# CHAPTER IV

# HYBRID METHOD FOR VARIATIONAL INEQUALITY PROBLEMS AND EQUILIBRIUM PROBLEMS

## 4.1 A hybrid iterative scheme for variational inequality problems for finite families of relatively weak quasi-nonexpansive mappings

In this section, we consider a hybrid projection algorithm basing on the shrinking projection method for two families of relatively weak quasi-nonexpansive mappings. We establish strong convergence theorems for approximating the common fixed point of the set of the common fixed points of such two families and the set of solutions of the variational inequality for an inverse-strongly monotone operator in Banach spaces.

**Theorem 4.1.1.** *Let $C$ be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space $E$, let $A$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $E^*$ with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in F$. Let $\{T_1, T_2, \dots, T_N\}$ and $\{S_1, S_2, \dots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from $C$ into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \cap VI(A, C)$. Assume that $T_i$ and $S_i$ are uniformly continuous for all $i \in \{1, 2, \dots, N\}$. Let $\{x_n\}$ be a sequence generated by the following* *algorithm:*

$$\begin{cases} x_0 = x \in C, \ \text{chosen arbitrary}, \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n A x_n), \\ z_n = J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n), \\ y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \le \delta_n \phi(u, x_1) + (1-\delta_n)[\alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)]\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases} \tag{4.1.1}$$

*where* $T_n = T_{n(\text{mod } N)}, S_n = S_{n(\text{mod } N)}$, *and* $J$ *is the normalized duality mapping on* $E$. *Assume that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ *and* $\{r_n\}$ *are the sequences in* $[0, 1]$ *satisfying the restrictions:*

*(C1)* $\lim_{n\to\infty} \delta_n = 0$;

*(C2)* $r_n \subset [a, b]$ *for some* $a, b$ *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the 2-uniformly convexity constant of* $E$;

*(C3)* $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

*(a)* $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ *and*

*(b)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

*Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_1$, *where* $\Pi_F$ *is the generalized projection from* $C$ *onto* $F$.

*Proof.* By the same method as in the proof of Cai and Hu [27], we can show that $C_n$ is closed and convex. Next, we show $F \subset C_n$ for all $n \ge 1$. In fact, $F \subset C_1 = C$

is obvious. For any $n \in \mathbb{N}$, suppose that $F \subset C_n$. Then, for all $q \in F \subset C_n$, we know from Lemma 2.1.66 that

$$\begin{aligned}
\phi(q, w_n) &= \phi(q, \Pi_C J^{-1}(Jx_n - r_nAx_n)) \\
&\leq \phi(q, J^{-1}(Jx_n - r_nAx_n)) \\
&= V(q, Jx_n - r_nAx_n) \\
&\leq V(q, (Jx_n - r_nAx_n) + r_nAx_n) - 2\langle J^{-1}(Jx_n - r_nAx_n) - q, r_nAx_n\rangle \\
&= V(q, Jx_n) - 2r_n\langle J^{-1}(Jx_n - r_nAx_n) - q, Ax_n\rangle \\
&= \phi(q, x_n) - 2r_n\langle x_n - q, Ax_n\rangle \\
&\quad + 2\langle J^{-1}(Jx_n - r_nAx_n) - x_n, -r_nAx_n\rangle. \qquad (4.1.2)
\end{aligned}$$

Since $q \in VI(A, C)$ and $A$ is $\alpha$-inverse-strongly monotone, we have

$$\begin{aligned}
-2r_n\langle x_n - q, Ax_n\rangle &= -2r_n\langle x_n - q, Ax_n - Aq\rangle - 2r_n\langle x_n - q, Aq\rangle \\
&\leq -2\alpha r_n\|Ax_n - Aq\|^2. \qquad (4.1.3)
\end{aligned}$$

Therefore, from Lemma 2.1.63 and the assumption that $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in F$, we obtain that

$$\begin{aligned}
2\langle J^{-1}(Jx_n - r_nAx_n) - x_n, -r_nAx_n\rangle &= 2\langle J^{-1}(Jx_n - r_nAx_n) - J^{-1}(Jx_n), -r_nAx_n\rangle \\
&\leq 2\|J^{-1}(Jx_n - r_nAx_n) - J^{-1}(Jx_n)\|\|r_nAx_n\| \\
&\leq \frac{4}{c^2}\|JJ^{-1}(Jx_n - r_nAx_n) - JJ^{-1}(Jx_n)\|\|r_nAx_n\| \\
&= \frac{4}{c^2}\|(Jx_n - r_nAx_n) - Jx_n\|\|r_nAx_n\| \\
&= \frac{4}{c^2}r_n^2\|Ax_n\|^2 \\
&\leq \frac{4}{c^2}r_n^2\|Ax_n - Aq\|^2. \qquad (4.1.4)
\end{aligned}$$

Substituting (4.1.3) and (4.1.4) into (4.1.2) and using the condition that $r_n < c^2\alpha/2$, we get

$$\phi(q, w_n) \leq \phi(q, x_n) + 2r_n \left( \frac{2}{c^2} r_n - \alpha \right) \|Ax_n - Aq\|^2 \leq \phi(q, x_n). \qquad (4.1.5)$$

Using (4.1.5) and the convexity of $\| \cdot \|^2$, for each $q \in F \subset C_n$, we obtain

$$\begin{aligned}
\phi(q, z_n) &= \phi(q, J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n)) \\
&= \|q\|^2 - 2\alpha_n \langle q, Jx_{n-1} \rangle - 2\beta_n \langle q, JT_n x_n \rangle - 2\gamma_n \langle q, JS_n w_n \rangle \\
&\quad + \|\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_n \langle q, Jx_{n-1} \rangle - 2\beta_n \langle q, JT_n x_n \rangle - 2\gamma_n \langle q, JS_n w_n \rangle \\
&\quad + \alpha_n \|Jx_{n-1}\|^2 + \beta_n \|JT_n x_n\|^2 + \gamma_n \|JS_n w_n\|^2 \\
&= \alpha_n \phi(q, x_{n-1}) + \beta_n \phi(q, T_n x_n) + \gamma_n \phi(q, S_n w_n) \\
&\leq \alpha_n \phi(q, x_{n-1}) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, w_n) \\
&\leq \alpha_n \phi(q, x_{n-1}) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, x_n) \\
&= \alpha_n \phi(q, x_{n-1}) + (1 - \alpha_n) \phi(q, x_n). \qquad (4.1.6)
\end{aligned}$$

It follows from (4.1.6) that

$$\begin{aligned}
\phi(q, y_n) &= \phi(q, J^{-1}(\delta_n J x_1 + (1 - \delta_n) J z_n)) \\
&= \|q\|^2 - 2\delta_n \langle q, Jx_1 \rangle - 2(1 - \delta_n) \langle q, Jz_n \rangle + \|\delta_n J x_1 + (1 - \delta_n) J z_n)\|^2 \\
&\leq \|q\|^2 - 2\delta_n \langle q, Jx_1 \rangle - 2(1 - \delta_n) \langle q, Jz_n \rangle + \delta_n \|x_1\|^2 + (1 - \delta_n) \|z_n\|^2 \\
&= \delta_n \phi(q, x_1) + (1 - \delta_n) \phi(q, z_n) \\
&\leq \delta_n \phi(q, x_1) + (1 - \delta_n) [\alpha_n \phi(q, x_{n-1}) + (1 - \alpha_n) \phi(q, x_n)]. \qquad (4.1.7)
\end{aligned}$$

So, $q \in C_{n+1}$. Then by induction, $F \subset C_n$ for all $n \geq 1$ and hence the sequence $\{x_n\}$ generated by (4.1.1) is well defined. Next, we show that $\{x_n\}$ is a convergent sequence in $C$. From $x_n = \Pi_{C_n} x_1$, we have

$$\langle x_n - u, Jx_1 - Jx_n \rangle \geq 0, \quad \forall u \in C_n. \qquad (4.1.8)$$

It follows from $F \subset C_n$ for all $n \geq 1$ that

$$\langle x_n - z, Jx_1 - Jx_n \rangle \geq 0, \quad \forall z \in F. \tag{4.1.9}$$

From Lemma 2.1.31, we have

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1),$$

for each $u \in F \subset C_n$ and for all $n \geq 1$. Therefore the sequence $\{\phi(x_n, x_1)\}$ is bounded. Furthermore, since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \text{for all } n \geq 1.$$

This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing and hence $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. Similarly, by Lemma 2.1.31, we have, for any positive integer $m$, that

$$\begin{aligned} \phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_1) \\ &\leq \phi(x_{n+m}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_{n+m}, x_1) - \phi(x_n, x_1), \text{ for all } n \geq 1. \end{aligned} \tag{4.1.10}$$

The existence of $\lim_{n\to\infty} \phi(x_n, x_1)$ implies that $\phi(x_{n+m}, x_n) \to 0$ as $n \to \infty$. From Lemma 2.1.64, we have

$$\|x_{n+m} - x_n\| \to 0, \text{ as } n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now, we will show that $p \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap VI(A, C)$.

(I) We first show that $p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i)$. Indeed, taking $m = 1$ in (4.1.10), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{4.1.11}$$

It follows from Lemma 2.1.64 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.1.12}$$

This implies that

$$\lim_{n\to\infty} \|x_{n+l} - x_n\| = 0, \qquad \text{for all } l \in \{1, 2, \ldots, N\}. \tag{4.1.13}$$

The property of the function $\phi$ implies that

$$\lim_{n\to\infty} \phi(x_{n+l}, x_n) = 0, \qquad \text{for all } l \in \{1, 2, \ldots, N\}. \tag{4.1.14}$$

Since $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, y_n) \leq \delta_n \phi(x_{n+1}, x_n) + (1 - \delta_n)[\alpha_n \phi(x_{n+1}, x_{n-1}) + (1 - \alpha_n)\phi(x_{n+1}, x_n)].$$

It follows from the condition (4.1.11) and (4.1.14) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

From Lemma 2.1.64, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{4.1.15}$$

Combining (4.1.12) and (4.1.15), we have

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \text{ as } n \to \infty. \tag{4.1.16}$$

Since $J$ is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{4.1.17}$$

On the other hand, noticing

$$\|Jy_n - Jz_n\| = \delta_n \|Jx_1 - Jz_n\| \to 0 \text{ as } n \to \infty. \tag{4.1.18}$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{4.1.19}$$

Using (4.1.12), (4.1.15) and (4.1.19) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{4.1.20}$$

Taking the constant $r = \sup_{n\geq 1}\{\|x_{n+1}\|, \|T_n x_n\|, \|S_n w_n\|\}$, we have, from Lemma 2.1.67, that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ satisfying the inequality (2.1.11) and $g(0) = 0$.

**Case I**. Assume that (a) holds. Applying (2.1.11) and (4.1.5), we can calculate

$$\begin{aligned}
\phi(u, z_n) &= \phi(u, J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT_n x_n + \gamma_n JS_n w_n)) \\
&= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&\quad + \|\alpha_n Jx_{n-1} + \beta_n JT_n x_n + \gamma_n JS_n w_n\|^2 \\
&\leq \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT_n x_n\|^2 + \gamma_n\|JS_n w_n\|^2 \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, T_n x_n) + \gamma_n\phi(u, S_n w_n) \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|)
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, w_n) \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT_nx_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, x_n) \\
&\quad + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 - \alpha_n\beta_n g(\|Jx_{n-1} - JT_nx_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT_nx_n\|). \qquad (4.1.21)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\alpha_n\beta_n g(\|Jx_{n-1} - JT_nx_n\|) &\leq \alpha_n[\phi(u, x_{n-1}) - \phi(u, x_n)] \\
&\quad + \phi(u, x_n) - \phi(u, z_n). \qquad (4.1.22)
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\alpha_n[\phi(u, x_{n-1}) - \phi(u, x_n)] + \phi(u, x_n) - \phi(u, z_n) \\
\leq\ & \alpha_n[\|x_{n-1}\|^2 - \|x_n\|^2 - 2\langle u, Jx_{n-1} - Jx_n\rangle] \\
& + \|x_n\|^2 - \|z_n\|^2 - 2\langle u, Jx_n - Jz_n\rangle \\
\leq\ & \alpha_n[\|x_{n-1} - x_n\|(\|x_{n-1}\| + \|x_n\|) + 2\|u\|\|Jx_{n-1} - Jx_n\|] \\
& + \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|u\|\|Jx_n - Jz_n\|.
\end{aligned}
$$

It follows from (4.1.12), (4.1.17), (4.1.18) and (4.1.20) that

$$
\lim_{n\to\infty} \alpha_n[\phi(u, x_{n-1}) - \phi(u, x_n)] + \phi(u, x_n) - \phi(u, z_n) = 0. \qquad (4.1.23)
$$

From $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ and (4.1.22), we get

$$
\lim_{n\to\infty} g(\|Jx_{n-1} - JT_nx_n\|) = 0.
$$

By the property of function $g$, we obtain that

$$\lim_{n\to\infty} \|Jx_{n-1} - JT_nx_n\| = 0. \tag{4.1.24}$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|x_{n-1} - T_nx_n\| = \lim_{n\to\infty} \|J^{-1}(Jx_{n-1}) - J^{-1}(JT_nx_n)\| = 0. \tag{4.1.25}$$

From (4.1.12) and (4.1.25) , we have

$$\lim_{n\to\infty} \|x_n - T_nx_n\| = 0. \tag{4.1.26}$$

Noticing that

$$\|x_n - T_{n+l}x_n\| \leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\|,$$

for all $l \in \{1, 2, \ldots, N\}$. By the uniformly continuity of $T_l$, (4.1.13) and (4.1.26), we obtain

$$\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0, \qquad \text{for all } l \in \{1, 2, \ldots, N\}. \tag{4.1.27}$$

Thus

$$\lim_{n\to\infty} \|x_n - T_lx_n\| = 0, \qquad \text{for all } l \in \{1, 2, \ldots, N\}. \tag{4.1.28}$$

From the closeness of $T_i$, we get $p = T_ip$. Therefore $p \in \cap_{i=1}^N F(T_i)$. In the same manner, we can apply the condition $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ to conclude that

$$\lim_{n\to\infty} \|x_n - S_nw_n\| = 0. \tag{4.1.29}$$

Again, by (C2) and (4.1.21), we have

$$2\gamma_n\left(\alpha - \frac{2}{c^2}b\right)\|Ax_n - Au\|^2 \leq \frac{1}{a}[\alpha_n\phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) - \phi(u, z_n)]$$

$$= \frac{1}{a}[\alpha_n(\phi(u, x_{n-1}) - \phi(u, x_n)) + \phi(u, x_n) - \phi(u, z_n)].$$

It follows from (4.1.23) and the assumption $\liminf_{n\to\infty} \gamma_n \geq \liminf_{n\to\infty} \beta_n\gamma_n > 0$ that

$$\liminf_{n\to\infty} \|Ax_n - Au\| \leq 0.$$

Since $\liminf_{n\to\infty} \|Ax_n - Au\| \geq 0$, we have

$$\lim_{n\to\infty} \|Ax_n - Au\| = 0. \tag{4.1.30}$$

From Lemma 2.1.31, Lemma 2.1.66, and (4.1.4), we have

$$\begin{aligned}
\phi(x_n, w_n) &= \phi(x_n, \Pi_C J^{-1}(Jx_n - r_nAx_n)) \\
&\leq \phi(x_n, J^{-1}(Jx_n - r_nAx_n)) \\
&= V(x_n, Jx_n - r_nAx_n) \\
&\leq V(x_n, (Jx_n - r_nAx_n) + r_nAx_n) \\
&\quad - 2\langle J^{-1}(Jx_n - r_nAx_n) - x_n, r_nAx_n\rangle \\
&= \phi(x_n, x_n) + 2\langle J^{-1}(Jx_n - r_nAx_n) - x_n, -r_nAx_n\rangle \\
&= 2\langle J^{-1}(Jx_n - r_nAx_n) - x_n, -r_nAx_n\rangle \\
&\leq \frac{4}{c^2}b^2\|Ax_n - Au\|^2.
\end{aligned} \tag{4.1.31}$$

It follows from (4.1.30) that

$$\lim_{n\to\infty} \phi(x_n, w_n) = 0. \tag{4.1.32}$$

Lemma 2.1.64 implies that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{4.1.33}$$

Since $J$ is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jw_n\| = 0. \tag{4.1.34}$$

Combining (4.1.29) and (4.1.33), we also obtain

$$\lim_{n\to\infty} \|w_n - S_n w_n\| = 0. \tag{4.1.35}$$

Moreover

$$\|w_n - w_{n+1}\| \leq \|w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - w_{n+1}\|.$$

By (4.1.33), (4.1.12), we have

$$\lim_{n\to\infty} \|w_n - w_{n+1}\| = 0. \tag{4.1.36}$$

This implies that

$$\lim_{n\to\infty} \|w_n - w_{n+l}\| = 0, \qquad \text{for all } l \in \{1, 2, \ldots, N\}. \tag{4.1.37}$$

Noticing that

$$\|w_n - S_{n+l}w_n\| \leq \|w_n - w_{n+l}\| + \|w_{n+l} - S_{n+l}w_{n+l}\| + \|S_{n+l}w_{n+l} - S_{n+l}w_n\|,$$

for all $l \in \{1, 2, \ldots, N\}$. Since $S_l$ is uniformly continuous, we can show that $\lim_{n\to\infty} \|w_n - S_l w_n\| = 0$. From the closeness of $S_l$, we get $p = S_l p$. Therefore $p \in \cap_{i=1}^{N} F(S_i)$. Hence $p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i)$.

**Case II**. Assume that (b) holds. Using the inequalities (2.1.11) and (4.1.5), we obtain

$$\begin{aligned} \phi(u, z_n) &= \phi(u, J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT_n x_n + \gamma_n JS_n w_n)) \\ &= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \|\alpha_n Jx_{n-1} + \beta_n JT_n x_n + \gamma_n JS_n w_n\|^2 \\
\le\ & \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&+ \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT_n x_n\|^2 + \gamma_n\|JS_n w_n\|^2 \\
&- \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \\
\le\ & \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, T_n x_n) + \gamma_n\phi(u, S_n w_n) \\
&- \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \\
\le\ & \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, w_n) \\
&- \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \\
\le\ & \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, x_n) \\
&+ 2r_n\gamma_n(\frac{2}{c^2}r_n - \alpha)\|Ax_n - Au\|^2 - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \\
\le\ & \alpha_n\phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) + 2r_n\gamma_n(\frac{2}{c^2}r_n - \alpha)\|Ax_n - Au\|^2 \\
&- \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|). \qquad (4.1.38)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|) \le\ & \alpha_n[\phi(u, x_{n-1}) - \phi(u, x_n)] + \phi(u, x_n) - \phi(u, z_n) \\
\le\ & \alpha_n[\|x_{n-1}\|^2 - \|x_n\|^2 - 2\langle u, Jx_{n-1} - Jx_n\rangle] \\
&+ \|x_n\|^2 - \|z_n\|^2 - 2\langle u, Jx_n - Jz_n\rangle \\
\le\ & \alpha_n[\|x_{n-1} - x_n\|(\|x_{n-1}\| + \|x_n\|) + 2\|u\|\|Jx_{n-1} - Jx_n\|] \\
&+ \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|u\|\|Jx_n - Jz_n\|.
\end{aligned}
$$

It follows from (4.1.16), (4.1.19) and the condition $\liminf_{n\to\infty}\beta_n\gamma_n > 0$ that

$$\lim_{n\to\infty} g(\|JT_n x_n - JS_n w_n\|) = 0.$$

By the property of function $g$, we obtain that

$$\lim_{n\to\infty} \|JT_n x_n - JS_n w_n\| = 0. \qquad (4.1.39)$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|T_n x_n - S_n w_n\| = \lim_{n\to\infty} \|J^{-1}(JT_n x_n) - J^{-1}(JS_n w_n)\| = 0. \tag{4.1.40}$$

On the other hand, we can calculate

$$\begin{aligned} \phi(T_n x_n, z_n) &= \phi(T_n x_n, J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n)) \\ &= \|T_n x_n\|^2 - 2\langle T_n x_n, \alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n)\rangle \\ &\quad + \|\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n)\|^2 \\ &\leq \|T_n x_n\|^2 - 2\alpha_n \langle T_n x_n, J x_n\rangle - 2\beta_n \langle T_n x_n, J T_n x_n\rangle - 2\gamma_n \langle T_n x_n, J S_n w_n\rangle \\ &\quad + \alpha_n \|x_n\|^2 + \beta_n \|T_n x_n\|^2 + \gamma_n \|S_n w_n\|^2 \\ &\leq \alpha_n \phi(T_n x_n, x_n) + \gamma_n \phi(T_n x_n, S_n w_n). \end{aligned} \tag{4.1.41}$$

Observe that

$$\begin{aligned} \phi(T_n x_n, S_n w_n) &= \|T_n x_n\|^2 - 2\langle T_n x_n, J S_n w_n\rangle + \|S_n w_n\|^2 \\ &= \|T_n x_n\|^2 - 2\langle T_n x_n, J T_n x_n\rangle + 2\langle T_n x_n, J T_n x_n - J S_n w_n\rangle + \|S_n w_n\|^2 \\ &\leq \|S_n w_n\|^2 - \|T_n x_n\|^2 + 2\|T_n x_n\| \|J T_n x_n - J S_n w_n\| \\ &\leq \|S_n w_n - T_n x_n\|(\|S_n w_n\| + \|T_n x_n\|) + 2\|T_n x_n\| \|J T_n x_n - J S_n w_n\|. \end{aligned}$$

It follows from (4.1.39) and (4.1.40) that

$$\lim_{n\to\infty} \phi(T_n x_n, S_n w_n) = 0. \tag{4.1.42}$$

Applying $\lim_{n\to\infty} \alpha_n = 0$ and (4.1.42) and the fact that $\{\phi(T_n x_n, x_n)\}$ is bounded to (4.1.41), we obtain

$$\lim_{n\to\infty} \phi(T_n x_n, z_n) = 0. \tag{4.1.43}$$

From Lemma 2.1.64, one obtains

$$\lim_{n\to\infty} \|T_n x_n - z_n\| = 0. \tag{4.1.44}$$

We observe that

$$\|T_n x_n - x_n\| \leq \|T_n x_n - z_n\| + \|z_n - x_n\|.$$

This together with (4.1.20) and (4.1.44), we obtain

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{4.1.45}$$

Noticing that

$$\|x_n - T_{n+l}x_n\| \leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\|,$$

for all $l \in \{1, 2, \ldots, N\}$. By the uniformly continuity of $T_l$, (4.1.13) and (4.1.45), we obtain

$$\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0, \qquad \text{for all } l \in \{1, 2, \ldots, N\}. \tag{4.1.46}$$

Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \qquad \text{for all } l \in \{1, 2, \ldots, N\}. \tag{4.1.47}$$

From the closeness of $T_i$, we get $p = T_i p$. Therefore $p \in \cap_{i=1}^{N} F(T_i)$. By the same proof as in Case I, we obtain that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{4.1.48}$$

Hence $w_n \to p$ as $n \to \infty$ for each $i \in I$ and

$$\lim_{n\to\infty} \|Jx_n - Jw_n\| = 0. \tag{4.1.49}$$

Combining (4.1.40), (4.1.45) and (4.1.48), we also have

$$\lim_{n\to\infty} \|S_n w_n - w_n\| = 0. \tag{4.1.50}$$

Moreover

$$\|w_n - w_{n+1}\| \leq \|w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - w_{n+1}\|.$$

By (4.1.33), (4.1.12), we have

$$\lim_{n\to\infty} \|w_n - w_{n+1}\| = 0. \tag{4.1.51}$$

This implies that

$$\lim_{n\to\infty} \|w_n - w_{n+l}\| = 0, \qquad \text{for all } l \in \{1, 2, \ldots, N\}. \tag{4.1.52}$$

Noticing that

$$\|w_n - S_{n+l}w_n\| \leq \|w_n - w_{n+l}\| + \|w_{n+l} - S_{n+l}w_{n+l}\| + \|S_{n+l}w_{n+l} - S_{n+l}w_n\|,$$

for all $l \in \{1, 2, \ldots, N\}$. Since $S_l$ is uniformly continuous, we can show that $\lim_{n\to\infty} \|w_n - S_l w_n\| = 0$. From the closeness of $S_i$, we get $p = S_i p$. Therefore $p \in \cap_{i=1}^N F(S_i)$. Hence $p \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$.

(II) We next show that $p \in VI(C, A)$.

Let $T \subset E \times E^*$ be an operator defined by :

$$Tv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{4.1.53}$$

By Lemma 2.1.69, $T$ is maximal monotone and $T^{-1}0 = VI(A, C)$. Let $(v, w) \in G(T)$, since $w \in Tv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $x_n = \Pi_{C_n} x \in C_n \subset C$, we get

$$\langle v - x_n, w - Av \rangle \;\geq\; 0. \tag{4.1.54}$$

Since $A$ is $\alpha$-inverse-strong monotone, we have

$$\begin{aligned} \langle v - x_n, w \rangle &\geq \langle v - x_n, Av \rangle \\ &= \langle v - x_n, Av - Ax_n \rangle + \langle v - x_n, Ax_n \rangle \\ &\geq \langle v - x_n, Ax_n \rangle. \end{aligned} \tag{4.1.55}$$

On other hand, from $w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n)$ and Lemma 2.1.65, we have $\langle v - w_n, Jw_n - (Jx_n - r_n Ax_n)\rangle \geq 0$, and hence

$$\langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n\rangle \leq 0. \tag{4.1.56}$$

Because $A$ is $\frac{1}{\alpha}$ constricted, it holds from (4.1.55) and (4.1.56) that

$$\begin{aligned}
\langle v - x_n, w\rangle &\geq \langle v - x_n, Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n\rangle \\
&= \langle v - w_n, Ax_n\rangle + \langle w_n - x_n, Ax_n\rangle - \langle v - w_n, Ax_n\rangle \\
&\quad + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\rangle \\
&= \langle w_n - x_n, Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\rangle \\
&\geq -\|w_n - x_n\|.\|Ax_n\| \\
&\quad - \|v - w_n\|.\frac{\|Jx_n - Jw_n\|}{a}, \forall n \in \mathbb{N} \cup \{0\}.
\end{aligned} \tag{4.1.57}$$

By taking the limit as $n \to \infty$ in (4.1.57) and from (4.1.33) and (4.1.34), we have $\langle v - p, w\rangle \geq 0$ as $n \to \infty$. By the maximality of $T$ we obtain $p \in T^{-1}0$ and hence $p \in VI(A, C)$. Hence we conclude that

$$p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \cap VI(A, C).$$

Finally, we show that $p \in \Pi_F x_1$. Indeed, taking the limit as $n \to \infty$ in (4.1.9), we obtain

$$\langle p - z, Jx_1 - Jp\rangle \geq 0, \quad \forall z \in F \tag{4.1.58}$$

and hence $p = \Pi_F x_1$ by Lemma 2.1.65. This complete the proof. □

**Remark 4.1.2.** Theorem 4.1.1 improves and extends main results of Iiduka and Takahashi [59], Xu and Ori [60], Qin, Cho, Kang and Zho [61], and Cai and Hu [27] because it can be applied to solving the problem of finding the common element of the set of common fixed points of two families of relatively weak quasi-nonexpansive mappings and the set of solutions of the variational inequality for an inverse-strongly monotone operator.

Strong convergence theorem for approximating a common fixed point of two finite families of closed relatively weak quasi-nonexpansive mappings in Banach spaces may not require that $E$ is 2-uniformly convex. In fact, we have the following theorem

**Corollary 4.1.3.** *Let $C$ be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from $C$ into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i)$. Assume that $T_i$ and $S_i$ are uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$\begin{cases} x_0 = x \in C, \text{ chosen arbitrary,} \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n x_n), \\ y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)[\alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)]\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.1.59}$$

*where $T_n = T_{n(\text{mod } N)}$, $S_n = S_{n(\text{mod } N)}$, and $J$ is the normalized duality mapping on $E$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0, 1]$ satisfying the restrictions:*

*(C1)* $\lim_{n \to \infty} \delta_n = 0$;

*(C2)* $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

*(a)* $\liminf_{n \to \infty} \alpha_n \beta_n > 0$ *and* $\liminf_{n \to \infty} \alpha_n \gamma_n > 0$ *and*

*(b)* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\liminf_{n\to\infty}\beta_n\gamma_n > 0$.

*Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_1$*, where* $\Pi_F$ *is the generalized projection from* $C$ *onto* $F$.

*Proof.* Put $A \equiv 0$ in Theorem 4.1.1. Then, we get that $w_n = x_n$. Thus, the method of the proof of Theorem 4.1.1 gives the required assertion without the requirement that $E$ is 2-uniformly convex. □

**Remark 4.1.4.** Corollary 4.1.3 improves Theorem 3.1 of Cai and Hu [27] from a finite family of of relatively weak quasi-nonexpansive mappings to two finite families of relatively weak quasi-nonexpansive mappings.

If $E = H$, a Hilbert space, then $E$ is 2-uniformly convex (we can choose $c = 1$) and uniformly smooth real Banach space and closed relatively weak quasi-nonexpansive map reduces to closed weak quasi-nonexpansive map. Furthermore, $J = I$, identity operator on $H$ and $\Pi_C = P_C$, projection mapping from $H$ into $C$. Thus, the following corollaries hold.

**Corollary 4.1.5.** *Let* $C$ *be a nonempty, closed and convex subset of a Hilbert space* $H$*. Let* $\{T_1, T_2, \ldots, T_N\}$ *and* $\{S_1, S_2, \ldots, S_N\}$ *be two finite families of closed weak quasi-nonexpansive mappings from* $C$ *into itself with* $F \neq \emptyset$*, where* $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap VI(A, C)$ *with* $\|Ay\| \leq \|Ay - Aq\|$ *for all* $y \in C$ *and* $q \in F$*. Assume that* $T_i$ *and* $S_i$ *are uniformly continuous for all* $i \in \{1, 2, \ldots, N\}$.

*Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 = x \in C, \text{ chosen arbitrary}, \\ C_1 = C, x_1 = P_{C_1}x_0, \\ w_n = P_C(x_n - r_n A x_n), \\ z_n = (\alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n S_n w_n), \\ y_n = (\delta_n x_1 + (1-\delta_n) z_n), \\ C_{n+1} = \{u \in C_n : \|u - y_n\|^2 \leq \delta_n \|u - x_1\|^2 + (1-\delta_n)[\alpha_n \|u - x_{n-1}\|^2 \\ \qquad\qquad + (1-\alpha_n)\|u - x_n\|^2]\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

*where $T_n = T_{n(\text{mod } N)}, S_n = S_{n(\text{mod } N)}$, and $J$ is the normalized duality mapping on $E$. Assume that $\{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:*

*(C1)* $\lim_{n\to\infty} \delta_n = 0$;

*(C2)* $r_n \subset [a,b]$ *for some* $a, b$ *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the 2-uniformly convexity constant of* $E$;

*(C3)* $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

*(a)* $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ *and*

*(b)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

*Then $\{x_n\}$ converges strongly to $P_F x_1$, where $P_F$ is the metric projection from $C$ onto $F$.*

## 4.2 Convergence theorems based on the shrinking projection method for variational inequality and equilibrium problems

In this section, we introduce a hybrid projection algorithm based on the shrinking projection method for two relatively weak nonexpansive mappings. We prove strong convergence theorem which approximate the common element in the fixed points of two such mappings, the solution set of the variational inequality and the solution set of the equilibrium problem in Banach spaces.

In [62, 15], Alber introduced the functional $V : E^* \times E \to \mathbb{R}$ defined by

$$V(\phi, x) \quad = \quad \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2,$$

The following properties of the generalized projection operator $\Pi_C$ and $V$ are useful for our paper. (See, for example, [16])



(i) $V : E^* \times E \to \mathbb{R}$ is continuous.

(ii) $V(\phi, x) = 0$ if and only if $\phi = Jx$.

(iii) $V(J\Pi_C(\phi), x) \leq V(\phi, x)$ for all $\phi \in E^*$ and $x \in E$.

(iv) The operator $\Pi_C$ is $J$ fixed at each point $x \in C$, *i.e.*, $\Pi_C(Jx) = x$.

(v) If $E$ is smooth, then for any given $\phi \in E^*$, $x \in C$, $x \in \Pi_C(\phi)$ if and only if $\langle \phi - Jx, x - y\rangle \geq 0$, for all $y \in C$.

(vi) The operator $\Pi_C : E^* \to C$ is single valued if and only if $E$ is strictly convex.

(vii) If $E$ is smooth, then for any given point $\phi \in E^*$, $x \in \Pi_C(\phi)$, the following inequality holds

$$V(Jx, y) \leq V(\phi, y) - V(\phi, x) \quad \forall y \in C.$$

(viii) $V(\phi, x)$ is convex with respect to $\phi$ when $x$ is fixed and with respect to $x$ when $\phi$ is fixed.

(ix) If $E$ is reflexive, then for any point $\phi \in E^*$, $\Pi_C(\phi)$ is a nonempty, closed, convex and bounded subset of $C$.

**Lemma 4.2.1.** [63] *Let $C$ be a closed and convex subset of a uniformly smooth, strictly convex, and reflexive Banach space $E$, and let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions $(A1) - (A4)$. For all $r > 0$ and $x \in E$, define the mapping*

$$T_r x = \Big\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C \Big\}. \tag{4.2.1}$$

*Then, the following hold:*

*(1) $T_r$ is single-valued;*

*(2) $T_r$ is a firmly nonexpansive-type mapping [64], that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \;\leq\; \langle T_r x - T_r y, Jx - Jy\rangle; \tag{4.2.2}$$

*(3) $F(T_r) = \hat{F}(T_r) = EP(f)$;*

*(4) $EP(f)$ is closed and convex.*

**Theorem 4.2.2.** *Let $E$ be a uniformly convex and uniformly smooth Banach space and $C$ be a nonempty closed and convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. Assume that $A$ is a continuous operator of $C$ into $E^*$ satisfying conditions (2.3.4) and (2.3.5) and $S, T : C \to C$ are relatively weak nonexpansive mappings with $F := F(S) \cap F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$. Let*

*$\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 = x \in C \textit{chosen arbitrary}, C_0 = C, \\ z_n = \Pi_C(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J \Pi_C(J z_n - \beta A z_n)), \\ u_n \in C \ \textit{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} J x \quad \forall n \geq 0. \end{cases} \tag{4.2.3}$$

*Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:*

*(C1)* $\alpha_n + \beta_n + \gamma_n = 1$*;*

*(C2)* $0 \leq \delta_n < 1, \limsup_{n\to\infty} \delta_n < 1$*;*

*(C3)* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$*; and*

*(C4)* $\liminf_{n\to\infty} \alpha_n\beta_n > 0, \liminf_{n\to\infty} \alpha_n\gamma_n > 0.$

*Then $\{x_n\}$ converges strongly to $\Pi_F x$.*

*Proof.* We divide the proof into five steps.

**Step 1**. $\Pi_F x$ and $\Pi_{C_{n+1}} x$ are well defined.

From Lemma 2.3.4, we know that $VI(A, C)$ is closed and convex. By the same argument as in the proof of [65, p. 260], one can show that $F(T) \cap F(S)$ is closed and convex. From Lemma 4.2.1 (4), we also have that $EP(f)$ is closed and convex. Hence $F$ is a nonempty, closed and convex subset of $C$. Consequently, $\Pi_F x$ is well defined.

Clearly, $C_0 = C$ is closed and convex. Suppose that $C_k$ is closed and convex for some $k \in \mathbb{N}$. For all $z \in C_{k+1}$, one obtains that

$$\phi(z, u_n) \leq \phi(z, x_n)$$

is equivalent to

$$2(\langle z, Jx_k\rangle - \langle z, Ju_k\rangle) \leq \|x_k\|^2 - \|u_k\|^2.$$

It is easy to see that $C_{k+1}$ is closed and convex. Then, for all $n \geq 0$, $C_n$ is closed and convex. Hence $\Pi_{C_{n+1}}x$ is well defined.

**Step 2.** $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

We observe that $F \subset C_0 = C$ is obvious. Suppose $F \subset C_k$ for some $k \in \mathbb{N}$. Let $w \in F \subset C_k$, then, from the definitions of $\phi$ and $V$, property (iii) of $V$, Lemma 2.1.70, conditions (2.3.4) and (2.3.5), we have

$$\begin{aligned}
\phi(w, \Pi_C(Jz_n - \beta Az_n)) &= V(J\Pi_C(Jz_n - \beta Az_n), w) \\
&\leq V(Jz_n - \beta Az_n, w) \\
&= \|Jz_n - \beta Az_n\|^2 - 2\langle Jz_n - \beta Az_n, w\rangle + \|w\|^2 \\
&\leq \|Jz_n\|^2 - 2\beta\langle Az_n, J^{-1}(Jz_n - \beta Az_n)\rangle \\
&\quad - 2\langle Jz_n - \beta Az_n, w\rangle + \|w\|^2 \\
&\leq \|Jz_n\|^2 - 2\langle Jz_n, w\rangle + \|w\|^2 \\
&= \phi(w, z_n), \qquad (4.2.4)
\end{aligned}$$

for each $n \in \mathbb{N} \cup \{0\}$. From Lemma 4.2.1(2), one has that $T_{r_n}$ is a relatively nonexpansive mapping. Therefore, by properties (viii) and (iii) of the operator $V$ and (4.2.4), we obtain

$$\begin{aligned}
\phi(w, u_k) &= \phi(w, T_{r_k}y_k) \\
&\leq \phi(w, y_k) \\
&= V(Jy_k, w) \\
&\leq \delta_k V(Jx_k, w) + (1 - \delta_k)V(J\Pi_C(Jz_k - \beta Az_k), w)
\end{aligned}$$

$$
\begin{aligned}
&= \delta_k\phi(w,x_k) + (1-\delta_k)\phi(w,\Pi_C(Jz_k-\beta Az_k)) \\
&\le \delta_k\phi(w,x_k) + (1-\delta_k)\phi(w,z_k) \\
&= \delta_k\phi(w,x_k) + (1-\delta_k)V(Jz_k,w) \\
&\le \delta_k\phi(w,x_k) + (1-\delta_k)V(\alpha_kJx_k+\beta_kJTx_k+\gamma_kJSx_k,w) \\
&= \delta_k\phi(w,x_k) + (1-\delta_k)\phi(w,J^{-1}(\alpha_kJx_k+\beta_kJTx_k+\gamma_kJSx_k)) \\
&= \delta_k\phi(w,x_k) + (1-\delta_k)[\|w\|^2 - 2\alpha_k\langle w,Jx_k\rangle - 2\beta_k\langle w,JTx_k\rangle \\
&\quad - 2\gamma_k\langle w,JSx_k\rangle + \|\alpha_kJx_k+\beta_kJTx_k+\gamma_kJSx_k\|^2] \\
&\le \delta_k\phi(w,x_k) + (1-\delta_k)[\|w\|^2 - 2\alpha_k\langle w,Jx_k\rangle - 2\beta_k\langle w,JTx_k\rangle \\
&\quad - 2\gamma_k\langle w,JSx_k\rangle + \alpha_k\|Jx_k\|^2 + \beta_k\|JTx_k\|^2 + \gamma_k\|JSx_k\|^2] \\
&= \delta_k\phi(w,x_k) + (1-\delta_k)[\alpha_k\phi(w,x_k) + \beta_k\phi(w,Tx_k) + \gamma_k\phi(w,Sx_k)] \\
&\le \delta_k\phi(w,x_k) + (1-\delta_k)\phi(w,x_k) \\
&= \phi(w,x_k), \qquad (4.2.5)
\end{aligned}
$$

which shows that $w \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \ge 0$.

**Step 3**. $\{x_n\}$ is a convergent sequence in C.

Since $x_n = \Pi_{C_n}Jx$ and $F \subset C_n$, we have $V(Jx,x_n) \le V(Jx,w)$ for each $w \in F$. Therefore, $\{V(Jx,x_n)\}$ is bounded. Moreover, from the definition of $V$, we have that $\{x_n\}$ is bounded. Since $x_{n+1} = \Pi_{C_{n+1}}Jx \in C_{n+1}$ and $x_n = \Pi_{C_n}Jx$, we have $V(Jx,x_n) \le V(Jx,x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Therefore $\{V(Jx,x_n)\}$ is nondecreasing. Hence

$\lim_{n\to\infty} V(Jx,x_n)$ exists.

By the construction of $C_n$, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m}Jx \in C_n$ for any positive integer $m \ge n$. From property (vii) of the operator $\Pi_C$, we have

$$V(Jx_n,x_m) \le V(Jx,x_m) - V(Jx,x_n), \qquad (4.2.6)$$

for each $n \in \mathbb{N} \cup \{0\}$ and any positive integer $m \geq n$. This implies that

$$V(Jx_n, x_m) \to 0 \quad \text{as} \quad n, m \to \infty. \tag{4.2.7}$$

The definition of $\phi$ implies that

$$\phi(x_m, x_n) \to 0 \quad \text{as} \quad n, m \to \infty. \tag{4.2.8}$$

Applying Lemma 2.1.64, we obtain

$$\|x_m - x_n\| \to 0 \quad \text{as} \quad n, m \to \infty. \tag{4.2.9}$$

Hence $\{x_n\}$ is a Cauchy sequence. The completeness of a Banach space $E$ and the closeness of $C$ imply that $\lim_{n\to\infty} x_n = p, \quad$ for some $p \in C$.

**Step 4**. We show that $p \in F$.

(I) First we show that $p \in F(S) \cap F(T)$.

Take $m = n + 1$ in (4.2.7), one arrives that

$$\lim_{n\to\infty} V(Jx_n, x_{n+1}) = 0.$$

By the definition of $\phi$, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{4.2.10}$$

Using Lemma 2.1.64, we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.2.11}$$

Note that $x_{n+1} = \Pi_{C_{n+1}} Jx \in C_{n+1}$ then $\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n)$.

It follows from (4.2.10) that $\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$.

Using Lemma 2.1.64, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{4.2.12}$$

Combining (4.2.15) with (4.2.12), one sees that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{4.2.13}$$

It follows from $x_n \to p$ as $n \to \infty$ that

$$u_n \to p \text{ as } n \to \infty. \tag{4.2.14}$$

On the other hand, since $J$ is uniformly norm-to-norm continuous an bounded sets, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{4.2.15}$$

Since $\{x_n\}$ is bounded, $\{Jx_n\}$, $\{JTx_n\}$ and $\{JSx_n\}$ are also bounded. Since $E$ is a uniformly smooth Banach space, one knows that $E^*$ is a uniformly convex Banach space. Let $r = \sup_{n\geq 0}\{\|Jx_n\|, \|JTx_n\|, \|JSx_n\|\}$. Therefore Lemma 2.1.67 implies that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ satisfying $g(0) = 0$ and inequality (2.1.11). It follows from the property (iii) of the operator $V$, (4.2.4) and the definition of $S$ and $T$, that

$$\begin{aligned}
\phi(p, z_n) &= V(Jz_n, p) \\
&\leq V(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n, p) \\
&= \phi(p, J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n) \\
&= \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, JTx_n\rangle - 2\gamma_n\langle p, JSx_n\rangle \\
&\quad + \|\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, JTx_n\rangle - 2\gamma_n\langle p, JSx_n\rangle \\
&\quad + \alpha_n\|Jx_n\|^2 + \beta_n\|JTx_n\|^2 + \gamma_n\|JSx_n\|^2 - \alpha_n\beta_n g(\|JTx_n - Jx_n\|)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\phi(p,x_n) + \beta_n\phi(p,Tx_n) + \gamma_n\phi(p,Sx_n) - \alpha_n\beta_n g(\|JTx_n - Jx_n\|) \\
&\leq \phi(p,x_n) - \alpha_n\beta_n g(\|JTx_n - Jx_n\|). \qquad (4.2.16)
\end{aligned}
$$

From property (viii) of the operator $V$, (4.2.4) and (4.2.16), we obtain

$$
\begin{aligned}
\phi(p,u_n) &= \phi(p,T_{r_n}y_n) \\
&\leq \phi(p,y_n) \\
&= V(Jy_n,p) \\
&\leq \delta_n V(Jx_n,p) + (1-\delta_n)V(J\Pi_C(Jz_n - \beta Az_n),p) \\
&= \delta_n\phi(p,x_n) + (1-\delta_n)\phi(p,\Pi_C(Jz_n - \beta Az_n)) \\
&\leq \delta_n\phi(p,x_n) + (1-\delta_n)\phi(p,z_n) \\
&\leq \delta_n\phi(p,x_n) + (1-\delta_n)[\phi(p,x_n) - \alpha_n\beta_n g(\|JTx_n - Jx_n\|)] \\
&= \phi(p,x_n) - (1-\delta_n)\alpha_n\beta_n g(\|JTx_n - Jx_n\|).
\end{aligned}
$$

Therefore,

$$
(1-\delta_n)\alpha_n\beta_n g(\|JTx_n - Jx_n\|) \leq \phi(p,x_n) - \phi(p,u_n). \qquad (4.2.17)
$$

On the other hand, we have

$$
\begin{aligned}
\phi(p,x_n) - \phi(p,u_n) &= 2\langle Ju_n - Jx_n, p\rangle + \|x_n\|^2 - \|u_n\|^2 \\
&= 2\langle Ju_n - Jx_n, p\rangle + (\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|) \\
&\leq 2\|Ju_n - Jx_n\|\|w\| + \|x_n - u_n\|(\|x_n\| + \|u_n\|).
\end{aligned}
$$

It follows from (4.2.13) and (4.2.15) that

$$
\lim_{n\to\infty}(\phi(p,x_n) - \phi(p,u_n)) = 0. \qquad (4.2.18)
$$

From the assumptions $\limsup_{n\to\infty}\delta_n < 1, \liminf_{n\to\infty}\alpha_n\beta_n > 0$, (4.2.17) and (4.2.22) we have

$$\lim_{n\to\infty} g(\|JTx_n - Jx_n\|) = 0. \tag{4.2.19}$$

It follows from the property of $g$ that

$$\lim_{n\to\infty} \|JTx_n - Jx_n\| = 0. \tag{4.2.20}$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|J^{-1}Jx_n - J^{-1}JTx_n\| = 0. \tag{4.2.21}$$

In a similar way, we can apply the condition $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ to get

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{4.2.22}$$

Since $x_n \to p$, we have $p \in \tilde{F}(S) \cap \tilde{F}(T) = F(S) \cap F(T)$. Moreover,

$$Sx_n \to p \text{ as } n \to \infty \text{ and } Tx_n \to p \text{ as } n \to \infty. \tag{4.2.23}$$

(II) $p \in EP(f)$.

From (4.2.5), we know that $\phi(u, y_n) \leq \phi(u, x_n)$.

From $u_n = T_{r_n}y_n$ and Lemma 4.2.1(2), one has

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(T_{r_n}y_n, y_n) \\
&\leq \phi(w, y_n) - \phi(w, T_{r_n}y_n) \\
&\leq \phi(w, x_n) - \phi(w, T_{r_n}y_n) \\
&= \phi(w, x_n) - \phi(w, u_n).
\end{aligned}$$

It follows from (4.2.18) that $\phi(u_n, y_n) \to 0$ as $n \to \infty$.

Applying Lemma 2.1.64, we obtain

$$\|u_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{4.2.24}$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{4.2.25}$$

From the assumption that $r_n \geq a$, one sees

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{4.2.26}$$

Observe that $u_n = T_{r_n} y_n$, one obtains

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy\rangle \;\geq\; 0, \quad \forall y \in C. \tag{4.2.27}$$

From (A2), one arrives that

$$\|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -f(u_n, y) \geq f(y, u_n), \; \forall y \in C.$$

Take $n \to \infty$ in the above inequality we get from (A4) and (4.2.14) that

$$f(y, p) \leq 0, \quad \forall y \in C.$$

For all $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Note that $y, p \in C$, one obtains $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from (A1) that

$$0 \;=\; f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

That is,

$$f(y_t, y) \;\geq\; 0. \tag{4.2.28}$$

Let $t \downarrow 0$, from (A3), we obtain $f(p, y) \geq 0$, for all $y \in C$. This implies that $p \in EP(f)$.

(III) $p \in VI(A, C)$.

From (4.2.13) and (4.2.24) we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.2.29}$$

Since $J$ is uniformly norm-to-norm continuous on bounded set, we have

$$\lim_{n\to\infty} \|Jy_n - Jx_n\| = 0. \tag{4.2.30}$$

Since $\|Jy_n - Jx_n\| = (1 - \delta_n)\|J\Pi_C(Jz_n - \beta Az_n) - Jx_n\|$ and $\limsup_{n\to\infty} \delta_n < 1$, we have

$$\lim_{n\to\infty} \|J\Pi_C(Jz_n - \beta Az_n) - Jx_n\| = 0.$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded set, we have

$$\|\Pi_C(Jz_n - \beta Az_n) - x_n\| = \lim_{n\to\infty} \|J^{-1}J\Pi_C(Jz_n - \beta Az_n) - J^{-1}Jx_n\| = 0. \tag{4.2.31}$$

From properties (iii) and (ii) of the operator V, we derive that



$$\begin{aligned}
\phi(x_n, z_n) &= V(Jz_n, x_n) \\
&\leq V(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n, x_n) \\
&= \phi(x_n, J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n)) \\
&= \|x_n\|^2 - 2\alpha_n\langle x_n, Jx_n\rangle - 2\beta_n\langle x_n, JTx_n\rangle \\
&\quad - 2\gamma_n\langle x_n, JSx_n\rangle + \|\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n\|^2 \\
&\leq \|x_n\|^2 - 2\alpha_n\langle x_n, Jx_n\rangle - 2\beta_n\langle x_n, JTx_n\rangle
\end{aligned}$$

$$\begin{aligned} & -2\gamma_n\langle x_n, JSx_n\rangle + \alpha_n\|Jx_n\|^2 + \beta_n\|JTx_n\|^2 + \gamma_n\|JSx_n\|^2 \\ = \;& \alpha_n\phi(x_n, x_n) + \beta_n\phi(x_n, Tx_n) + \gamma_n\phi(x_n, Sx_n). \end{aligned}$$

By the continuity of the function $\phi$ and (4.2.23), we have

$$\lim_{n\to\infty} \phi(x_n, z_n) = 0.$$

From Lemma 2.1.64, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{4.2.32}$$

Using inequalities (4.2.31) and (4.2.32) we obtain

$$\begin{aligned} \|\Pi_C(Jz_n - \beta Az_n) - z_n\| \;\leq\; & \|\Pi_C(Jz_n - \beta Az_n) - x_n\| \\ & + \|x_n - z_n\| \to 0. \end{aligned} \tag{4.2.33}$$

Since $x_n \to p$ we get that $z_n \to p$. By the continuity of the operator $J$, $A$ and $\Pi_C$, we have

$$\lim_{n\to\infty} \|\Pi_C(Jz_n - \beta Az_n) - \Pi_C(Jp - \beta Ap)\| = 0. \tag{4.2.34}$$

Note that

$$\begin{aligned} \|\Pi_C(Jz_n - \beta Az_n) - p\| \;\leq\; & \|\Pi_C(Jz_n - \beta Az_n) - z_n\| \\ & + \|z_n - q\| \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{4.2.35}$$

Hence, it follows from the uniqueness of the limit that $p = \Pi_C(Jp - \beta Ap)$. From Lemma 2.3.6, we have $p \in VI(A, C)$. By cases I, II and III, we conclude that $p \in F$.

**Step 5**. $p = \Pi_F Jx$.

Since $p \in F$, then from property(vii) of the operator $\Pi_C$, we have

$$V(J\Pi_F Jx, p) + V(Jx, \Pi_F Jx) \leq V(Jx, p). \tag{4.2.36}$$

On the other hand, since $x_{n+1} = \Pi_{C_{n+1}} Jx$, and $F \subset C_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, then it follows from property (vii) of the operator $\Pi_C$ that

$$V(Jx_{n+1}, \Pi_F Jx) + V(Jx, x_{n+1}) \leq V(Jx, \Pi_F Jx). \tag{4.2.37}$$

Moreover, by the continuity of the operator $V$, we get that

$$\lim_{n \to \infty} V(Jx, x_{n+1}) = V(Jx, p). \tag{4.2.38}$$

Combining (4.2.36), (4.2.37) with (4.2.38), we obtain that $V(Jx, p) = V(Jx, \Pi_F Jx)$. Therefore, it follows from the uniqueness of $\Pi_F Jx$ that $p = \Pi_F Jx$. This completes the proof. □

**Remark 4.2.3.** The following sequences of parameters are examples which support our main result:

$$\alpha_n = \frac{1}{3} - \frac{1}{n+1}, \beta_n = \frac{1}{3} \text{ and } \gamma_n = \frac{1}{3} + \frac{1}{n+1}$$

$$r_n = n + 3 \text{ and } \delta_n = \frac{n}{2n+1}$$

for all $n \in \mathbb{N}$.

Setting $S = T$ in Theorem 4.2.2, we obtain the following result.

**Corollary 4.2.4.** *Let $E$ be a uniformly convex and uniformly smooth Banach space and $C$ be a nonempty, closed and convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4). Assume that $A$ is a continuous operator of $C$ into $E^*$ satisfying conditions (2.3.4) and (2.3.5) and $T : C \to C$ is a relatively weak nonexpansive mapping with $F := F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$*

*be a sequence generated by the following manner:*

$$\begin{cases} x_0 = x \in C \textit{chosen arbitrary}, C_0 = C, \\ z_n = \Pi_C(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J \Pi_C (J z_n - \beta A z_n)), \\ u_n \in C \quad \textit{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} J x \quad \forall n \geq 0. \end{cases} \tag{4.2.39}$$

*Assume that* $\{\alpha_n\}$ *and* $\{\delta_n\}$ *are the sequences in* $[0,1]$ *satisfying the restrictions:*

*(C1)* $0 \leq \delta_n < 1, \limsup_{n\to\infty} \delta_n < 1$;

*(C2)* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$; *and*

*(C3)* $0 < \alpha_n < 1$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

*Then* $\{x_n\}$ *converges strongly to* $\Pi_F x$.

**Corollary 4.2.5.** *Let* $E$ *be a uniformly convex and uniformly smooth Banach space and* $C$ *be a nonempty, closed and convex subset of* $E$. *Assume that* $A$ *is an continuous operator of* $C$ *into* $E^*$ *satisfying conditions (2.3.4) and (2.3.5) and* $T : C \to C$ *is a relatively weak nonexpansive mapping with* $F := F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following manner:*

$$\begin{cases} x_0 = x \in C \textit{chosen arbitrary}, C_0 = C, \\ z_n = \Pi_C(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J \Pi_C (J z_n - \beta A z_n)), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} J x \quad \forall n \geq 0. \end{cases} \tag{4.2.40}$$

*Assume that* $\{\alpha_n\}$ *and* $\{\delta_n\}$ *are the sequences in* $[0,1]$ *satisfying the restrictions:*

*(C1)* $0 \leq \delta_n < 1, \limsup_{n\to\infty} \delta_n < 1$;

*(C2)* $0 < \alpha_n < 1$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

*Then* $\{x_n\}$ *converges strongly to* $\Pi_F x$.

*Proof.* Setting $S = T$, $f(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \geq 0$ in Theorem 4.2.2, we obtain that $\{x_n\}$ defined by (4.2.40) converges strongly to $\Pi_F x$. □

Now, we present two examples of mappings which are relatively weak nonexpansive mappings and can be found in Kim and Lee's results [66].

**Example 4.2.6.** [66, Example 3.13] Let $U$ denote the unit ball in the space $E = l^p$, where $1 < p < \infty$. Obviously, $E$ is uniformly convex and uniformly smooth. Let $T : E \to E$ be defined by

$$Tx = (0, x_1^2, \lambda_2 x_2, \lambda_3 x_3, \ldots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in U$, where $\lambda_n = 1 - \frac{1}{n^2}$ for $n \geq 2$ (hence $\prod_{n=2}^{\infty} \lambda_n = \frac{1}{2}$). Therefore,

(1). $F(T) = \{0 = (0, 0, 0, \ldots)\}$

(2). $T$ is relatively nonexpansive and hence it is relatively weak nonexpansive.

Next, consider an example where $F(T)$ is not singleton.

**Example 4.2.7.** [66, Example 3.14] Let $E = l^p$, where $2 < p < \infty$, and $C = \{x = (x_1, x_2, \ldots) \in X; 0 \leq x_n \leq 1\}$. Then $C$ is a closed convex subset of $X$. Note that $C$ is not bounded. Let $S : C \to C$ be defined by

$$Sx = (x_1, 0, x_2^2, \lambda_2 x_3, \lambda_2 x_4, \ldots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in C$, where $\lambda_n = 1 - \frac{1}{n^2}$ for $n \geq 2$ as in Example 4.2.6. Then

(1). $F(S) = \{p = (p_1, 0, 0, \ldots) : 0 \leq p_1 \leq 1\}$

(2). $S$ is relatively nonexpansive and hence it is relatively weak nonexpansive.

**Remark 4.2.8.** We observe that $0 = (0, 0, \ldots, )$ is a common fixed point of the mapping $T$ in Example 4.2.6 and the mapping $S$ in Example 4.2.7.