# THESIS APPROVAL

## GRADUATE SCHOOL, KASETSART UNIVERSITY

Master of Science (Mathematics)

**DEGREE**

Mathematics

**FIELD**

Mathematics

**DEPARTMENT**

**TITLE:** Positivity and Periodicity of Linear Recurrence Relation

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**THESIS**

# POSITIVITY AND PERIODICITY OF LINEAR RECURRENCE RELATION

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**A Thesis Submitted in Partial Fulfillment of**
**the Requirements for the Degree of**
**Master of Science (Mathematics)**
**Graduate School, Kasetsart University**
**2009**

Pinthira Tangsupphathawat 2009: Positivity and Periodicity of Linear Recurrence Relation. Master of Science (Mathematics), Major Field: Mathematics, Department of Mathematics. Thesis Advisor: Professor Vichian Laohakosol, Ph.D. 29 pages.

A (homogeneous) **linear recurrence** over the integers has the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k}, \tag{1}$$

for $n \geq k \in \mathbb{N}$, where $a_1, a_2, \cdots, a_k \in \mathbb{Z}$ are integer constants.The linear reccurrence equation (1) defines a unique integer sequence $(u_n)_{n=0}^{\infty}$ after the first $k$ initial terms $u_0, u_1, \cdots, u_{k-1}$ are given. A sequence $(u_n)_{n=0}^{\infty}$ is said to be *recurrent* if it is defined by a linear recurrence. The integer $k$ in (1) is called the *order* of the recurrence and also of the defined recurrent sequent.

We shall consider the following problem of recurrent sequences.
**Positivity Problem:**Let a linear recurrence (1) be given together with the initial term $u_i$ for $i = 0, 1, \cdots, k$. Is the recurrent sequence $(u_n)_{n=0}^{\infty}$ nonnegative,i.e.,does it hold that $u_n \geq 0$ for all $n$?

In 2006, Vesa Halava, Tero Harju and Mika Hirvensalo proved that the second order Positivity Problem is decidable.

In this thesis, we consider that the Positivity Problem for a sequence satisfying a third order linear recurrence with integer coefficients, i.e., the problem whether each element of this sequence is nonnegative, is decidable.

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Student's signature Thesis Advisor's signature

**ACKNOWLEDGEMENTS**

I would like to express my gratitude to all those who gave me the possibility to complete this thesis. I would like to thank Professor Dr. Vichian Laohakosol, my thesis advisor, whose encouragement helped me through all the research for and the writing of this thesis.

Especially, I would like to give special thanks to my family whose patient love enabled me to complete this work.

Pinthira Tangsupphathawat

July 2009

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| $\bar{\lambda}_2$ | 13 |
| $\Omega$ | 11 |
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# POSITIVITY AND PERIODICITY OF LINEAR RECURRENCE RELATION

## INTRODUCTION

Consider a third order linear recurrence of the form

$$u_n = a_1u_{n-1} + a_2u_{n-2} + a_3u_{n-3} \quad (n \geq 3), \tag{1}$$

where $a_1, a_2, a_3 (\neq 0)$ are given integers. The recurrence (1) defines a unique sequence of integers provided the initial integers $u_0, u_1, u_2$ are given. We are interested in the *Positivity Problem*: Is it possible to decide whether the sequence $(u_n)_{n\geq 0}$ is nonnegative ? Equivalently, is it decidable whether $u_n \geq 0$ for all $n \geq 0$ ? Let us emphasize that the Positivity Problem considered here is to decide whether *all* elements of a sequence are nonnegative. In contrast, the *Eventual Positivity Problem*, which asks whether the terms $u_n$ are nonnegative for all sufficiently large $n$ is not of interest here. For third order recurrences with integral coefficients, the Eventual Positivity Problem is rather trivial to discern. Deciding the positivity of all sequence elements requires some further considerations to take care of the initial tail of the sequence.

As mentioned in (Halava *et al*, 2006), the Positivity Problem is a natural question and well-known in a number of situations, such as for matrices and in the Mortality Problem (Salomaa, 1976; Salomaa and Soittola, 1978; Halava and Harju, 2001; Niven, 1967). A closely related result, known as the Skolem-Mahler-Lech theorem, provides information about when the terms of a linear recurrence sequence are zero (Berstel and Mignotte, 1976; Halava *et al*, 2005; section 2.1.1 of Everest, 2003). Another related problem, known as the Orbit Problem, is, in an equivalent form, a question about simultaneous solvability of a system of equations in linear recurrence sequences (see section 14.2.3 of Everest *et al*, 2003).

The Positivity Problem for sequences satisfying a second order linear recurrence has already been shown to be decidable by Halava, Harju and Hirvensalo, in 2006. This means that there is an algorithmic procedure to decide whether elements in a sequence satisfying a second order linear recurrence with integer coefficients are nonnegative. We show here that the same conclusion holds for each sequence satisfying a third order linear recurrence with integer coefficients. Our underlying methodology is based on the fact that the roots of a third degree algebraic equation with integer coefficients can be explicitly computed, and in one instance on a result in Diophantine approximation; such approximation result has been used in this context before e.g. in Asymptotic behaviour of linear recurrences (Burke and Webb, 1981) and Point lattices and oscillating recurrence sequences (Gerhold, 2005).

# OBJECTIVE

To show that the Positivity Problem for a sequence satisfying a third order linear recurrence with integer coefficients is decidable, i.e., the problem whether each element of this sequence is nonnegative, is decidable.

# LITERATURE REVIEW

**The Kronecker Approximation Theorem.**

**Theorem 1.** (***Dirichlet's Approximation Theorem.***) *For each real number $\alpha$ and natural number $\mathbb{N}$ one can find a natural number $n < N$ and integer $p$ with*

$$\left|\alpha - \frac{p}{n}\right| < \frac{1}{nN}.$$

*In particular,*

$$|n\alpha - p| < \frac{1}{N}.$$

If instead of the real number $\alpha$ and the real line $\mathbb{R}$ one considers only the fractional part $\{\alpha\} = \alpha - [\alpha]$ and the half-open interval $[0, 1)$.

Topologically open set on the circle are identified as open sets on $[0, 1)$, i.e. neighbourhoods in $[0, 1)$ are defined by the quotient topology in $\mathbb{R}/\mathbb{Z}$. Consider in this way Dirichlet's Theorem state

**Proposition 1.** *In each neighbourhood of zero there are infinitely many numbers of the from $\alpha n (mod 1)$ where $n$ runs through the numbers and $\alpha$ is real and arbitrary.*

**Theorem 2.** (***Kronecker's Approximation Theorem.***) *For each irrational number $\alpha$, each real number $\beta$, each preassigned arbitrarity small number $\epsilon > 0$, and arbitrarity large number $\Omega$, there exist integers $p$ and $n$ with $|n| \geq \Omega$ and $|\alpha n - \beta - p| < \epsilon$.*

*Proof.* Choosing $N > 1/\epsilon$. By the Dirichlet theorem there exists an integer $g$ and a natural number $q$ with

$$0 < |\alpha q - g| < \epsilon.$$

Now from the equations $n = kq$, $p = kg + c$, where the integers $k$ and $c$ are to be determind in the course of the proof. Since one must have

$$|\alpha n - \beta - p| = |k(\alpha q - g) - \beta - c| = |\alpha q - g| \cdot |k - \frac{\beta + c}{\alpha q - g}|$$

remaining smaller than $\epsilon$, fix $k$ by means of the equation

$$k = \left[\frac{\beta + c}{\alpha q - g}\right] + 1.$$

Finally choose $c$ so that $|n| \geq \Omega$. Because $|n| = q|k|$ it suffices to restrict oneself to $|k| \geq \Omega$, that is,

$$\left|\frac{\beta + c}{\alpha q - g}\right| \geq \Omega + 1.$$

The inequality

$$\left|\frac{\beta + c}{\alpha q - g}\right| \geq \frac{|c|}{|\alpha q - g|} - \frac{|\beta|}{|\alpha q - g|}$$

shows that the condition above is guaranteed, if one ensures that

$$|c| \geq (\Omega + 1)|\alpha q - g| + |\beta|.$$

In this way not only is

$$|\alpha n - \beta - p| < \epsilon$$

guaranteed, but also $|n| \geq \Omega$, which was to be proved. □

If we reduce modulo 1 the integeral part $p$ falls away from $\alpha n - \beta - p$, and we can formulate the Kronecker approximation theorem in the following simpler manner.

**Corollary 1.** *For each irrational number $\alpha$ infinitely many of the numbers $\alpha n$ modulo 1 lie in each arbitrarity small $\epsilon$-neighbourhood of an arbitrary element $\beta$ from $[0, 1)$.*

**Corollary 2.** *For each irrational number $\alpha$ the set of numbers $\alpha n$ reduced modulo 1 is dense in the whole interval $[0, 1)$.*

**The work of Vesa Halava, Tero Harju and Mika Hirvensalo.**

They give a decision method for the Positivity Problem for second order recurrence sequences answering the question: it is decidable whether or not a recurrence sequence defined by $u_n = au_{n-1} + bu_{n-2}$ has only nonnegative terms.

**Lemma 1.** *Let*

$$u_n = au_{n-1} + bu_{n-2} \tag{2}$$

*be a linear recurrence equation with $a, b \neq 0$ and let $p(x) = x^2 - ax - b$ be its characteristic polynomial with discriminant $D = a^2 + 4b$.*

*I. If $D > 0$, then $u_n = A\lambda_1^n + B\lambda_2^n$, where $\lambda_1 \neq \lambda_2$ are the real roots of $p(x)$, $A = \frac{u_1 - u_0\lambda_2}{\sqrt{D}}$ and $B = \frac{u_0\lambda_1 - u_1}{\sqrt{D}}$.*

*II. If $D = 0$, then $u_n = (A + Bn)\lambda^n$, where $\lambda = \frac{a}{2}$ is the double root of $p(x)$, $A = u_0$, and $B = \frac{2u_1 - u_0 a}{a}$.*

*III. If $D < 0$, then $u_n = A\lambda^n + \bar{A}\bar{\lambda}^n$, where $\lambda$ and $\bar{\lambda}$ are the complex roots of $p(x)$, and $A$ is as in Case 1.*

**Lemma 2.** *If $a = 0$ or $b = 0$ in (2), then the Positivity Problem can be effectively solved.*

**Lemma 3.** *Assume that, for all $n \geq 0$, $u_n = A\lambda_1^n + B\lambda_2^n$, where $\lambda_1 \neq \lambda_2$ are real and nonzero, and $A, B \in \mathbb{R}$. Then the Positivity Problem of $(u_n)_{n=0}^{\infty}$ can be effectivity solved.*

**Lemma 4.** *Assume that $u_n = (A + Bn)\lambda^n$ for all $n \geq 0$. Then the Positivity Problem of the sequence $(u_n)_{n=0}^{\infty}$ can be effectively solved.*

**Lemma 5.** *If $u_n = A\lambda^n + \bar{A}\bar{\lambda}^n$ for $n \geq 0$ and $\lambda \notin \mathbb{R}$ then $(u_n)_{n=0}^{\infty}$ has negative elements.*

**Theorem 3.** *The Positivity Problem is decidable for second order recurrent sequences.*

## MATERIALS AND METHODS

The characteristic polynomial associated with the recurrence (1) is

$$p(x) = x^3 - a_1x^2 - a_2x - a_3 \in \mathbb{Z}[x].$$

Following the derivation on pages 169-170 of (Grillet, 1999), we start by reviewing some facts. Write

$$p\left(x + \frac{a_1}{3}\right) = x^3 + \alpha x + \beta, \quad \alpha := \frac{-a_1^2 - 3a_2}{3}, \ \beta := \frac{-2a_1^3 - 9a_2a_1 - 27a_3}{27}.$$

Let $x_1, x_2$, and $x_3$ be all the three roots of $p\,(x + a_1/3)$. The discriminant of $p\,(x + a_1/3)$ is

$$D = -4\alpha^3 - 27\beta^2 = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2.$$

**Proposition 2.** *Let $(u_n)_{n\geq 0}$ be a sequence of integers satisfying (1) with integer coefficients $a_1, a_2, a_3(\neq 0)$, and initial integral values $u_0, u_1, u_2$. Let $p(x)$ be the characteristic polynomial of (1) whose roots are $\lambda_i$ $(i = 1, 2, 3)$, and whose discriminant is $D = -4\alpha^3 - 27\beta^2$, where $\alpha = (-a_1^2 - 3a_2)/3$ and $\beta = (-2a_1^3 - 9a_2a_1 - 27a_3)/27$.*

*I. If $D > 0$, then $\lambda_1, \lambda_2, \lambda_3$ are distinct nonzero real numbers and*

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (n \geq 0),$$

*where*

$$A = \frac{\lambda_2\lambda_3u_0 - (\lambda_3 + \lambda_2)u_1 + u_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \in \mathbb{R}, \ B = \frac{-\lambda_1\lambda_3u_0 + (\lambda_3 + \lambda_1)u_1 - u_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} \in \mathbb{R},$$
$$C = \frac{\lambda_1\lambda_2u_0 - (\lambda_2 + \lambda_1)u_1 + u_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \in \mathbb{R}.$$

*II. (II.1) If $D = 0$ and $\alpha = 0$, then $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda \in \mathbb{R} \setminus \{0\}$ and*

$$u_n = (A + Bn + Cn^2)\lambda^n \quad (n \geq 0),$$

*where*

$$A = u_0 \in \mathbb{R}, \ B = \frac{-3u_0}{2} + \frac{2u_1}{\lambda} - \frac{u_2}{2\lambda^2} \in \mathbb{R}, \ C = \frac{u_0}{2} - \frac{u_1}{\lambda} + \frac{u_2}{2\lambda^2} \in \mathbb{R}.$$

*(II.2) If $D = 0$ and $\alpha \neq 0$, then $p(x)$ has two distinct real roots $\lambda_1 (\neq 0)$ of multiplicity 1, $\lambda_2 = \lambda_3 (\neq 0)$ of multiplicity 2, and*

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (n \geq 0),$$

*where*

$$A = \frac{\lambda_2^2 u_0 - 2\lambda_2 u_1 + u_2}{(\lambda_2 - \lambda_1)^2} \in \mathbb{R}, \; B = \frac{-\lambda_1(2\lambda_2 - \lambda_1)u_0 + 2\lambda_2 u_1 - u_2}{(\lambda_2 - \lambda_1)^2} \in \mathbb{R},$$
$$C = \frac{\lambda_1 \lambda_2 u_0 - (\lambda_2 + \lambda_1)u_1 + u_2}{\lambda_2(\lambda_2 - \lambda_1)} \in \mathbb{R}.$$

*III. If $D < 0$, then $p(x)$ has one real root $\lambda_1 (\neq 0)$, two complex conjugate roots $\lambda_2, \lambda_3 = \bar{\lambda}_2 \in \mathbb{C} \setminus \mathbb{R}$ and*

$$u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (n \geq 0),$$

*where*

$$A = \frac{\lambda_2 \bar{\lambda}_2 u_0 - (\bar{\lambda}_2 + \lambda_2)u_1 + u_2}{(\lambda_2 - \lambda_1)(\bar{\lambda}_2 - \lambda_1)} \in \mathbb{R}, \; B = \frac{-\lambda_1 \bar{\lambda}_2 u_0 + (\bar{\lambda}_2 + \lambda_1)u_1 - u_2}{(\lambda_2 - \lambda_1)(\bar{\lambda}_2 - \lambda_1)} \in \mathbb{C}.$$

*Proof.* Let $\lambda_1, \lambda_2$ and $\lambda_3$ be roots of $p(x)$.

I. Assume that $D > 0$ , then $\lambda_1, \lambda_2$ and $\lambda_3$ are distinct nonzero real numbers and

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (n \geq 0).$$

Thus,

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix},$$

and

$$\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2).$$

Therefore

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{\lambda_2\lambda_3 u_0 - (\lambda_3 + \lambda_2)u_1 + u_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \\ \frac{-\lambda_1\lambda_3 u_0 + (\lambda_3 + \lambda_1)u_1 - u_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} \\ \frac{\lambda_1\lambda_2 u_0 - (\lambda_2 + \lambda_1)u_1 + u_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \end{bmatrix}.$$

II. (II.1) Assume that $D = 0$ and $\alpha = 0$, then $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda \in \mathbb{R} \setminus \{0\}$ and

$$u_n = (A + Bn + Cn^2)\lambda^n \quad (n \geq 0).$$

Thus,

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \lambda & \lambda & \lambda \\ \lambda^2 & 2\lambda^2 & 4\lambda^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix},$$

and

$$\begin{vmatrix} 1 & 0 & 0 \\ \lambda & \lambda & \lambda \\ \lambda^2 & 2\lambda^2 & 4\lambda^2 \end{vmatrix} = 2\lambda^3.$$

Therefore

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} u_0 \\ \frac{-3u_0}{2} + \frac{2u_1}{\lambda} - \frac{u_2}{2\lambda^2} \\ \frac{u_0}{2} - \frac{u_1}{\lambda} + \frac{u_2}{2\lambda^2} \end{bmatrix}.$$

(II.2) Assume that $D = 0$ and $\alpha \neq 0$, then $p(x)$ has two distinct real roots $\lambda_1(\neq 0)$ of multiplicity 1, $\lambda_2 = \lambda_3(\neq 0)$ of multiplicity 2, and

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (n \geq 0).$$

Thus,

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda^2 & \lambda_2^2 & 2\lambda_2^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix},$$

and

$$\begin{vmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda^2 & \lambda_2^2 & 2\lambda_2^2 \end{vmatrix} = \lambda_2(\lambda_2 - \lambda_1)^2.$$

Therefore

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{\lambda_2^2 u_0 - 2\lambda_2 u_1 + u_2}{(\lambda_2 - \lambda_1)^2} \\ \frac{-\lambda_1(2\lambda_2 - \lambda_1)u_0 + 2\lambda_2 u_1 - u_2}{(\lambda_2 - \lambda_1)^2} \\ \frac{\lambda_1\lambda_2 u_0 - (\lambda_2 + \lambda_1)u_1 + u_2}{\lambda_2(\lambda_2 - \lambda_1)} \end{bmatrix}.$$

III. Assume $D < 0$, then $p(x)$ has one real root $\lambda_1(\neq 0)$, two complex conjugate roots $\lambda_2, \lambda_3 = \bar{\lambda}_2 \in \mathbb{C} \setminus \mathbb{R}$ and

$$u_n = A\lambda_1^n + B\lambda_2^n + C\bar{\lambda}_2^n \quad (n \geq 0).$$

Thus,

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \bar{\lambda_2} \\ \lambda_1^2 & \lambda_2^2 & \bar{\lambda_2}^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix},$$

and

$$\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \bar{\lambda_2} \\ \lambda_1^2 & \lambda_2^2 & \bar{\lambda_2}^2 \end{vmatrix} = (\lambda_2 - \lambda_1)(\bar{\lambda_2} - \lambda_1)(\bar{\lambda_2} - \lambda_2).$$

Therefore

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{\lambda_2\bar{\lambda_2}u_0 - (\bar{\lambda_2}+\lambda_2)u_1 + u_2}{(\lambda_2-\lambda_1)(\bar{\lambda_2}-\lambda_1)} \\ \frac{-\lambda_1\bar{\lambda_2}u_0 + (\bar{\lambda_2}+\lambda_1)u_1 - u_2}{(\lambda_2-\lambda_1)(\bar{\lambda_2}-\lambda_2)} \\ \frac{\lambda_1\lambda_2 u_0 - (\lambda_2+\lambda_1)u_1 + u_2}{(\bar{\lambda_2}-\lambda_1)(\bar{\lambda_2}-\lambda_2)} \end{bmatrix},$$

so $B = \bar{C}$. □

We treat each of the three cases according to $D > 0,\ D = 0,\ D < 0$ separately in the remaining sections. In order to facilitate the flow of the proof of the main result we single out two auxiliary lemmas.

**Lemma 6.** *Let $a, b$ be positive real numbers belonging to the interval $(0, 1)$ and let $\mathcal{B}$ and $\mathcal{C}$ be two positive real numbers. Consider the following three exponential polynomials with argument $n \in \mathbb{N} \cup \{0\}$,*

$$P(n) = a^n \left(\mathcal{B} - \mathcal{C}b^n\right), \quad Q(n) = a^n \left(\mathcal{B} + \mathcal{C}b^n\right), \quad R(n) = a^n \left(-\mathcal{B} + \mathcal{C}b^n\right).$$

*The following conclusions hold:*
***(Type F-Z)*** *there are explicitly computable integers $N_0 \in \mathbb{N} \cup \{0\}$ and $N_1 \in \mathbb{N}$ such that*

$$\sup\{P(n)\ ;\ n \geq 0\} = \max\{P(n)\ ;\ N_0 \leq n \leq N_1\};$$

***(Type F+Z)*** $\qquad \sup\{Q(n)\ ;\ n \geq 0\} = Q(0)\ ;$
***(Type -F+Z)*** *there is an explicitly computable integer $N_2 \in \mathbb{N} \cup \{0\}$ such that*

$$\sup\{R(n)\ ;\ n \geq 0\} = \max\{0, R(n)\ ;\ 0 \leq n \leq N_2\}.$$

*Proof.* (Type F-Z): Since $b^n \downarrow 0$ $(n \to \infty)$, there exists $N_0 \in \mathbb{N} \cup \{0\}$ such that

$$\mathcal{B} - \mathcal{C}b^i \leq 0 < \mathcal{B} - \mathcal{C}b^n \quad (n \geq N_0,\ 0 \leq i < N_0)$$

so that $P(n) > 0$ for all $n \geq N_0$ and $P(n) \leq 0$ for $0 \leq n < N_0$. Since $P(n) \to 0$ $(n \to \infty)$, there exists a positive integer $N_1 > N_0$ such that $P(n) < P(N_0)$ for all $n > N_1$, and the desired result follows.

(Type F+Z): Since $Q(n) > 0$ and $Q(n) \downarrow 0$ $(n \to \infty)$, the conclusion is immediate in this case.

(Type -F+Z): Since $b^n \downarrow 0$ $(n \to \infty)$, there exists $N_2 \in \mathbb{N} \cup \{0\}$ such that $-\mathcal{B}+\mathcal{C}b^n < 0$ for all $n > N_2$ and the desired result follows at once noting that the 0 on the right-hand maximal value is to take care of the situation when $R(n) \leq 0$ for all $n \geq 0$. □

**Lemma 7.** *Let $\varphi, \theta \in [-\pi, \pi)$ with $\theta \notin \{-\pi, 0\}$.*

*I. If $\theta$ is a rational multiple of $\pi$, then $\cos(\varphi + n\theta)$ is periodic and takes only finitely many explicitly computable values as $n$ varies over $\mathbb{N} \cup \{0\}$.*

*II. If $\theta$ is not a rational multiple of $\pi$, then as $n$ varies over $\mathbb{N} \cup \{0\}$ the range of values of $\cos(\varphi + n\theta)$ is dense in $[-1, 1]$.*

*Proof.* I. Assume that $\theta$ is a rational multiple of $\pi$, say $\theta = s\pi/t$, where $s, t(> 0) \in \mathbb{Z} \setminus \{0\}$ and $\gcd(s,t) = 1$. Since the cosine function is periodic of period $2\pi$, it is easily checked that $\cos(\varphi + n\theta) = \cos(\varphi + ns\pi/t)$ takes at most $2t$ distinct values corresponding to $n = 0, 1, 2, \ldots, 2t - 1$.

II. Assume that $\theta$ is not a rational multiple of $\pi$, say $\theta = \vartheta\pi$, where $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$. Writing $\vartheta = [\vartheta] + \xi$ where $[\vartheta]$ denotes its integer part, and $\xi := \{\vartheta\} \in (0, 1)$ denotes its fractional part which must be irrational, for even $n = 2k$ $(k \in \mathbb{N} \cup \{0\})$ we have

$$\cos(\varphi + n\theta) = \cos(\varphi + 2k\vartheta\pi) = \cos(\varphi + 2k\xi\pi) = \cos\left(\varphi + \{k\xi\}\, 2\pi\right). \qquad (3)$$

Since $\xi$ is irrational, by the Kronecker's approximation theorem, we know the set $\{\{k\xi\}\ ;\ k \in \mathbb{N} \cup \{0\}\}$ is dense in [0,1]. Consequently, the set $\{\{k\xi\}\, 2\pi\ ;\ n \in \mathbb{N}\}$ is dense in $[0, 2\pi]$, implying that the range of values of $\cos\left(\varphi + \{k\xi\}\, 2\pi\right)$ $(k \in \mathbb{N} \cup \{0\}$ is dense in $[-1, 1]$. □

## Location and Duration of Research

Location Department of Mathematics, Kasetsart University.

Duration of Research September 2008-July 2009.

# RESULTS AND DISCUSSION

**Lemma 8.** *Assume that*

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (n \geq 0),$$

*where* $\lambda_1, \lambda_2, \lambda_3$ *are distinct nonzero real numbers. Then the Positivity Problem of the sequence* $(u_n)_{n=0}^{\infty}$ *can be effectively solved.*

*Proof.* There are two possibilities depending on whether there are two $\lambda_i$'s having the same absolute value.

1.There are two roots $\lambda_i, \lambda_j \ \ (i, j \in \{1, 2, 3\}\,,\ i \neq j)$ such that $|\lambda_i| = |\lambda_j|$.
Without loss of generality, let the two roots be $\lambda_1 > 0$ and $\lambda_2 = -\lambda_1$. Thus,

$$u_n = \{A + (-1)^n B\}\, \lambda_1^n + C\lambda_3^n \quad (n \geq 0).$$

Since $\lambda_3 \neq \lambda_1$, we subdivide into two further subcases depending on whether $\lambda_1 > |\lambda_3|$.

1.1. $\lambda_1 > |\lambda_3| > 0$.

Rewrite the general term of the sequence as

$$u_n = \lambda_1^n \left\{A + (-1)^n B + C\,(\lambda_3/\lambda_1)^n\right\} \quad (n \geq 0).$$

We consider two possibilities corresponding to the signs of $\lambda_3$.

- $\lambda_3 < 0$.

  For $k \in \mathbb{N} \cup \{0\}$, we have

$$u_n = \begin{cases} \lambda_1^n \left(A + B + C\left(|\lambda_3|/\lambda_1\right)^n\right) & ; \quad n \ \text{ even} \\ \lambda_1^n \left(A - B - C\left(|\lambda_3|/\lambda_1\right)^n\right) & ; \quad n \ \text{ odd.} \end{cases}$$

  If $C \geq 0$, then the sequence $(u_n)$ is nonnegative if and only if

$$A \geq \max\{-B, B + C\,|\lambda_3|/\lambda_1\}.$$

  If $C < 0$, then the sequence $(u_n)$ is nonnegative if and only if

$$A \geq \max\{B, -B + |C|\}.$$

- $\lambda_3 > 0$.

  For $k \in \mathbb{N} \cup \{0\}$, we have

$$u_n = \begin{cases} \lambda_1^n \left(A + B + C\left(\lambda_3/\lambda_1\right)^n\right) & ; \quad n \ \text{ even} \\ \lambda_1^n \left(A - B + C\left(\lambda_3/\lambda_1\right)^n\right) & ; \quad n \ \text{ odd.} \end{cases}$$

  If $C \geq 0$, then the sequence $(u_n)$ is nonnegative if and only if $A \geq |B|$.

  If $C < 0$, then the sequence $(u_n)$ is nonnegative if and only if

$$A \geq \max\{-B + |C|\,, B + |C|\,\lambda_3/\lambda_1\}.$$

1.2. $\lambda_1 < |\lambda_3|$.

Rewriting the general term of the sequence as

$$u_n = \lambda_3^n \left\{(A + (-1)^n B)\,(\lambda_1/\lambda_3)^n + C\right\} \quad (n \geq 0),$$

we have two choices according to the sign of $\lambda_3$.

- $\lambda_3 < 0$.

  For $k \in \mathbb{N} \cup \{0\}$, we have

$$u_n = \begin{cases} |\lambda_3^n|\left((A + B)\,|\lambda_1/\lambda_3|^n + C\right) & ; \quad n \ \text{ even} \\ -\,|\lambda_3^n|\left((B - A)\,|\lambda_1/\lambda_3|^n + C\right) & ; \quad n \ \text{ odd.} \end{cases}$$

  The sequence $(u_n)$ is nonnegative if and only if, for each $k \geq 0$,

$$(A + B)\,|\lambda_1/\lambda_3|^{2k} \geq -C \text{ and } (B - A)\,|\lambda_1/\lambda_3|^{2k+1} \leq -C$$
$$\Longleftrightarrow C = 0,\ B < A,\ A + B > 0.$$

- $\lambda_3 > 0$.

  For $k \in \mathbb{N} \cup \{0\}$, we have

$$u_n = \begin{cases} \lambda_3^n \left((A + B)\,(\lambda_1/\lambda_3)^n + C\right) & ; \quad n \ \text{ even} \\ \lambda_3^n \left((A - B)\,(\lambda_1/\lambda_3)^n + C\right) & ; \quad n \ \text{ odd.} \end{cases}$$

  The sequence $(u_n)$ is nonnegative if and only if, for each $k \geq 0$,

$$C \geq -(A + B)\,(\lambda_1/\lambda_3)^{2k} \ \text{ and } \ C \geq -(A - B)\,(\lambda_1/\lambda_3)^{2k+1}. \tag{4}$$

  If $A \geq |B|$, then (4) holds if and only if $C \geq 0$. If $A < |B|$, then (4) holds if and only if $\begin{cases} C \geq -(A - B)\,(\lambda_1/\lambda_3) & \text{provided } B > 0 \\ C \geq -(A + B) & \text{provided } B \leq 0. \end{cases}$

2. All three roots $\lambda_1, \lambda_2$ and $\lambda_3$ have different absolute values.
Without loss of generality, assume $|\lambda_1| > |\lambda_2| > |\lambda_3| > 0$. Here,

$$u_n = \lambda_1^n \{A + B(\lambda_2/\lambda_1)^n + C(\lambda_3/\lambda_1)^n\} \quad (n \geq 0). \tag{5}$$

We assume that $A, B$ and $C$ *do not vanish simultaneously*, for otherwise $u_n \equiv 0$. We treat two separate subcases depending on the sign of $\lambda_1$.

2.1. $\lambda_1 < 0$.

Note that $\lambda_1^n$ has alternating signs. Since $(\lambda_2/\lambda_1)^n$ and $(\lambda_3/\lambda_1)^n \to 0$ $(n \to \infty)$, for $n$ sufficiently large, $u_n \geq 0$ is possible only when $A = 0$. Thus,

$$u_n = B\lambda_2^n + C\lambda_3^n = \lambda_2^n \{B + C(\lambda_3/\lambda_2)^n\}.$$

This particular case thus reduces to a sequence satisfying a second order linear recurrence and by the result in Positivity of the second order linear recurrent sequences ( Halava *et al*, 2006) it is decidable.

2.2. $\lambda_1 > 0$.

- If $B = C = 0$, then $u_n = A\lambda_1^n \geq 0$ for all $n \geq 0$ if and only if $A \geq 0$.
- If $B = 0$, then for $C > 0$ the sequence $(u_n)$ is nonnegative if and only if

  $$A \geq -C(\lambda_3/\lambda_1)^n \quad (n \geq 0) \Longleftrightarrow \begin{cases} A/C \geq 0 & \text{provided } \lambda_3 > 0 \\ A/C \geq |\lambda_3/\lambda_1| & \text{provided } \lambda_3 < 0. \end{cases}$$

  For $C < 0$, the sequence $(u_n)$ is nonnegative if and only if

  $$A \geq -C(\lambda_3/\lambda_1)^n \quad (n \geq 0) \Longleftrightarrow A \geq |C|.$$

- If $C = 0$, then for $B > 0$, the sequence $(u_n)$ is nonnegative if and only if

  $$A \geq -B(\lambda_2/\lambda_1)^n \quad (n \geq 0) \Longleftrightarrow \begin{cases} A/B \geq 0 & \text{provided } \lambda_2 > 0 \\ A/B \geq |\lambda_2/\lambda_1| & \text{provided } \lambda_2 < 0. \end{cases}$$

  For $B < 0$, the sequence $(u_n)$ is nonnegative if and only if

  $$A \geq -B(\lambda_2/\lambda_1)^n \quad (n \geq 0) \Longleftrightarrow A \geq |B|.$$

- If $B < 0, C < 0$, then the sequence $(u_n)$ is nonnegative if and only if $A \geq |B| + |C|$.

- If $B<0, C>0$, then the sequence $(u_n)$ is nonnegative if and only if

$$A \geq |B|\,(\lambda_2/\lambda_1)^n - |C|\,(\lambda_3/\lambda_1)^n \quad (n \geq 0).$$

•• For $\lambda_2>0, \lambda_3>0$, consider the exponential polynomial

$$P_1(n) := (\lambda_2/\lambda_1)^n \left\{|B| - |C|\,(\lambda_3/\lambda_2)^n\right\} \;\; (n \geq 0).$$

This exponential polynomial is of the type F-Z, and so Lemma 6 shows that $(u_n)$ is nonnegative if and only if

$$\begin{aligned} A &\geq \max\left\{P_1(n); N_0 \leq n \leq N_1\right\} \\ &= \max\left\{|B|\,(\lambda_2/\lambda_1)^n - |C|\,(\lambda_3/\lambda_1)^n\,; N_0 \leq n \leq N_1\right\} > 0, \end{aligned}$$

for some computable integers $N_1 > N_0 \in \mathbb{N} \cup \{0\}$.

•• For $\lambda_2>0, \lambda_3<0$, we have

$$|B|\,(\lambda_2/\lambda_1)^n - |C|\,(\lambda_3/\lambda_1)^n = \begin{cases} |B|\,(\lambda_2/\lambda_1)^n - |C|\,(|\lambda_3|\,/\lambda_1)^n & ;\; n \text{ even} \\ |B|\,(\lambda_2/\lambda_1)^n + |C|\,(|\lambda_3|\,/\lambda_1)^n & ;\; n \text{ odd.} \end{cases}$$

For even $n=2k$, since the exponential polynomial

$$P_2(k) := (\lambda_2/\lambda_1)^{2k}\{|B| - |C|\,(|\lambda_3|\,/\lambda_2)^{2k}\} \;\; (k \geq 0)$$

is of Type F-Z, by Lemma 6 there are computable integers $K_1 > K_0 \in \mathbb{N} \cup \{0\}$ such that

$$\sup\left\{P_2(k); k \geq 0\right\} = \max\left\{P_2(k); K_0 \leq k \leq K_1\right\}.$$

Next consider $n = 2k+1$, the exponential polynomial

$$P_3(k) := (\lambda_2/\lambda_1)^{2k+1}\{|B| + |C|\,(|\lambda_3|\,/\lambda_2)^{2k+1}\} \;\; (k \geq 0)$$

is of Type F+Z. Thus, by Lemma 6 $(u_n)$ is nonnegative if and only if

$$A \geq \max\left\{\; P_2(k_1), P_3(0)\;;\; K_0 \leq k_1 \leq K_1\right\}.$$

•• For $\lambda_2<0, \lambda_3>0$, we have

$$|B|\,(\lambda_2/\lambda_1)^n - |C|\,(\lambda_3/\lambda_1)^n = \begin{cases} |B|\,(|\lambda_2|\,/\lambda_1)^n - |C|\,(\lambda_3/\lambda_1)^n & ;\; n \text{ even} \\ -\,|B|\,(|\lambda_2|\,/\lambda_1)^n - |C|\,(\lambda_3/\lambda_1)^n & ;\; n \text{ odd.} \end{cases}$$

The case of odd $n = 2k+1$ can be ignored as the terms are negative. For even $n = 2k$, whose exponential polynomial is of Type F-Z, by Lemma 6 there are computable integers $K_4 > K_3 \in \mathbb{N} \cup \{0\}$ so that $(u_n)$ is nonnegative if and only if

$$A \geq \max\left\{ |B| \left( |\lambda_2| / \lambda_1 \right)^{2k} - |C| \left( \lambda_3 / \lambda_1 \right)^{2k}; K_3 \leq k \leq K_4 \right\}.$$

•• For $\lambda_2 < 0, \lambda_3 < 0$, we have

$$|B| \left( \lambda_2 / \lambda_1 \right)^n - |C| \left( \lambda_3 / \lambda_1 \right)^n = \begin{cases} |B| \left( |\lambda_2| / \lambda_1 \right)^n - |C| \left( |\lambda_3| / \lambda_1 \right)^n & ; \; n \text{ even} \\ - |B| \left( |\lambda_2| / \lambda_1 \right)^n + |C| \left( |\lambda_3| / \lambda_1 \right)^n & ; \; n \text{ odd.} \end{cases}$$

In the case of even $n = 2k$, the exponential polynomial

$$P_4(k) := \left( |\lambda_2| / \lambda_1 \right)^{2k} \{ |B| - |C| \left( |\lambda_3| / |\lambda_2| \right)^{2k} \} \quad (k \geq 0)$$

is of Type F-Z, and so by Lemma 6 there are computable integers $K_6 > K_5 \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{ P_4(k); k \geq 0 \} = \max \{ P_4(k); K_5 \leq k \leq K_6 \} > 0.$$

Next for the case of odd $n = 2k+1$, the exponential polynomial

$$P_5(k) := \left( |\lambda_2| / \lambda_1 \right)^{2k+1} \{ - |B| + |C| \left( |\lambda_3| / |\lambda_2| \right)^{2k+1} \} \quad (k \geq 0)$$

is of Type -F+Z. By Lemma 6, there exists $K_7 \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{ P_5(k); \; k \geq 0 \} = \max \{ 0, P_5(k) \; ; \; 0 \leq k \leq K_7 \}. \tag{6}$$

Thus, $(u_n)$ is nonnegative if and only if

$$A \geq \max \{ \; P_4(k_1), P_5(k_2) \; ; \; K_5 \leq k_1 \leq K_6, \; 0 \leq k_2 \leq K_7 \}.$$

• If $B > 0, C < 0$ the sequence $(u_n)$ is nonnegative if and only if

$$A \geq - |B| \left( \lambda_2 / \lambda_1 \right)^n + |C| \left( \lambda_3 / \lambda_1 \right)^n.$$

•• For $\lambda_2 > 0, \lambda_3 > 0$, the exponential polynomial

$$P_6(n) = \left( \lambda_2 / \lambda_1 \right)^n \{ - |B| + |C| \left( \lambda_3 / \lambda_2 \right)^n \} \quad (n \geq 0)$$

is of Type -F+Z, and so by Lemma 6, there is a computable integer $K_8 \in \mathbb{N} \cup \{0\}$ such that $(u_n)$ is nonnegative if and only if

$$A \geq \max\{0, P_6(n)\ ;\ 0 \leq n \leq K_8\}.$$

•• For $\lambda_2 > 0, \lambda_3 < 0$, we have

$$-|B|(\lambda_2/\lambda_1)^n + |C|(\lambda_3/\lambda_1)^n = \begin{cases} -|B|(\lambda_2/\lambda_1)^n + |C|(|\lambda_3|/\lambda_1)^n & ;\ n \text{ even} \\ -|B|(\lambda_2/\lambda_1)^n - |C|(|\lambda_3|/\lambda_1)^n & ;\ n \text{ odd.} \end{cases}$$

The case of odd $n = 2k+1$ is ignored because of negative terms. In the case of even $n = 2k$, the corresponding exponential polynomial is of Type -F+Z and so by Lemma 6, $(u_n)$ is nonnegative if and only if

$$A \geq \max\left\{0, -|B|(\lambda_2/\lambda_1)^{2k} + |C|(|\lambda_3|/\lambda_1)^{2k}\ ;\ 0 \leq k \leq K_9\right\},$$

where $K_9 \in \mathbb{N} \cup \{0\}$ is a computable integer.

•• For $\lambda_2 < 0, \lambda_3 > 0$, we have

$$-|B|(\lambda_2/\lambda_1)^n + |C|(\lambda_3/\lambda_1)^n = \begin{cases} -|B|(|\lambda_2|/\lambda_1)^n + |C|(\lambda_3/\lambda_1)^n & ;\ n \text{ even} \\ |B|(|\lambda_2|/\lambda_1)^n + |C|(\lambda_3/\lambda_1)^n & ;\ n \text{ odd.} \end{cases}$$

In the case of even $n = 2k$, the exponential polynomial

$$P_7(k) := (|\lambda_2|/\lambda_1)^{2k}\{-|B| + |C|(\lambda_3/|\lambda_2|)^{2k}\}\quad (k \geq 0)$$

is of Type -F+Z and so by Lemma 6, there is a computable integer $K_{10} \in \mathbb{N} \cup \{0\}$ such that

$$\sup\{P_7(k); k \geq 0\} = \max\{0, P_7(k); 0 \leq k \leq K_{10}\}.$$

In the case of odd $n = 2k+1$, the exponential polynomial

$$P_8(k) := (|\lambda_2|/\lambda_1)^{2k+1}\{|B| + |C|(\lambda_3/|\lambda_2|)^{2k+1}\}\quad (k \geq 0)$$

is of Type F+Z, and so by Lemma 6

$$\sup\{P_8(k); k \geq 0\} = P_8(0) = (|\lambda_2|/\lambda_1)\{|B| + |C|(\lambda_3/|\lambda_2|) > 0.$$

Consequently, the sequence $(u_n)$ is nonnegative if and only if

$$A \geq \max\{\, P_7(k_1), P_8(0)\ ;\ 0 \leq k_1 \leq K_{10}\}\,.$$

•• For $\lambda_2 < 0, \lambda_3 < 0$, we have

$$-|B|\,(\lambda_2/\lambda_1)^n + |C|\,(\lambda_3/\lambda_1)^n = \begin{cases} -|B|\,(|\lambda_2|\,/\lambda_1)^n + |C|\,(|\lambda_3|\,/\lambda_1)^n & ;\ n \text{ even} \\ |B|\,(|\lambda_2|\,/\lambda_1)^n - |C|\,(|\lambda_3|\,/\lambda_1)^n & ;\ n \text{ odd.} \end{cases}$$

For even $n = 2k$, the exponential polynomial

$$P_9(k) := (|\lambda_2|\,/\lambda_1)^{2k}\,\{-|B| + |C|\,(|\lambda_3|\,/\,|\lambda_2|)^{2k}\}\quad (k \geq 0)$$

is of Type -F+Z, and so by Lemma 6, there is a computable integer $K_{12} \in \mathbb{N} \cup \{0\}$ such that

$$\sup\{P_9(k); k \geq 0\} = \max\{0, P_9(k); 0 \leq k \leq K_{12}\}\,.$$

For odd $n$, the exponential polynomial

$$P_{10}(k) := (|\lambda_2|\,/\lambda_1)^{2k+1}\,\{|B| - |C|\,(|\lambda_3|\,/\,|\lambda_2|)^{2k+1}\}\quad (k \geq 0)$$

is of Type F-Z and so by Lemma 6, there are computable integers $K_{14} > K_{13} \in \mathbb{N} \cup \{0\}$ such that

$$\sup\{P_{10}(k); k \geq 0\} = \max\{P_{10}(k); K_{13} \leq k \leq K_{14}\} > 0.$$

The sequence $(u_n)$ is nonnegative if and only if

$$A \geq \max\{\, P_9(k_1), P_{10}(k_2)\ ;\ 0 \leq k_1 \leq K_{12},\ K_{13} \leq k_2 \leq K_{14}\}\,.$$

- If $B > 0, C > 0$ the sequence $(u_n)$ is nonnegative if and only if

$$A \geq -B\,(\lambda_2/\lambda_1)^n - C\,(\lambda_3/\lambda_1)^n \quad (n \geq 0). \tag{7}$$

•• If $\lambda_2 > 0$ and $\lambda_3 > 0$, then (7) holds if and only if $A \geq 0$.

•• If $\lambda_2 > 0$ and $\lambda_3 < 0$, then

$$-B\,(\lambda_2/\lambda_1)^n - C\,(\lambda_3/\lambda_1)^n = \begin{cases} -B(\lambda_2/\lambda_1)^n - C\,(|\lambda_3|\,/\lambda_1)^n & n \text{ even} \\ -B(\lambda_2/\lambda_1)^n + C\,(|\lambda_3|\,/\lambda_1)^n & n \text{ odd.} \end{cases}$$

The case of even $n$ is ignored because of negative terms. For odd $n = 2k+1$, this is of Type -F+Z, and so by Lemma 6, there is a computable integer $K_{15} \in \mathbb{N} \cup \{0\}$ such that $(u_n)$ is nonnegative if and only if

$$A \geq \max\left\{0, -B\left(\lambda_2/\lambda_1\right)^{2k+1} + C\left(\left|\lambda_3\right|/\lambda_1\right)^{2k+1} \ ; \ 0 \leq k \leq K_{15}\right\}.$$

•• For $\lambda_2 < 0, \lambda_3 > 0$, then

$$-B\left(\lambda_2/\lambda_1\right)^n - C\left(\lambda_3/\lambda_1\right)^n \begin{cases} -B(\left|\lambda_2\right|/\lambda_1)^n - C\left(\lambda_3/\lambda_1\right)^n & ; \ n \text{ even} \\ B(\left|\lambda_2\right|/\lambda_1)^n - C\left(\lambda_3/\lambda_1\right)^n & ; \ n \text{ odd.} \end{cases}$$

The case of even $n$ is ignored because of negative terms. For odd $n = 2k+1$, this is of Type F-Z, and so by Lemma 6, there are computable integers $K_{17} > K_{16} \in \mathbb{N} \cup \{0\}$ such that $(u_n)$ is nonnegative if and only if

$$A \geq \max\left\{B\left(\left|\lambda_2\right|/\lambda_1\right)^{2k+1} - C\left(\lambda_3/\lambda_1\right)^{2k+1} \ ; \ K_{16} \leq k \leq K_{17}\right\}.$$

•• For $\lambda_2 < 0, \lambda_3 < 0$, then

$$-B\left(\lambda_2/\lambda_1\right)^n - C\left(\lambda_3/\lambda_1\right)^n = \begin{cases} -B(\left|\lambda_2\right|/\lambda_1)^n - C\left(\left|\lambda_3\right|/\lambda_1\right)^n & ; \ n \text{ even} \\ B(\left|\lambda_2\right|/\lambda_1)^n + C\left(\left|\lambda_3\right|/\lambda_1\right)^n & ; \ n \text{ odd.} \end{cases}$$

The case of even $n$ is ignored because of negative terms. For odd $n$, this is of Type F+Z, and so by Lemma 6, the sequence $(u_n)$ is nonnegative if and only if

$$A \geq B\left(\left|\lambda_2\right|/\lambda_1\right) + C\left(\left|\lambda_3\right|/\lambda_1\right).$$

□

**Lemma 9.** *Assume that*

$$u_n = (A + Bn + Cn^2)\lambda^n \quad (n \geq 0),$$

*where* $\lambda \in \mathbb{R} \setminus \{0\}$. *Then the Positivity Problem of the sequence* $(u_n)_{n=0}^{\infty}$ *can be effectively solved.*

*Proof.* We have two subcases depending on the sign of $\lambda$.

1. $\lambda < 0$.

The sequence $(u_n)$ is nonnegative if and only if, for each $n$, either $A+Bn+Cn^2 = 0$

or $\mathrm{sign}(A + Bn + Cn^2) = \mathrm{sign}(\lambda^n)$. Since $\mathrm{sign}(A + Bn + Cn^2)$ changes at most twice, the sequence $(u_n)$ is nonnegative if and only if $A + Bn + Cn^2 \equiv 0$, i.e., if and only if $A = B = C = 0$.

2. $\lambda > 0$.

The sequence $(u_n)$ is nonnegative if and only if $A + Bn + Cn^2 \geq 0$ for all $n \geq 0$. Since $A + Bn + Cn^2$ is a quadratic polynomial in $n$, the sequence $(u_n)$ is nonnegative if and only if $C \geq 0$ and either

- the quadratic polynomial $A+Bn+Cn^2$ has no real root, which is equivalent to $B^2 - 4AC < 0$, or
- the quadratic polynomial $A+Bn+Cn^2$ has two (possibly equal) real roots which is equivalent to $B^2 - 4AC \geq 0$, and there are no nonnegative integers in the open interval between the two roots $r_1 = (-B + \sqrt{B^2 - 4AC})/2C$ and $r_2 = (-B - \sqrt{B^2 - 4AC})/2C$.

□

**Lemma 10.** *Assume that*

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (n \geq 0),$$

*where* $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, $\lambda_1 \neq \lambda_2$. *Then the Positivity Problem of the sequence* $(u_n)_{n=0}^{\infty}$ *can be effectively solved.*

*Proof.* We distinguish three possibilities according to the absolute values of the two roots.

1. $|\lambda_1| > |\lambda_2| > 0$.

Rewriting

$$u_n = \lambda_1^n \{A + (B + Cn) (\lambda_2/\lambda_1)^n\},$$

we further subdivide into two subcases depending on the sign of $\lambda_1$.

- If $\lambda_1 < 0$, since $(B + Cn) (\lambda_2/\lambda_1)^n \to 0 \;\; (n \to \infty)$ and $\mathrm{sign}(\lambda_1^n)$ oscillates, the sequence $(u_n)$ is nonnegative only when $A = 0$, which yields $u_n = (B + Cn)\lambda_2^n$. This is exactly the same as Lemma 4 of Positivity of the second order linear recurrent sequences ( Halava *et al*, 2006), which has already been shown to be decidable.

- If $\lambda_1 > 0$, since sign$\{A + (B + Cn)\,(\lambda_2/\lambda_1)^n\}$= sign$(A)$ when $n$ is large, for the sequence $(u_n)$ to be nonnegative we must have $A \geq 0$ and

$$A \geq -(B + Cn)\,(\lambda_2/\lambda_1)^n \quad (n \geq 0). \tag{8}$$

Clearly, there is a computable integer $T \in \mathbb{N} \cup \{0\}$ such that (8) holds if and only if $A \geq \max\{-(B + Cn)\,(\lambda_2/\lambda_1)^n\ ;\ 0 \leq n \leq T\}$. Consequently, the sequence $(u_n)$ is nonnegative if and only if

$$A \geq \max\{0, -(B + Cn)\,(\lambda_2/\lambda_1)^n\ ;\ 0 \leq n \leq T\}.$$

2. $|\lambda_1| < |\lambda_2|$.
Write

$$u_n = \lambda_2^n\,\{A\,(\lambda_1/\lambda_2)^n + (B + Cn)\}.$$

We distinguish two subcases according to the sign of $\lambda_2$.

- $\lambda_2 < 0$.
Since sign$(A\,(\lambda_1/\lambda_2)^n + (B + Cn))$ = sign$(C)$ when $n$ is large enough, and sign$(\lambda_2^n)$ oscillates, the sequence $(u_n)$ is nonnegative only when $C = 0$, and so $u_n = \lambda_2^n\,(A\,(\lambda_1/\lambda_2)^n + B)$. Since sign$(A\,(\lambda_1/\lambda_2)^n + B)$ = sign$(B)$ when $n$ is sufficiently large, we conclude that the sequence $(u_n)$ is nonnegative only when $B = 0$ and so $u_n = A\lambda_1^n$. Hence, the sequence $(u_n)$ is nonnegative if and only if $A \geq 0$ and $\lambda_1 > 0$.

- $\lambda_2 > 0$.
The sequence $(u_n)$ is nonnegative if and only if

$$B \geq R(n) := -A\,(\lambda_1/\lambda_2)^n - Cn \quad (n \geq 0). \tag{9}$$

It is now a matter of finding $\sup_{n \geq 0} R(n)$. Clearly, we can rule out the situation where $C < 0$ because no real number $B$ satisfies (9) for all $n \geq 0$.
•• If $C = 0$, then $R(n) = -A\,(\lambda_1/\lambda_2)^n \to 0 \quad (n \to \infty)$ and so there is a computable integer $T_2 \in \mathbb{N} \cup \{0\}$ such that $\sup\{R(n); n \geq 0\} = \max\{0, R(n); 0 \leq n \leq T_2\}$; the 0 on the right-hand maximum is to take care of the case when $R(n) \leq 0$ for all $n \geq 0$. Thus, (9) holds if and only if

$$B \geq \max\{0, R(n)\ ;\ 0 \leq n \leq T_2\}.$$

•• If $C > 0$, since $R(n) \to -\infty \quad (n \to \infty)$, there is a computable integer $T_3 \in \mathbb{N} \cup \{0\}$ such that $\max\{R(n); n \geq 0\} = \max\{R(n); 0 \leq n \leq T_3\}$ and so (9) holds if and only if

$$B \geq \max\{R(n); 0 \leq n \leq T_3\}.$$

3. $|\lambda_1| = |\lambda_2|$.

- $\lambda_1 = -\lambda_2 > 0$. The general term of the sequence is

$$\begin{aligned} u_n &= \{A + (-1)^n(B + Cn)\}\,\lambda_1^n \\ &= \begin{cases} \{A + (B + 2Ck)\}\,\lambda_1^{2k} & \text{if } n = 2k \\ \{A - (B + C + 2Ck)\}\,\lambda_1^{2k+1} & \text{if } n = 2k + 1. \end{cases} \end{aligned}$$

Then the sequence $(u_n)$ is nonnegative if and only if for all $k \geq 0$ we have

$$A - B - C \geq \ 2Ck \ \geq -A - B.$$

Observe that $C$ must vanish for if $C > 0$, the left-hand inequality cannnot hold for sufficiently large $k$, while for $C < 0$, the right-hand inequality is untenable for large $k$. Thus, the sequence $(u_n)$is nonnegative if and only if

$$A - B \geq 0 \geq -A - B \Longleftrightarrow A \geq |B|.$$

- $\lambda_1 = -\lambda_2 < 0$. Here,

$$\begin{aligned} u_n &= \{(-1)^n A + B + Cn\}\,\lambda_2^n \\ &= \begin{cases} (A + B + 2Ck)\lambda_2^{2k} & \text{if } n = 2k \\ (-A + B + C + 2Ck)\lambda_2^{2k+1} & \text{if } n = 2k + 1. \end{cases} \end{aligned}$$

The sequence $(u_n)$ is nonnegative if and only if for all $k \geq 0$, we must have $A + B + 2kC \geq 0$ and $-A + B + (2k+1)C \geq 0$. Now, $C \geq 0$ since both inequalities do not hold if $C < 0$ when $k$ is large enough. Thus, the two inequalities hold for all $k \geq 0$ if and only if $C \geq 0$, $A + B \geq 0$ and $C \geq A - B$.

□

**Lemma 11.** *Let*

$$u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n,$$

*where* $A, \lambda_1 \in \mathbb{R},\ B \in \mathbb{C}$ *and* $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. *Then*

*I. if* $\lambda_1 < 0$, *then* $(u_n)_{n=0}^{\infty}$ *has negative elements,*

*II. if* $\lambda_1 > 0$, *then the Positivity Problem of the sequence* $(u_n)_{n=0}^{\infty}$ *can be effectively solved.*

*Proof.* Let $\lambda_2 = |\lambda_2|e^{i\theta},\ B = |B|e^{i\varphi}$, where $\theta, \varphi \in [-\pi, \pi),\ \theta \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + 2|B||\lambda_2|^n \cos(\varphi + n\theta).$$

If $A = 0$, then $u_n = 2|B||\lambda_2|^n \cos(\varphi + n\theta)$. Since $\theta \neq 0$, when $n$ varies over $N \cup \{0\}$, by the same arguments as in the proof of Lemma 5 in Positivity of second order linear recurrent sequences ( Halava *et al*, 2006), $\cos(\varphi + n\theta)$ takes both positive and negative values implying that the sequence $(u_n)$ is nonnegative only when $B = 0$, i.e. $u_n \equiv 0$ for every $n$. *Assume henceforth that* $A \neq 0$. Our two main cases correspond to the signs of $\lambda_1$.

I. $\lambda_1 < 0$.

If $B = 0$, then $u_n = A\lambda_1^n$ oscillates between positive and negative values and so the sequence $(u_n)$ is never nonnegative. *For the rest of this subsection we assume* $B \neq 0$.

I(i). $|\lambda_1| > |\lambda_2|$. Here,

$$u_n = \lambda_1^n \left\{A + 2\,|B|\,(|\lambda_2|\,/\lambda_1)^n \cos(\varphi + n\theta)\right\}.$$

Since $\text{sign}\{A + 2\,|B|\,(|\lambda_2|\,/\lambda_1)^n \cos(\varphi + n\theta)\} = \text{sign}(A)$ when $n$ is large enough, and $\lambda_1^n$ oscillates between $\pm\,|\lambda_1|$, the sequence $(u_n)$ is nonnegative only when $A = 0$, which is not the case under consideration.

I(ii). $|\lambda_1| = |\lambda_2|$. Here,

$$u_n = |\lambda_2|^n \left\{(-1)^n A + 2\,|B| \cos(\varphi + n\theta)\right\}.$$

The sequence $(u_n)$ is nonnegative if and only if

$$-2\,|B| \cos(\varphi + 2k\theta) \leq A \leq 2\,|B| \cos(\varphi + (2k+1)\theta) \quad (k \geq 0).$$

By the same arguments as in the proof of Lemma 5 in Positivity of second order linear recurrent sequences ( Halava *et al*, 2006), as $k$ varies over the nonnegative integers, since $\theta \neq 0$, the functions $\cos(\varphi + 2k\theta)$ and $\cos(\varphi + (2k+1)\theta)$ take both positive and negative values. Thus, the two inequalities hold only when $A = B = 0$ which is not the case here.

I(iii). $|\lambda_1| < |\lambda_2|$. Here,

$$u_n = A\lambda_1^n + 2\left|B\lambda_2^n\right|\cos(\varphi + n\theta) = \left|\lambda_2^n\right|\left\{A(\lambda_1/\left|\lambda_2\right|)^n + 2\left|B\right|\cos(\varphi + n\theta)\right\}.$$

Observe that $A(\lambda_1/|\lambda_2|)^n \to 0$ $(n \to \infty)$. Next, by Lemma 7, $\cos(\varphi + n\theta)$ is either periodic or interval-filling. The periodic case occurs when $\theta = s\pi/t \ \ (s, t(>0) \in \mathbb{Z} \setminus \{0\},\ \gcd(s,t) = 1)$ is a multiple of $\pi$. Since $\theta \in [-\pi, \pi) \setminus \{-\pi, 0\}$, we have $t \geq 2$, and so in the periodic case $\cos(\varphi + n\theta)$ takes both positive and negative values. Thus, the sequence $(u_n)$ is nonnegative only when $B = 0$ yielding $u_n = A\lambda_1^n$ which is oscillating (as $\lambda_1 < 0$) and so is nonnegative only when $A = 0$, which is not tenable here.

II. $\lambda_1 > 0$ If $B = 0$, then $u_n = A\lambda_1^n$ and so the sequence $(u_n)$ is nonnegative if and only if $A \geq 0$. From now on, *we assume that* $B \neq 0$.

II(i). $|\lambda_1| > |\lambda_2|$. Here,

$$u_n = \lambda_1^n\left\{A + 2\left|B\right|\left(\left|\lambda_2\right|/\lambda_1\right)^n\cos(\varphi + n\theta)\right\}.$$

Since $\operatorname{sign}\{A + 2|B|(|\lambda_2|/\lambda_1)^n\cos(\varphi + n\theta)\} = \operatorname{sign}(A)$ when $n$ is sufficiently large, for the sequence $(u_n)$ to be nonnegative we must have $A \geq 0$ and

$$A \geq -2\left|B\right|\left(\left|\lambda_2\right|/\lambda_1\right)^n\cos(\varphi + n\theta) \quad (n \geq 0).$$

By Lemma 5 of Positivity of second order linear recurrent sequences ( Halava *et al*, 2006), there is a least $N_L \in \mathbb{N} \cup \{0\}$ such that $\cos(\varphi + N_L\theta) < 0$. Since $(|\lambda_2|/\lambda_1)^n \to 0$ $(n \to \infty)$ there is $N_M > N_L$ such that for all $n \geq N_M$ we have

$$-2\left|B\right|\left(\left|\lambda_2\right|/\lambda_1\right)^n\cos(\varphi + n\theta) < -2\left|B\right|\left(\left|\lambda_2\right|/\lambda_1\right)^{N_L}\cos(\varphi + N_L\theta).$$

Consequently, the sequence $(u_n)$ is nonnegative if and only if

$$A \geq \max\left\{-2\left|B\right|\left(\left|\lambda_2\right|/\lambda_1\right)^n\cos(\varphi + n\theta)\ ;\ N_L \leq n \leq N_M\right\}.$$

II(ii). $|\lambda_1| = |\lambda_2|$. Here,

$$u_n = \lambda_1^n\left\{A + 2\left|B\right|\cos(\varphi + n\theta)\right\}.$$

The sequence $(u_n)$ is nonnegative if and only if

$$A \geq \max\{-2|B|\cos(\varphi + n\theta)\ ;\ n \geq 0\}. \tag{10}$$

If $\theta$ is a rational multiple of $\pi$, by Lemma 7 (I), $\cos(\varphi + n\theta)$ takes on only finitely many explicitly computable values, say $C_1, \ldots, C_m$, and so (10) holds if and only if

$$A \geq \max\{-2|B|C_1, \ldots, -2|B|C_m\}.$$

If $\theta$ is not a rational multiple of $\pi$, by Lemma 7 (II), (10) holds if and only if $A \geq 2|B|$.

II(iii). $|\lambda_1| < |\lambda_2|$. Here,

$$u_n = |\lambda_2^n|\{A(\lambda_1/|\lambda_2|)^n + 2|B|\cos(\varphi + n\theta)\}.$$

The situation in this case is similar but simpler than that in Subsection 1.3, and by analogous arguments, we deduce that the sequence $(u_n)$ is never nonnegative. □

From the previous lemmas, our main theorem follows:

**Theorem 4.** *The Positivity Problem is decidable for each sequence of integers satisfying a linear third order recurrence with integer coefficients.*

# CONCLUSION

The Positivity Problem is decidable for each sequence of integers satisfying a linear third order recurrence with integer coefficients.

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# CURRICULUM VITAE

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