CHAPTER II

PRELIMINARIES

In this chapter, we give some notations, definitions, and some useful results that will be used in the later chapter.

2.1 Normed spaces and Banach spaces.

Definition 2.1.1. [10] Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{K}$ is said to be *a norm on* X if it satisfies the following conditions:

1) $\|x\| \geq 0, \forall x \in X;$

2) $\|x\| = 0 \Leftrightarrow x = 0;$

3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in E;$

4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in E$ and $\forall \alpha \in \mathbb{K}$.

We use the notation $\|\cdot\|$ for norm.

Definition 2.1.2. [10] Let $(X, \|\cdot\|)$ be a normed space.

1) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. That is, if for any $\varepsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \varepsilon, \forall n \geq N$. We often write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ to mean that x is the limit of the sequence $\{x_n\}$.

2) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\varepsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \varepsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$.

3) A sequence $\{x_n\} \subset X$ is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leqslant M, \forall n \in \mathbb{N}$.

Definition 2.1.3. [10] A normed space X is called to be *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.1.4. [10] A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$

Definition 2.1.5. [10] Let X and Y be linear spaces over the field $\mathbb{K}$.

1) A mapping $T : X \to Y$ is called *a linear operator* if $T(x+y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in X$, and $\forall \alpha \in \mathbb{K}$.

2) A mapping $T : X \to \mathbb{K}$ is called *a linear functional on* X if T is a linear operator.

Definition 2.1.6. [10] Let X and Y be normed spaces over the field $\mathbb{K}$ and $T : X \to Y$ a linear operator. T is said to be *bounded* on X, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Definition 2.1.7. [10] Let X and Y be normed spaces over the field $\mathbb{K}$, $T : X \to Y$ an operator and $x_0 \in X$. We say that T is *continuous at* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$ and $x \in X$. If T is continuous at each $x \in X$, then T is said to be *continuous on* X.

Definition 2.1.8. [11] Let X be a normed space, $\{x_n\} \subset X$ and $f : X \to (-\infty, \infty]$. Then f is said to be

1) *lower semicontinuous* on X if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x_0$.

2) *upper semi (or hemi) continuous* on X if for any $x_0 \in X$, $\limsup_{n\to\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \to x_0$.

3) *weakly lower semicontinuous* on X if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\to\infty} f(x_n)$ whenever $x_n \rightharpoonup x_0$.

4) *weakly upper semicontinuous* on X if for any $x_0 \in X$, $\limsup_{n\to\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \rightharpoonup x_0$.

Definition 2.1.9. [10] Let X be a normed space. A mapping $T : X \to X$ is said to be *Lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in X$,

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.1.1}$$

The smallest number k for which (2.1.1) holds is called the *Lipschitz constant* of T and T is called *a contraction (nonexpansive mapping)* if $k \in (0,1)$ (k=1).

Definition 2.1.10. [10] An element $x \in X$ is said to be

1) a *fixed point* of a mapping $T : X \to X$ provided $Tx = x$.

2) a *common fixed point* of two mappings $S, T : X \to X$ provided $Sx = x = Tx$. The set of all fixed points of T is denoted by $F(T)$.

Theorem 2.1.11. (Banach contraction principle, [10]) *Every contraction mapping T defined on a Banach space X into itself has a unique fixed point $x^* \in X$.*

Definition 2.1.12. [10] Let X be a normed space. Then the set of all bounded linear functionals on X is called *a dual space* of X and is denoted by X^*.

Definition 2.1.13. [10] A normed space X is said to be *reflexive* if the *canonical mapping* $G : X \to X^{**}$ (i.e. $G(x) = g_x$ for all $x \in X$ where $g_x(f) = f(x)$ for all $f \in X^*$) is surjective.

Definition 2.1.14. [11] A Banach space X is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$

Definition 2.1.15. [12] A Banach space X is said to be *uniformly convex* if for each $0 < \varepsilon \leqslant 2$, there is $\delta > 0$ such that $\forall x, y \in X$, the condition $\|x\| = \|y\| = 1$, and $\|x - y\| \geqslant \varepsilon$ imply $\|\frac{x+y}{2}\| \leqslant 1 - \delta$.

Definition 2.1.16. [12] Let X be a Banach space. Then *the modulus of convexity of E* is $\delta : (0,2] \to [0,1]$ defined as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant \varepsilon\right\}.$$

Theorem 2.1.17. [12] *Let X be a Banach space. Then X is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.*

Definition 2.1.18. [11] Let X be a Banach space and $S = \{x \in X : \|x\| = 1\}$. Then X is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1.2}$$

exists for all $x, y \in S$. It is also said to be *uniformly smooth* if the limit (2.1.2) is attained uniformly for $x, y \in S$.

Remark 2.1.19. [11] 1) X is uniformly convex if and only if X^* is uniformly smooth.

2) X is smooth if and only if X^* is strictly convex.

Definition 2.1.20. [11] Let X^* be dual space of a Banach space X. The mapping $J : X \to X^*$ defined by

$$J(x) = \{x^* \in X : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}, \text{for all } x \in X,$$

is called the *duality mapping of* X.

Lemma 2.1.21. [11] *Let X be a strictly convex, smooth, and reflexive Banach space, and let J be the duality mapping from X into X^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from X^* into X.*

Lemma 2.1.22. [13] *Let X be a reflexive Banach space and X^* be strictly convex.*

(i) The duality mapping $J : X \to X^*$ is single-valued, surjective and bounded.

(ii) If X and X^* are locally uniformly convex, then J is a homeomorphism, that is, J and J^{-1} are continuous single-valued mappings.

Definition 2.1.23. [14] Let p be a fixed real number with $p \geq 1$. A Banach space X is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in (0, 2]$.

Definition 2.1.24. [11] For each $p > 1$, the *generalized duality mapping* $J_p : X \to 2^{X^*}$ is defined by

$$J_p(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \tag{2.1.3}$$

for all $x \in X$.

Remark 2.1.25. [11] 1) $J = J_2$ is called the normalized duality mapping. If X is a Hilbert space (the next section), then $J = I$, where I is the identity mapping.

2) If X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Definition 2.1.26. [11] Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space E. A Banach space E is said to have

- *a Gâteaux differentiable norm* (we also say that E is smooth), if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1.4}$$

  exists for each $x, y \in S(E)$;

- *a uniformly Gâteaux differentiable norm* , if for each y in $S(E)$, the limit (2.1.4) is uniformly attained for $x \in S(E)$;

- *a Fréchet differentiable norm,* if for each $x \in S(E)$, the limit (2.1.4) is attained uniformly for $y \in S(E)$;

- *a uniformly Fréchet differentiable norm* (we also say that E is uniformly smooth), if the limit (2.1.4) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

Definition 2.1.27. [11] A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

It is known that if E is uniformly convex, then E has the Kadec-Klee property.

Definition 2.1.28. [15] Let X be a smooth Banach space. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.1.5}$$

for all $x, y \in X$.

Remark 2.1.29. (1) $(\|y\| - \|x\|)^2 \leqslant \phi(y,x) \leqslant (\|y\| + \|x\|)^2$, for all $x, y \in X$.

(2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$, for all $x, y, z \in X$.

(3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$, for all $x, y \in X$.

(4) In a Hilbert space H, we have $\phi(x,y) = \|x - y\|^2$ for all $x, y \in H$.

Definition 2.1.30. [16] Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space X, for any $x \in X$, there exists a point $x_0 \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$. The mapping $\Pi_C : X \to C$ defined by $\Pi_C x = x_0$ is called the *generalized projection.*

The following are well-known results.

Lemma 2.1.31. [17] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$$

for all $y \in C$.

Lemma 2.1.32. [16] *Let C be a nonempty closed convex subset of a smooth Banach space X, let $x \in X$, and let $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for all $y \in C$.*

Definition 2.1.33. [17] Let X be reflexive Banach space with its dual X^* and K be a nonempty, closed and convex subset of X. The operator $\pi_K : X^* \to K$ defined by

$$\pi_K(x^*) = \{x \in K : V(x^*, x) = \inf_{y \in K} V(x^*, y)\}, \text{ for all } x^* \in X^*, \tag{2.1.6}$$

is said to be a *generalized projection operator.* For each $\varphi \in X^*$, the set $\pi_K(x^*)$ is called the *generalized projection of x^* on K.*

Lemma 2.1.34. [17] *Let X be a reflexive Banach space with its dual X^* and K be a nonempty closed convex subset of X, then the following properties hold:*

(i) The operator $\pi_K : X^* \to 2^K$ is single-valued if and only if X is strictly convex.

(ii) If X is smooth, then for any given $\varphi \in X^*, x \in \pi_K x^*$ if and only if $\langle x^* - J(x), x - y\rangle \geq 0$, $\forall y \in K$.

(iii) If X is strictly convex, then the generalized projection operator $\pi_K : X^* \to K$ is continuous.

Definition 2.1.35. [18] Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space X, let T be a mapping from C into itself, and let $F(T)$ be the set of all fixed points of T. Then a point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\}$ in C converging weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Definition 2.1.36. [18] A mapping T from C into itself is called *relatively nonexpansive mapping* if the following conditions are satisfied:

(R1) $F(T)$ is nonempty;

(R2) $\phi(u, Tx) \leqslant \phi(u, x)$ for all $u \in F(T)$ and $x \in C$;

(R3) $\hat{F}(T) = F(T)$.

Definition 2.1.37. [18] A mapping T from C into itself is called *relatively weak nonexpansive* if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Definition 2.1.38. [18] A mapping T from C into itself is called *relatively weak quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Definition 2.1.39. [18] A mapping $T : C \to E^*$ is said to be *relaxed η-ξ monotone* if there exist a mapping $\eta : C \times C \to E$ and a function $\xi : E \to \mathbb{R}$ positively homogeneous of degree p, that is, $\xi(tz) = t^p \xi(z)$ for all $t > 0$ and $z \in E$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq \xi(x - y), \quad \forall x, y \in C,$$

Definition 2.1.40. [11] A sequence $\{x_n\}$ in a normed space is said to *converge weakly* to some vector x if $\lim_{n\to\infty} f(x_n) = f(x)$ holds for every continuous linear functional f.

Definition 2.1.41. [11] Let X be a normed space and let C be a convex subset of X. A function $f : C \to (-\infty, \infty]$ is *convex* on X if for any $x_1, x_2 \in X$ and $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leqslant tf(x_1) + (1-t)f(x_2)$$

Definition 2.1.42. [19] A Banach space X is said to satisfy *Opial's condition* if any sequence $\{x_n\}$ in C, $x_n \rightharpoonup x$ as $n \to \infty$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in C$ with $y \neq x$.

Lemma 2.1.43. [20] *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) For all $x, y \in E$, the following inequality holds:

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

In particular, in a smooth Banach space E, for all $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

(ii) Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E.$$

Theorem 2.1.44. [21] *Let C be a nonempty convex subset of a smooth Banach space E and let $x \in E$ and $y \in C$. Then the following are equivalent:*

(a) y is a best approximation to x: $y = P_C x$.

(b) y is a solution of the variational inequality:

$$\langle y - z, J(x-y)\rangle \geq 0 \textit{ for all } z \in C,$$

where J is a duality mapping. The mapping $P_C : E \to C$ defined by $P_C x = y$ is called the metric projection from E onto C.

Lemma 2.1.45. [13] *Let E be a uniformly convex Banach space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < b \leq \alpha_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\limsup_{n\to\infty} \|x_n\| \leq d$, $\limsup_{n\to\infty} \|y_n\| \leq d$ and $\lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = d$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.1.46. [22] *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing, convex and continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\left\|T\left(\sum_{i=1}^{n} \lambda_i x_i\right) - \sum_{i=1}^{n} \lambda_i T x_i\right\|\right) \leq \max_{1\leq j\leq k\leq n} (\|x_j - x_k\| - \|Tx_j - Tx_k\|)$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, \ldots, x_n\} \subset C$, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ and nonexpansive mapping T of C into E.

Definition 2.1.47. [23] Let $\{S_n\}$ be a sequence of mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Then $\{S_n\}$ is said to satisfy the NST-condition if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} \|z_n - S_n z_n\| = 0$$

implies $\omega_w(z_n) \subset \bigcap_{n=1}^{\infty} F(S_n)$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$.

Lemma 2.1.48. [24] *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and let $\{S_n\}$ be a family of nonexpansive mappings of C into itself such that $F = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Let $\{\beta_n^k\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that*

(i) $\sum_{k=1}^{n} \beta_n^k = 1$ *for every* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \beta_n^k > 0$ *for every* $k \in \mathbb{N}$

and let $T_n = \alpha_n I + (1-\alpha_n)\sum_{k=1}^{n} \beta_n^k S_k$ *for all* $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset [a,b]$ *for some* $a, b \in (0,1)$ *with* $a \leq b$. *Then,* $\{T_n\}$ *is a family of nonexpansive mappings of* C *into itself with* $\bigcap_{n=1}^{\infty} F(T_n) = F$ *and satisfies the NST-condition.*

Definition 2.1.49. [17] Let C be a nonempty, closed and convex subset of a Hilbert space H. A mapping $T : C \to C$ is said to be *λ-strictly pseudocontractive mapping* if there exists a constant $0 \leq \lambda < 1$, such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \lambda\|(I-T)x - (I-T)y\|^2, \tag{2.1.7}$$

for all $x, y \in C$.

Lemma 2.1.50. [25] *Let E be a real 2-uniformly smooth Banach space and $T : E \to E$ a λ-strict pseudo-contraction. Then $S := (1-\lambda/K^2)I + \lambda/K^2 T$ is nonexpansive and $F(T) = F(S)$.*

Definition 2.1.51. [17] A countable family of mapping $\{T_n : C \to C\}_{i=1}^{\infty}$ is called a *family of uniformly ε-strict pseudo-contractions*, if there exists a constant $\varepsilon \in [0,1)$ such that

$$\|T_n x - T_n y\|^2 \leq \|x-y\|^2 + \varepsilon\|(I-T_n)x - (I-T_n)y\|^2, \quad \forall x, y \in C, \quad \forall n \geq 1.$$

Lemma 2.1.52. [26] *Let E be a real 2-uniformly smooth Banach space with the best smoothness constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 2.1.53. [27] *Assume A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.1.54. [28] *Let E be a real p-uniformly convex Banach space and C a nonempty closed convex subset of E. let $T : C \to C$ be a λ-strict pseudocontraction* *with respect to p, and* $\{\xi_n\}$ *a real sequence in* $[0,1]$. *If* $T_n : C \to C$ *is defined by* $T_n x := (1-\xi_n)x + \xi_n T x$, $\forall x \in C$, *then for all* $x, y \in C$, *the inequality holds*

$$\|T_n x - T_n y\|^p \leq \|x-y\|^p - (w_p(\xi_n)c_p - \xi_n\lambda)\|(I-T)x - (I-T)y\|^p,$$

where c_p *is a constant in* [29, Theorem 1]. In addition, if $0 \leq \lambda < \min\{1, 2^{-(p-2)}c_p\}$, $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$, and $\xi_n \in [0, \xi]$, then $\|T_n x - T_n y\| \leq \|x - y\|$, for all $x, y \in C$.

Lemma 2.1.55. [30, 31] *Let* C *be a nonempty closed convex subset of a Banach space* E *which has uniformly Gâteaux differentiable norm,* $T : C \to C$ *a nonexpansive mapping with* $F(T) \neq \emptyset$ *and* $f : C \to C$ *a k-contraction. Assume that every nonempty closed convex bounded subset of* C *has the fixed points property for nonexpansive mappings. Then there exists a continuous path:* $t \to x_t$, $t \in (0,1)$ *satisfying* $x_t = tf(x_t) + (1-t)Tx_t$, *which converges to a fixed point of* T *as* $t \to 0^+$.

Lemma 2.1.56. [32] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and* $\{\beta_n\}$ *a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Definition 2.1.57. [33] Let $\{S_n\}$ be a family of mappings from a subset C of a Banach space E into E with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$. We say that $\{S_n\}$ satisfies the AKTT-condition if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup_{z\in B} \|S_{n+1}z - S_n z\| < \infty. \qquad (2.1.8)$$

Lemma 2.1.58. [34] *Let* E *be a strictly convex Banach space. Let* T_1 *and* T_2 *be two nonexpansive mappings from* E *into itself with a common fixed point. Define a mapping* S *by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad \forall x \in E,$$

where λ *is a constant in* $(0,1)$. *Then* S *is nonexpansive and* $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.1.59. [35] *Let C be a nonempty closed convex subset of reflexive Banach space E which satisfies Opial's condition, and suppose that $T : C \to E$ is nonexpansive. Then the mapping $I-T$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ implies $x = Tx$.*

Definition 2.1.60. Let $M : E \to 2^E$ be a multi-valued maximal accretive mapping. The single-valued mapping $J_{(M,\rho)} : E \to E$ defined by

$$J_{(M,\rho)}(u) = (I + \rho M)^{-1}(u), \quad \forall u \in E$$

is called the resolvent operator associated with M, where ρ is any positive number and I is the identity mapping.

Lemma 2.1.61. [36] *The resolvent operator $J_{(M,\rho)}$ associated with M is single-valued and nonexpansive for all $\rho > 0$*

Lemma 2.1.62. [37] *If E is a reflexive, strictly convex and smooth Banach space, then $\Pi_C = J^{-1}$.*

Lemma 2.1.63. [38, 39] *Let E be a 2-uniformly convex Banach, then for all x, y from any bounded set of E and $jx \in Jx, jy \in Jy$,*

$$\langle x - y, jx - jy \rangle \geq \frac{c^2}{2} \|x - y\|^2 \tag{2.1.9}$$

where $\frac{1}{c}$ is the 2-uniformly convexity constant of E.

Lemma 2.1.64. [40] *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, then $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

Lemma 2.1.65. [17] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0 \rangle \geqslant 0$ for any $y \in C$.*

Let E be a reflexive strictly convex, smooth and uniformly Banach space and the duality mapping J from E to E^*. Then J^{-1} is also single-valued, one to one, surjective, and it is the duality mapping from E^* to E. We need the following mapping V which studied in Alber [17],

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x\|^2 \tag{2.1.10}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma:

Lemma 2.1.66. [40] *Let E be a reflexive, strictly convex and smooth Banach space, and let V be as in* (2.1.10). *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.1.67. [41] *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|), \tag{2.1.11}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping F of $[0, 1)$ into E^* defined by $F(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \forall y \in C\}.$$

Definition 2.1.68. [17] Let E be a Banach space with the dual space E^* and let K be a nonempty subset of E. Let $T : K \to E^*$ and $\eta : K \times K \to E$ be two mappings. The mapping $T : K \to E^*$ is said to be *η-hemicontinuous* if, for any fixed $x, y \in K$, the function $f : [0, 1] \to (-\infty, \infty)$ defined by $f(t) = \langle T((1 - t)x + ty, \eta(x, y)\rangle$ is continuous at 0^+.

Lemma 2.1.69. [42] *Let C be a nonempty closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(A, C)$.

Lemma 2.1.70. [43] *Let E be a uniformly convex and uniformly smooth Banach space. We have*

$$\|\phi + \Phi\|^2 \leq \|\phi\|^2 + 2\langle \Phi, J(\phi + \Phi)\rangle, \quad \forall\, \phi, \Phi \in E^*.$$

Lemma 2.1.71. [44] *Let E be a strictly convex Banach space and C be a closed convex subset of E. Let $S_1, S_2, \ldots, S_N : C \to C$ be a finite family of nonexpansive mappings of C into itself such that the set of common fixed points of $S_1, S_2, \ldots, S_N$ is nonempty. Let $T_1, T_2, \ldots, T_N : C \to C$ be mappings given by*

$$T_i = (1 - \alpha_i)I + \alpha_i S_i, \text{ for all } i = 1, 2, \ldots, N, \tag{2.1.12}$$

where I denotes the identity mapping on C. Then, the finite family $\{T_1, T_2, \ldots, T_N\}$ satisfies the following : $\cap_{i=1}^N F(T_i) = \cap_{i=1}^N F(S_i)$ and

$$\cap_{i=1}^N F(T_i) = F(T_N T_{N-1} T_{N-2} \cdots T_1) = F(T_1 T_N \cdots T_2) = F(T_{N-1} T_{N-2} \cdots T_1 T_N).$$

Lemma 2.1.72. [45]*(Demiclosedness Principle) Let X be a uniformly convex Banach space, let C be a nonempty closed convex subset of X and let $T : C \to X$ be a nonexpansive mapping. Then, the mapping $(I - T)$ is demiclosed on C, i.e., if $\{x_n\}$ is weakly convergent to x and $\{(I - T)(x_n)\}$ is strongly convergent to y, then $(I - T)x = y$.*

Definition 2.1.73. [21] Let C be a nonempty closed convex subset of Banach space E. Given a mapping $\Psi : C \to E$.

1. Ψ is said to be *accretive* if

$$\langle \Psi x - \Psi y, J(x-y)\rangle \geq 0$$

for all $x, y \in C$.

2. Ψ is said to be *α-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle \Psi x - \Psi y, J(x-y)\rangle \geq \alpha\|x-y\|^2$$

for all $x; y \in C$.

3. Ψ is said to be *α-inverse-strongly accretive* or *α-cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle \Psi x - \Psi y, J(x-y)\rangle \geq \alpha\|\Psi x-\Psi y\|^2$$

for all $x, y \in C$.

4. Ψ is said to be *α-relaxed cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle \Psi x - \Psi y, J(x-y)\rangle \geq -\alpha\|\Psi x-\Psi y\|^2$$

for all $x, y \in C$.

5. Ψ is said to be *(α, β)-relaxed cocoercive* if there exists positive constants α and β such that

$$\langle \Psi x - \Psi y, J(x-y)\rangle \geq (-\alpha)\|\Psi x-\Psi y\|^2 + \beta\|x-y\|^2$$

for all $x, y \in C$.

Definition 2.1.74. [11] Let D be a subset of C and P be a mapping of C into D. Then P is said to be *sunny if*

$$P(Px + t(x - Px)) = Px,$$

whenever $Px + t(x - Px) \in C$ *for* $x \in C$ *and* $t \geq 0$.

Definition 2.1.75. [11] A mapping P of C into itself is called a *retraction if $P^2 = P$. If a mapping P of C into itself is a retraction, then $Pz = z$ for all $z \in R(P)$, where $R(P)$ is the range of P. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.*

Lemma 2.1.76. [46] *Let B be a nonempty subset of a Hausdorff topological vector space X and let $G : B \to 2^X$ be a KKM mapping. If $G(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x \in B} G(x) \neq \emptyset$.*

Definition 2.1.77. [21] Let τ^* be the norm topology of X^* generated by the norm $\|\cdot\|_*$ (of X^*). Then there exists a topology denoted by $\sigma(X^*, X)$ on X^* such that $\sigma(X^*, X) \subseteq \tau^*$. The topology $\sigma(X^*, X)$ is called the *weak* topology* on X^*.

2.2 Inner product spaces and Hilbert spaces.

Definition 2.2.1. [10] The real-valued function of two variables $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called *inner product* on a real vector space X if it satisfies the following conditions:

1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and all real number α and β;

2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$; and

3) $\langle x, x \rangle \geq 0$ for each $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

A *real inner product space* is a real vector space equipped with an inner product.

Remark 2.2.2. [10] Every inner product space is a normed space with respect to the norm $\|x\| = |\langle x, x \rangle|^{\frac{1}{2}}, x, y \in X$.

Definition 2.2.3. [10] A *Hilbert space* is an inner product space which is complete under the norm induced by its inner product.

Definition 2.2.4. [10] A sequence $\{x_n\}$ in a Hilbert space H is said to *converge weakly* to a point x in H if $\lim_{n \to \infty} \langle x_n, y \rangle = \langle x, y \rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

Lemma 2.2.5. [47] *An inner product and the corresponding norm satisfy the Schwarz inequality and the triangle inequality as follows.*

(i) *we have*

$$|\langle x, y\rangle| \leq \|x\|\|y\| \qquad \textit{(Schwarz inequality)}$$

(ii) *That norm also satisfies*

$$\|x + y\| \leq \|x\| + \|y\| \qquad \textit{(Triangle inequality)}$$

Definition 2.2.6. A mapping $B : H \to H$ is called *strongly positive bounded linear operator* on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2 \text{ for all } x \in H. \tag{2.2.1}$$

Definition 2.2.7. Let C be a subset of an inner product space X. A mapping $A : C \to C$ is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \;\; \forall x, y \in C.$$

Definition 2.2.8. [10] A mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

It is clear that any α-inverse-strongly monotone mapping is $\frac{1}{\alpha}$-Lipschitz continuous.

Lemma 2.2.9. [33] *Suppose that $\{T_n\}$ satisfies AKTT-condition. Then, for each $y \in C$, $\{T_n y\}$ converses strongly to a point in C. Moreover, let the mapping T be defined by*

$$Ty = \lim_{n\to\infty} T_n y \textit{ for all } y \in C.$$

Then for each bounded subset B of C, $\lim_{n\to\infty} \sup_{z\in B} \|Tz - T_n z\| = 0$.

Lemma 2.2.10. [26] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Definition 2.2.11. A mapping $F : H \to H$ is called *δ-strongly monotone* if there exists a positive constant δ such that

$$\langle Fx - Fy, x - y\rangle \geq \delta\|x-y\|^2, \forall x, y \in H. \tag{2.2.2}$$

Definition 2.2.12. A mapping $F : H \to H$ is called *λ-strictly pseudo-contractive* if there exists a positive constant λ such that

$$\langle Fx - Fy, x - y\rangle \leq \|x-y\|^2 - \lambda\|(x-y) - (Fx - Fy)\|^2, \forall x, y \in H. \tag{2.2.3}$$

The following lemma can be found in [48, Lemma 2.7]. For the sake of the completeness, we include its proof in a Hilbert space'version.

Lemma 2.2.13. *Let* H *be a real Hilbert space and* $F : H \to H$ *a mapping.*

(i) *If* F *is δ-strongly monotone and λ-strictly pseudo-contractive with* $\delta + \lambda > 1$, *then* $I - F$ *is contractive with constant* $\sqrt{(1-\delta)/\lambda}$.

(ii) *If* F *is δ-strongly monotone and λ-strictly pseudo-contractive with* $\delta + \lambda > 1$, *then for any fixed number* $\tau \in (0,1)$, $I - \tau F$ *is contractive with constant* $1 - \tau(1 - \sqrt{(1-\delta)/\lambda})$.

Proof. (i) For any $x, y \in H$, we have

$$\lambda\|(I-F)x-(I-F)y\|^2 \leq \|x-y\|^2-\langle Fx-Fy, x-y\rangle \leq (1-\delta)\|x-y\|^2, \forall x, y \in H.$$

Thus

$$\|(I-F)x-(I-F)y\| \leq \sqrt{\frac{1-\delta}{\lambda}}\|x-y\|, \text{ for all } x, y \in H.$$

Since $\delta + \lambda > 1$, we have $\frac{1-\delta}{\lambda} \in (0,1)$. Hence $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$.

(ii) Since $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$, we have for any $\tau \in (0, 1)$,

$$\begin{aligned}
\|x-y-\tau(Fx-Fy)\| &= \|(1-\tau)(x-y)+\tau[(I-F)x-(I-F)y]\| \\
&\leq (1-\tau)\|x-y\|+\tau\|(I-F)x-(I-F)y\| \\
&\leq (1-\tau)\|x-y\|+\tau\sqrt{\frac{1-\delta}{\lambda}}\|x-y\| \\
&= \left(1-\tau\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|, \text{ for all } x, y \in H.
\end{aligned}$$

Hence $I - \tau F$ is contractive with constant $1-\tau(1-\sqrt{(1-\delta)/\lambda})$. □

Lemma 2.2.14. *Let $S_1, S_2, \ldots, S_N : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{1+k_{p(n)}^{i(n)}\}$ respectively, such that $k_{p(n)}^{i(n)} \to 0$ as $n \to \infty$. Then, there exists a sequence $\{h_n\} \subset [0, \infty)$ with $h_n \to 0$ as $n \to \infty$ such that*

$$\|S_{i(n)}^{p(n)}x - S_{i(n)}^{p(n)}y\| \leq (1+h_n)\|x-y\|, \forall x, y \in C,$$

where $p(n) = j+1$ if $jN < n \leq (j+1)N, j = 1, 2, \ldots$ and $n = jN + i(n); i(n) \in \{1, 2, \ldots, N\}$.

Proof. Define the sequence $\{h_n\}$ by $h_n := \max\{k_{p(n)}^{i(n)} : 1 \leq i(n) \leq N\}$ and the result follows immediately. □

Lemma 2.2.15. [49] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $r > 0$ be a constant. Then, we have*

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \forall x, y \in C.$$

In particular, if $0 \leq r \leq 2\alpha$, then $I - rA$ is nonexpansive.

Lemma 2.2.16. [50] *Let S be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $\|Sx_n - x_n\| \to 0$ as $n \to \infty$, then $x \in F(S)$.*

Lemma 2.2.17. [51] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0$$

where $\{\alpha_n\}, \{\sigma_n\}$ and $\{\gamma_n\}$ are nonnegative real sequences satisfying the following conditions

(i) $\{\alpha_n\} \subset [0, 1], \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Definition 2.2.18. Let H be a real Hilbert space. A family $\mathcal{S} := \{T(s) : 0 \leq s < \infty\}$ of mappings of H into itself is called a *nonexpansive semigroup* on H if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in H$;

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in H$ and $s \geq 0$;

(iv) for all $x \in H, s \mapsto T(s)x$ is continuous.

We denote by $F(T(s)) = \{x \in C : T(s)x = x\}$ the set of fixed points of $T(s)$ and by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, i.e. $F(\mathcal{S}) = \cap_{s \geq 0} F(T(s))$. It is known that $F(\mathcal{S})$ is closed and convex.

Lemma 2.2.19. [52] *Let C be a nonempty bounded closed convex subset of Hilbert space H and let $(T(s))_{s\geq 0}$ be a nonexpansive semigroup on C. Then, for every $h \geq 0$,*

$$\lim_{t\to+\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x ds - T(h)\frac{1}{t}\int_0^t T(s)x ds \right\| = 0.$$

Lemma 2.2.20. [47]*For all $x, y \in H$, the inequality $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ holds.*

We recall that, if C is a closed convex subset of real Hilbert space H, the metric projection $P_C : H \to C$ is the mapping defined as follows: for each $x \in H$, $P_C x$ is the only point in C with the property that $\|x - P_C x\| = \inf_{y\in C} \|x - y\|$.

Lemma 2.2.21. [45] *Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$, $z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0$ for all $y \in C$.*

Lemma 2.2.22. [53] *Let H be a Hilbert space and let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. If $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.2.23. [53] *Let C be a nonempty closed convex subset of a real Hilbert space H, let $f : H \to H$ be an α-contraction $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$. Then, for every $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, $(A - \gamma f)$ is a strongly monotone with coefficient $(\bar{\gamma} - \alpha\gamma)$, i.e.*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in H.$$

2.3 Equilibrium problems and variational inequality problem.

Definition 2.3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$. The *equilibrium problem* is to find $x \in C$ such that

$$f(x,y) \geq 0 \text{ for all } y \in C. \tag{2.3.1}$$

The set of solutions of (2.3.1) is denoted by $EP(f)$.

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \quad \leq \quad f(x,y); \tag{2.3.2}$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x,y) = \langle Ax, y - x \rangle, \forall x, y \in C.$$

Then, f satisfies (A1)-(A4).

Lemma 2.3.2. [8] *Let C be a closed and convex subset of a smooth, strictly convex, and reflexive Banach spaces E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{2.3.3}$$

Lemma 2.3.3. [54] *Let C be a nonempty closed and convex subset of a real Hilbert space H and $G : C \times C \to \mathbb{R}$ a function satisfying the condition (E1)-(E4). For $r > 0$ and $x \in H$, A mapping $T_r : H \to C$ defined by*

$$T_r(x) = \left\{ z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}, \quad x \in H.$$

Then :

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e.

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H;$$

(iii) $F(T_r) = EP(G)$;

(iv) $EP(G)$ is closed and convex.

Lemma 2.3.4. [37] *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty, closed and convex subset of E. Suppose A is an operator of C into E^* and that there exists a positive number β such that*

$$\langle Ax, J^{-1}(Jx - \beta Ax)\rangle \geq 0, \quad \text{for all } x \in C, \tag{2.3.4}$$

and

$$\langle Ax, y\rangle \leq 0, \quad \forall x \in C, \ y \in VI(A, C). \tag{2.3.5}$$

Then $VI(A, C)$ is closed and convex.

Definition 2.3.5. Let $A : C \to E^*$ be an operator. We consider the following variational inequality:

$$\text{Find } x \in C, \text{ such that } \langle Ax, y - x\rangle \geq 0, \quad \text{for all } y \in C. \tag{2.3.6}$$

A point $x_0 \in C$ is called a solution of the variational inequality (2.3.6) if for every $y \in C$, $\langle Ax_0, y - x_0 \rangle \geq 0$. The solution set of the variational inequality (2.3.6) is denoted by $VI(A, C)$.

Lemma 2.3.6. [37] *Let E be a reflexive, strictly convex and smooth Banach space with dual space E^*. Let A be an arbitrary operator from Banach space E to E^* and β an arbitrary fixed positive number. Then $x \in C \subset E$ is a solution of variational inequality (2.3.6) if and only if x is a solution of the operator equation in E*

$$x = \Pi_C(Jx - \beta Ax).$$