CHAPTER III

APPROXIMATION METHODS FOR EQUILIBRIUM PROBLEMS, AND FIXED POINT PROBLEMS IN HILBERT SPACES

3.1 Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces

In this section, we introduce two iterative schemes (one implicit and one explicit) for finding a common element of the set of solutions of the generalized equilibrium problems and the set of all common fixed points of a nonexpansive semigroup in a real Hilbert space.

Let $G : H \times H \to \mathbb{R}$ be an equilibrium bifunction, i.e., $G(u,u) = 0$ for each $u \in H$ and $\Psi : H \to H$ is a mapping. Then, we consider the following generalized equilibrium problem (for short, GEP):

$$\text{Finding } x^* \in H \text{ such that } G(x^*, y) + \langle \Psi x^*, y - x^* \rangle \geq 0, \forall y \in H. \qquad (3.1.1)$$

The problem (3.1.1) was studied by Moudafi [55]. The set of solutions for the problem GEP (3.1.1) is denoted by $GEP(G, \Psi)$.

Theorem 3.1.1. *(Implicit Method) Let* $\mathcal{S} = (T(s))_{s \geq 0}$ *be a nonexpansive semigroup on a real Hilbert space* H. *Let* $f : H \to H$ *be an* α*-contraction, let* $A : H \to H$ *be a strongly positive linear bounded self adjoint operator with coefficient* $\bar{\gamma}$, $G : H \times H \to \mathbb{R}$ *a mapping satisfying the hypotheses (A1)-(A4),* $\Psi : H \to H$ *an inverse-strongly monotone mapping with a coefficients* δ *such that* $F(\mathcal{S}) \cap GEP(G, \Psi) \neq \emptyset$. *Let* $\{\alpha_n\} \subset (0,1)$, $\{r_n\} \subset (0, 2\delta)$ *and* $\{s_n\} \subset (0, \infty)$ *be the real sequences. Then*

(i) *For any* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, *there exists a unique sequence* $\{x_n\} \subset H$ *such that*

$$\begin{cases} G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \forall n \geq 1. \end{cases} \tag{3.1.2}$$

(ii) *If* $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} r_n > 0$, *and* $\lim_{n\to\infty} s_n = +\infty$, *the sequence* $\{x_n\}$ *defined by* (3.1.2) *converges strongly to* z, *which is a unique solution in* $F(S) \cap GEP(G, \Psi)$ *of the variational inequality*

$$\langle (\gamma f - A)z, p - z \rangle \leq 0, \quad \forall p \in F(S) \cap GEP(G, \Psi). \tag{3.1.3}$$

Proof. First of all, we will prove that $\{x_n\}$ is well defined. Indeed, consider the mapping $S_n : H \to H$ defined by

$$S_n x := \alpha_n \gamma f(x) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)x ds. \tag{3.1.4}$$

for all $x \in H$ and $n \geq 1$. We claim that S_n is $(1 - \alpha_n(\bar{\gamma} - \gamma\alpha))$-contraction. We observe that $T_{r_n}(I - r_n\Psi)$ is nonexpansive for all $n \geq 1$. Indeed, for any $x, y \in H$,

$$\begin{aligned} \|T_{r_n}(I - r_n\Psi)x - T_{r_n}(I - r_n\Psi)y\|^2 &\leq \|(I - r_n\Psi)x - (I - r_n\Psi)y\|^2 \\ &= \|(x - y) - r_n(\Psi x - \Psi y)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, \Psi x - \Psi y \rangle + r_n^2\|\Psi x - \Psi y\|^2 \\ &\leq \|x - y\|^2 - 2r_n\delta\|\Psi x - \Psi y\|^2 + r_n^2\|\Psi x - \Psi y\|^2 \\ &= \|x - y\|^2 + r_n(r_n - 2\delta)\|\Psi x - \Psi y\|^2 \\ &= \|x - y\|^2. \end{aligned} \tag{3.1.5}$$

By Lemma 2.2.22 and (3.1.5), we have

$$\|S_n x - S_n y\| \leq \left\| \alpha_n \gamma f(x) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)x ds \right.$$

$$
\begin{aligned}
&-\alpha_n\gamma f(y)-(I-\alpha_nA)\frac{1}{s_n}\int_0^{s_n}T(s)T_{r_n}(I-r_n\Psi)yds\Big\| \\
\leq\ & \alpha_n\gamma\|f(x)-f(y)\|+(1-\alpha_n\bar{\gamma})\Big\|\frac{1}{s_n}\int_0^{s_n}[T(s)T_{r_n}(I-r_n\Psi)x \\
&-T(s)T_{r_n}(I-r_n\Psi)y]ds\Big\| \\
\leq\ & \alpha_n\gamma\alpha\|x-y\|+(1-\alpha_n\bar{\gamma})\|T_{r_n}(I-r_n\Psi)x-T_{r_n}(I-r_n\Psi)y\| \\
\leq\ & \alpha_n\gamma\alpha\|x-y\|+(1-\alpha_n\bar{\gamma})\|x-y\| \\
=\ & (1-\alpha_n(\bar{\gamma}-\gamma\alpha))\|x-y\|.
\end{aligned}
$$

Since $0<1-\alpha_n(\bar{\gamma}-\gamma\alpha)<1$, it follows that S_n is a contraction. Therefore by Banach contraction principle, S_n has a unique fixed point $x_n\in H$ such that

$$x_n=\alpha_n\gamma f(x_n)+(I-\alpha_nA)\frac{1}{s_n}\int_0^{s_n}T(s)T_{r_n}(I-r_n\Psi)x_nds.$$

Next, we will show that $\{x_n\}$ is bounded. Since $\alpha_n\to 0$ as $n\to\infty$, we may assume, with no loss of generality, that $\alpha_n<\|A\|^{-1}$, for all $n\geq 1$. Note that u_n can be written as $u_n=T_{r_n}(x_n-r_n\Psi x_n)$ for all $n\geq 1$. Take $p\in F(S)\cap GEP(G,\Psi)$. Applying $p=T_{r_n}(p-r_n\Psi p)$ and (3.1.5), we obtain the following

$$
\begin{aligned}
\|u_n-p\|^2 &\leq \|T_{r_n}(x_n-r_n\Psi x_n)-T_{r_n}(p-r_n\Psi p)\|^2 \\
&\leq \|x_n-p\|^2+r_n(r_n-2\delta)\|\Psi x_n-\Psi p\|^2\leq\|x_n-p\|^2. \qquad (3.1.6)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_n-p\| &\leq \alpha_n\|\gamma f(x_n)-Ap\|+(1-\alpha_n\bar{\gamma})\frac{1}{s_n}\int_0^{s_n}\|T(s)u_n-T(s)p\|ds \\
&\leq \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|+(1-\alpha_n\bar{\gamma})\|x_n-p\|.
\end{aligned}
$$

Hence,

$$\|x_n-p\| \leq \frac{1}{\bar{\gamma}-\gamma\alpha}\|\gamma f(p)-Ap\|,$$

i.e., $\{x_n\}$ is bounded and so is $\{u_n\}$. Now, we will prove that

$$\lim_{n\to\infty}\|x_n-u_n\|=0.$$

From Lemma 2.2.20 and (3.1.6), we have

$$\begin{aligned}
\|x_n - p\|^2 &\leq (1-\alpha_n\bar{\gamma})^2\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ap + \gamma f(p) - \gamma f(p), x_n - p\rangle \\
&\leq (1+\alpha_n^2\bar{\gamma}^2 - 2\alpha_n\bar{\gamma})\|u_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle \\
&\leq (1+\alpha_n^2\bar{\gamma}^2)\|u_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle \\
&\leq (1+\alpha_n^2\bar{\gamma}^2)(\|x_n - p\|^2 + r_n(r_n - 2\delta)\|\Psi x_n - \Psi p\|^2) \\
&\quad - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle \\
&= (1+\alpha_n^2\bar{\gamma}^2)\|x_n - p\|^2 + (1+\alpha_n^2\bar{\gamma}^2)r_n(r_n - 2\delta)\|\Psi x_n - \Psi p\|^2 \\
&\quad - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle
\end{aligned}$$

and hence

$$\begin{aligned}
(1+\alpha_n^2\bar{\gamma}^2)r_n(2\delta - r_n)\|\Psi x_n - \Psi p\|^2 &\leq \alpha_n(\alpha_n\bar{\gamma}^2 - 2\gamma\alpha)\|x_n - p\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle.
\end{aligned}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and $\liminf_{n\to\infty} r_n > 0$, we have

$$\|\Psi x_n - \Psi p\| \to 0, \text{ as } n \to \infty. \tag{3.1.7}$$

On the other hand, using Lemma 4.2.1 and (3.1.6), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n\Psi x_n) - T_{r_n}(p - r_n\Psi p)\|^2 \\
&\leq \langle x_n - r_n\Psi x_n - (p - r_n\Psi p), u_n - p\rangle \\
&= \frac{1}{2}(\|(x_n - r_n\Psi x_n) - (p - r_n\Psi p)\|^2 + \|u_n - p\|^2 \\
&\quad - \|(x_n - r_n\Psi x_n) - (p - r_n\Psi p) - (u_n - p)\|^2 \\
&\leq \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(\Psi x_n - \Psi p)\|^2) \\
&= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2
\end{aligned}$$

$$+2r_n\langle x_n - u_n, \Psi x_n - \Psi p\rangle - r_n^2\|\Psi x_n - \Psi p\|^2).$$

So, we have

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, \Psi x_n - \Psi p\rangle - r_n^2\|\Psi x_n - \Psi p\|^2. \qquad (3.1.8)\end{aligned}$$

It follows from Lemma 2.2.20 and (3.1.8), for any $p \in F(\mathcal{S}) \cap GEP(G, \Psi)$,

$$\begin{aligned}\|x_n - p\|^2 &\leq (1 - \alpha_n\bar{\gamma})^2\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\|^2 \\ &\quad + 2\alpha_n\langle\gamma f(x_n) - Ap + \gamma f(p) - \gamma f(p), x_n - p\rangle \\ &\leq (1 + \alpha_n^2\bar{\gamma}^2 - 2\alpha_n\bar{\gamma})\|u_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 \\ &\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle \\ &\leq (1 + \alpha_n^2\bar{\gamma}^2)\|u_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle \\ &\leq \|u_n - p\|^2 + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 \\ &\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle \\ &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, \Psi x_n - \Psi p\rangle - r_n^2\|\Psi x_n - \Psi p\|^2 \\ &\quad + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle \\ &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|\Psi x_n - \Psi p\| - r_n^2\|\Psi x_n - \Psi p\|^2 \\ &\quad + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 \\ &\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle. \qquad (3.1.9)\end{aligned}$$

So, we have

$$\begin{aligned}\|x_n - u_n\|^2 &\leq \|\Psi x_n - \Psi p\|[2r_n\|x_n - u_n\| - r_n^2\|\Psi x_n - \Psi p\|] \\ &\quad + \alpha_n[\bar{\gamma}^2\|x_n - p\|^2 + 2\gamma\alpha\|x_n - p\|^2 \\ &\quad + 2\langle\gamma f(p) - Ap, x_n - p\rangle]. \qquad (3.1.10)\end{aligned}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and (3.1.7), we can conclude that

$$\|x_n - u_n\| \to 0, \text{ as } n \to \infty. \tag{3.1.11}$$

On the other hand, let $z_1 = P_{F(\mathcal{S})}x_1$ and $D = \{z \in H : \|z - z_1\| \leq \frac{1}{\bar{\gamma}-\gamma\alpha}\|\gamma f(z_1) - Az_1\|\}$. Then D is a nonempty closed bounded convex subset of H which is $T(s)$-invariant for each $s \in [0,\infty)$ and contains $\{x_n\}$ and $\{u_n\}$. We may assume, without loss of generality, that $\mathcal{S} = (T(s))_{s\geq 0}$ is a nonexpansive semigroup on D. In view of Lemma 2.2.19, we can obtain that

$$\begin{aligned} &\lim_{n\to\infty}\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - T(t)(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds)\right\| \\ \leq\ &\lim_{n\to\infty}\sup_{z\in D}\left\|\frac{1}{s_n}\int_0^{s_n} T(s)z ds - T(t)(\frac{1}{s_n}\int_0^{s_n} T(s)z ds)\right\| = 0 \end{aligned} \tag{3.1.12}$$

for every $t \in [0,\infty)$. We observe that, for any $0 \leq t < \infty$,

$$\begin{aligned} \|T(s)x_n - x_n\| &\leq \left\|T(s)x_n - T(t)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &\quad + \left\|T(t)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &\quad + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - x_n\right\| \\ &\leq 2\left\|x_n - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &\quad + \left\|T(t)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &= 2\alpha_n\left\|\gamma f(x_n) - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &\quad + \left\|T(t)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|. \end{aligned}$$

Applying (3.1.12), Lemma 2.2.19 and the boundedness of $\{x_n\}$, $\{u_n\}$, we obtain that

$$\|T(s)x_n - x_n\| \to 0, \text{ as } n \to \infty \text{ for all } 0 \leq s < \infty. \tag{3.1.13}$$

Consider a subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $z \in H$. Next, we show that $z \in F(\mathcal{S}) \cap GEP(G, \Psi)$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup z$ as $i \to \infty$. From $\|T(s)x_{n_i} - x_{n_i}\| \to 0$ and the demiclosedness principle of $I - T(s)$ for all $0 \leq s < \infty$, one sees that

$$T(s)z = z \text{ for all } 0 \leq s < \infty \text{ that is } z \in F(\mathcal{S}).$$

Next, we show that $z \in GEP(G, \Psi)$. From $\|x_{n_i} - u_{n_i}\| \to 0$, one sees that

$$u_{n_i} \rightharpoonup z \text{ and } T(s)x_{n_i} \rightharpoonup z, \text{ as } i \to \infty \text{ for all } 0 \leq s < \infty.$$

Putting $\{x_i\} := \{x_{n_i}\}$, $\{u_i\} := \{u_{n_i}\}$ and $\{r_i\} := \{r_{n_i}\}$. Since $u_n = T_{r_n}(x_n - r_n\Psi x_n)$, for any $y \in H$ we have

$$G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0.$$

From (A2), we have

$$\langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq G(y, u_n).$$

Replacing n by n_i, we have

$$\langle \Psi x_i, y - u_i \rangle + \frac{1}{r_i}\langle y - u_i, u_k - x_i \rangle \geq G(y, u_i), \qquad \text{for all } y \in H. \tag{3.1.14}$$

Put $u_t = ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in H$. Then, we have $u_t \in H$. So from (3.1.14) we have

$$\begin{aligned} \langle u_t - u_i, \Psi u_t \rangle &\geq \langle u_t - u_i, \Psi u_t \rangle - \langle u_t - u_i, \Psi x_i \rangle - \langle u_t - u_i, \frac{u_i - x_i}{r_i} \rangle + G(u_t, u_i) \\ &= \langle u_t - u_i, \Psi u_t - \Psi u_i \rangle + \langle u_t - u_i, \Psi u_i - \Psi x_i \rangle - \langle u_t - u_i, \frac{u_i - x_i}{r_i} \rangle \\ &\quad + G(u_t, u_i). \end{aligned}$$

Since $\|u_i - x_i\| \to 0$, we have $\|\Psi u_i - \Psi x_i\| \to 0$. Further, from monotonicity of Ψ, we have $\langle u_t - u_i, \Psi u_t - \Psi u_i \rangle \geq 0$. So, from (A4) we have

$$\langle u_t - z, \Psi u_t \rangle \geq G(u_t, z), \tag{3.1.15}$$

as $i \to \infty$. From (A1) and (A4) and (3.1.15), we also have

$$\begin{aligned} 0 &= G(u_t, u_t) \leq tG(u_t, y) + (1-t)G(u_t, z) \\ &\leq tG(u_t, y) + (1-t)\langle u_t - z, \Psi u_t \rangle \\ &= tG(u_t, y) + (1-t)\langle y - z, \Psi u_t \rangle \end{aligned}$$

and hence

$$0 \leq G(u_t, y) + (1-t)\langle y - z, \Psi u_t \rangle.$$

Letting $t \to \infty$, we have, for each $y \in C$,

$$0 \leq G(z, y) + \langle y - z, \Psi z \rangle.$$

This implies $z \in GEP(G, \Psi)$. Hence $z \in F(\mathcal{S}) \cap GEP(G, \Psi)$ is proved. Next, we show that z solves the variational inequality (3.1.3). We observe that

$$\begin{aligned} \|x_n - z\|^2 &= \alpha_n \langle \gamma f(x_n) - Az, x_n - z \rangle \\ &\quad + \left\langle (I - \alpha_n A)\left(\frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - z \right), x_n - z \right\rangle \\ &\leq (1 - \alpha_n \bar{\gamma})\|x_n - z\|^2 + \alpha_n \langle \gamma f(x_n) - Az, x_n - z \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})\|x_n - z\|^2 + \alpha_n \gamma \alpha \|x_n - z\|^2 + \alpha_n \langle \alpha f(z) - Az, x_n - z \rangle. \end{aligned}$$

This implies that

$$\|x_n - z\|^2 \leq \frac{1}{\bar{\gamma} - \gamma\alpha} \langle \gamma f(z) - Az, x_n - z \rangle.$$

In particular, we have

$$\|x_i - z\|^2 \leq \frac{1}{\bar{\gamma} - \gamma\alpha} \langle \gamma f(z) - Az, x_i - z \rangle. \qquad (3.1.16)$$

Since $x_i \rightharpoonup z$, it follows from (3.1.16) that $x_i \to z$ as $i \to \infty$. We rewrite $(A - \gamma f)x_n$ as

$$(A - \gamma f)x_n = -\frac{1}{\alpha_n}(I - \alpha_n A)\left[x_n - \frac{1}{s_n} \int_0^{s_n} T(s) T_{r_n}(I - r_n \Psi) x_n ds \right]$$

and utilize the fact that $(I-T)$ is monotone if T is nonexpansive. Hence, for any $p \in F(\mathcal{S}) \cap GEP(G, \Psi)$, we have

$$\begin{aligned}
\langle (A-\gamma f)x_n, x_n - p\rangle &= -\frac{1}{\alpha_n}\langle (I-\alpha_n A)\left[x_n - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)x_n ds\right], x_n - p\rangle \\
&= -\frac{1}{\alpha_n}\Big[\langle \Big(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\Big)x_n \\
&\quad - \Big(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\Big)p, x_n - p\rangle\Big] \\
&\quad + \frac{1}{s_n}\langle A\int_0^{s_n}[x_n - T(s)u_n]ds, x_n - p\rangle. \\
&= -\frac{1}{\alpha_n}\Big[\langle \Big(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\Big)x_n \\
&\quad - \Big(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\Big)p, x_n - p\rangle ds\Big] \\
&\quad + \frac{1}{s_n}\langle A\int_0^{s_n}[x_n - T(s)u_n]ds, x_n - p\rangle. \qquad (3.1.17)
\end{aligned}$$

Since the map $\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)ds$ is nonexpansive, then $I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)ds$ is monotone. This implies that

$$\Big\langle \Big(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\Big)x_n - \Big(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\Big)p, x_n - p\Big\rangle \geq 0.$$

This together with (3.1.17), we obtain that

$$\langle (A-\gamma f)x_n, x_n - p\rangle \leq \Big\langle Ax_n - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds, x_n - p\Big\rangle.$$

By the definition of x_n, we obtain that

$$Ax_n - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds = \alpha_n A\Big(\gamma f(x_n) - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds\Big).$$

Then,

$$\langle (A-\gamma f)x_n, x_n - p\rangle \leq \alpha_n\Big\langle A\Big(\gamma f(x_n) - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds\Big), x_n - p\Big\rangle. \qquad (3.1.18)$$

In particular, we have

$$\langle (A-\gamma f)x_i, x_i - p\rangle \leq \alpha_i\Big\langle A\Big(\gamma f(x_i) - \frac{A}{s_i}\int_0^{s_i} T(s)u_i ds\Big), x_i - p\Big\rangle. (3.1.19)$$

where $\alpha_i := \alpha_{n_i}$. Passing to the limit $i \to \infty$, by the boundedness of x_i and u_i we obtain

$$\langle (A - \gamma f)z, z - p\rangle = \lim_{i\to\infty} \langle (A - \gamma f)x_i, x_i - p\rangle \leq 0, \quad \forall p \in F(\mathcal{S}) \cap GEP(G, \Psi). \tag{3.1.20}$$

That is , $z \in F(\mathcal{S}) \cap GEP(G, \Psi)$ is a solution of the variational inequality (3.1.3). Finally, we will show that the sequence $\{x_n\}$ converges strongly to z. Assume that there exits a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^*$ as $k \to \infty$. By the same methods as in the above proof, we obtain $x^* \in F(\mathcal{S}) \cap GEP(G, \Psi)$. If follows from the inequality (3.1.20) that

$$\langle (A - \gamma f)z, z - x^*\rangle \leq 0. \tag{3.1.21}$$

Interchange z and x^* to obtain

$$\langle (A - \gamma f)x^*, x^* - z\rangle \leq 0. \tag{3.1.22}$$

Adding the inequalities (3.1.21) and (3.1.22), yields

$$(\bar{\gamma} - \gamma\alpha)\|z - x^*\|^2 \leq \langle z - x^*, (A - \gamma f)z - (A - \gamma f)x^*\rangle \leq 0$$

by Lemma 2.2.23. Hence $z = x^*$ and therefore $x_n \to z$ as $n \to \infty$. This completes the proof. □

Setting $G \equiv 0$, $\Psi \equiv 0$, $r_n \equiv 1$ in Theorem 3.1.1, we have the following result.

Corollary 3.1.2. [56, Theorem 3.1] Let C be nonempty closed convex subset of real Hilbert space H. Suppose that $f : C \to C$ is a fixed contractive mapping with coefficient $0 < \alpha < 1$, and $\mathcal{S} = \{T(s) : s \geq 0\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty, and A a strong positive linear bounded operator with coefficient $\bar{\gamma} > 0$, $\{\alpha_n\} \subset (0,1)$, $\{s_n\} \subset (0,\infty)$ are real sequences such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} s_n = \infty$, then for any $0 < \gamma < \bar{\gamma}/\alpha$, there is a unique sequence $\{x_n\} \subset C$ such that

$$x_n = (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds + \alpha_n \gamma f(x_n)$$

and the sequence $\{x_n\}$ converges strongly to the unique solution $z \in F(\mathcal{S})$ of the variational inequality $\langle(\gamma f - A)z, p - z\rangle \leq 0$, $\forall p \in F(\mathcal{S})$.

Theorem 3.1.3. *(Explicit Iterative Method) Let $\mathcal{S} = (T(s))_{s\geq 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \to H$ be an α-contraction, let $A : H \to H$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$ and let γ be a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $G : H \times H \to \mathbb{R}$ be a mapping satisfying the hypotheses (A1)-(A4) and $\Psi : H \to H$ an inverse-strongly monotone mapping with coefficient δ such that $F(\mathcal{S}) \cap GEP(G, \Psi) \neq \emptyset$. Let the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1 \in H \text{ chosen arbitrary}, \\ G(u_n, y) + \langle \Psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ y_n = \beta_n x_n + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \forall n \geq 1, \end{cases} \tag{3.1.23}$$

where the real sequences $\{r_n\} \subset (0, 2\delta)$, $\{s_n\} \subset (0,\infty)$ and $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ satisfy the following conditions:

(D1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$,

(D2) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(D3) $\lim_{n\to\infty} s_n = +\infty$, $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$, *and*

(D4) $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0$.

Then the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to z which is a unique solution in $F(S) \cap GEP(G, \Psi)$ of the variational inequality (3.1.3).

Proof. We divide the proof of Theorem 3.1.3 into five steps:

Step 1. Firstly, we show that $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are bounded.
Note that u_n can be written as $u_n = T_{r_n}(x_n - r_n\Psi x_n)$ for all $n \geq 1$. Take $p \in F(S) \cap GEP(G, \Psi)$. Since $p = T_{r_n}(p - r_n\Psi p)$, $\Psi : H \to H$ is an inverse-strongly monotone mappings with coefficients δ satisfying $0 \leq r_n \leq 2\delta$, we obtain the following

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n\Psi x_n) - T_{r_n}(p - r_n\Psi p)\|^2 \\
&\leq \|(x_n - r_n\Psi x_n) - (p - r_n\Psi p)\|^2 \\
&= \|(x_n - p) - r_n(\Psi x_n - \Psi p)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, \Psi x_n - \Psi p\rangle + r_n^2\|\Psi x_n - \Psi p\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n\delta\|\Psi x_n - \Psi p\|^2 + r_n^2\|\Psi x_n - \Psi p\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\delta)\|\Psi x_n - \Psi p\|^2 \\
&= \|x_n - p\|^2. \qquad (3.1.24)
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ we may assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ for all $n \in \mathbb{N}$. Applying Lemma 2.2.22 and (3.1.24), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|\alpha_n\gamma f(x_n) + (I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds) - p\right\| \\
&= \Bigg\|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(p) + \alpha_n\gamma f(p) - \alpha_n Ap \\
&\quad + (I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p)\Bigg\| \\
&\leq \|\alpha_n\gamma(f(x_n) - f(p))\| + \|\alpha_n(\gamma f(p) - Ap)\| \\
&\quad + (1 - \alpha_n\bar{\gamma})\left\|\beta_n x_n + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&= \|\alpha_n\gamma(f(x_n) - f(p))\| + \|\alpha_n(\gamma f(p) - Ap)\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\alpha_n\bar{\gamma})\left\|\beta_n(x_n - p) + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n}[T(s)u_n - T(s)p]ds\right\| \\
&\leq \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| \\
&\quad + (1-\alpha_n\bar{\gamma})(\beta_n\|x_n - p\| + (1-\beta_n)\|u_n - p\|) \\
&\leq \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| \\
&\quad + (1-\alpha_n\bar{\gamma})(\beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\|) \\
&\leq (1-\alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|. \qquad (3.1.25)
\end{aligned}
$$

From a simple inductive process, it follows that

$$
\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}\right\}, \ n \geq 1,
$$

which yields that $\{x_n\}$ is bounded, so is $\{u_n\}$. Moreover, since

$$
\begin{aligned}
\|y_n - p\| &= \left\|\beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&= \left\|\beta_n x_n - \beta_n p + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - (1-\beta_n)p\right\| \\
&= \left\|\beta_n(x_n - p) + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n}[T(s)u_n - T(s)p]ds\right\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\|u_n - p\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\
&= \|x_n - p\|, \qquad (3.1.26)
\end{aligned}
$$

$\{y_n\}$ is also bounded.

Step 2. Now we show that

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \ \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0 \text{ and } \lim_{n\to\infty}\|u_{n+1} - u_n\| = 0.
$$

We rewrite x_{n+1} in the form:

$$
x_{n+1} = (1-\lambda_n)x_n + \lambda_n z_n, \ \text{ where } \lambda_n = 1 - (1-\alpha_n)\beta_n, \qquad (3.1.27)
$$

and

$$
\begin{aligned}
z_n &= \frac{\alpha_n\beta_n}{\lambda_n}(I-A)x_n + \frac{(1-\beta_n)}{\lambda_n}(I-\alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \\
&+ \frac{\alpha_n}{\lambda_n}\gamma f(x_n).
\end{aligned} \tag{3.1.28}
$$

Since $\alpha_n \to 0$ and $0 < a \le \beta_n \le b < 1$, then

$$
0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1.
$$

Next some manipulations give us that

$$
\begin{aligned}
z_{n+1} - z_n &= \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}(I-A)x_{n+1} - \frac{\beta_n\alpha_n}{\lambda_n}(I-A)x_n \\
&+ \frac{1-\beta_{n+1}}{\lambda_{n+1}}\Big(\frac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)u_{n+1}ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\Big) \\
&- \frac{(1-\beta_{n+1})\alpha_{n+1}}{\lambda_{n+1}}A\Big(\frac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)u_{n+1}ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\Big) \\
&+ \Big(\frac{1-\beta_{n+1}}{\lambda_{n+1}} - \frac{1-\beta_n}{\lambda_n}\Big)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \\
&- \Big(\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\Big)(1-\beta_n)A\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \\
&+ \frac{\alpha_{n+1}}{\lambda_{n+1}}(\beta_n - \beta_{n+1})A\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \\
&+ \frac{\alpha_{n+1}}{\lambda_{n+1}}(\gamma f(x_{n+1}) - \gamma f(x_n)) + \Big(\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\Big)\gamma f(x_n).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
\le\ & \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}\|(I-A)x_{n+1}\| + \frac{\beta_n\alpha_n}{\lambda_n}\|(I-A)x_n\| + \Big|\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\Big|\|\gamma f(x_n)\| \\
&+ \Big(\frac{1-\beta_{n+1}}{\lambda_{n+1}} - 1\Big)\left\|\frac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)u_{n+1}ds - Tx_{n+1} + \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&+ \frac{(1-\beta_{n+1})\alpha_{n+1}}{\lambda_{n+1}}\|A\|\left\|\frac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)u_{n+1}ds - Tx_{n+1} + \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&+ \Big|\frac{1-\beta_{n+1}}{\lambda_{n+1}} - \frac{1-\beta_n}{\lambda_n}\Big|\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&+ \Big|\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\Big|\left\|(1-\beta_n)A\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|
\end{aligned}
$$

$$+\frac{\alpha_{n+1}}{\lambda_{n+1}}|\beta_n-\beta_{n+1}|\left\|A\frac{1}{s_n}\int_0^{s_n}T(s)u_nds\right\|$$
$$+\frac{\alpha_{n+1}}{\lambda_{n+1}}\|(\gamma f(x_{n+1})-\gamma f(x_n))\|+|\frac{\alpha_{n+1}}{\lambda_{n+1}}-\frac{\alpha_n}{\lambda_n}|\|\gamma f(x_n)\|.$$

Since $\lambda_n = 1-(1-\alpha_n)\beta_n$ and $\alpha_n \to 0$, then

$$\lim_{n\to\infty}\frac{1-\beta_n}{\lambda_n}=\lim_{n\to\infty}(1-\frac{\alpha_n\beta_n}{\lambda_n})=1.$$

Then last inequality implies

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0$$

and so an application of Lemma 2.1.56 asserts that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.1.29}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\lambda_n)\|z_n-x_n\|=0. \tag{3.1.30}$$

From the fact that

$$\left(\frac{1}{a}-\frac{1}{b}\right)b=-\frac{a-b}{a},$$

for all nonzero real numbers a, b, we obtain that, for any $p \in F(\mathcal{S})$,

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \left\|\beta_n x_n+(1-\beta_n)\frac{1}{s_n}\int_0^{s_n}T(s)u_nds\right.\\
&\quad\left.-\beta_{n-1}x_{n-1}-(1-\beta_{n-1})\frac{1}{s_{n-1}}\int_0^{s_{n-1}}T(s)u_{n-1}ds\right\|\\
&= \left\|\beta_n x_n-\beta_{n-1}x_{n-1}+\frac{1}{s_n}\int_0^{s_n}T(s)u_nds-\frac{1}{s_{n-1}}\int_0^{s_{n-1}}T(s)u_{n-1}ds\right.\\
&\quad\left.-\frac{\beta_n}{s_n}\int_0^{s_n}T(s)u_nds+\frac{\beta_{n-1}}{s_{n-1}}\int_0^{s_{n-1}}T(s)u_{n-1}ds\right\|\\
&\leq \left\|\beta_n(x_n-x_{n-1})+(\beta_n-\beta_{n-1})x_{n-1}+(1-\beta_n)(\frac{1}{s_n}\int_0^{s_n}T(s)u_nds\right.\\
&\quad\left.-\frac{1}{s_{n-1}}\int_0^{s_{n-1}}T(s)u_{n-1}ds)-(\beta_n-\beta_{n-1})\frac{1}{s_{n-1}}\int_0^{s_{n-1}}T(s)u_{n-1}ds\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg\| \beta_n(x_n - x_{n-1}) + (\beta_n - \beta_{n-1})x_{n-1} \\
&\quad + (1-\beta_n)\Bigg\{ \frac{1}{s_n}\int_0^{s_n} [T(s)u_n - T(s)u_{n-1}]ds + (\frac{1}{s_n} - \frac{1}{s_{n-1}}) \\
&\quad \times \int_0^{s_{n-1}} [T(s)u_{n-1} - T(s)p]ds + \frac{1}{s_n}\int_{s_{n-1}}^{s_n} [T(s)u_{n-1} - T(s)p]ds \Bigg\} \\
&\quad - (\beta_n - \beta_{n-1})\frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds \Bigg\| \\
&\le \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1-\beta_n)\Bigg\{ \|u_n - u_{n-1}\| \\
&\quad + \left(\frac{2|s_n - s_{n-1}|}{s_n}\right)\|u_{n-1} - p\| \Bigg\} \\
&\quad + |\beta_n - \beta_{n-1}| \left\| \frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds \right\|. \qquad (3.1.31)
\end{aligned}
$$

On the other hand, we observe that

$$u_n = T_{r_n}(x_n - r_n\Psi x_n) \text{ and } u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1}\Psi x_{n+1})$$

we have

$$F(u_n, y) + \langle \Psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \text{ for all } y \in H \qquad (3.1.32)$$

and

$$F(u_{n+1}, y) + \langle \Psi x_{n+1}, y - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0 \text{ for all } y \in H. \qquad (3.1.33)$$

Putting $y = u_{n+1}$ in (3.1.32) and $y = u_n$ in (3.1.33), we have

$$F(u_n, u_{n+1}) + \langle \Psi x_n, u_{n+1} - u_n\rangle + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0$$

and

$$F(u_{n+1}, u_n) + \langle \Psi x_{n+1}, u_n - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0.$$

Adding the above two inequalities, the monotonicity of F implies that

$$\langle \Psi x_{n+1} - \Psi x_n, u_n - u_{n+1}\rangle + \langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \ge 0.$$

Hence

$$\begin{aligned} 0 &\le \langle u_n - u_{n+1}, r_n(\Psi x_{n+1} - \Psi x_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n)\rangle \\ &= \langle u_{n+1} - u_n, u_n - u_{n+1} + (1 - \frac{r_n}{r_{n+1}})u_{n+1} + (x_{n+1} - r_n\Psi x_{n+1}) \\ &\quad - (x_n - r_n\Psi x_n) - x_{n+1} + \frac{r_n}{r_{n+1}}x_{n+1}\rangle \\ &= \langle u_{n+1} - u_n, u_n - u_{n+1} + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) + (x_{n+1} - r_n\Psi x_{n+1}) \\ &\quad - (x_n - r_n\Psi x_n)\rangle. \end{aligned}$$

It follows that

$$\|u_{n+1} - u_n\|^2 \le \|u_{n+1} - u_n\| \left\{\left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|\right\}$$

and hence

$$\|u_{n+1} - u_n\| \le \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|. \tag{3.1.34}$$

Since $\liminf_{n\to\infty} r_n$ is strictly positive, there exists $b > 0$ such that $r_n > b$ for large $n \in \mathbb{N}$. Then,

$$\|u_{n+1} - u_n\| \le \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{b}\|u_{n+1} - x_{n+1}\|. \tag{3.1.35}$$

Using (3.1.31) and (3.1.35), we can obtain

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n)\Big\{\|x_n - x_{n-1}\| \\ &\quad + \frac{|r_n - r_{n-1}|}{b}\|u_n - x_n\| + \left(\frac{2|s_n - s_{n-1}|}{s_n}\right)\|u_{n-1} - p\|\Big\} \\ &\quad + |\beta_n - \beta_{n-1}|\left\|\frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| \\ &= \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1 - \beta_n)\Big\{\frac{|r_n - r_{n-1}|}{b}\|u_n - x_n\| \\ &\quad + \left(\frac{2|s_n - s_{n-1}|}{s_n}\right)\|u_{n-1} - p\|\Big\} \\ &\quad + |\beta_n - \beta_{n-1}|\left\|\frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\|. \end{aligned} \tag{3.1.36}$$

From (3.1.30) and (D2)-(D4), it follows that also

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = \lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.1.37}$$

Step 3. Now we will prove that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|u_n - y_n\| = \lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.1.38}$$

In fact, since

$$\|x_n - y_n\| \le \|y_n - y_{n-1}\| + \alpha_{n-1}\|\gamma f(x_{n-1}) - Ay_{n-1}\|,$$

and from the boundedness of $\{f(x_{n-1})\}$, $\{A(y_{n-1})\}$ and $\lim_{n\to\infty} \|y_n - y_{n-1}\| = 0$, it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

From Lemma 2.2.20, it follows that

$$\|x_{n+1} - p\|^2 \le \|(I - \alpha_n A)(y_n - p)\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ap, x_{n+1} - p\rangle. \tag{3.1.39}$$

i.e.,

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n\bar{\gamma})^2\|y_n - p\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ap, x_{n+1} - p\rangle. \tag{3.1.40}$$

Using (3.1.26) and (3.1.6), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1 - \alpha_n\bar{\gamma})^2(\beta_n\|x_n - p\| + (1 - \beta_n)\|u_n - p\|)^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ap, x_{n+1} - p\rangle \\
&\le (1 - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2)(\beta_n\|x_n - p\|^2 + (1 - \beta_n)\|u_n - p\|^2) \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ap, x_{n+1} - p\rangle \\
&\le (1 + \alpha_n^2\bar{\gamma}^2)(\beta_n\|x_n - p\|^2 + (1 - \beta_n)\|u_n - p\|^2) \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ap, x_{n+1} - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&= (1+\alpha_n^2\bar{\gamma}^2)\beta_n\|x_n-p\|^2+(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)\|u_n-p\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle \\
&\leq (1+\alpha_n^2\bar{\gamma}^2)\beta_n\|x_n-p\|^2+(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)(\|x_n-p\|^2 \\
&\quad +r_n(r_n-2\delta)\|\Psi x_n-\Psi p\|^2) \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle \\
&= (1+\alpha_n^2\bar{\gamma}^2)\|x_n-p\|^2+(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)r_n(r_n-2\delta)\|\Psi x_n-\Psi p\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle \\
&\leq \|x_n-p\|^2+\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle \\
&\quad +(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)r_n(r_n-2\delta)\|\Psi x_n-\Psi p\|^2 \qquad (3.1.41)
\end{aligned}
$$

and hence

$$
\begin{aligned}
(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)r_n(2\delta-r_n)\|\Psi x_n-\Psi p\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle \\
&\quad +\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2 \\
&\leq \|x_n-x_{n+1}\|(\|x_{n+1}-p\|+\|x_n-p\|) \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle \\
&\quad +\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2.
\end{aligned}
$$

From $\lim_{n\to\infty}\alpha_n=0$, $\liminf_{n\to\infty} r_n>0$ and (3.1.30) , we have

$$
\|\Psi x_n-\Psi p\|\to 0, \text{ as } n\to\infty. \qquad (3.1.42)
$$

Moreover, for $p\in F(S)\cap GEP(G,\Psi)$, we have that,

$$
\begin{aligned}
\|u_n-p\|^2 &\leq \|x_n-p\|^2-\|x_n-u_n\|^2 \\
&\quad +2r_n\langle x_n-u_n,\Psi x_n-\Psi p\rangle-r_n^2\|\Psi x_n-\Psi p\|^2. \qquad (3.1.43)
\end{aligned}
$$

From (3.1.26), we obtain

$$\begin{aligned}\|y_n - p\|^2 &\leq \left\|\beta_n(x_n - p) + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n}[T(s)u_n - T(s)p]ds\right\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2. \qquad (3.1.44)\end{aligned}$$

From (3.1.40), (3.1.44)and (3.1.43), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n\bar{\gamma})^2\beta_n\|x_n - p\|^2 + (1-\alpha_n\bar{\gamma})^2(1-\beta_n)\|u_n - p\|^2 \\ &\quad + 2\alpha_n\gamma\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \gamma f(p) - Ap, x_{n+1} - p\rangle \\ &\leq (1-\alpha_n\bar{\gamma})^2\beta_n\|x_n - p\|^2 + (1-\alpha_n\bar{\gamma})^2(1-\beta_n)(\|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, \Psi x_n - \Psi p\rangle - r_n^2\|\Psi x_n - \Psi p\|^2) \\ &\quad + 2\alpha_n\gamma\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \gamma f(p) - Ap, x_{n+1} - p\rangle \\ &\leq (1-\alpha_n\bar{\gamma})^2\|x_n - p\|^2 - (1-\alpha_n\bar{\gamma})^2(1-\beta_n)\|x_n - u_n\|^2 \\ &\quad + (1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n - u_n\|\|\Psi x_n - \Psi p\| \\ &\quad - (1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n - \Psi p\|^2 \\ &\quad + 2\alpha_n\gamma\langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle \gamma f(p) - Ap, x_{n+1} - p\rangle \qquad (3.1.45)\end{aligned}$$

and hence,

$$\begin{aligned}&(1-\alpha_n\bar{\gamma})^2(1-b)\|x_n - u_n\|^2 \\ &\leq (1-\alpha_n\bar{\gamma})^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + (1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n - u_n\|\|\Psi x_n - \Psi p\| \\ &\quad - (1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n - \Psi p\|^2 \\ &\quad + 2\alpha_n\gamma\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \gamma f(p) - Ap, x_{n+1} - p\rangle \\ &\leq (1 - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + (1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n - u_n\|\|\Psi x_n - \Psi p\| \\ &\quad - (1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n - \Psi p\|^2 + 2\alpha_n\gamma\langle f(x_n) - f(p), x_{n+1} - p\rangle\end{aligned}$$

$$
\begin{aligned}
&+2\alpha_n\langle\gamma f(p)-Ap,x_{n+1}-p\rangle\\
\leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2-2\alpha_n\bar{\gamma}\|x_n-p\|^2+\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2\\
&+(1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n-u_n\|\|\Psi x_n-\Psi p\|\\
&-(1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n-\Psi p\|^2\\
&+2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\alpha_n\langle\gamma f(p)-Ap,x_{n+1}-p\rangle\\
\leq\ & \|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&+\alpha_n[\alpha_n\bar{\gamma}^2\|x_n-p\|^2-2\bar{\gamma}\|x_n-p\|^2+2\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle\\
&+2\langle\gamma f(p)-Ap,x_{n+1}-p\rangle]\\
&+\|\Psi x_n-\Psi p\|[(1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n-u_n\|\\
&-(1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n-\Psi p\|]
\end{aligned}
$$

From (3.1.7), $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, the boundedness of $\{x_n\}$ and hypothesis (D1), it follows that

$$\lim_{n\to\infty}\|x_n-u_n\|=0, \tag{3.1.46}$$

and consequently

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.1.47}$$

From (3.1.46) and (3.1.47), we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.1.48}$$

Putting $t_n=\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds$, since

$$
\begin{aligned}
\|x_n-t_n\| &\leq \|x_n-y_n\|+\|y_n-t_n\|\\
&\leq \|x_n-y_n\|+\|\beta_n x_n+(1-\beta_n)t_n-t_n\|\\
&\leq \|x_n-y_n\|+\beta_n\|x_n-t_n\|,
\end{aligned}
$$

we have

$$(1-\beta_n)\|x_n - t_n\| \leq \|x_n - y_n\|.$$

From (C4) and (3.1.48), it follows that

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{3.1.49}$$

By (3.1.46) and (3.1.49), we have

$$\lim_{n\to\infty} \|t_n - u_n\| = 0. \tag{3.1.50}$$

Step 4. Next, we show that

$$\limsup_{n\to\infty} \langle (A-\gamma f)z, z - x_n\rangle \leq 0, \tag{3.1.51}$$

where $z = P_{F(S)\cap GEP(G,\Psi)}(I - A + \gamma f)(z)$ is a unique solution of the variational inequality (3.1.3). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle (A-\gamma f)z, z - x_{n_i}\rangle = \limsup_{n\to\infty} \langle (A-\gamma f)z, z - x_n\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. From $\lim_{n\to\infty}\|x_n - t_n\| = 0$, we obtain $t_{n_i} \rightharpoonup w$. Let $z_1 = P_{F(S)}x_1$ and $D = \{z \in H : \|z - z_1\| \leq \|x_1 - z_1\| + \frac{1}{\bar{\gamma}-\gamma\alpha}\|\gamma f(z_1) - Az_1\|\}$. Then D is a nonempty closed bounded convex subset of H which is $T(s)$-invariant for each $s \in [0,\infty)$ and contains $\{x_n\}$, $\{u_n\}$. We may assume, without loss of generality, that $S = (T(s))_{s\geq 0}$ is a nonexpansive semigroup on D. In view of Lemma 2.2.19, we can obtain that , for every $s \geq 0$,

$$\lim_{n\to\infty} \|t_n - T(s)t_n\| = 0.$$

By the same argument as in the proof of Theorem 3.1.1, we conclude that $w \in F(\mathcal{S}) \cap GEP(G, \Psi)$. This implies that

$$\begin{aligned}
\limsup_{n\to\infty}\langle (A-\gamma f)z, z-x_n\rangle &= \lim_{i\to\infty}\langle (A-\gamma f)z, z-x_{n_i}\rangle \\
&= \langle (A-\gamma f)z, z-w\rangle \leq 0. \qquad (3.1.52)
\end{aligned}$$

Step 5. Finally, we prove that $x_n \to z$ and $u_n \to z$ as $n \to \infty$. From (3.1.40), we obtain

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq (1-\alpha_n\bar{\gamma})^2\|x_n-z\|^2 + 2\alpha_n\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1}-z\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n-z\|^2 + 2\alpha_n\gamma\alpha\|x_n-z\|\|x_{n+1}-z\| \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1}-z\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n-z\|^2 + 2\alpha_n\gamma\alpha(\|x_n-z\|^2 + \|x_{n+1}-z\|^2) \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1}-z\rangle \\
&\leq ((1-\alpha_n\bar{\gamma})^2 + \alpha_n\gamma\alpha)\|x_n-z\|^2 + \alpha_n\gamma\alpha\|x_{n+1}-z\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1}-z\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-z\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(z) - Az, x_{n+1}-z\rangle, \\
&= \left(1-\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right)\|x_n-z\|^2 + \frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\alpha}\|x_n-z\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(z) - Az, x_{n+1}-z\rangle.
\end{aligned}$$

Setting

$$M := \sup_{n\in\mathbb{N}}\|x_n-z\|^2, \qquad (3.1.53)$$

we obtain

$$\|x_{n+1} - z\|^2 \le \left(1 - \frac{2(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\right)\|x_n - z\|^2 + \frac{2(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha} \times \left(\frac{\alpha_n\bar{\gamma}^2 M}{2(\bar{\gamma} - \gamma\alpha)} + \frac{1}{(\bar{\gamma} - \gamma\alpha)}\langle \gamma f(z) - Az, x_{n+1} - z\rangle\right) \quad (3.1.54)$$

Setting $\gamma_n = \frac{2(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}$ and $\beta_n := \frac{(\alpha_n\bar{\gamma}^2)M}{2(\bar{\gamma} - \gamma\alpha)} + \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(z) - Az, x_{n+1} - z\rangle$. It is easily to see that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \beta_n \le 0$ by (3.1.51). Hence, by Lemma 2.2.17, the sequence $\{x_n\}$ converges strongly to z. From $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim_{n\to\infty} \|x_n - u_n\| = 0$, we conclude that $\{y_n\}$ and $\{u_n\}$ also converge strongly to z as $n \to \infty$. This completes the proof of Theorem 3.1.3. □

Setting $\Psi \equiv 0$ in Theorem 3.1.3, we obtain the following results

Corollary 3.1.4. *Let $\mathcal{S} = (T(s))_{s\ge 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \to H$ be an α-contraction, let $A : H \to H$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$ and let γ be a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $G : H \times H \to \mathbb{R}$ be a mapping satisfying hypotheses (A1)-(A4). Assume that $F(\mathcal{S}) \cap EP(G) \neq \emptyset$ and the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1 \in H \text{ chosen arbitrary}, \\ G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in H, \\ y_n = \beta_n x_n + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n, \forall n \ge 1, \end{cases}$$

where the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{s_n\}$, $\{r_n\}$ satisfy the following conditions:

(D1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$,

(D2) $\liminf_{n\to\infty} r_n \ge 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(D3) $\lim_{n\to\infty} s_n = +\infty$, $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$, *and*

(D4) $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0$.

Then the sequences $\{x_n\}, \{u_n\}$ *and* $\{y_n\}$ *converge strongly to* z *which is a unique solution in* $F(S) \cap EP(G)$ *of the variational inequality (3.1.3).*

Remark 3.1.5. Theorem 3.1.3 and Corollary 3.1.4 generalize and improve [57, Theorem 4.1] . In fact,

1. the conditions (C1) and (C2) can be replaced by the weaker conditions (D1) and (D2) respectively.
2. The control condition $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n}\frac{1}{\alpha_n} = 0$ on (C3) is placed by the *strictly weaker* condition : $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$ in (D3) as shown in the next example.

Example 3.1.6. *(a) If* $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n}\frac{1}{\alpha_n} = 0$, *then* $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$.

(b) The converse of (a) is not true.

Proof. Since $\{s_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$, we obtain

$$\frac{|s_n - s_{n-1}|}{s_n} \leq \frac{|s_n - s_{n-1}|}{s_n}\frac{1}{\alpha_n}.$$

Then it is easy to see that (a) is true. Let $s_n = n$ and $\alpha_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = \lim_{n\to\infty} \frac{1}{n} = 0$ but $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n}\frac{1}{\alpha_n} = 1$. Then converse of (a) is not true. Hence (b) is proved. □

3.2 A General composite algorithms for solving generalized equilibrium problems and fixed point problems in Hilbert spaces

In this section, we introduce a general composite algorithm for finding a common element of the set of solutions of an generalized equilibrium problem and the common fixed point set of a finite family of asymptotically nonexpansive mappings in Hilbert spaces. Strong convergence of such iterative scheme is obtained which solving some variational inequalities for a strongly monotone and strictly pseudo-contractive mapping.

Let $A : C \to H$ be a nonlinear mapping and $\phi : C \times C \to \mathbb{R}$ be a bifunction. Consider a generalized equilibrium problem:

$$\text{Find } z \in C \text{ such that } \phi(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{3.2.1}$$

The set of all solutions of the generalized equilibrium problem (3.2.1) is denoted by EP, i.e.,

$$EP = \{z \in C : \phi(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C\}.$$

In the rest of our discussion in this section we shall assume that $p(n) = j+1$ if $jN < n \leq (j+1)N, j = 1, 2, \ldots$ and $n = jN + i(n)$; $i(n) \in \{1, 2, \ldots, N\}$ and $h_n := \max\{k_{p(n)}^{i(n)} : 1 \leq i(n) \leq N\}$ for all $n \geq 1$, and for each $n \geq 1$, $n = (p(n) - 1)N + i(n)$.

Theorem 3.2.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $S_1, S_2, \ldots, S_N : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{1 + k_{p(n)}^{i(n)}\}$ respectively, such that $k_{p(n)}^{i(n)} \to 0$ as $n \to \infty$, $h_n := \max_{1 \leq i(n) \leq N}\{k_{p(n)}^{i(n)}\}$ and $\Gamma := \cap_{i=1}^{N} F(S_i)$,*

$$\Gamma = F(S_N S_{N-1} S_{N-2} \ldots S_1) = F(S_1 S_N \ldots S_2) = \cdots = F(S_{N-1} S_{N-2} \cdots S_1 S_N).$$

Let $A : C \to H$ be an α-inverse strongly monotone mapping. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4) such that $\Omega := EP \cap \Gamma$ is nonempty. Let $F : C \to H$ be δ-strongly monotone and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, $f : C \to H$ a ρ-contraction, γ a positive real number such that $\gamma < (1 - \sqrt{(1-\delta)/\lambda})/\rho$ and r a constant such that $r \in (0, 2\alpha)$. For given $x_0 \in C$

arbitrarily, let the sequence $\{x_n\}$ *be generated iteratively by*

$$\begin{cases} \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n F) S_{i(n+1)}^{p(n+1)} x_n, \\ x_{n+1} = \mu_n P_C[y_n] + (1-\mu_n) u_n, n \geq 0. \end{cases} \tag{3.2.2}$$

where $p(n) = j+1$ *if* $jN < n \leq (j+1)N, j = 1, 2, \ldots$ *and* $n = jN + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$. *Suppose that* $\{\alpha_n\}$ *and* $\{\mu_n\}$ *are two sequences in* $[0,1]$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \frac{h_n}{\alpha_n} = 0$;

(C2) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$ *and* $\lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\alpha_{n+1}} = 0$.

Assume that $\sum_{n=1}^{\infty} \sup_{z\in B} \|S_{i(n+1)}^{p(n+1)} z - S_{i(n)}^{p(n)} z\| < \infty$, *for each bounded subset* B *of* C. *Then, the sequence* $\{x_n\}$ *converges strongly to* x^* *of the following variational inequality*

$$\langle (F - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in \Omega \tag{3.2.3}$$

or equivalently $\tilde{x} = P_\Omega(I - F + \gamma f)\tilde{x}$, *where* P_Ω *is the metric projection of* H *onto* Ω.

Proof. First, we rewrite the sequence $\{x_n\}$ by the following :

$$x_{n+1} = \mu_n P_C[y_n] + (1-\mu_n) T_r(x_n - rAx_n), n \geq 0, \tag{3.2.4}$$

where the mapping T_r is defined in Lemma 2.3.3. Pick $z \in \Omega$ and $u_n = T_r(x_n - rAx_n)$. The nonexpansivity of T_r and $I - rA$ implies that

$$\begin{aligned} \|u_n - z\| &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\| \\ &\leq \|x_n - z\| \text{ for all } z \in \Omega. \end{aligned} \tag{3.2.5}$$

Setting $\bar{\gamma} := \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$ and using Lemma 2.2.13 (ii), we have

$$\begin{aligned}\|y_n - z\| &= \|\alpha_n\gamma(f(x_n) - Fz) + (I - \alpha_n F)(S_{i(n+1)}^{p(n+1)}x_n - z)\| \\ &\le \alpha\alpha_n\gamma\|x_n - z\| + \alpha_n\|\gamma f(x_n) - Fz\| + (1 - \alpha_n\bar{\gamma})(1 + h_{n+1})\|x_n - z\| \\ &= [1 - \alpha_n(\bar{\gamma} - \alpha\gamma) + (1 - \alpha_n\bar{\gamma})h_{n+1}]\|x_n - z\| \\ &\quad + \alpha_n\|\gamma f(x_n) - Fz\|. \end{aligned} \tag{3.2.6}$$

By our assumptions, we have $(1 - \alpha_n\bar{\gamma})\frac{h_{n+1}}{\alpha_n} \to 0$ as $n \to \infty$. We can assume, without loss of generality, that $(1 - \alpha_n\bar{\gamma})\frac{h_{n+1}}{\alpha_n} < \frac{1}{2}(\bar{\gamma} - \alpha\gamma)$. Applying Lemma 2.2.13, we can calculate the following

$$\begin{aligned}\|x_{n+1} - z\| &= \|\mu_n(P_C[y_n] - z) + (1 - \mu_n)(u_n - z)\| \\ &\le \mu\|P_C[y_n] - z\| + (1 - \mu_n)\|u_n - z\| \\ &\le \mu_n\|y_n - z\| + (1 - \mu_n)\|x_n - z\| \\ &\le \mu_n[1 - \alpha_n(\bar{\gamma} - \alpha\gamma) + (1 - \alpha_n\bar{\gamma})h_{n+1}]\|x_n - z\| \\ &\quad + \mu_n\alpha_n\|\gamma f(x_n) - Fz\| + (1 - \mu_n)\|x_n - z\| \\ &\le \left[1 - \mu_n\alpha_n[(\bar{\gamma} - \alpha\gamma) - (1 - \alpha_n\bar{\gamma})\frac{h_{n+1}}{\alpha_n}]\right]\|x_n - z\| + \mu_n\alpha_n\|\gamma f(x_n) - Fz\| \\ &\le \left[1 - \frac{1}{2}\mu_n\alpha_n(\bar{\gamma} - \alpha\gamma)\right]\|x_n - z\| \\ &\quad + \frac{\mu_n\alpha_n\frac{1}{2}(\bar{\gamma} - \alpha\gamma)}{\frac{1}{2}(\bar{\gamma} - \alpha\gamma)}\|\gamma f(x_n) - Fz\|. \end{aligned} \tag{3.2.7}$$

By induction, we obtain, for all $n \ge 0$,

$$\|x_n - z\| \le \max\left\{\|x_0 - z\|, \frac{2\|\gamma f(z) - F(z)\|}{\bar{\gamma} - \alpha\gamma}\right\}.$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$, $\{f(x_n)\}$, and $\{y_n\}$ are all bounded.

Next we show that

$$\lim_{n\to\infty}\|x_{n+N} - x_n\| = 0.$$

From (3.2.2), we have

$$\begin{aligned}
\|y_{n+N} - y_{n+N-1}\| &= \Big\|\alpha_{n+N}\gamma f(x_{n+N}) + (I - \alpha_{n+N}F)S_{i(n+N+1)}^{p(n+N+1)}x_{n+N} \\
&\quad - \alpha_{n+N}\gamma f(x_{n+N}) - (I - \alpha_{n+N}F)S_{i(n+N+1)}^{p(n+N+1)}x_{n+N}\Big\| \\
&= \Big\|\alpha_{n+N}\gamma(f(x_{n+N}) - f(x_{n+N-1}) + (\alpha_{n+N} - \alpha_{n+N-1})\gamma f(x_{n+N-1}) \\
&\quad + (I - \alpha_{n+N}F)(S_{i(n+N+1)}^{p(n+N+1)}x_{n+N} - S_{i(n+N+1)}^{p(n+N+1)}x_{n+N-1}) \\
&\quad + [(I - \alpha_{n+N}F) - (I - \alpha_{n+N-1}F)]S_{i(n+N+1)}^{p(n+N+1)}x_{n+N-1} \\
&\quad + (I - \alpha_{n+N-1}F)(S_{i(n+N+1)}^{p(n+N+1)}x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1})\Big\| \\
&\leq \alpha_{n+N}\gamma\alpha\|x_{n+N} - x_{n+N-1}\| + |\alpha_{n+N} - \alpha_{n+N-1}|\gamma\|f(x_{n+N-1})\| \\
&\quad + (1 - \alpha_{n+N}\bar{\gamma})(1 + h_{n+N+1})\|x_{n+N} - x_{n+N-1}\| \\
&\quad + |\alpha_{n+N-1} - \alpha_{n+N}|\|F\|\|S_{i(n+N+1)}^{p(n+N+1)}x_{n+N-1}\| \\
&\quad + (1 - \alpha_{n+N-1}\bar{\gamma})\|S_{i(n+N+1)}^{p(n+N+1)}x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\| \\
&\leq \alpha_{n+N}\gamma\alpha\|x_{n+N} - x_{n+N-1}\| + |\alpha_{n+N} - \alpha_{n+N-1}|\gamma\|f(x_{n+N-1})\| \\
&\quad + (1 - \alpha_{n+N}\bar{\gamma})(1 + h_{n+N+1})\|x_{n+N} - x_{n+N-1}\| \\
&\quad + |\alpha_{n+N-1} - \alpha_{n+N}|\|F\|\|S_{i(n+N+1)}^{p(n+N+1)}x_{n+N-1}\| \\
&\quad + \sup_{x\in\{x_n : n\in\mathbb{N}\}}\|S_{i(n+N+1)}^{p(n+N+1)}x - S_{i(n+N)}^{p(n+N)}x\| \qquad (3.2.8)
\end{aligned}$$

and from (3.2.4), we have

$$\begin{aligned}
\|x_{n+N+1} - x_{n+N}\| &= \|\mu_{n+N}P_C[y_{n+N}] + (1 - \mu_{n+N})u_{n+N} - \mu_{n+N-1}P_C[y_{n+N-1}] \\
&\quad - (1 - \mu_{n+N-1})u_{n+N-1}\| \\
&= \|\mu_{n+N}(P_C[y_{n+N}] - P_C[y_{n+N-1}]) + (\mu_{n+N} - \mu_{n+N-1})P_C[y_{n+N-1}] \\
&\quad + (1 - \mu_{n+N})(u_{n+N} - u_{n+N-1}) + (\mu_{n+N-1} - \mu_{n+N})u_{n+N-1}\| \\
&\leq \mu_{n+N}\|y_{n+N} - y_{n+N-1}\| + (1 - \mu_{n+N})\|u_{n+N} - u_{n+N-1}\| \\
&\quad + |\mu_{n+N} - \mu_{n+N-1}|(\|P_C[y_{n+N-1}]\| + \|u_{n+N-1}\|)
\end{aligned}$$

and

$$\begin{aligned}
\|u_{n+N} - u_{n+N-1}\| &= \|T_r(x_{n+N} - rAx_{n+N}) - T_r(x_{n+N-1} - rAx_{n+N-1})\| \\
&\leq \|(x_{n+N} - rAx_{n+N}) - (x_{n+N-1} - rAx_{n+N-1})\| \\
&\leq \|x_{n+N} - x_{n+N-1}\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|x_{n+N+1} - x_{n+N}\| \\
\leq\ & \mu_{n+N}\alpha_{n+N}\gamma\alpha\|x_{n+N} - x_{n+N-1}\| + \mu_{n+N}|\alpha_{n+N} - \alpha_{n+N-1}|\gamma\|f(x_{n+N-1})\| \\
& + \mu_{n+N}(1 - \alpha_{n+N}\bar{\gamma})(1 + h_{n+N+1})\|x_{n+N} - x_{n+N-1}\| \\
& + \mu_{n+N}|\alpha_{n+N-1} - \alpha_{n+N}|\|F\|\|S^{p(n+N+1)}_{i(n+N+1)}x_{n+N-1}\| \\
& + \mu_{n+N} \sup_{x\in\{x_n:n\in\mathbb{N}\}} \|S^{p(n+N+1)}_{i(n+N+1)}x - S^{p(n+N)}_{i(n+N)}x\| \\
& + (1 - \mu_{n+N})\|x_{n+N} - x_{n+N-1}\| \\
& + |\mu_{n+N} - \mu_{n+N-1}|(\|P_C[y_{n+N-1}]\| + \|u_{n+N-1}\|) \\
\leq\ & (1 - \mu_{n+N}\alpha_{n+N}(\bar{\gamma} - \gamma\alpha))\|x_{n+N} - x_{n+N-1}\| \\
& + \mu_{n+N}\alpha_{n+N}\Bigg[\left(\frac{h_{n+N+1}}{\alpha_{n+N}} + h_{n+N+1}\bar{\gamma}\right)M \\
& + \left|1 - \frac{\alpha_{n+N-1}}{\alpha_{n+N}}\right|\gamma\|f(x_{n+N-1})\| + \left|\frac{\alpha_{n+N-1}}{\alpha_{n+N}} - 1\right|\|F\|\|S^{p(n+N+1)}_{i(n+N+1)}x_{n+N-1}\| \\
& + \frac{1}{\mu_{n+N}}\left|\frac{\mu_{n+N} - \mu_{n+N-1}}{\alpha_{n+N}}\right|(\|P_C[y_{n+N-1}]\| + \|u_{n+N-1}\|)\Bigg] \\
& + \sup_{x\in\{x_n:n\in\mathbb{N}\}} \|S^{p(n+N+1)}_{i(n+N+1)}x - S^{p(n+N)}_{i(n+N)}x\|. \qquad (3.2.9)
\end{aligned}$$

By Lemma 2.2.17, we obtain that

$$\lim_{n\to\infty} \|x_{n+N+1} - x_{n+N}\| = 0.$$

Furthermore,

$$\|x_{n+N} - x_n\| \leq \|x_{n+N} - x_{n+N-1}\| + \|x_{n+N-1} - x_{n+N-2}\| + \cdots + \|x_{n+1} - x_n\| \to 0, \text{ as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0. \tag{3.2.10}$$

Next, we show that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.2.11}$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\mu_n(P_C[y_n] - z) + (1-\mu_n)(u_n - z)\|^2 \\
&\leq \mu_n\|P_C[y_n] - z\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&\leq \mu_n\|y_n - z\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&= \mu_n\|\alpha_n\gamma f(x_n) + (I - \alpha_n F)S_{i(n+1)}^{p(n+1)}x_n - z\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&= \mu_n\|\alpha_n\gamma f(x_n) - \alpha_n F(z) + (I - \alpha_n F)S_{i(n+1)}^{p(n+1)}x_n - (I - \alpha_n F)z\|^2 \\
&\quad + (1-\mu_n)\|u_n - z\|^2 \\
&\leq \mu_n\|(I - \alpha_n F)S_{i(n+1)}^{p(n+1)}x_n - (I - \alpha_n F)z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\mu_n\alpha_n\langle (I - \alpha_n F)S_{i(n+1)}^{p(n+1)}x_n - (I - \alpha_n F)z, \gamma f(x_n) - F(z)\rangle \\
&\quad + (1-\mu_n)\|u_n - z\|^2 \\
&\leq \mu_n(1 - \alpha_n\bar{\gamma})^2(1 + h_{n+1})^2\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)\|u_n - z\|^2 \\
&\leq \mu_n(1 - \alpha_n\bar{\gamma})(1 + 2h_{n+1} + h_{n+1}^2)\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)\|u_n - z\|^2 \\
&= \mu_n(1 - \alpha_n\bar{\gamma})(1 + h_{n+1}^*)\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1-\mu_n)\|u_n - z\|^2,
\end{aligned} \tag{3.2.12}$$

where $h^*_{n+1} = 2h_{n+1} + h^2_{n+1}$. From Lemma 2.2.15, we get

$$\begin{aligned}\|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\ &\le \|(x_n - rAx_n) - (z - rAz)\|^2 \\ &\le \|x_n - z\|^2 + r(r - 2\alpha)\|Ax_n - Az\|^2. \end{aligned} \tag{3.2.13}$$

Substituting (3.2.13) into (3.2.12), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \mu_n(1 - \alpha_n\bar{\gamma})(1 + h^*_{n+1})\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\ &\quad + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\ &\quad + (1 - \mu_n)[\|x_n - z\|^2 + r(r - 2\alpha)\|Ax_n - Az\|^2] \\ &= \left(1 - \alpha_n\mu_n(\bar{\gamma} - \frac{h^*_{n+1}}{\alpha_n})\right)\|x_n - z\|^2 - \alpha_n\mu_n h_{n+1}\bar{\gamma}\|x_n - z\|^2 \\ &\quad + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\ &\quad + (1 - \mu_n)r(r - 2\alpha)\|Ax_n - Az\|^2. \end{aligned} \tag{3.2.14}$$

Therefore,

$$\begin{aligned} &(1 - \mu_n)r(2\alpha - r)\|Ax_n - Az\|^2 \\ &\le \left(1 - \alpha_n\mu_n(\bar{\gamma} - \frac{h^*_{n+1}}{\alpha_n})\right)\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\quad + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\ &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\ &\quad + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\ &\le (\|x_n - z\| + \|x_{n+1} - z\|)\|x_n - x_{n+1}\| + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\ &\quad + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \end{aligned} \tag{3.2.15}$$

Since $\liminf_{n\to\infty}(1 - \mu_n)r(2\alpha - r) > 0$, $\|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we derive

$$\lim_{n\to\infty} \|Ax_n - Az\| = 0.$$

From Lemma 2.3.3, we obtain

$$
\begin{aligned}
\|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\
&\leq \langle (x_n - rAx_n) - (z - rAz), u_n - z\rangle \\
&= \frac{1}{2}\Big(\|(x_n - rAx_n) - (z - rAz)\|^2 + \|u_n - z\|^2 \\
&\quad - \|(x_n - z) - r(Ax_n - Az) - (u_n - z)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 \\
&\quad - \|(x_n - u_n) - r(Ax_n - Az)\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r\langle x_n - u_n, Ax_n - Az\rangle - r^2\|Ax_n - Az\|^2\Big).
\end{aligned}
$$

Thus, we deduce

$$
\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|. \quad (3.2.16)
$$

By (3.2.12) and (3.2.16), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \mu_n(1 - \alpha_n\bar{\gamma})(1 + h^*_{n+1})\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1 - \mu_n)[\|x_n - z\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|] \\
&\leq \Big(1 - \alpha_n\mu_n(\bar{\gamma} - \frac{h^*_{n+1}}{\alpha_n})\Big)\|x_n - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\| \\
&\quad + (1 - \mu_n)[-\|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(1 - \mu_n)\|x_n - u_n\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\|
\end{aligned}
$$

$$+ (1-\mu_n)[2r\|x_n - u_n\|\|Ax_n - Az\|]$$

$$\leq (\|x_n - z\| - \|x_{n+1} - z\|)\|x_n - x_{n+1}\| + \mu_n\alpha_n^2\|\gamma f(x_n) - F(z)\|^2$$

$$+ 2\alpha_n\mu_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n) - F(z)\|\|x_n - z\|$$

$$+ 2r(1-\mu_n)\|x_n - u_n\|\|Ax_n - Az\|.$$

Since $\liminf_{n\to\infty}(1-\mu_n) > 0$, $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Ax_n - Az\| \to 0$, we derive that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \quad (3.2.17)$$

Next, we show that

$$\lim_{n\to\infty} \|x_n - S_{i(n+N)}S_{i(n+N-1)}S_{i(n+N-2)}\cdots S_{i(n+2)}S_{i(n+1)}x_n\| = 0. \quad (3.2.18)$$

By using (3.2.10), it suffices to show that

$$\lim_{n\to\infty} \|x_{n+N} - S_{i(n+N)}S_{i(n+N-1)}S_{i(n+N-2)}\cdots S_{i(n+2)}S_{i(n+1)}x_n\| = 0.$$

Observe that

$$\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\|$$

$$\leq \|x_{n+N} - x_{n+N-1}\| + \|x_{n+N} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\|$$

$$\leq \|x_{n+N} - x_{n+N-1}\| + \mu_{n+N-1}\|P_C[y_{n+N-1}] - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\|$$

$$+ (1-\mu_{n+N-1})\|u_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\|$$

$$\leq \|x_{n+N} - x_{n+N-1}\| + \mu_{n+N-1}\|y_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\|$$

$$+ (1-\mu_{n+N-1})\|u_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\|$$

$$\leq \|x_{n+N} - x_{n+N-1}\| + \mu_{n+N-1}\alpha_{n+N-1}(\|\gamma f(x_{n+N-1})\| + \|FS_{i(n+N)}^{p(n+N)}x_{n+N-1}\|)$$

$$+ (1-\mu_{n+N-1})\|u_{n+N-1} - x_{n+N-1}\|$$

$$+ (1-\mu_{n+N-1})\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\|. \quad (3.2.19)$$

Hence,

$$\begin{aligned}\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)} x_{n+N-1}\| &\leq \frac{1}{\mu_{n+N-1}}\|x_{n+N} - x_{n+N-1}\| \\ &+ \alpha_{n+N-1}(\|\gamma f(x_{n+N-1})\| + \|FS_{i(n+N)}^{p(n+N)} x_{n+N-1}\|) \\ &+ \frac{(1-\mu_{n+N-1})}{\mu_{n+N-1}}\|u_{n+N-1} - x_{n+N-1}\|. \qquad (3.2.20)\end{aligned}$$

From (3.2.10), (3.2.17), $\lim_{n\to\infty} \alpha_n = 0$ and (C2), we have

$$\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)} x_{n+N-1}\| \to 0 \text{ as } n \to \infty. \qquad (3.2.21)$$

Since $S_{i(n)}$ is Lipschitz with constant $L_{i(n)}$ for each $i(n) \in \{1, 2, ..., N\}$ and for $L = \max_{1\leq i\leq N}\{L_{i(n)}\}$, and for any positive number $n \geq 1, n = (p(n) - 1)N + i(n)$, we have

$$\begin{aligned}&\|x_{n+N-1} - S_{i(n+N)} x_{n+N-1}\| \\ \leq\ &\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)} x_{n+N-1}\| + \|S_{i(n+N)}^{p(n+N)} x_{n+N-1} - S_{i(n+N)} x_{n+N-1}\| \\ \leq\ &\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)} x_{n+N-1}\| + L\|S_{i(n+N)}^{p(n+N)-1} x_{n+N-1} - x_{n+N-1}\| \\ \leq\ &\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)} x_{n+N-1}\| + L\Big(\|S_{i(n+N)}^{p(n+N)-1} x_{n+N-1} - S_{i(n)}^{p(n+N)-1} x_{n-1}\| \\ &+\|S_{i(n)}^{p(n+N)-1} x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_{n+N-1}\|\Big). \qquad (3.2.22)\end{aligned}$$

Since for each $n > N, n + N = n(\mod N)$, and also $n = (p(n) - 1)N + i(n)$ so,

$$n + N = (p(n) - 1 + 1)N + i(n) = (p(n + N) - 1)N + i(n + N),$$

that is,

$$p(n + N) - 1 = p(n) \text{ and } i(n + N) = i(n).$$

Hence,

$$\begin{aligned}\|S_{i(n+N)}^{p(n+N)-1} x_{n+N-1} - S_{i(n)}^{p(n+N)-1} x_{n-1}\| &= \|S_{i(n)}^{p(n)} x_{n+N-1} - S_{i(n)}^{p(n)} x_{n-1}\| \\ &\leq L\|x_{n+N-1} - x_{n-1}\|. \qquad (3.2.23)\end{aligned}$$

Also,

$$\|S_{i(n)}^{p(n+N)-1}x_{n-1} - x_{n-1}\| = \|S_{i(n)}^{p(n)}x_{n-1} - x_{n-1}\|. \tag{3.2.24}$$

Therefore, substituting (3.2.23) and (3.2.24) into (3.2.22), we have

$$\begin{aligned} \|x_{n+N-1} - S_{i(n+N)}x_{n+N-1}\| \leq\ & \|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\| \\ & + L^2\|x_{n+N-1} - x_{n-1}\| \\ & + L\|S_{i(n)}^{p(n)}x_{n-1} - x_{n-1}\| \\ & + L\|x_{n-1} - x_{n+N-1}\|. \end{aligned} \tag{3.2.25}$$

From (3.2.21) and (3.2.10), we have

$$\lim_{n\to\infty} \|x_{n+N-1} - S_{i(n+N)}x_{n+N-1}\| = 0. \tag{3.2.26}$$

Also,

$$\|x_{n+N} - S_{i(n+N)}x_{n+N-1}\| \leq \|x_{n+N} - x_{n+N-1}\| + \|x_{n+N-1} - S_{i(n+N)}x_{n+N-1}\|,$$

so that

$$x_{n+N-1} - S_{i(n+N)}x_{n+N-1} \to 0 \text{ as } n \to \infty. \tag{3.2.27}$$

Indeed, noting that each $S_{i(n)}$ is Lipschitzian and using (3.2.27), we can calculate the following

$$\begin{aligned} & x_{n+N} - S_{i(n+N)x_{n+N-1}} \to 0 \text{ as } n \to \infty; \\ & S_{i(n+N)}x_{n+N-1} - S_{i(n+N)}S_{i(n+N-1)}x_{n+N-2} \text{ as } n \to \infty; \\ & \vdots \\ & S_{i(n+N)}S_{i(n+N-1)}\cdots S_{i(n+2)}x_{n+1} - S_{i(n+N)}S_{i(n+N-1)}\cdots S_{i(n+2)}S_{i(n+1)}x_n \to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows from the above table that

$$x_{n+N} - S_{i(n+N)}S_{i(n+N-1)}\cdots S_{i(n+1)}x_n \to 0 \text{ as } n \to \infty.$$

Using (3.2.10), we have

$$x_n - S_{i(n+N)}S_{i(n+N-1)}\cdots S_{i(n+1)}x_n \to 0 \text{ as } n \to \infty.$$

Hence (3.2.18) is proved. Let $\Phi = P_\Omega$. Then $\Phi(I - F - \gamma f)$ is a contraction on C. In fact, from Lemma 2.2.13 (i), we have

$$\begin{aligned}
& \|\Phi(I - F - \gamma f)x - \Phi(I - F - \gamma f)y\| \\
\leq\ & \|(I - F - \gamma f)x - (I - F - \gamma f)y\| \\
\leq\ & \|(I - F)x - (I - F)y\| + \gamma\|f(x) - f(y)\| \\
\leq\ & \sqrt{\frac{1-\delta}{\lambda}}\|x - y\| + \alpha\gamma\|x - y\| \\
=\ & \left(\sqrt{\frac{1-\delta}{\lambda}} + \alpha\gamma\right)\|x - y\|, \text{ for all } x, y \in C. \qquad (3.2.28)
\end{aligned}$$

Therefore, $\Phi(I - F - \gamma f)$ is a contraction on C with coefficient $\left(\sqrt{\frac{1-\delta}{\lambda}} + \alpha\gamma\right) \in (0,1)$. Thus, by Banach contraction principal, $P_\Omega(I - F - \gamma f)$ has a uninique fixed point x^*. That is $P_\Omega(I - F - \gamma f)x^* = x^*$ which mean that x^* is the unique solution in Ω of the variational inequality (3.2.3). Next we show that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Fx^*, x_n - x^*\rangle \leq 0.$$

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Fx^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle \gamma f(x^*) - Fx^*, x_{n_j} - x^*\rangle.$$

Since $\{x_n\}$ is bounded, we may also assume that there exists some $\tilde{x} \in H$ such that $x_{n_j} \rightharpoonup \tilde{x}$. Since the family $\{S_i\}_{i=1}^N$ is finite, passing to a further subsequence if necessary, we may further assume that, for some $i(n) \in \{1, 2, ..., N\}$, it follows that

$$x_{n_j} - S_{i(n+N)}S_{i(n+N-1)}\cdots S_{i(n+1)}x_{n_j} \to 0 \text{ as } j \to \infty.$$

By Lemma 2.2.16, we obtain

$$\tilde{x} \in F(S_{i(n+N)}S_{i(n+N-1)}\cdots S_{i(n+1)}),$$

so this implies that $\tilde{x} \in \Gamma$. Next we show $\tilde{x} \in EP$. Since $u_n = T_r(x_n - rAx_n)$, for any $y \in C$ we have

$$\phi(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq 0.$$

From the monotonicity of F, we have

$$\frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq \phi(y, u_n), \forall y \in C.$$

Hence,

$$\langle y - u_n, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\rangle \geq \phi(y, u_{n_i}), \forall y \in C. \tag{3.2.29}$$

Put $z_t = ty + (1-t)\tilde{x}$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.2.29) we have

$$\begin{aligned}
\langle z_t - u_{n_i}, Az_t\rangle &\geq \langle z_t - u_{n_i}, Az_t\rangle - \langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\rangle \\
&\quad + \phi(z_t, u_{n_i}) \\
&= \langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle + \langle z_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\
&\quad + \langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\rangle + \phi(z_t, u_{n_i}).
\end{aligned} \tag{3.2.30}$$

Note that $\|Au_{n_i} - Ax_{n_i}\| \leq \frac{1}{\alpha}\|u_{n_i} - x_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle \geq 0$. Letting $i \to \infty$ in (3.2.30), we have

$$\langle z_t - \tilde{x}, Az_t\rangle \geq \phi(z_t, \tilde{x}). \tag{3.2.31}$$

From (A1), (A4) and (3.2.31), we also have

$$\begin{aligned}
0 &= \phi(z_t, z_t) \leq t\phi(z_t, y) + (1-t)\phi(z_t, \tilde{x}) \\
&\leq t\phi(z_t, y) + (1-t)\langle z_t - \tilde{x}, Az_t\rangle \\
&= t\phi(z_t, y) + (1-t)t\langle y - \tilde{x}, Az_t\rangle
\end{aligned}$$

and hence

$$0 \leq \phi(z_t, y) + (1-t)\langle Az_t, y - \tilde{x}\rangle. \tag{3.2.32}$$

Letting $t \to 0$ in (3.2.32) and using (A3), we have, for each $y \in C$,

$$0 \leq \phi(\tilde{x}, y) + \langle y - \tilde{x}, A\tilde{x}\rangle. \tag{3.2.33}$$

This implies that $\tilde{x} \in EP$. Therefore, $\tilde{x} \in \Omega$. Therefore,

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Fx^*, x_n - x^*\rangle = \langle \gamma f(x^*) - Fx^*, \tilde{x} - x^*\rangle \leq 0.$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. From Lemma 2.2.13 and (3.2.2), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
= \ & \|\mu_n(P_C[y_n] - x^*) + (1-\mu_n)(u_n - x^*)\|^2 \\
\leq \ & \mu_n\|P_C[y_n] - x^*\|^2 + (1-\mu_n)\|u_n - x^*\|^2 \\
\leq \ & \mu_n\|y_n - x^*\|^2 + (1-\mu_n)\|u_n - x^*\|^2 \\
= \ & \mu\|\alpha_n\gamma f(x_n) - \alpha_n F(x^*) + (I - \alpha_n F)S_{i(n+1)}^{p(n+1)}x_n - (I - \alpha_n F)x^*\|^2 \\
& +(1-\mu_n)\|u_n - x^*\|^2 \\
= \ & (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\|\alpha_n(\gamma f(x_n) - F(x^*)) + (I - \alpha_n F)(S_{i(n+1)}^{p(n+1)}x_n - x^*)\|^2 \\
\leq \ & (1-\mu_n)\|x_n - x^*\|^2 + \mu_n(1-\alpha_n\bar{\gamma})^2(1+h_{n+1})^2\|x_n - x^*\|^2 \\
& +2\mu_n\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\
\leq \ & (1-\mu_n)\|x_n - x^*\|^2 + \mu_n(1-\alpha_n\bar{\gamma})(1+2h_{n+1}+h_{n+1}^2)\|x_n - x^*\|^2 \\
& +2\mu_n\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\
= \ & (1-\mu_n)\|x_n - x^*\|^2 + \mu_n(1-\alpha_n\bar{\gamma})(1+h_{n+1}^*)\|x_n - x^*\|^2 \\
& +2\mu_n\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\
\leq \ & \left(1 - \mu_n\alpha_n(\bar{\gamma} - \frac{h_{n+1}^*}{\alpha_n})\right)\|x_n - x^*\|^2
\end{aligned}$$

$$+2\mu_n\alpha_n\left(\bar{\gamma}-\frac{h^*_{n+1}}{\alpha_n}\right)\left[\frac{1}{\left(\bar{\gamma}-\frac{h^*_{n+1}}{\alpha_n}\right)}\langle\gamma f(x_n)-F(x^*),x_{n+1}-x^*\rangle\right], \quad (3.2.34)$$

where $h^*_{n+1} = 2h_{n+1} + h^2_{n+1}$. Hence, all conditions of Lemma 2.2.17 are satisfied. Therefore, $x_n \to x^*$. This completes the proof. □

The following example shows that there exist the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfying the conditions (C1) and (C2) of Theorem 3.2.1.

Example 3.2.2. *For each $n \geq 0$, let $\alpha_n = \frac{1}{n+1}$ and $\mu_n = \frac{1}{2} + \frac{1}{n+1}$. Then, it is easily to obtain $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1$, $0 < \frac{1}{2} = \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n = \frac{1}{2} < 1$ and $\lim_{n\to\infty}\frac{\mu_{n+1}-\mu_n}{\alpha_{n+1}} = 0$. Hence conditions (C1) and (C2) of Theorem 3.2.1 are satisfied.*

Corollary 3.2.3. *Let $C, H, A, \phi, \Omega, f, F, r$ be as in Theorem 3.2.1. Let $S_1, S_2, \ldots, S_N : C \to C$ be a family of nonexpansive mappings. Let $T_1, T_2, \ldots, T_N : C \to C$ be mappings defined by (2.1.12). For $T_n := T_{n \bmod N}$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} \phi(u_n, y) + \langle Ax_n, y-u_n\rangle + \frac{1}{r}\langle y-u_n, u_n-x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \mu_n P_C[\alpha_n\gamma f(x_n) + (1-\alpha_n F)T_n x_n] + (1-\mu_n)u_n, n \geq 0. \end{cases}$$

Assume that $\sum_{n=1}^{\infty}\sup_{z\in B}\|T_{n+1}z - T_n z\| < \infty$ for each bounded subset B of C and the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the following conditions :

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1$;

(C2) $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < 1$ and $\lim_{n\to\infty}\frac{\mu_{n+1}-\mu_n}{\alpha_{n+1}} = 0$.

Then the sequence $\{x_n\}$ converges strongly to x^* of the following variational inequality

$$\langle (F-\gamma f)x^*, x-x^*\rangle \geq 0, \quad x \in \Omega$$

or equivalently $\tilde{x} = P_\Omega(I - F + \gamma f)\tilde{x}$, *where* P_Ω *is the metric projection of* H *onto* Ω.

Proof. By Lemma 2.1.71, we have

$$\cap_{i=1}^N F(T_i) = F(T_N T_{N-1} T_{N-2} \cdots T_1) = F(T_1 T_N \cdots T_2) = F(T_{N-1} T_{N-2} \cdots T_1 T_N).$$

Therefore, the result follows from Theorem 3.2.1. □

Remark 3.2.4. As in [33, Theorem 4.1], we can generate a sequence $\{S_n\}$ of nonexpansive mappings satisfying the condition $\sum_{n=1}^\infty \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C by using convex combination of a general sequence $\{T_k\}$ of nonexpansive mappings with a common fixed point.

Setting $\gamma = 1$, $F = I$ and $S_n \equiv S$, a nonexpansive mapping, in Corollary 3.2.3, we obtain the following result.

Corollary 3.2.5. [58, Theorem 3.7] Let C, H, A, ϕ, f, r be as in Theorem 3.2.1. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := EP \cap F(S) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \mu_n P_C[\alpha_n f(x_n) + (1 - \alpha_n)Sx_n] + (1 - \mu_n)u_n, n \geq 0. \end{cases}$$

Assume the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the following conditions :

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$;

(C2) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$ and $\lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\alpha_{n+1}} = 0$.

Then the sequence $\{x_n\}$ converges strongly to x^* of the following variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \quad x \in \Omega.$$