - x_n\| + \|Sx_n - x_n\|)^2 \\ &\quad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2\end{aligned}$$

and therefore

$$\langle f(x_t)-x_t, J(x_n-x_t)\rangle \leq \frac{t}{2}\|x_t-x_n\|^2+\frac{(1+t)^2\|x_n - Sx_n\|}{2t}(2\|x_t-x_n\|+\|x_n-Sx_n\|).$$

Since $\{x_n\}$, $\{x_t\}$ and $\{Sx_n\}$ are bounded and $\lim_{n\to\infty}\frac{\|x_n - Sx_n\|}{2t} = 0$, we obtain

$$\limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle \leq \frac{t}{2}M, \tag{5.2.30}$$

where $M = \sup_{n\geq 1, t\in(0,1)}\{\|x_t - x_n\|^2\}$. We also know that

$$\begin{aligned}\langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x})\rangle &= \langle f(x_t) - x_t, J(x_n - x_t)\rangle + \langle f(\tilde{x}) - f(x_t) + x_t - \tilde{x}, J(x_n - x_t)\rangle \\ &\quad + \langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x}) - J(x_n - x_t)\rangle. \qquad (5.2.31)\end{aligned}$$

From the fact that $x_t \to \tilde{x} \in F(T)$, as $t \to 0$, $\{x_n\}$ is bounded and the duality mapping J is norm-to-weak* uniformly continuous on bounded subset of E, it follows that as $t \to 0$,

$$\langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x}) - J(x_n - x_t)\rangle \to 0, \text{ for all } n \in \mathbb{N}$$

and

$$\langle f(\tilde{x}) - f(x_t) + x_t - \tilde{x}, J(x_n - x_t)\rangle \to 0, \text{ for all } n \in \mathbb{N}.$$

Combining (5.2.30), (5.2.31) and two results mentioned above, we get

$$\limsup_{n\to\infty}\langle f(\tilde{x})-\tilde{x}, J(x_n-\tilde{x})\rangle \leq 0. \tag{5.2.32}$$

From (5.2.15) and Lemma 2.1.43, we get

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &\leq \|\alpha_n(f(x_n)-f(\tilde{x}))+\beta_n'(x_n-\tilde{x})+\gamma_n'(S_nx_n-\tilde{x})\|^2\\ &\quad +2\alpha_n\langle f(\tilde{x})-\tilde{x}, J(x_{n+1}-\tilde{x})\rangle\\ &\leq (1-\alpha_n(1-k))\|x_n-\tilde{x}\|^2\\ &\quad +2\alpha_n\langle f(\tilde{x})-\tilde{x}, J(x_{n+1}-\tilde{x})\rangle. \end{aligned} \tag{5.2.33}$$

Hence applying in Lemma 2.2.10 to (5.2.33), we conclude that $\lim_{n\to\infty}\|x_n-\tilde{x}\| = 0$. □

Corollary 5.2.7. *Let E be a real p-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings. Let $\{T_n : C \to C\}$ be a family of nonexpansive mappings such that $\cap_{n=1}^{\infty}F(T_n)\neq\emptyset$. Let $f : C\to C$ be a k-contraction with $k\in(0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:*

1. *$\alpha_n+\beta_n+\gamma_n = 1$ for all $n\in\mathbb{N}$;*
2. *$\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = +\infty$;*
3. *$0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<1$.*

Let $\{x_n\}$ be the sequence generated by the following

$$\begin{cases} x_1 = x\in C,\\ x_{n+1} = \alpha_n f(x_n)+\beta_n x_n+\gamma_n T_n x_n, \quad n\geq 1. \end{cases}$$

Suppose that $\{T_n\}$ *satisfies the AKTT-condition. Let* T *be a mapping of* C *into itself defined by* $Tz = \lim_{n\to\infty} T_n z$ *for all* $z \in C$ *and suppose that* $F(T) = \cap_{n=1}^{\infty} F(T_n)$*. Then the sequence* $\{x_n\}$ *converges strongly to a common fixed point* $\tilde{x}$ *of* $\{T_n\}$*.*

Corollary 5.2.8. *Let* E *be a real p-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and* C *a nonempty closed convex subset of* E *which has the fixed point property for nonexpansive mappings. Let* $T : C \to C$ *be a* λ*-strict pseudo-contractions with respect to p,* $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ *and* $F(T) \neq \emptyset$*. Let* $f : C \to C$ *be a k-contraction with* $k \in (0,1)$*. Assume that real sequences* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy the following conditions:*

1. $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$*;*
2. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$*;*
3. $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \xi$*, where* $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$*.*

Let $\{x_n\}$ *be the sequence generated by the following*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n, \quad n \geq 1. \end{cases} \tag{5.2.34}$$

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point* $\tilde{x}$ *of* $\{T_n\}$*.*