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FUNCTIONAL DECOMPOSITION OF STATE INDUCED C^*-MATRIX SPACES

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Abstract. A theorem of Dixmier states that each bounded linear functional f on the algebra of bounded linear operators on a separable Hilbert space is a direct sum of a trace functional g and a singular functional h, vanishing on the compact operators, such that $\|f\| = \|g\| + \|h\|$. We use elementary methods to construct, via the state space of a C^*-algebra, a Banach space of C^* matrices that contains a closed subspace on which a version of Dixmier's theorem is proved. When the C^*-algebra is taken to be the complex numbers our approach gives elementary and transparent proofs of Dixmier's theorem and the trace formula tr(AB) = tr(BA), without using the operator theoretical machineries used in the known proofs.

1. Introduction and notation

Let f be a bounded linear functional on $\mathcal{B}(\ell^2)$ (the space of bounded linear operators on the Hilbert sequence space ℓ^2). Then f defines a bounded linear functional on $\mathcal{K}(\ell^2)$, the ideal of compact operators on ℓ^2. Thus there is a trace class operator (or matrix) A_f such that $f(B) = \mathrm{tr}(A_f B)$, where tr denotes the trace function, for all $B \in \mathcal{K}(\ell^2)$ [5, p. 46, Theorem 1]. Since the trace class operators form an ideal in $\mathcal{B}(\ell^2)$ [5, p. 42, Theorem 5], the function $g(B) = \mathrm{tr}(A_f B)$ for $B \in \mathcal{B}(\ell^2)$ defines a bounded linear functional on $\mathcal{B}(\ell^2)$. The functional $h = f - g$ vanishes on $\mathcal{K}(\ell^2)$ is also known as a *singular linear functional*. Dixmier's theorem

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([2], [5, p. 50, Theorem 1]), which has also been attributed to Schatten, states that this decomposition is unique and satisfies the norm equality $\|f\| = \|g\| + \|h\|$.

As defined in the 1976 paper [1] of Alfsen and Effros, a closed subspace J of a Banach space X is an M-ideal if the annihilator $J^\perp$ is complemented as an ℓ^1 summand in the dual space $X^\#$ of X, i.e., $X^\# = J^\perp \oplus_1 E$ for some closed subspace E of $X^\#$. This theorem of Dixmier can now be restated as the compact operators form an M-ideal in $\mathcal{B}(\ell^2)$ (later it is also known as the only nontrivial one [6, 7]). See also [3]. Most spaces with known M-ideal structures are Banach algebras, mainly bounded operators on certain Banach spaces.

Since a C^*-algebra resemble the complex field in many ways, here we will use a fixed C^*-algebra $\mathcal{A}$ with identity 1 and state space $s(\mathcal{A})$, together with the pair $\mathcal{K}(\ell^2)$ and $\mathcal{B}(\ell^2)$, to build a Banach space of matrices over $\mathcal{A}$ with an M-ideal that corresponds to $\mathcal{K}(\ell^2)$. The resulting space is not a Banach algebra. When the C^*-algebra is taken to be $\mathbb{C}$, the space is exactly $\mathcal{B}(\ell^2)$. Since there is no parallel machinery available for our setting, this approach also gives elementary alternate proofs of Dixmier's theorem and the trace formula tr(AB) = tr(BA), without using the theory of trace class operators and other machineries.

Let $\mathcal{A}$ be a C^*-algebra with identity 1 and state space $s(\mathcal{A})$ (consisting of all states, i.e., bounded positive linear functionals of norm 1, on $\mathcal{A}$) with the weak* topology (as a subspace of the dual space $\mathcal{A}^\#$ of $\mathcal{A}$). For each matrix $B = \left[b_{jk}\right]$ with entries $b_{jk} \in \mathcal{A}$, and each $\psi \in s(\mathcal{A})$, denote by $\widetilde{\psi}(B)$ the complex matrix $\left[\psi(b_{jk})\right]$. Let $\mathcal{M}$ be the space of all matrices $A = \left[a_{jk}\right]$ over $\mathcal{A}$ such that (the scalar matrix)

$$\begin{aligned}&\widetilde{\varphi}(A) := \left[\varphi(a_{jk})\right] \in \mathcal{B}(\ell^2) \quad \text{for all } \varphi \in s(\mathcal{A}) \qquad \text{and}\\ &\text{the map} \quad \varphi \mapsto \widetilde{\varphi}(A) = \left[\varphi\left(a_{jk}\right)\right] \quad \text{is continuous from}\\ &s(\mathcal{A}) \text{ with the weak}^* \text{ topology to } \mathcal{B}(\ell^2) \text{ with the norm topology.}\end{aligned}$$

Thus each $A \in \mathcal{M}$ defines a continuous map, $\varphi \mapsto \widetilde{\varphi}(A)$, from $s(\mathcal{A})$ to $\mathcal{B}(\ell^2)$. Since $s(\mathcal{A})$ with the weak* topology is a compact Hausdorff space [4, p. 257], it is well known that $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$ is a Banach space with the norm

$$\|A\| = \sup_{\varphi \in s(\mathcal{A})} \|\widetilde{\varphi}(A)\|_{\mathcal{B}(\ell^2)}.$$

Each $A \in \mathcal{M}$ induces an element $\widetilde{A}$ in $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$:

$$\widetilde{A}(\varphi) = \widetilde{\varphi}(A), \qquad \varphi \in s(\mathcal{A}).$$

So $\mathcal{M}$ can be considered as a subspace of the Banach space $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$. The map $A \mapsto \widetilde{A}$ does not map $\mathcal{M}$ onto $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$, even when ℓ^2 is replaced by the one dimensional $\mathbb{C}$ and in the very simple case of $\mathcal{A} = C([0,1])$ (the algebra of continuous complex-valued functions on the interval $[0,1]$).

Example 1.1. *With $\mathcal{A} = C[0,1]$ there is a continuous map $\Psi : s(\mathcal{A}) \to \mathbb{C}$ such that there does not exist $a \in \mathcal{A}$ that satisfies $\Psi(\varphi) = \varphi(a)$ for all $\varphi \in s(\mathcal{A})$.*

Proof. Each $t \in [0,1]$ induces a state φ_t on $\mathcal{A}$: the evaluation functional $\varphi_t(a) = a(t)$ for all $a \in \mathcal{A}$. Let $a_1 \in \mathcal{A}$ be given by $a_1(t) = t$ for all $t \in [0,1]$. Let

$$\mathcal{V} = \left\{ \varphi \in s(\mathcal{A}) : \left| \varphi(a_1) - \varphi_{1/2}(a_1) \right| < \frac{1}{4} \right\},$$

a weak* neighborhood of $\varphi_{1/2}$. Since $s(\mathcal{A})$, with the relative weak* topology, being compact and Hausdorff [4, p. 257], is normal, there is a continuous map $\Psi : s(\mathcal{A}) \to \mathbb{C}$ such that $\Psi(\varphi_{1/2}) = 1$ and $\Psi(\varphi) = 0$ for all $\varphi \in s(\mathcal{A}) \setminus \mathcal{V}$. In particular $\Psi(\varphi_0) = 0$. Suppose there is an $a \in \mathcal{A}$ such that

$$\Psi(\varphi) = \varphi(a) \quad \text{for all } \varphi \in s(\mathcal{A}).$$

Then $1 = \Psi(\varphi_{1/2}) = a(1/2)$ and $0 = \Psi(\varphi_0) = a(0)$. Let $\hat{\varphi} := \frac{1}{5}\varphi_{1/2} + \frac{4}{5}\varphi_0$. Then $\hat{\varphi} \in s(\mathcal{A})$ and

$$\left| \hat{\varphi}(a_1) - \varphi_{1/2}(a_1) \right| = \left| \frac{1}{5}\varphi_{1/2}(a_1) + \frac{4}{5}\varphi_0(a_1) - \varphi_{1/2}(a_1) \right| = \frac{2}{5} > \frac{1}{4}.$$

Thus $\hat{\varphi} \in s(\mathcal{A}) \setminus \mathcal{V}$, and hence,

$$0 = \Psi(\hat{\varphi}) = \hat{\varphi}(a) = \frac{1}{5} a\,(1/2) + \frac{4}{5} a\,(0) = \frac{1}{5},$$

which is a contradiction. ☐

It will be shown in Proposition 2.1 that the image of $\mathcal{M}$ under the map $A \mapsto \widetilde{A}$ is a closed subspace of $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$, and $\mathcal{M}$ is a Banach space with the norm

$$\|A\| = \sup_{\varphi \in s(\mathcal{A})} \|\widetilde{\varphi}(A)\|_{\mathcal{B}(\ell^2)}.$$

Let $A \in \mathcal{M}$. For each $n \in \mathbb{N}$, $A_{n_\lrcorner}$ denotes the $n-$th compression matrix of A; that is, the (j,k)-th entry of $A_{n_\lrcorner}$ is exactly the same as that of A for $1 \le j,k \le n$, and is zero otherwise. Denote by $A_{\underline{n}}$ [respectively, $A_{n|}$] the matrix whose first n rows [respectively, columns] coincide with that of A and all other rows [respectively, columns] are zero. Dually, $A_{\overline{n}}$ [respectively, $A_{|n}$] is the matrix whose first n rows [respectively, columns] are zero and all other rows [respectively, columns] coincide with that of A. Denote by $\mathcal{K}$ the space of all $A \in \mathcal{M}$ with the property that

$$\left\| A - A_{\underline{n}} \right\| = \|A_{\overline{n}}\| \to 0 \quad \text{as } n \to \infty.$$

Note that this is equivalent to the compactness of A (i.e., $A \in \mathcal{K}(\ell^2)$) when $\mathcal{A}$ is the complex field $\mathbb{C}$.

We will show that the annihilator $\mathcal{K}^\perp$ of $\mathcal{K}$ behaves in the dual space $\mathcal{M}^\#$ of $\mathcal{M}$ just like $[\mathcal{K}(\ell^2)]^\perp$ in $[\mathcal{B}(\ell^2)]^\#$, as in Dixmier's theorem. That is $\mathcal{K}$ is an M-ideal in $\mathcal{M}$.

2. Preliminary results

We begin the section by showing that $\mathcal{M}$ is a Banach space.

Proposition 2.1. *$\mathcal{M}$ is a Banach space with the norm*

$$\|A\| = \sup_{\varphi\in s(\mathcal{A})} \|\widetilde{\varphi}(A)\|_{\mathcal{B}(\ell^2)} .$$

The *state norm* $\|\cdot\|_s$ on $\mathcal{A}$ is defined by

$$\|a\|_s = \sup_{\varphi\in s(\mathcal{A})} |\varphi(a)| , \qquad a \in \mathcal{A}.$$

The state norm is a norm and [8, Proposition 2.3]

$$\|a\|_s \le \|a\| \le 2\,\|a\|_s \qquad \text{for all } a \in \mathcal{A}.$$

The state norm and the C^*-norm on $\mathcal{A}$ are equivalent.

Proof. It suffices to show that the image of $\mathcal{M}$ under the map $A \mapsto \widetilde{A}$ is closed in $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$. Let $\{A_n\}$ be a sequence in $\mathcal{M}$ such that $\widetilde{A}_n \to \Psi$ for some $\Psi \in C(s(\mathcal{A}), \mathcal{B}(\ell^2))$. Let $A_n = \left[a^{(n)}_{jk}\right]$. For each $j, k \in \mathbb{N}$,

$$\begin{aligned}\left\|a^{(n)}_{jk} - a^{(m)}_{jk}\right\|_s &= \sup_{\varphi\in s(\mathcal{A})} \left|\varphi(a^{(n)}_{jk}) - \varphi(a^{(m)}_{jk})\right| \\ &\le \sup_{\varphi\in s(\mathcal{A})} \|\widetilde{\varphi}(A_n) - \widetilde{\varphi}(A_m)\|_{\mathcal{B}(\ell^2)} \to 0\end{aligned}$$

as $n, m \to \infty$. By the equivalence of the state norm and the norm on $\mathcal{A}$, the sequence $\left\{a^{(n)}_{jk}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}$. Thus there is an $a_{jk} \in \mathcal{A}$ such that $\left\|a^{(n)}_{jk} - a_{jk}\right\| \to 0$. We also have

$$\left\|\widetilde{A}_n(\varphi) - \Psi(\varphi)\right\|_{\mathcal{B}(\ell^2)} \to 0 \qquad \text{for all } \varphi \in s(\mathcal{A}).$$

For each $\varphi \in s(\mathcal{A})$, let $\Psi(\varphi) = \left[\psi_{jk}(\varphi)\right]$. It follows that

$$\varphi(a^{(n)}_{jk}) \to \psi_{jk}(\varphi) \qquad \text{for all } \varphi \in s(\mathcal{A}).$$

But we also have

$$\varphi(a^{(n)}_{jk}) \to \varphi(a_{jk}) \qquad \text{for all } \varphi \in s(\mathcal{A}),$$

and hence

$$\varphi(a_{jk}) = \psi_{jk}(\varphi) \qquad \text{for all } \varphi \in s(\mathcal{A}).$$

Let $A = \left[a_{jk}\right]$. Then

$$\Psi(\varphi) = \left[\psi_{jk}(\varphi)\right] = \left[\varphi(a_{jk})\right] = \widetilde{\varphi}(A) = \widetilde{A}(\varphi) \qquad \text{for all } \varphi \in s(\mathcal{A}).$$

That is $\widetilde{A} = \Psi \in C(s(\mathcal{A}), \mathcal{B}(\ell^2))$, and $A \in \mathcal{M}$. □

Now we prove some properties of $\mathcal{K}$ that are parallel to well-known properties of compact operators.

Proposition 2.2. *$\mathcal{K}$ is a closed proper subspace of $\mathcal{M}$.*

Proof. Let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{K}$ such that $\|A_k - A\| \to 0$ for some $A \in \mathcal{M}$. Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that

$$\|A_k - A\| < \frac{\epsilon}{4} \quad \text{for all } k \geq N.$$

Since $A_N \in \mathcal{K}$, there is an $n_0 \in \mathbb{N}$ such that

$$\left\|(A_N)_{\underline{n}} - A_N\right\| < \frac{\epsilon}{4} \quad \text{for all } n \geq n_0.$$

Let $n \geq n_0$.

$$\begin{aligned}\left\|A_{\underline{n}} - A\right\| &\leq \left\|A_{\underline{n}} - (A_N)_{\underline{n}}\right\| + \left\|(A_N)_{\underline{n}} - A_N\right\| + \|A_N - A\| \\ &< \left\|(A - A_N)_{\underline{n}}\right\| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \|A_N - A\| + \frac{\epsilon}{2} < \epsilon\end{aligned}$$

That is $\left\|A - A_{\underline{n}}\right\| \to 0$ as $n \to \infty$, and hence $A \in \mathcal{K}$.

By definition, we have $\mathcal{K} \subseteq \mathcal{M}$. To see that the inclusion is proper, we note that the matrix A with 1 (the identity of $\mathcal{A}$) on the diagonal and 0 elsewhere (i.e., $A(j,k) = \delta_{jk}1$) is in $\mathcal{M}$ but not in $\mathcal{K}$. Weak* to norm continuity of the map $\varphi \mapsto \widetilde{\varphi}(A)$ follows immediately from the fact that $\widetilde{\varphi}(A)$ is the identity matrix in $\mathcal{B}(\ell^2)$ for each $\varphi \in s(\mathcal{A})$. Thus $A \in \mathcal{M}$. But $\left\|\widetilde{\varphi}(A - A_{\underline{n}})\right\| = 1$ for all $\varphi \in s(\mathcal{A})$ and all $n \in \mathbb{N}$, which implies that $A \notin \mathcal{K}$. □

Proposition 2.3. *Let $A \in \mathcal{M}$ satisfy $A = A_{N\rceil}$ (respectively, $A = A_{\underline{N}}$) for some fixed $N \in \mathbb{N}$. Then $A \in \mathcal{K}$, and $\left\|A - A_{\nu\lrcorner}\right\| \to 0$ as $\nu \to \infty$.*

Proof. Suppose $A = A_{\underline{N}} \in \mathcal{M}$. For $n \geq N$, we have $A_{\underline{n}} = A_{\underline{N}} = A$. Thus $\left\|A - A_{\underline{n}}\right\| = 0$ for all $n \geq N$, and hence $A \in \mathcal{K}$.

If $A = A_{N\rceil} \in \mathcal{M}$, then the transpose of A,

$$B = A^T \qquad \left(B_{jk} = (A^T)_{jk} = A_{kj} \;\forall\; j,\ k \in \mathbb{N}\right),$$

satisfies

$$B = A^T = [A_{N\rceil}]^T = B_{\underline{N}},$$

and hence,

$$\left\|B - B_{\underline{n}}\right\| = 0 \quad \text{for all } n \geq N.$$

For each $n \geq N$ we have

$$\begin{aligned}\left\|A - A_{n\rceil}\right\| &= \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}\left(A - A_{n\rceil}\right)\right\|_{\mathcal{B}(\ell^2)} = \sup_{\varphi \in s(\mathcal{A})} \left\|\left(\widetilde{\varphi}\left(A - A_{n\rceil}\right)\right)^T\right\|_{\mathcal{B}(\ell^2)} \\ &= \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}\left(A^T - \left(A_{n\rceil}\right)^T\right)\right\|_{\mathcal{B}(\ell^2)} = \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}\left(B - B_{\underline{n}}\right)\right\|_{\mathcal{B}(\ell^2)} \\ &= \left\|B - B_{\underline{n}}\right\| = 0.\end{aligned}$$

Since A is assumed to be in $\mathcal{M}$, this shows that $A = A_{N\rceil} \in \mathcal{K}$, and hence

$$\left\|A - A_{\nu\lrcorner}\right\| = \left\|A - A_{\underline{\nu}}\right\| \to 0 \quad \text{as } \nu \to \infty.$$

For the case $A = A_{\underline{N}}$, we see as above that $C = A^T$ satisfies $C = C_{N|} \in \mathcal{K}$, and hence

$$\|A - A_{\nu_\lrcorner}\| = \left\|\left(A - A_{\nu_\lrcorner}\right)^T\right\| = \|C - C_{\underline{\nu}}\| \to 0 \quad \text{as } \nu \to \infty.$$

□

For each $A = \left[a_{jk}\right] \in \mathcal{M}$, A^* is defined by

$$(A^*)_{jk} = a^*_{kj} \qquad \text{for all } j,\ k \in \mathbb{N}.$$

It is easy to see that $A^* \in \mathcal{M}$ whenever $A \in \mathcal{M}$.

Proposition 2.4. *Let $A = \left[a_{jk}\right]$ be a matrix over $\mathcal{A}$.*

(1) *$A \in \mathcal{K}$ iff the map $\varphi \mapsto \widetilde{\varphi}(A)$ is continuous form $s(\mathcal{A})$ with the weak* topology to $\mathcal{K}(\ell^2)$ with the operator norm topology.*

(2) *$A \in \mathcal{K}$ iff $A^* = \left[a^*_{jk}\right]^T \in \mathcal{K}$.*

(3) *If $A \in \mathcal{M}$, then $A \in \mathcal{K}$ iff $\left\|A - A_{n|}\right\| = \left\|A_{|n}\right\| \to 0$ as $n \to \infty$.*

Proof. (1) [⇒] Suppose $A \in \mathcal{K}$. Then $A \in \mathcal{M}$. Thus $\varphi \mapsto \widetilde{\varphi}(A)$ is continuous from $s(\mathcal{A})$ with weak* topology to $\mathcal{B}(\ell^2)$ with norm topology. It suffices to show that $\widetilde{\varphi}(A) \in \mathcal{K}(\ell^2)$ for all $\varphi \in s(\mathcal{A})$. Let $\varphi \in s(\mathcal{A})$. We have

$$\left\|\widetilde{\varphi}(A) - [\widetilde{\varphi}(A)]_{\underline{n}}\right\|_{\mathcal{B}(\ell^2)} = \left\|\widetilde{\varphi}(A - A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \le \left\|A - A_{\underline{n}}\right\| \to 0 \quad \text{as } n \to \infty$$

and hence $\widetilde{\varphi}(A) \in \mathcal{K}(\ell^2)$.

(1) [⇐] Let $\epsilon > 0$. By continuity, for each $\varphi \in s(\mathcal{A})$, there is a weak* open set $V_\varphi \subseteq s(\mathcal{A})$ such that

$$\varphi \in V_\varphi \quad \text{and} \quad \left\|\widetilde{\varphi}(A) - \widetilde{\psi}(A)\right\|_{\mathcal{K}(\ell^2)} = \left\|\widetilde{\varphi}(A) - \widetilde{\psi}(A)\right\|_{\mathcal{B}(\ell^2)} < \frac{\epsilon}{4} \ \ \forall \ \ \psi \in V_\varphi.$$

Since $s(\mathcal{A})$ with the weak* topology is a compact Hausdorff space [4, p. 257], and

$$s(\mathcal{A}) \subseteq \bigcup_{\varphi \in s(\mathcal{A})} V_\varphi,$$

there are $\varphi_1, \ldots \varphi_k \in s(\mathcal{A})$ such that

$$s(\mathcal{A}) \subseteq \bigcup_{j=1}^{k} V_{\varphi_j}.$$

For each $j = 1, \ldots, k$, since $\widetilde{\varphi}_j(A) \in \mathcal{K}(\ell^2)$, there is an $N_j \in \mathbb{N}$ such that

$$\left\|\widetilde{\varphi}_j(A) - [\widetilde{\varphi}_j(A)]_{\underline{n}}\right\|_{\mathcal{B}(\ell^2)} = \left\|\widetilde{\varphi}_j(A) - [\widetilde{\varphi}_j(A_{\underline{n}})]\right\|_{\mathcal{B}(\ell^2)} < \frac{\epsilon}{4} \quad \text{for all } n \ge N_j.$$

Put $N = \max\{N_j : j = 1, \ldots, k\}$. Then for $n \geq N$ and $\varphi \in s(\mathcal{A})$, we have $\varphi \in V_{\varphi_j}$ for some $j = 1, \ldots, k$, and thus

$$\begin{aligned}
&\left\|\widetilde{\varphi}(A) - \widetilde{\varphi}(A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \\
\leq &\left\|\widetilde{\varphi}(A) - \widetilde{\varphi}_j(A)\right\|_{\mathcal{B}(\ell^2)} + \left\|\widetilde{\varphi}_j(A) - \widetilde{\varphi}_j(A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} + \left\|\widetilde{\varphi}_j(A_{\underline{n}}) - \widetilde{\varphi}(A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \\
< &\frac{\epsilon}{4} + \frac{\epsilon}{4} + \left\|\left[\widetilde{\varphi}_j(A) - \widetilde{\varphi}(A)\right]_{\underline{n}}\right\|_{\mathcal{B}(\ell^2)} < \frac{\epsilon}{2} + \left\|\widetilde{\varphi}_j(A) - \widetilde{\varphi}(A)\right\|_{\mathcal{B}(\ell^2)} < \frac{3\epsilon}{4}
\end{aligned}$$

Since $\varphi \in s(\mathcal{A})$ is arbitrary,

$$\begin{aligned}
\left\|A - A_{\underline{n}}\right\| &= \sup_{\varphi\in s(\mathcal{A})} \left\|\widetilde{\varphi}(A - A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} = \sup_{\varphi\in s(\mathcal{A})} \left\|\widetilde{\varphi}(A) - \widetilde{\varphi}(A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \\
&\leq \frac{3\epsilon}{4} < \epsilon \qquad \text{for all } \ n \geq N.
\end{aligned}$$

(2) [$\Rightarrow$] Suppose that $A \in \mathcal{K}$. Then $\varphi \mapsto \widetilde{\varphi}(A)$ is weak* to norm continuous from $s(\mathcal{A})$ to $\mathcal{K}(\ell^2)$. Let $\epsilon > 0$. For each $\varphi \in s(\mathcal{A})$, there is a weak* neighborhood U_φ of φ such that

$$\text{for all } \psi \in U_\varphi, \ \ \widetilde{\psi}(A) \in \mathcal{K} \quad \text{ and } \quad \left\|\widetilde{\varphi}(A) - \widetilde{\psi}(A)\right\|_{\mathcal{B}(\ell^2)} < \epsilon.$$

Since ψ is a positive linear functional, $\psi(a^*) = \overline{\psi(a)}$ for all $a \in \mathcal{A}$ [4, p. 255]. From $\widetilde{\psi}(A) \in \mathcal{K}(\ell^2)$, we have $\widetilde{\psi}(A^*) = [\widetilde{\psi}(A)]^* \in \mathcal{K}(\ell^2)$, and

$$\left\|\widetilde{\varphi}(A^*) - \widetilde{\psi}(A^*)\right\|_{\mathcal{B}(\ell^2)} = \left\|[\widetilde{\varphi}(A)]^* - [\widetilde{\psi}(A)]^*\right\|_{\mathcal{B}(\ell^2)} = \left\|\widetilde{\varphi}(A) - \widetilde{\psi}(A)\right\|_{\mathcal{B}(\ell^2)} < \epsilon.$$

Thus the map $\varphi \mapsto \widetilde{\varphi}(A^*)$ is continuous from $s(\mathcal{A})$ with weak* topology to $\mathcal{K}(\ell^2)$ with norm topology. Hence $A^* \in \mathcal{K}$ by part (1).

(2) [$\Leftarrow$] Suppose that $A^* \in \mathcal{K}$. Then $A = (A^*)^* \in \mathcal{K}$.

(3) [$\Rightarrow$] Suppose $A \in \mathcal{K}$. Then $A^* \in \mathcal{K}$ and hence

$$\left\|A^* - (A^*)_{\underline{n}}\right\| \to 0 \quad \text{ as } n \to \infty.$$

Thus

$$\left\|A - A_{n|}\right\| = \left\|(A - A_{n|})^*\right\| = \left\|A^* - (A^*)_{\underline{n}}\right\| \to 0 \quad \text{ as } n \to \infty.$$

(3) [$\Leftarrow$] Suppose $\left\|A - A_{n|}\right\| \to 0$. Since each $A_{n|} \in \mathcal{K}$ by Proposition 2.3, and since $\mathcal{K}$ is closed under the operator norm, $A \in \mathcal{K}$. $\square$

3. The dual of $\mathcal{K}$

In this section we will obtain a functional matrix representation of the dual $\mathcal{K}^\#$ of $\mathcal{K}$. First note that for $A = [a_{jk}] \in \mathcal{M}$, and each $j, k \in \mathbb{N}$, we have

$$\left\|a_{jk}\right\| \leq 2\left\|a_{jk}\right\|_s = 2 \sup_{\varphi\in s(\mathcal{A})} \left|\varphi(a_{jk})\right| \leq 2 \sup_{\varphi\in s(\mathcal{A})} \left\|\widetilde{\varphi}(A)\right\|_{\mathcal{B}(\ell^2)} = 2\left\|A\right\|.$$

We will need the following lemma in the proofs of Propositions 3.2 and 3.3

Lemma 3.1. *Let $\{f_n\}$ be a sequence in the dual space $X^\#$ of a Banach space X such that $f(x) = \sum_{k=1}^{\infty} f_k(x)$ converges for all $x \in X$. Then $f \in X^\#$.*

Proof. A routine argument shows that f is linear. For the boundedness of f, let $g_n = \sum_{k=1}^{n} f_k$ for each $n \in \mathbb{N}$. Then $g_n \in X^\#$. For each $x \in X$, since $\sum_{k=1}^{\infty} f_k(x)$ converges, there is an $\alpha_x \geq 0$ such that $|g_n(x)| \leq \alpha_x$ for all $n \in \mathbb{N}$. So $\{g_n\}$ is a sequence in $X^\#$ that is pointwise bounded. The uniform boundedness principle implies that $\{g_n\}$ is uniformly bounded; i.e., there is a β such that $\|g_n\| \leq \beta$ for all $n \in \mathbb{N}$. For each $x \in X$, we have

$$|f(x)| = \lim_{n\to\infty} \left| \sum_{k=1}^{n} f_k(x) \right| = \lim_{n\to\infty} |g_n(x)| \leq \limsup_{n\to\infty} \|g_n\| \, \|x\| \leq \beta \, \|x\| \, .$$

Thus $f \in X^\#$ with $\|f\| \leq \beta$. □

Proposition 3.2. *For each $f \in \mathcal{K}^\#$, there exists a unique matrix $[f_{jk}]$, with $f_{jk} \in \mathcal{A}^\#$, such that*

$$f(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{for all} \;\; A = [a_{jk}] \in \mathcal{K}.$$

Conversely, each matrix $[g_{jk}]$ over $\mathcal{A}^\#$ with the property that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \;\; \text{converges for every} \;\; A = [a_{jk}] \in \mathcal{K},$$

defines a bounded linear functional

$$g(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \quad \left(A = [a_{jk}] \in \mathcal{K}\right) \quad \text{on } \mathcal{K}.$$

Moreover, in this case,

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} g_{jk}(a_{jk}) \;\; \text{converges,} \quad \text{and,}$$

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} g_{jk}(a_{jk}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \qquad \text{for all} \;\; [a_{jk}] \in \mathcal{K}.$$

Thus $\mathcal{K}^\#$ is identified with the space of all such matrices. The norm of such a matrix is defined to be the norm of the bounded linear functional it represents, i.e., $\|[f_{jk}]\| = \|f\|$ if $[f_{jk}]$ represents $f \in \mathcal{K}^\#$.

Proof. Let $f \in \mathcal{K}^\#$. For each $(j,k) \in \mathbb{N} \times \mathbb{N}$ and each $a \in \mathcal{A}$, since the matrix $E_{jk}(a)$ with (j,k) entry a and all others 0 is easily seen from Proposition 2.3 to be in $\mathcal{K}$ with

$$\|E_{jk}(a)\| = \|a\|_s \leq \|a\| \, ,$$

we define f_{jk} by

$$f_{jk}(a) = f(E_{jk}(a)) \qquad \text{for all } a \in \mathcal{A}.$$

It is readily seen that f_{jk} is linear, and

$$\left|f_{jk}(a)\right| = \left|f(E_{jk}(a))\right| \leq \|f\| \left\|E_{jk}(a)\right\| \leq \|f\| \|a\|.$$

Hence $f_{jk} \in \mathcal{A}^{\#}$ with $\left\|f_{jk}\right\| \leq \|f\|$. Let $A = \left[a_{jk}\right] \in \mathcal{K}$. For each $n \in \mathbb{N}$, $A_{\underline{n}} \in \mathcal{K}$, and, by Proposition 2.3,

$$\left\|A_{\underline{n}} - [A_{\underline{n}}]_{\nu\lrcorner}\right\| \to 0 \quad \text{as} \quad \nu \to \infty.$$

Thus, by linearity,

$$\sum_{j=1}^{n} \sum_{k=1}^{\nu} f_{jk}(a_{jk}) = f([A_{\underline{n}}]_{\nu\lrcorner}) \to f(A_{\underline{n}}) \quad \text{as } \nu \to \infty.$$

That is

$$f(A_{\underline{n}}) = \sum_{j=1}^{n} \sum_{k=1}^{\infty} f_{jk}(a_{jk}).$$

Since $\left\|A - A_{\underline{n}}\right\| \to 0$ as $n \to \infty$, $f(A_{\underline{n}}) \to f(A)$, and hence

$$f(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}).$$

Now suppose $\left[g_{jk}\right]$ is a matrix over $\mathcal{A}^{\#}$ such that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \text{ converges for every } A = \left[a_{jk}\right] \in \mathcal{K}.$$

For each fixed $m, n \in \mathbb{N}$, define $\hat{g}_{mn} : \mathcal{K} \to \mathbb{C}$ by

$$\hat{g}_{mn}(A) = g_{mn}(a_{mn}) \quad \text{for each } A = \left[a_{jk}\right] \in \mathcal{K}.$$

Then

$$|\hat{g}_{mn}(A)| \leq \|g_{mn}\| \|a_{mn}\| \leq 2 \|A\| \|g_{mn}\|$$

i.e., $\hat{g}_{mn} \in \mathcal{K}^{\#}$. Since by assumption

$$g_m(A) := \sum_{k=1}^{\infty} \hat{g}_{mk}(A) = \sum_{k=1}^{\infty} g_{mk}(a_{mk}) \text{ converges for every } A = \left[a_{jk}\right] \in \mathcal{K},$$

by Lemma 3.1, $g_m \in \mathcal{K}^{\#}$. Since we also assume that

$$g(A) := \sum_{m=1}^{\infty} g_m(A) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} g_{mk}(a_{mk}) \quad \text{converges for every } A = \left[a_{jk}\right] \in \mathcal{K},$$

by Lemma 3.1 again, the functional g is bounded, i.e., $g \in \mathcal{K}^{\#}$.

For each $A = \left[a_{jk}\right] \in \mathcal{K}$, since the matrix $A_{|k|} = A_{k|} - A_{(k-1)|}$, with the k-th column the same as that of A and all others 0, is in $\mathcal{K}$,

$$\sum_{j=1}^{\infty} g_{jk}(a_{jk}) = g(A_{|k|}) \text{ converges, for all } k \in \mathbb{N}.$$

Since $\left\|A - A_{m|}\right\| \to 0$ as $m \to \infty$,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} g_{jk}(a_{jk}) = g(A) = \lim_{m\to\infty} g(A_{m|}) = \lim_{m\to\infty}\left[\sum_{k=1}^{m} g(A_{|k|})\right]$$
$$= \lim_{m\to\infty}\left[\sum_{k=1}^{m}\sum_{j=1}^{\infty} g_{jk}(a_{jk})\right] = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} g_{jk}(a_{jk}).$$

□

Next we show that if $[f_{jk}] \in \mathcal{K}^{\#}$, then the two double sums both converge and are equal for each $A = [a_{jk}] \in \mathcal{M}$, not just for elements in $\mathcal{K}$.

Proposition 3.3. *For each $f = [f_{jk}] \in \mathcal{K}^{\#}$ and each $A = [a_{jk}] \in \mathcal{M}$, both*

$$\hat{f}(A) := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk}) \qquad \text{and} \qquad g(A) := \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f_{jk}(a_{jk})$$

converge, and they have the same sum. Furthermore $\hat{f}$ is a bounded linear functional on $\mathcal{M}$ with norm $\left\|\hat{f}\right\|_{\mathcal{M}^{\#}} = \|f\|_{\mathcal{K}^{\#}}$.

Proof. Let $A = [a_{jk}] \in \mathcal{M}$. Then for each $j \in \mathbb{N}$, the row j matrix $A_{\underline{\bar{j}}} = A_{\underline{j}} - A_{\underline{j-1}} \in \mathcal{K}$. Thus

$$\sum_{k=1}^{\infty} f_{jk}(a_{jk}) \ \text{ converges for every } \ j \in \mathbb{N}.$$

Suppose

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk}) \ \text{ does not converge.}$$

Then there are an $\epsilon > 0$ and two sequences $\{j_\nu\}$, $\{l_\nu\}$ in $\mathbb{N}$ such that

$$1 \le j_1 < l_1 < j_2 < l_2 < \ldots < j_\nu < l_\nu < \ldots, \quad \text{ and}$$
$$\left|\sum_{j=j_\nu}^{l_\nu}\sum_{k=1}^{\infty} f_{jk}(a_{jk})\right| > \epsilon \quad \text{ for all } \ \nu \in \mathbb{N}.$$

Let $A_\nu = A_{\underline{l_\nu}} - A_{\underline{j_\nu - 1}}$, the matrix whose rows from j_ν-th through l_ν-th coincide with that of A and all others are 0; let

$$\alpha_\nu = \frac{1}{\nu}\ \mathrm{sgn}\left[\sum_{j=j_\nu}^{l_\nu}\sum_{k=1}^{\infty} f_{jk}(a_{jk})\right]; \quad \text{ and } \quad B = \sum_{\nu=1}^{\infty}\alpha_\nu A_\nu.$$

We show that $B \in \mathcal{K}$ but the sum for $f(B)$ diverges. Let $\eta > 0$. There is a $\nu_0 \in \mathbb{N}$ such that

$$\sum_{\nu=\nu_0}^{\infty}\frac{\|A\|^2}{\nu^2} < \frac{\eta^2}{4}.$$

For $n \geq j_{\nu_0}$, $\varphi \in s(\mathcal{A})$, and $x = \{x_k\} \in \ell^2$, let ν_1 be the largest ν such that $j_\nu \leq n$. Thus $\nu_1 \geq \nu_0$, and hence,

$$\left\|\widetilde{\varphi}\left(B - B_{\underline{n}}\right)x\right\|^2_{\ell^2} = \left\|\left[\widetilde{\varphi}(B) - \widetilde{\varphi}(B_{\underline{n}})\right]x\right\|^2_{\ell^2}$$

$$= \sum_{j=n+1}^{l_{\nu_1}} \left|\alpha_{\nu_1} \sum_{k=1}^{\infty} \varphi(a_{jk})x_k\right|^2 + \sum_{\nu=\nu_1+1}^{\infty} \sum_{j=j_\nu}^{l_\nu} \left|\alpha_\nu \sum_{k=1}^{\infty} \varphi(a_{jk})x_k\right|^2$$

$$= \left|\alpha_{\nu_1}\right|^2 \sum_{j=n+1}^{l_{\nu_1}} \left|\sum_{k=1}^{\infty} \varphi(a_{jk})x_k\right|^2 + \sum_{\nu=\nu_1+1}^{\infty} \left|\alpha_\nu\right|^2 \sum_{j=j_\nu}^{l_\nu} \left|\sum_{k=1}^{\infty} \varphi(a_{jk})x_k\right|^2$$

$$\leq \frac{1}{\nu_1^2} \sum_{j=1}^{\infty} \left|\sum_{k=1}^{\infty} \varphi(a_{jk})x_k\right|^2 + \sum_{\nu=\nu_1+1}^{\infty} \frac{1}{\nu^2} \sum_{j=1}^{\infty} \left|\sum_{k=1}^{\infty} \varphi(a_{jk})x_k\right|^2$$

$$\leq \frac{\|A\|^2}{\nu_1^2} \|x\|^2_{\ell^2} + \sum_{\nu=\nu_2}^{\infty} \frac{\|A\|^2}{\nu^2} \|x\|^2_{\ell^2} < \frac{\eta^2}{4} \|x\|^2_{\ell^2}.$$

Since this is true for all $x \in \ell^2$, we see that

$$\left\|\widetilde{\varphi}(B - B_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \leq \frac{\eta}{2}.$$

But $\varphi \in s(\mathcal{A})$ is also arbitrary,

$$\left\|B - B_{\underline{n}}\right\| \leq \frac{\eta}{2} < \eta.$$

Since this is true for all $n \geq j_{\nu_0}$, we conclude that $B \in \mathcal{K}$.
On the other hand we also have

$$f(B) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{\nu=1}^{\infty} \alpha_\nu \sum_{j=j_\nu}^{l_\nu} \sum_{k=1}^{\infty} f_{jk}(a_{jk})$$

$$= \sum_{\nu=1}^{\infty} \frac{1}{\nu} \left|\sum_{j=j_\nu}^{l_\nu} \sum_{k=1}^{\infty} f_{jk}(a_{jk})\right| \geq \sum_{\nu=1}^{\infty} \frac{\epsilon}{\nu} = \infty,$$

contradicting $B \in \mathcal{K}$ and $f \in \mathcal{K}^\#$. Therefore

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \text{ converges.}$$

A similar argument shows that the sum in the other order for g also converges.

Uniform boundedness arguments similar to that used in the proof of Proposition 3.2 show that $\hat{f}$ and g are both bounded linear functionals on $\mathcal{M}$.

For $A \in \mathcal{M}$, since $A_{n|} \in \mathcal{K}$, for each $n \in \mathbb{N}$, by last part of the preceding proposition,

$$|g(A)| = \lim_{n\to\infty} \left|g(A_{n|})\right| = \lim_{n\to\infty} \left|f(A_{n|})\right| \leq \limsup_{n\to\infty} \|f\| \left\|A_{n|}\right\| \leq \|f\| \|A\|,$$

thus $\|g\| \le \|f\|$. Also $g|_{\mathcal{K}} = f$, we see that $\|g\| \ge \|f\|$, and thus $\|f\| = \|g\|$. Similarly $\left\|\hat{f}\right\| = \|f\|$.

To see that the two sums are equal, we first show that the sequence $\{g_n\}$ defined by

$$g_n(A) := \sum_{k=1}^{n}\sum_{j=1}^{\infty} f_{jk}(a_{jk}) \qquad \left(A = [a_{jk}] \in \mathcal{K}\right)$$

is a Cauchy sequence in $\mathcal{K}^{\#}$. Suppose $\{g_n\}$ is not a Cauchy sequence in $\mathcal{K}^{\#}$. Then there exist an $\epsilon > 0$ and sequences $\{k_\nu\}_{\nu\in\mathbb{N}}$, $\{l_\nu\}_{\nu\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$l_{\nu-1} + 1 \le k_\nu < l_\nu \ \ (\text{where } l_0 = 0), \ \text{ and } \quad \left\| g_{l_\nu} - g_{k_\nu} \right\| > 2\epsilon \quad \text{for all } \nu \in \mathbb{N}.$$

Thus there are elements $A_\nu \in \mathcal{K}$ such that

$$\|A_\nu\| = 1 \ \text{ and } \ \left| g_{l_\nu}(A_\nu) - g_{k_\nu}(A_\nu) \right| > 2\epsilon.$$

Let

$$\alpha_\nu = \frac{1}{\nu}\,\operatorname{sgn}\left[g_{l_\nu}(A_\nu) - g_{k_\nu}(A_\nu)\right] \quad \text{ and } \quad B = \sum_{\nu=1}^{\infty} \alpha_\nu A_\nu .$$

Then an argument similar to that used above shows that

$$B \in \mathcal{K} \qquad \text{but} \quad g(B) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f_{jk}(B(j,k)) \ \text{ diverges,}$$

which is a contradiction. Therefore $\{g_n\}$ is a Cauchy sequence in $\mathcal{K}^{\#}$. Thus there is an $h \in \mathcal{K}^{\#}$ such that

$$\|g_n - h\|_{\mathcal{K}^{\#}} \to 0.$$

But since each $A \in \mathcal{K}$ has $\left\| A - A_{n|} \right\| \to 0$, also $g \in \mathcal{K}^{\#}$ and $g_n(A) = g(A_{n|})$, we have

$$g_n(A) \to g(A) \ \text{ for each } A \in \mathcal{K}.$$

Thus $g = h$ and hence

$$\|g_n - g\|_{\mathcal{K}^{\#}} \to 0.$$

For each $A = [a_{jk}] \in \mathcal{M}$, since

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{ and } \quad \sum_{j=1}^{\infty}\sum_{k=1}^{n} f_{jk}(a_{jk}) \ \text{ converge for all } \ n \in \mathbb{N},$$

$$\begin{aligned}(\hat{f} - g_n)(A) =& \hat{f}(A) - g_n(A) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk}) - \sum_{j=1}^{\infty}\sum_{k=1}^{n} f_{jk}(a_{jk}) \\ =& \sum_{j=1}^{\infty}\sum_{k=n+1}^{\infty} f_{jk}(a_{jk}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \tilde{f}_{jk}(a_{jk})\end{aligned}$$

where

$$\tilde{f}_{jk} = \begin{cases} f_{jk} & \text{for } k > n \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\widehat{(f - g_n)} = \hat{f} - g_n$, and, by Proposition 3.2, that $f = g$ on $\mathcal{K}$. Thus, from the first part, we have

$$\begin{aligned} \lim_{n\to\infty} \left\| \hat{f} - g_n \right\|_{\mathcal{M}^\#} &= \lim_{n\to\infty} \left\| \widehat{(f - g_n)} \right\|_{\mathcal{M}^\#} = \lim_{n\to\infty} \left\| f - g_n \right\|_{\mathcal{K}^\#} \\ &= \lim_{n\to\infty} \left\| g - g_n \right\|_{\mathcal{K}^\#} = 0. \end{aligned}$$

Therefore

$$\hat{f}(A) = \lim_{n\to\infty} g_n(A) \qquad \text{for all } \ A \in \mathcal{M},$$

and hence, for each $A = [a_{jk}] \in \mathcal{M}$,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk}) = \hat{f}(A) = \lim_{n\to\infty} g_n(A) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f_{jk}(a_{jk}).$$

□

Note that this proposition corresponds to the fact that the trace functional satisfies $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for a trace class A and bounded B on a Hilbert space. The proof of this proposition can easily be adapted to a proof of the trace identity. Since each $[f_{jk}] \in \mathcal{K}^\#$ defines a bounded linear functional

$$\hat{f}(A) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk}) \qquad (A = [a_{jk}] \in \mathcal{M})$$

on $\mathcal{M}$ with the same norm $\left\|\hat{f}\right\|_{\mathcal{M}^\#} = \left\| [f_{jk}] \right\|_{\mathcal{K}^\#}$. The space of all such linear functionals $\hat{f}$ will be denoted by $\widehat{\mathcal{K}^\#}$

4. The main theorem

Now we are ready for the main Dixmier's theorem. Denote by $\mathcal{K}^\perp$ the subspace of $\mathcal{M}^\#$ consisting of bounded linear functionals on $\mathcal{M}$ that vanish on $\mathcal{K}$.

Theorem 4.1. *For each $f \in \mathcal{M}^\#$, there is a unique pair $g \in \widehat{\mathcal{K}^\#}$ and $h \in \mathcal{K}^\perp$ such that*

$$f = g + h \qquad \textit{and} \qquad \|f\| = \|g\| + \|h\|.$$

Proof. For each $(j,k) \in \mathbb{N}\times\mathbb{N}$, define f_{jk} by $f_{jk}(a) = f(E_{jk}(a))$ for all $a \in \mathcal{A}$. Then $f_{jk} \in \mathcal{A}^\#$ with $\left\|f_{jk}\right\| \le \|f\|$. Then as in the proof of Proposition 3.2 the matrix $[f_{jk}]$ represents a bounded linear functional $\widetilde{f} = f|_{\mathcal{K}}$ on $\mathcal{K}$. By Proposition 3.3, $[f_{jk}]$ defines a bounded linear functional $g = \widehat{(\widetilde{f})} = \widehat{(f|_{\mathcal{K}})}$ on $\mathcal{M}$, where

$$g(A) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk}) \qquad \text{for all } \ A = [a_{jk}] \in \mathcal{M},$$

and

$$\|g\|_{\mathcal{M}^\#} = \left\|\widetilde{f}\right\|_{\mathcal{K}^\#}.$$

Let $h = f - g$. It is clear that $h \in \mathcal{K}^\perp$. The uniqueness of the decomposition follows from the fact that $\widehat{\mathcal{K}^\#} \oplus \mathcal{K}^\perp = \mathcal{M}^\#$ is a direct sum.

Since $\|f\| \leq \|g\| + \|h\|$, it suffices to prove that $\|f\| \geq \|g\| + \|h\|$. Let $\epsilon > 0$. Since $\|g\|_{\mathcal{M}^\#} = \left\|g|_{\mathcal{K}}\right\|$, there is an $A = \left[a_{jk}\right] \in \mathcal{K}$ such that

$$\|A\| = 1 \quad \text{and} \quad g(A) > \|g\| - \frac{\epsilon}{8}.$$

There is also a $B = \left[b_{jk}\right] \in \mathcal{M}$ such that

$$\|B\| = 1 \quad \text{and} \quad h(B) > \|h\| - \frac{\epsilon}{8}.$$

Form the convergence of the double sum, there is a j_0 such that

$$\left|\sum_{j=n}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk})\right| < \frac{\epsilon}{8} \quad \forall \ n > j_0.$$

There is also a k_0 such that

$$\left|\sum_{j=1}^{j_0}\sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk})\right| < \frac{\epsilon}{8}.$$

By Proposition 3.3,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f_{jk}(b_{jk}),$$

thus there is a $j_1 \geq j_0$ such that

$$\left|\sum_{j=j_1+1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(b_{jk})\right| < \frac{\epsilon}{8}.$$

Put

$$\hat{f}_{jk} = \begin{cases} 0 & \text{if } 1 \leq j \leq j_1 \\ f_{jk} & \text{if } j_1 < j \end{cases}$$

Then $\left[\hat{f}_{jk}\right] \in \mathcal{K}^\#$. Thus

$$\sum_{j=j_1+1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \hat{f}_{jk}(b_{jk}) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \hat{f}_{jk}(b_{jk})$$

$$= \sum_{k=1}^{\infty}\sum_{j=j_1+1}^{\infty} f_{jk}(b_{jk})$$

converges, and hence there is a $k_1 \geq k_0$ such that

$$\left| \sum_{k=k_1+1}^{\infty} \sum_{j=j_1+1}^{\infty} f_{jk}(b_{jk}) \right| < \frac{\epsilon}{8}.$$

Let

$$A_0(j,k) = \begin{cases} a_{jk} & \text{if } 1 \leq j \leq j_0, \text{ and } 1 \leq k \leq k_0 \\ 0 & \text{otherwise,} \end{cases}$$

$$B_0(j,k) = \begin{cases} b_{jk} & \text{if } j_1 < j, \text{ and } k_1 < k \\ 0 & \text{otherwise;} \end{cases}$$

and let $C = [c_{jk}] = A_0 + B_0$. Then $\|C\| = \max\{\|A_0\|, \|B_0\|\} \leq 1$. Since $h \in \mathcal{K}^\perp$, and $A_0, B - B_0 \in \mathcal{K}$, we have $h(A_0) = 0$, and hence $h(B) = h(B_0)$. Therefore

$$\begin{aligned}
\|f\| &\geq |f(C)| = |g(A_0) + g(B_0) + h(A_0) + h(B_0)| \\
&\geq |g(A_0) + h(B_0)| - |g(B_0)| > \mathrm{Re}[g(A_0)] + \mathrm{Re}[h(B_0)] - \frac{\epsilon}{8} \\
&= \mathrm{Re}\left[g(A) - \sum_{j=j_0+1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) - \sum_{j=1}^{j_0} \sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk}) \right] + h(B) - \frac{\epsilon}{8} \\
&> \|g\| - \frac{\epsilon}{8} - \left| \sum_{j=j_0+1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \right| - \left| \sum_{j=1}^{j_0} \sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk}) \right| + \|h\| - \frac{\epsilon}{4} \\
&> \|g\| + \|h\| - \frac{5\epsilon}{8} > \|g\| + \|h\| - \epsilon.
\end{aligned}$$

Since the preceding argument holds for every $\epsilon > 0$, we conclude that

$$\|f\| \geq \|g\| + \|h\|.$$

□

We note that when $\mathcal{A}$ is the complex field $\mathbb{C}$, then $s(\mathcal{A})$ consists of the identity map alone. So a matrix A over $\mathbb{C}$ is in $\mathcal{M}$ iff A is in $\mathcal{B}(\ell^2)$ and A is in $\mathcal{K}$ iff A is in $\mathcal{K}(\ell^2)$. A matrix defines a bounded linear functional on $\mathcal{K}(\ell^2)$ iff it is represented by a trace class matrix and hence it is a trace class matrix itself. Thus Dixmier's Theorem is an immediate consequence of this result.

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