

จะได้ $u \in F$ ให้ลำดับย่อย $\{x_{n_k}\}$ และ $\{x_{m_k}\}$ ของลำดับ $\{x_n\}$ เข้าสู่แบบอ่อนสู่ u และ v ตามลำดับ เนื่องจาก $I - T_i$, ($i = 1, 2, 3$) เป็นการส่งครึ่งปิดที่ 0 จะได้ u และ $v \in F$ โดยบทตั้ง 4 จะได้ $\lim_{n \rightarrow \infty} \|x_n - u\|$ และ $\lim_{n \rightarrow \infty} \|x_n - v\|$ มีจริง และจากบทตั้ง 3 จะได้ $u = v$ เพราะฉะนั้น $\{x_n\}$ เข้าสู่แบบอ่อนสู่จุดตรึงร่วมของ T_1, T_2 และ T_3 \square

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ภาคผนวก**ก. บทความวิจัยที่ได้รับการตีพิมพ์ในวารสารคณิตศาสตร์ระดับนานาชาติ**

ได้ตีพิมพ์บทความวิจัยเรื่อง Common Fixed Points of a New Three-Step Iteration with Errors of Asymptotically Quasi-Nonexpansive Nonself-Mappings in Banach spaces ในวารสารคณิตศาสตร์ระดับนานาชาติ ชื่อ Journal of Nonlinear Analysis and Optimization, Vol. 1 No. 1 (2010) หน้า 169-182 (สำเนาบทความวิจัยตามที่แนบ)



ข. ตารางเปรียบเทียบวัตถุประสงค์ กิจกรรมที่วางแผนไว้ กิจกรรมที่ดำเนินการมา และผลที่ได้รับตลอดโครงการ

วัตถุประสงค์	กิจกรรมที่วางแผน	กิจกรรมที่ดำเนินการ	ผลที่ได้รับ
1. สร้างระเบียบวิธีทำซ้ำสำหรับหาจุดตรงร่วมของการส่งแบบกึ่งไม่ขยายเชิงเส้นกำกับที่ไม่ส่งไปยังตัวมันเอง	<p>1. ศึกษาหาความรู้เพิ่มเติมของการส่งแบบกึ่งไม่ขยายเชิงเส้นกำกับจากเอกสารอ้างอิงและจากผู้เชี่ยวชาญ</p> <p>2. ศึกษาหาความรู้เพิ่มเติมในเรื่องทฤษฎีจุดตรงจากเอกสารอ้างอิงและจากผู้เชี่ยวชาญ</p> <p>3. ศึกษาระเบียบวิธีทำซ้ำสำหรับหาจุดตรงร่วมของการส่งแบบกึ่งไม่ขยายเชิงเส้นกำกับ</p> <p>4. สร้างระเบียบวิธีทำซ้ำสำหรับหาจุดตรงร่วมของการส่งแบบกึ่งไม่ขยายเชิงเส้นกำกับที่ไม่ส่งไปยังตัวมันเอง</p>	ได้ดำเนินการตามแผนที่วางไว้ทุกข้อ	ได้ศึกษาเอกสารที่เกี่ยวข้องและได้ความรู้เกี่ยวกับขั้นตอนการทำซ้ำแบบต่างๆของการส่งที่สนใจและสร้างเป็นขั้นตอนการทำซ้ำแบบใหม่ที่สามารถขยายผลขั้นตอนการทำซ้ำที่มีอยู่ก่อน

วัตถุประสงค์	กิจกรรมที่วางแผน	กิจกรรมที่ดำเนินการ	ผลที่ได้รับ
<p>2. หาเงื่อนไขที่เพียงพอสำหรับการสู่เข้าแบบอ่อนและแบบเข้ม</p> <p>สู่จุดตรงร่วมของการส่งแบบกึ่งไม่ขยายเชิงเส้นกำกับที่ไม่ส่งไปยังตัวมันเอง โดยใช้ระเบียบวิธีทำซ้ำที่สร้างขึ้นตามข้อ 1.</p>	<p>1. หาเงื่อนไขที่เพียงพอสำหรับการสู่เข้าแบบอ่อนและแบบเข้มสู่จุดตรงร่วมของการส่งแบบกึ่งไม่ขยายเชิงเส้นกำกับที่ไม่ส่งไปยังตัวมันเอง</p> <p>2. ปรึกษาผู้เชี่ยวชาญเกี่ยวกับผลลัพธ์ที่ได้และปรับปรุงแก้ไขให้ถูกต้อง</p> <p>3. เขียนบทความเพื่อตีพิมพ์ในวารสารคณิตศาสตร์ระดับนานาชาติ</p>	<p>ได้ดำเนินการตามแผนที่วางไว้ทุกข้อ</p>	<p>1. หาเงื่อนไขที่เพียงพอสำหรับการสู่เข้าแบบอ่อน และแบบเข้มสู่จุดตรงร่วมของการส่งแบบกึ่งไม่ขยายเชิงเส้นกำกับที่ไม่ส่งไปยังตัวมันเอง</p> <p>2. ได้เขียนบทความวิจัย 1 เรื่อง และได้รับการตีพิมพ์ในวารสารคณิตศาสตร์ระดับนานาชาติ</p>

ค. รายงานการเงิน

งบประมาณที่ได้รับ 112,200 บาท ใช้จ่ายจริง 112,200 บาท รายละเอียดดังตารางข้างล่างนี้

รายการ	แผนการใช้จ่าย (บาท)	ใช้จ่ายจริง (บาท)
1. งบดำเนินงาน		
1.1 หมวดค่าตอบแทน ใช้น้อยและวัสดุ		
1.1.1 หมวดค่าตอบแทน		
- ค่าตอบแทนสำหรับผู้รับทุนวิจัย 20%	22,440	22,400
1.1.2 หมวดค่าใช้น้อย		
- ค่าใช้จ่ายสำหรับการเข้าร่วมประชุมวิชาการกับกลุ่มเครือข่ายวิจัยและเพื่อปริญญางานวิจัยกับผู้เชี่ยวชาญหรือนักวิจัยที่ปรึกษา	25,000	12,250
- ค่าใช้จ่ายสำหรับการเผยแพร่ผลงานวิจัย	40,000	1,500
1.1.3 หมวดค่าวัสดุ		
- ค่าหนังสือ ตำรา และวารสารประกอบการทำวิจัย	10,000	19,520
- ค่าวัสดุสำนักงาน	4,560	46,330
2. งบค่าสาธารณูปโภค		
- ค่าสาธารณูปโภค 10%	10,200	10,200
รวม	112,200	112,200



Common Fixed Points of a New Three-Step Iteration with Errors of Asymptotically Quasi-Nonexpansive Nonself-Mappings in Banach spaces

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ABSTRACT: In this paper, we introduce an iterative method for finding a common element of the set of solutions of a generalized mixed equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Then, we prove that the sequence converges strongly to a common element of the above two sets. Furthermore, we apply our result to prove three new strong convergence theorems in fixed point problems, mixed equilibrium problems, generalized equilibrium problems and equilibrium problems.

1. Introduction

We assume that X is a normed space and C is a nonempty subset of X . A mapping $T : C \rightarrow C$ is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n \rightarrow \infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and each $n \geq 1$. The class of asymptotically nonexpansive mappings is a natural generalization of the important class of nonexpansive mappings. Goebel and Kirk [3] proved that if C is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point. A mapping $T : C \rightarrow C$ is called *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n \rightarrow \infty} k_n = 1$ such that $\|T^n x - q\| \leq k_n \|x - q\|$ for all $x \in C, q \in F(T), n \geq 1$, where $F(T)$ is the set of fixed points of T . The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y \in C$ and each $n \geq 1$. It is easy to see that an asymptotically nonexpansive mapping must be uniformly L-Lipschitzian as well as asymptotically quasi-nonexpansive but the converse does not hold.

In 2000, Noor [9] introduced a three-step iterative sequence and studied the approximate solutions of variational inclusions in Hilbert spaces. Glowinski and Le Tallec [4] applied three-step iterative sequences for finding the approximate solutions of the elastoviscoplasticity problem, eigenvalue problems and in the liquid crystal theory. It has been shown in [1], that three-step method performs better than two-step and one-step methods for solving variational inequalities. The three-step schemes are natural generalization of the splitting methods to solve partial differential equations; see, Noor [9, 10, 11]. This signifies that Noor three-step

methods are robust and more efficient than the Mann (one-step) and Ishikawa (two-step) type iterative methods to solve problems of pure and applied sciences.

In 2001, Khan and Takahashi [5] have approximated common fixed points of two asymptotically nonexpansive mappings by the modified Ishikawa iteration. Recently Shahzad and Udomene [15] established convergence theorems for the modified Ishikawa iteration process of two asymptotically quasi-non expansive mappings to a common fixed point of the mappings. For related results with error terms, we refer to [2, 6, 13] and [15].

The purpose of this paper is to establish strong and weak convergence theorems of a new three-step iteration for three asymptotically quasi-nonexpansive non-self mappings in a uniformly convex Banach space. This scheme can be viewed as an extension of Xu and Noor [18], Suantai [16] and Nilsrakoo and Saejung [8].

Let X be a normed space. A subset C of X is said to be a *retract* of X if there exists a continuous map $P : X \rightarrow C$ such that $Px = x$ for all $x \in C$. Every closed convex set of a uniformly convex Banach space is a retract. A map $P : X \rightarrow C$ is said to be a *retraction* if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P . A mapping $T : C \rightarrow X$ is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|T(PT)^{n-1}x - q\| \leq k_n \|x - q\|$$

for all $x \in C, q \in F(T), n \geq 1$, where $F(T)$ is the set of fixed points of T and $(PT)^0 = I$, the identity operator on C .

The mapping $T : C \rightarrow X$ is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L \|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Let C be a nonempty closed convex subset of X and $P : X \rightarrow C$ a nonexpansive retraction of X onto C , and let $T_1, T_2, T_3 : C \rightarrow X$ be asymptotically quasi-nonexpansive mappings and F is the set of all common fixed points of T_i i.e., $F = \bigcap_{i=1}^3 F(T_i)$, where $F(T_i) = \{x \in C : T_i x = x\}$ for all $i = 1, 2, 3$. Then, for arbitrary $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P[a_n T_1 (PT_1)^{n-1} x_n + (1 - a_n - \delta_n) x_n + \delta_n u_n], \\ y_n &= P[b_n T_2 (PT_2)^{n-1} z_n + c_n T_1 (PT_1)^{n-1} x_n + (1 - b_n - c_n - \sigma_n) x_n + \sigma_n v_n], \\ x_{n+1} &= P[\alpha_n T_3 (PT_3)^{n-1} y_n + \beta_n T_2 (PT_2)^{n-1} z_n + \gamma_n T_1 (PT_1)^{n-1} x_n \\ &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) x_n + \rho_n w_n] \end{aligned} \tag{1}$$

for all $n \geq 1$, where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C .

Without errors ($\delta_n = \sigma_n = \rho_n \equiv 0$), and T_1, T_2, T_3 are self-maps of C , the iterative scheme (1) reduces to the following iterative scheme:

$$\begin{aligned} z_n &= a_n T_1^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T_2^n z_n + c_n T_1^n x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} &= \alpha_n T_3^n y_n + \beta_n T_2^n z_n + \gamma_n T_1^n x_n + (1 - \alpha_n - \beta_n - \gamma_n) x_n, \quad n \geq 1, \end{aligned} \tag{2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

If $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Nilsrakoo and Saejung [8].

If $\gamma_n \equiv 0$ and $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Suantai [16].

If $c_n = \beta_n = \gamma_n \equiv 0$ and $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Xu and Noor [18].

If $a_n = b_n = c_n \equiv 0$, then (2) reduces to the following iterative scheme:

$$(3) \quad x_{n+1} = \alpha_n T_3^n x_n + \beta_n T_2^n x_n + \gamma_n T_1^n x_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

To study strong and weak convergence theorems of the iterative scheme 1, we recall some useful well-known concepts and results.

Recall that a Banach space X is said to satisfy *Opial's condition* [12] if for each sequence $\{x_n\}$ and $x, y \in X$ with $x_n \rightarrow x$ weakly as $n \rightarrow \infty$ and $x \neq y$ imply that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

In what follows, we shall make use of the following lemmas.

Lemma 1.1. [17, Lemma 1]. *Let $\{a_n\}, \{b_n\}, \{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n \quad \text{for all } n = 1, 2, \dots$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

- (i) $\lim_{n \rightarrow \infty} a_n$ exists, and
- (ii) $\lim_{n \rightarrow \infty} a_n = 0$ whenever $\liminf_{n \rightarrow \infty} a_n = 0$.

Lemma 1.2. [7, Lemma 1.4]. *Let X be a uniformly convex Banach space and let $B_r = \{x \in X : \|x\| \leq r\}, r > 0$ be a closed ball of X . Then there exists a continuous, strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty), g(0) = 0$ such that*

$$\|\lambda x + \mu y + \zeta z + \vartheta w\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \zeta \|z\|^2 + \vartheta \|w\|^2 - \lambda \mu g(\|x - y\|)$$

for all $x, y, z, w \in B_r$ and all $\lambda, \mu, \zeta, \vartheta \in [0, 1]$ with $\lambda + \mu + \zeta + \vartheta = 1$.

Similar to Lemma 1.2, we can prove the next lemma.

Lemma 1.3. *Let X be a uniformly convex Banach space and let B_r be a closed ball of X . Then there exists a continuous, strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty), g(0) = 0$ such that*

$$\|\lambda x + \mu y + \zeta z + \vartheta w + \zeta s\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \zeta \|z\|^2 + \vartheta \|w\|^2 + \zeta \|s\|^2 - \lambda \mu g(\|x - y\|)$$

for all $x, y, z, w, s \in B_r$ and all $\lambda, \mu, \zeta, \vartheta, \zeta \in [0, 1]$ with $\lambda + \mu + \zeta + \vartheta + \zeta = 1$.

Lemma 1.4. [16, Lemma 2.7]. *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be so that $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v , respectively, then $u = v$.*

2. Main Results

In this section, we prove strong and weak convergence theorems for the iterative scheme (1) for asymptotically quasi-nonexpansive nonself-mappings in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \rightarrow X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1, l_n \geq 1, m_n \geq 1, \sum_{n=1}^{\infty} (k_n - 1) < \infty, \sum_{n=1}^{\infty} (l_n - 1) < \infty, \sum_{n=1}^{\infty} (m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n, b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1, \sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C . For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the*

sequences defined as in (1). Then

- (i) $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$.
- (ii) If one of the following conditions (a), (b), (c) and (d) holds, then $\lim_{n \rightarrow \infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.
 - (a) $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $0 < \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} (a_n + \delta_n) < 1$.
 - (b) $\liminf_{n \rightarrow \infty} \alpha_n, \liminf_{n \rightarrow \infty} b_n > 0$ and $0 < \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} (a_n + \delta_n) < 1$.
 - (c) $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.
 - (d) $0 < \liminf_{n \rightarrow \infty} \alpha_n$ and $0 < \liminf_{n \rightarrow \infty} c_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n + \sigma_n) < 1$.
- (iii) If either (a) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$ or (b) $\liminf_{n \rightarrow \infty} \alpha_n > 0$ and $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n + \sigma_n) < 1$, then $\lim_{n \rightarrow \infty} \|T_2(PT_2)^{n-1}z_n - x_n\| = 0$.
- (iv) If $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$, then $\lim_{n \rightarrow \infty} \|T_3(PT_3)^{n-1}y_n - x_n\| = 0$.



Proof. (i) Let $q \in F$. By (1), we obtain

$$\begin{aligned}
 \|z_n - q\| &= \|P[a_n T_1(PT_1)^{n-1}x_n + (1 - a_n - \delta_n)x_n + \delta_n u_n] - P(q)\| \\
 &\leq a_n \|T_1(PT_1)^{n-1}x_n - q\| + (1 - a_n - \delta_n)\|x_n - q\| + \delta_n \|u_n - q\| \\
 (4) \quad &\leq (1 + a_n(k_n - 1) - \delta_n)\|x_n - q\| + \delta_n \|u_n - q\|
 \end{aligned}$$

and

$$\begin{aligned}
 \|y_n - q\| &= \|P[b_n T_2(PT_2)^{n-1}z_n + c_n T_1(PT_1)^{n-1}x_n + (1 - b_n - c_n - \sigma_n)x_n + \sigma_n v_n] - P(q)\| \\
 &\leq b_n \|T_2(PT_2)^{n-1}z_n - q\| + c_n \|T_1(PT_1)^{n-1}x_n - q\| \\
 &\quad + (1 - b_n - c_n - \sigma_n)\|x_n - q\| + \sigma_n \|v_n - q\| \\
 (5) \quad &\leq b_n l_n \|z_n - q\| + c_n k_n \|x_n - q\| + (1 - b_n - c_n - \sigma_n)\|x_n - q\| \\
 &\quad + \sigma_n \|v_n - q\|.
 \end{aligned}$$

By (4) and (5), we obtain

$$\begin{aligned}
 \|x_{n+1} - q\| &= \|P[\alpha_n T_3(PT_3)^{n-1}y_n + \beta_n T_2(PT_2)^{n-1}z_n + \gamma_n T_1(PT_1)^{n-1}x_n + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] - P(q)\| \\
 &\leq \alpha_n \|T_3(PT_3)^{n-1}y_n - q\| + \beta_n \|T_2(PT_2)^{n-1}z_n - q\| \\
 &\quad + \gamma_n \|T_1(PT_1)^{n-1}x_n - q\| + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| \\
 &\quad + \rho_n \|w_n - q\| \\
 &\leq \alpha_n m_n \|y_n - q\| + \beta_n l_n \|z_n - q\| + \gamma_n k_n \|x_n - q\| \\
 &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| + \rho_n \|w_n - q\| \\
 &\leq (\alpha_n m_n b_n l_n + \beta_n l_n)\|z_n - q\| + \alpha_n m_n c_n k_n \|x_n - q\| \\
 &\quad + (\alpha_n m_n - \alpha_n m_n b_n - \alpha_n m_n c_n - \alpha_n m_n \sigma_n)\|x_n - q\| \\
 &\quad + \alpha_n m_n \sigma_n \|v_n - q\| + \gamma_n k_n \|x_n - q\| \\
 &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| + \rho_n \|w_n - q\| \\
 &\leq \|x_n - q\| + ((l_n - 1)(\alpha_n m_n b_n + \beta_n) + (k_n - 1)(\gamma_n + \alpha_n m_n c_n) \\
 &\quad + (\alpha_n m_n b_n l_n + \beta_n l_n) a_n + \alpha_n (m_n - 1))\|x_n - q\| \\
 &\quad + (m_n l_n + l_n) \delta_n \|u_n - q\| + m_n \sigma_n \|v_n - q\| + \rho_n \|w_n - q\|.
 \end{aligned}$$

Since $\{l_n\}, \{m_n\}, \{u_n\}, \{v_n\}, \{w_n\}$ are bounded, there exists a constant $K > 0$ such that $\alpha_n m_n b_n + \beta_n \leq K, \gamma_n + \alpha_n m_n c_n + (\alpha_n m_n b_n l_n + \beta_n l_n) a_n \leq K, (m_n l_n + l_n) \|u_n - q\| \leq K, m_n \|v_n - q\| \leq K, \|w_n - q\| \leq K$ and $\alpha_n \leq K$ for all $n \geq 1$. Then

$$(6) \quad \begin{aligned} \|x_{n+1} - q\| &\leq \left(1 + K((k_n - 1) + (l_n - 1) + (m_n - 1))\right) \|x_n - q\| \\ &\quad + K(\delta_n + \sigma_n + \rho_n) \end{aligned}$$

By Lemma 1.1, we obtain $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists.

Next, we want to prove (ii), (iii) and (iv). It follows from (i) that $\{x_n - q\}, \{T_1(PT_1)^{n-1}x_n - q\}, \{y_n - q\}, \{T_3(PT_3)^{n-1}y_n - q\}, \{z_n - q\}$ and $\{T_2(PT_2)^{n-1}z_n - q\}$ are all bounded. Let

$$M = \max \left\{ \sup_{n \geq 1} \|x_n - q\|, \sup_{n \geq 1} \|T_1(PT_1)^{n-1}x_n - q\|, \sup_{n \geq 1} \|y_n - q\|, \right. \\ \left. \sup_{n \geq 1} \|T_3(PT_3)^{n-1}y_n - q\|, \sup_{n \geq 1} \|z_n - q\|, \sup_{n \geq 1} \|u_n - q\|, \right. \\ \left. \sup_{n \geq 1} \|T_2(PT_2)^{n-1}z_n - q\|, \sup_{n \geq 1} \|v_n - q\|, \sup_{n \geq 1} \|w_n - q\| \right\}.$$

By Lemma 1.3, there exists a continuous, strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$(7) \quad \begin{aligned} \|\lambda x + \mu y + \xi z + \theta w + \zeta s\|^2 &\leq \lambda \|x\|^2 + \mu \|y\|^2 + \xi \|z\|^2 + \theta \|w\|^2 + \zeta \|s\|^2 \\ &\quad - \lambda \mu g(\|x - y\|) \end{aligned}$$

for all $x, y, z, w, s \in B_r$ and all $\lambda, \mu, \xi, \theta, \zeta \in [0, 1]$ with $\lambda + \mu + \xi + \theta + \zeta = 1$. By (7), we have

$$(8) \quad \begin{aligned} \|z_n - q\|^2 &= \|P[a_n T_1(PT_1)^{n-1}x_n + (1 - a_n - \delta_n)x_n + \delta_n u_n] - P(q)\|^2 \\ &\leq \|a_n(T_1(PT_1)^{n-1}x_n - q) + (1 - a_n - \delta_n)(x_n - q) + \delta_n(u_n - q)\|^2 \\ &\leq a_n \|T_1(PT_1)^{n-1}x_n - q\|^2 + (1 - a_n - \delta_n) \|x_n - q\|^2 \\ &\quad + \delta_n \|u_n - q\|^2 - a_n(1 - a_n - \delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\ &\leq a_n k_n^2 \|x_n - q\|^2 + (1 - a_n - \delta_n) \|x_n - q\|^2 + \delta_n \|u_n - q\|^2 \\ &\quad - a_n(1 - a_n - \delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\ &\leq (1 + a_n(k_n^2 - 1) - \delta_n) \|x_n - q\|^2 + \delta_n \|u_n - q\|^2 \end{aligned}$$

and

$$(9) \quad \begin{aligned} \|y_n - q\|^2 &= \|P[b_n T_2(PT_2)^{n-1}z_n + c_n T_1(PT_1)^{n-1}x_n + (1 - b_n - c_n - \sigma_n) \\ &\quad x_n + \sigma_n v_n] - P(q)\|^2 \\ &\leq \|b_n(T_2(PT_2)^{n-1}z_n - q) + c_n(T_1(PT_1)^{n-1}x_n - q) \\ &\quad + (1 - b_n - c_n - \sigma_n)(x_n - q) + \sigma_n(x_n - q)\|^2 \\ &\leq b_n \|T_2(PT_2)^{n-1}z_n - q\|^2 + (1 - b_n - c_n - \sigma_n) \|x_n - q\|^2 \\ &\quad + c_n \|T_1(PT_1)^{n-1}x_n - q\|^2 + \sigma_n \|v_n - q\|^2 \\ &\quad - b_n(1 - b_n - c_n - \sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\ &\leq b_n l_n^2 \|z_n - q\|^2 + (1 - b_n - c_n - \sigma_n) \|x_n - q\|^2 + c_n k_n^2 \|x_n - q\|^2 \\ &\quad + \sigma_n \|v_n - q\|^2 - b_n(1 - b_n - c_n - \sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \end{aligned}$$

By (7) , (8) and (9), we obtain

$$\begin{aligned}
 \|x_{n+1} - q\|^2 &= \|P[\alpha_n T_3(P T_3)^{n-1} y_n + \beta_n T_2(P T_2)^{n-1} z_n + \gamma_n T_1(P T_1)^{n-1} x_n \\
 &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] - P(q)\|^2 \\
 &\leq \alpha_n \|T_3(P T_3)^{n-1} y_n - q\|^2 + \beta_n \|T_2(P T_2)^{n-1} z_n - q\|^2 \\
 &\quad + \gamma_n \|T_1(P T_1)^{n-1} x_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 \\
 &\quad - \alpha_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_3(P T_3)^{n-1} y_n - x_n\|) \\
 &\quad + \rho_n \|w_n - q\|^2 \\
 &\leq \alpha_n m_n^2 \|y_n - q\|^2 + \beta_n l_n^2 \|z_n - q\|^2 + \gamma_n k_n^2 \|x_n - q\|^2 + \rho_n \|w_n - q\|^2 \\
 &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 \\
 &\quad - \alpha_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_3(P T_3)^{n-1} y_n - x_n\|) \\
 &\leq \alpha_n m_n^2 c_n k_n^2 \|x_n - q\|^2 + \alpha_n m_n^2 (1 - b_n - c_n - \sigma_n) \|x_n - q\|^2 \\
 &\quad + \gamma_n k_n^2 \|x_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 \\
 &\quad + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) \|z_n - q\|^2 + \alpha_n m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 \\
 &\quad - \alpha_n m_n^2 b_n (1 - b_n - c_n - \sigma_n) g(\|T_2(P T_2)^{n-1} z_n - x_n\|) \\
 &\quad - \alpha_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_3(P T_3)^{n-1} y_n - x_n\|) \\
 &\leq \alpha_n m_n^2 c_n k_n^2 \|x_n - q\|^2 + \alpha_n m_n^2 (1 - b_n - c_n) \|x_n - q\|^2 \\
 &\quad + \gamma_n k_n^2 \|x_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n) \|x_n - q\|^2 + (\alpha_n m_n^2 b_n l_n^2 \\
 &\quad + \beta_n l_n^2) \|x_n - q\|^2 + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) (a_n (k_n^2 - 1)) \|x_n - q\|^2 \\
 &\quad - (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) a_n (1 - a_n - \delta_n) g(\|T_1(P T_1)^{n-1} x_n - x_n\|) \\
 &\quad + (m_n^2 l_n^2 + l_n^2) \delta_n \|u_n - q\|^2 + m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 \\
 &\quad - \alpha_n m_n^2 b_n (1 - b_n - c_n - \sigma_n) g(\|T_2(P T_2)^{n-1} z_n - x_n\|) \\
 &\quad - \alpha_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_3(P T_3)^{n-1} y_n - x_n\|) \\
 &= \|x_n - q\|^2 + ((k_n^2 - 1) (\alpha_n m_n^2 c_n + \gamma_n + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) a_n) \\
 &\quad + (l_n^2 - 1) (\alpha_n m_n^2 b_n + \beta_n) + \alpha_n (m_n^2 - 1)) \|x_n - q\|^2 \\
 &\quad + (m_n^2 l_n^2 + l_n^2) \delta_n \|u_n - q\|^2 + m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 \\
 &\quad - \alpha_n m_n^2 b_n l_n^2 a_n (1 - a_n - \delta_n) g(\|T_1(P T_1)^{n-1} x_n - x_n\|) \\
 &\quad - \beta_n l_n^2 a_n (1 - a_n - \delta_n) g(\|T_1(P T_1)^{n-1} x_n - x_n\|) \\
 &\quad - \alpha_n m_n^2 b_n (1 - b_n - c_n - \sigma_n) g(\|T_2(P T_2)^{n-1} z_n - x_n\|) \\
 &\quad - \alpha_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_3(P T_3)^{n-1} y_n - x_n\|)
 \end{aligned}$$

Since $\{k_n\}, \{l_n\}, \{m_n\}, \{u_n\}, \{v_n\}, \{w_n\}$ are bounded and $\{x_n\}$ is bounded, there exist constants $K_0 > 0$ such that

$$\begin{aligned}
 (\alpha_n m_n^2 c_n + \gamma_n + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) a_n) \|x_n - q\|^2 &\leq K_0, \\
 (\alpha_n m_n^2 b_n + \beta_n) \|x_n - q\|^2 \leq K_0, \alpha_n \|x_n - q\|^2 &\leq K_0, (m_n^2 l_n^2 + l_n^2) \|u_n - q\|^2 \leq K_0, \\
 m_n^2 \|v_n - q\|^2 \leq K_0 \text{ and } \|w_n - q\|^2 &\leq K_0 \text{ for all } n \geq 1. \text{ Thus}
 \end{aligned}$$

$$\begin{aligned}
 \alpha_n m_n^2 b_n l_n^2 a_n (1 - a_n - \delta_n) g(\|T_1(P T_1)^{n-1} x_n - x_n\|) &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
 &\quad + K_0 ((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) \\
 (10) \quad &\quad + K_0 (\delta_n + \sigma_n + \rho_n).
 \end{aligned}$$

$$\begin{aligned}
 \beta_n l_n^2 a_n (1 - a_n - \delta_n) g(\|T_1(P T_1)^{n-1} x_n - x_n\|) &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
 &\quad + K_0 ((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) \\
 (11) \quad &\quad + K_0 (\delta_n + \sigma_n + \rho_n).
 \end{aligned}$$

$$\begin{aligned} \alpha_n m_n^2 b_n (1 - b_n - c_n - \sigma_n) g(\|T_2(PT_2)^{n-1} z_n - x_n\|) &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\quad + K_0((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) \\ &\quad + K_0(\delta_n + \sigma_n + \rho_n). \end{aligned} \tag{12}$$

$$\begin{aligned} \alpha_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_3(PT_3)^{n-1} y_n - x_n\|) &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\quad + K_0((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) \\ &\quad + K_0(\delta_n + \sigma_n + \rho_n). \end{aligned} \tag{13}$$

Again (7), we obtain

$$\begin{aligned} \|y_n - q\|^2 &= \|P[b_n T_2(PT_2)^{n-1} z_n + c_n T_1(PT_1)^{n-1} x_n + (1 - b_n - c_n - \sigma_n)x_n \\ &\quad + \sigma_n v_n] - P(q)\|^2 \\ &\leq c_n \|T_1(PT_1)^{n-1} x_n - q\|^2 + (1 - b_n - c_n - \sigma_n) \|x_n - q\|^2 \\ &\quad + b_n \|T_2(PT_2)^{n-1} z_n - q\|^2 + \sigma_n \|v_n - q\|^2 \\ &\quad - c_n (1 - b_n - c_n - \sigma_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \\ &\leq c_n k_n^2 \|x_n - q\|^2 + (1 - b_n - c_n - \sigma_n) \|x_n - q\|^2 + b_n l_n^2 \|z_n - q\|^2 \\ &\quad + \sigma_n \|v_n - q\|^2 - c_n (1 - b_n - c_n - \sigma_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \end{aligned} \tag{14}$$

By (7), (8) and (14), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|P[\alpha_n T_3(PT_3)^{n-1} y_n + \beta_n T_2(PT_2)^{n-1} z_n + \gamma_n T_1(PT_1)^{n-1} x_n \\ &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] - P(q)\|^2 \\ &\leq \alpha_n \|T_3(PT_3)^{n-1} y_n - q\|^2 + \beta_n \|T_2(PT_2)^{n-1} z_n - q\|^2 + \rho_n \|w_n - q\|^2 \\ &\quad + \gamma_n \|T_1(PT_1)^{n-1} x_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 \\ &\quad - \beta_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_2(PT_2)^{n-1} z_n - x_n\|) \\ &\leq \alpha_n m_n^2 \|y_n - q\|^2 + \beta_n l_n^2 \|z_n - q\|^2 + \gamma_n k_n^2 \|x_n - q\|^2 \\ &\quad + \rho_n \|w_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 \\ &\quad + \beta_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_2(PT_2)^{n-1} z_n - x_n\|) \\ &\leq \alpha_n m_n^2 c_n k_n^2 \|x_n - q\|^2 + \alpha_n m_n^2 (1 - b_n - c_n - \alpha_n) \|x_n - q\|^2 \\ &\quad + \gamma_n k_n^2 \|x_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 \\ &\quad + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) \|z_n - q\|^2 + \alpha_n m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 \\ &\quad - \alpha_n m_n^2 c_n (1 - b_n - c_n - \sigma_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \\ &\quad - \beta_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_2(PT_2)^{n-1} z_n - x_n\|) \\ &\leq \alpha_n m_n^2 c_n k_n^2 \|x_n - q\|^2 + \alpha_n m_n^2 (1 - b_n - c_n) \|x_n - q\|^2 \\ &\quad + \gamma_n k_n^2 \|x_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n) \|x_n - q\|^2 + (\alpha_n m_n^2 b_n l_n^2 \\ &\quad + \beta_n l_n^2) \|x_n - q\|^2 + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) (a_n (k_n^2 - 1)) \|x_n - q\|^2 \\ &\quad + (m_n^2 l_n^2 + l_n^2) \delta_n \|u_n - q\|^2 + m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 \\ &\quad - \alpha_n m_n^2 c_n (1 - b_n - c_n - \sigma_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \\ &\quad - \beta_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_2(PT_2)^{n-1} z_n - x_n\|) \\ &= \|x_n - q\|^2 + ((k_n^2 - 1)(\alpha_n m_n^2 c_n + \gamma_n + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) a_n) \\ &\quad + (l_n^2 - 1)(\alpha_n m_n^2 b_n + \beta_n) + \alpha_n (m_n^2 - 1)) \|x_n - q\|^2 \\ &\quad + (m_n^2 l_n^2 + l_n^2) \delta_n \|u_n - q\|^2 + m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 \\ &\quad - \alpha_n m_n^2 c_n (1 - b_n - c_n - \sigma_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \\ &\quad - \beta_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_2(PT_2)^{n-1} z_n - x_n\|) \end{aligned}$$

Thus

$$\alpha_n m_n^2 c_n (1 - b_n - c_n - \sigma_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + K_0((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) + K_0(\delta_n + \sigma_n + \rho_n).$$

(15)

$$\beta_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_2(PT_2)^{n-1} z_n - z_n\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + K_0((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) + K_0(\delta_n + \sigma_n + \rho_n).$$

(16)

By (7), (8) and (9), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|P[\alpha_n T_3(PT_3)^{n-1} y_n + \beta_n T_2(PT_2)^{n-1} z_n + \gamma_n T_1(PT_1)^{n-1} x_n + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] - P(q)\|^2 \\ &\leq \gamma_n \|T_1(PT_1)^{n-1} x_n - q\|^2 + \alpha_n \|T_3(PT_3)^{n-1} y_n - q\|^2 + \rho_n \|w_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 + \beta_n \|T_2(PT_2)^{n-1} z_n - q\|^2 - \gamma_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \\ &\leq \gamma_n k_n^2 \|x_n - q\|^2 + \alpha_n m_n^2 \|y_n - q\|^2 + \beta_n l_n^2 \|z_n - q\|^2 + \rho_n \|w_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 - \gamma_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \\ &\leq \gamma_n k_n^2 \|x_n - q\|^2 + \beta_n l_n^2 \|z_n - q\|^2 + \alpha_n m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 + \alpha_n m_n^2 b_n l_n^2 \|z_n - q\|^2 + \alpha_n m_n^2 (1 - b_n - c_n - \sigma_n) \|x_n - q\|^2 + \alpha_n m_n^2 c_n k_n^2 \|x_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \|x_n - q\|^2 - \gamma_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \\ &\leq \gamma_n k_n^2 \|x_n - q\|^2 + m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) \|x_n - q\|^2 + (m_n^2 l_n^2 + l_n^2) \delta_n \|u_n - q\|^2 + ((\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) a_n (k_n^2 - 1)) \|x_n - q\|^2 + \alpha_n m_n^2 c_n k_n^2 \|x_n - q\|^2 + (1 - \alpha_n - \beta_n - \gamma_n) \|x_n - q\|^2 + \alpha_n m_n^2 (1 - b_n - c_n) \|x_n - q\|^2 - \gamma_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \\ &= \|x_n - q\|^2 + ((k_n^2 - 1)(\alpha_n m_n^2 c_n + \gamma_n + (\alpha_n m_n^2 b_n l_n^2 + \beta_n l_n^2) a_n) + (l_n^2 - 1)(\alpha_n m_n^2 b_n + \beta_n) + \alpha_n (m_n^2 - 1)) \|x_n - q\|^2 + (m_n^2 l_n^2 + l_n^2) \delta_n \|u_n - q\|^2 + m_n^2 \sigma_n \|v_n - q\|^2 + \rho_n \|w_n - q\|^2 - \gamma_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|). \end{aligned}$$

Thus

$$\gamma_n (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) g(\|T_1(PT_1)^{n-1} x_n - x_n\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + K_0((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) + K_0(\delta_n + \sigma_n + \rho_n).$$

(17)

(ii) (a) Let $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $0 < \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} (a_n + \delta_n) < 1$. Then there exists a positive integer n_0 and $\eta, \eta' \in (0, 1)$ such that $0 < \eta < \beta_n, 0 < \eta < a_n$ and $a_n + \delta_n < \eta' < 1$

for all $n \geq n_0$. This implies by (11) that

$$\begin{aligned}
 \eta^2(1 - \eta')g(\|T_1(PT_1)^{n-1}x_n - x_n\|) &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
 &\quad + K_0((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) \\
 &\quad + K_0(\delta_n + \sigma_n + \rho_n)
 \end{aligned}
 \tag{18}$$

for all $n \geq n_0$. It follows from (18) that for $r \geq n_0$,

$$\begin{aligned}
 \sum_{n=n_0}^r g(\|T_1(PT_1)^{n-1}x_n - x_n\|) &\leq \frac{1}{\eta^2(1 - \eta')} \left(\sum_{n=n_0}^r (\|x_n - q\|^2 - \|x_{n+1} - q\|^2) \right. \\
 &\quad \left. + K_0 \sum_{n=n_0}^r ((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1) \right. \\
 &\quad \left. + \delta_n + \sigma_n + \rho_n) \right) \\
 &\leq \frac{1}{\eta^2(1 - \eta')} \left(\|x_{n_0} - q\|^2 + K_0 \sum_{n=n_0}^r ((k_n^2 - 1) \right. \\
 &\quad \left. + (l_n^2 - 1) + (m_n^2 - 1)) \right).
 \end{aligned}
 \tag{19}$$

Since $0 \leq t^2 - 1 \leq 2t(t - 1)$ for all $t \geq 1$ and $\sum_{n=1}^\infty (k_n - 1) < \infty$, $\sum_{n=1}^\infty (l_n - 1) < \infty$, $\sum_{n=1}^\infty (m_n - 1) < \infty$, we get $\sum_{n=1}^\infty (k_n^2 - 1) < \infty$, $\sum_{n=1}^\infty (l_n^2 - 1) < \infty$, $\sum_{n=1}^\infty (m_n^2 - 1) < \infty$. By inequality (19), let $r \rightarrow \infty$.

We get $\sum_{n=n_0}^\infty g(\|T_1(PT_1)^{n-1}x_n - x_n\|) < \infty$. Thus $\lim_{n \rightarrow \infty} g(\|T_1(PT_1)^{n-1}x_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n \rightarrow \infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

By using a similar method as in (ii) part (a) together with (10), (17), (15), (16), (12) and (13), the results in (ii) (b,c,d), (iii) (a,b) and (iv), respectively, can be proved. \square

Lemma 2.2. Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \rightarrow X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1, l_n \geq 1, m_n \geq 1, \sum_{n=1}^\infty (k_n - 1) < \infty, \sum_{n=1}^\infty (l_n - 1) < \infty, \sum_{n=1}^\infty (m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n, b_n + c_n + \sigma_n, \alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1, \sum_{n=1}^\infty \delta_n < \infty, \sum_{n=1}^\infty \sigma_n < \infty, \sum_{n=1}^\infty \rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C . For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (1). Suppose T_1, T_2, T_3 are uniformly L -Lipschitzian. If $\lim_{n \rightarrow \infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0, \lim_{n \rightarrow \infty} \|T_2(PT_2)^{n-1}z_n - x_n\| = 0, \lim_{n \rightarrow \infty} \|T_3(PT_3)^{n-1}y_n - x_n\| = 0$, then

- (i) $\lim_{n \rightarrow \infty} \|T_1x_n - x_n\| = 0,$
- (ii) $\lim_{n \rightarrow \infty} \|T_2x_n - x_n\| = 0,$ and
- (iii) $\lim_{n \rightarrow \infty} \|T_3x_n - x_n\| = 0.$

Proof. Since

$$\begin{aligned}
 \|x_{n+1} - x_n\| &\leq \alpha_n \|T_3(PT_3)^{n-1}y_n - x_n\| + \beta_n \|T_2(PT_2)^{n-1}z_n - x_n\| \\
 &\quad + \gamma_n \|T_1(PT_1)^{n-1}x_n - x_n\| + \rho_n \|w_n - x_n\| \\
 &\rightarrow 0 \text{ as } n \rightarrow \infty,
 \end{aligned}$$

we obtain

$$\begin{aligned}
 \|T_1(PT_1)^{n-1}x_{n+1} - x_{n+1}\| &\leq \|T_1(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{n-1}x_n\| \\
 &\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| + \|x_{n+1} - x_n\| \\
 &\leq L\|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \\
 &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}
 \tag{20}$$

By (20), we get

$$\begin{aligned}
 \|T_1x_n - x_n\| &\leq \|T_1(PT_1)^{n-1}x_n - x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1x_n\| \\
 &\leq \|T_1(PT_1)^{n-1}x_n - x_n\| + L\|T_1(PT_1)^{n-2}x_n - x_n\| \\
 &\longrightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \|T_1x_n - x_n\| = 0$. Next, we want to prove (ii). Since

$$\|z_n - x_n\| \leq a_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \delta_n\|u_n - x_n\| \longrightarrow 0 \text{ as } n \rightarrow \infty,$$

we obtain

$$\begin{aligned}
 \|T_2(PT_2)^{n-1}x_{n+1} - x_{n+1}\| &\leq \|T_2(PT_2)^{n-1}x_{n+1} - T_2(PT_2)^{n-1}x_n\| \\
 &\quad + \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| \\
 &\quad + \|T_2(PT_2)^{n-1}z_n - x_n\| + \|x_{n+1} - x_n\| \\
 &\leq L\|x_{n+1} - x_n\| + L\|z_n - x_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\| \\
 &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 \|T_2x_n - x_n\| &\leq \|T_2(PT_2)^{n-1}x_n - x_n\| + \|T_2(PT_2)^{n-1}x_n - T_2x_n\| \\
 &\leq \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\| \\
 &\quad + L\|T_2(PT_2)^{n-2}x_n - x_n\| \longrightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \|T_2x_n - x_n\| = 0$, so (ii) is obtained. Since

$$\begin{aligned}
 \|y_n - x_n\| &\leq b_n\|T_2(PT_2)^{n-1}z_n - x_n\| + c_n\|T_1(PT_1)^{n-1}x_n - x_n\| \\
 &\quad + \sigma_n\|v_n - x_n\| \longrightarrow 0
 \end{aligned}$$

and $\|T_3(PT_3)^{n-1}y_n - x_n\| \longrightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\begin{aligned}
 \|T_3(PT_3)^{n-1}x_n - x_n\| &\leq \|T_3(PT_3)^{n-1}y_n - T_3(PT_3)^{n-1}x_n\| \\
 &\quad + \|T_3(PT_3)^{n-1}y_n - x_n\| \\
 &\leq L\|y_n - x_n\| + \|T_3(PT_3)^{n-1}y_n - x_n\| \\
 &\longrightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Thus

$$\begin{aligned}
 \|T_3(PT_3)^{n-1}x_{n+1} - x_{n+1}\| &\leq \|T_3(PT_3)^{n-1}x_{n+1} - T_3(PT_3)^{n-1}x_n\| \\
 &\quad + \|T_3(PT_3)^{n-1}y_n - T_3(PT_3)^{n-1}x_n\| \\
 &\quad + \|T_3(PT_3)^{n-1}y_n - x_n\| + \|x_{n+1} - x_n\| \\
 &\leq L\|x_{n+1} - x_n\| + L\|y_n - x_n\| + \|T_3(PT_3)^{n-1}y_n - x_n\| \\
 &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

It follows that,

$$\begin{aligned} \|T_3x_n - x_n\| &\leq \|T_3(PT_3)^{n-1}x_n - x_n\| + \|T_3(PT_3)^{n-1}x_n - T_3x_n\| \\ &\leq \|T_3(PT_3)^{n-1}x_n - x_n\| + L\|T_3(PT_3)^{n-2}x_n - x_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence (iii) is satisfied. □

Theorem 2.3. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \rightarrow X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1, l_n \geq 1, m_n \geq 1, \sum_{n=1}^{\infty} (k_n - 1) < \infty, \sum_{n=1}^{\infty} (l_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n, b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C . Assume that T_1, T_2, T_3 are uniformly L -Lipschitzian. If one of $T_i (i = 1, 2, 3)$ is a completely continuous and one of the following conditions (C1)-(C5) is satisfied:*

- (C1) $0 < \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} (a_n + \delta_n) < 1,$
 $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n + \sigma_n) < 1,$ and
 $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$
- (C2) $0 < \liminf_{n \rightarrow \infty} b_n, \liminf_{n \rightarrow \infty} c_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n + \sigma_n) < 1,$ and
 $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$
- (C3) $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n + \sigma_n) < 1,$ and
 $0 < \liminf_{n \rightarrow \infty} \alpha_n, \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$
- (C4) $\liminf_{n \rightarrow \infty} b_n > 0,$ and $0 < \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} (a_n + \delta_n) < 1,$ and
 $0 < \liminf_{n \rightarrow \infty} \alpha_n, \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$
- (C5) $0 < \liminf_{n \rightarrow \infty} \alpha_n, \liminf_{n \rightarrow \infty} \beta_n, \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ defined as in (1) converge strongly to a common fixed point of T_1, T_2 and T_3 .

Proof. Suppose one of the conditions (C1)-(C5) is satisfied. By Lemma 2.2, we obtain $\lim_{n \rightarrow \infty} \|T_i x_n - x_n\| = 0$ for $i = 1, 2, 3$. Assume one of T_1, T_2 and T_3 says T_1 is completely continuous. Since $\{x_n\}$ is a bounded sequence in C , there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_1 x_{n_k}\}$ converges to $q \in C$. Since $\|x_{n_k} - q\| \leq \|T_1 x_{n_k} - x_{n_k}\| + \|T_1 x_{n_k} - q\|$, we get $\lim_{k \rightarrow \infty} \|x_{n_k} - q\| = 0$.

Thus $\{x_{n_k}\}$ converges to $q \in C$. By continuity of T_i , we have $T_i x_{n_k} \rightarrow T_i q$ as $k \rightarrow \infty$. Since

$$\|T_i q - q\| \leq \|T_i x_{n_k} - T_i q\| + \|T_i x_{n_k} - x_{n_k}\| + \|x_{n_k} - q\| \rightarrow 0 \text{ as } k \rightarrow \infty,$$

we obtain $T_i q = q (i = 1, 2, 3)$. Thus $q \in F$. By Lemma 2.1 (i), $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists. This implies

$\lim_{n \rightarrow \infty} \|x_n - q\| = 0$. By Lemma 2.1, we have

$$\|T_1(PT_1)^{n-1}x_n - x_n\| \rightarrow 0 \text{ and } \|T_2(PT_2)^{n-1}z_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It follows that

$$\begin{aligned} \|y_n - x_n\| &\leq b_n \|T_2(PT_2)^{n-1}z_n - x_n\| + c_n \|T_1(PT_1)^{n-1}x_n - x_n\| + \sigma_n \|v_n - x_n\| \\ &\rightarrow 0 \text{ and } \|z_n - x_n\| \leq a_n \|T_1(PT_1)^{n-1}x_n - x_n\| + \delta_n \|u_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

These imply $\lim_{n \rightarrow \infty} y_n = q$ and $\lim_{n \rightarrow \infty} z_n = q$. □

Remark 2.4. In Theorem 2.3, assume that T_1, T_2 and T_3 are asymptotically nonexpansive self-mappings of C such that one of them is completely continuous and thus T_1, T_2, T_3 are uniformly

L-Lipschitzian and $\delta_n = \sigma_n = \rho_n \equiv 0$. We obtain the following results.

- (1) If one of the conditions (C1) – (C5) is satisfied, then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ defined as in (2) converge strongly to a common fixed point of T_1, T_2 and T_3 .
- (2) If $T := T_1 = T_2 = T_3$ and one of the conditions (C1) – (C5) is satisfied, then we obtain the results of Nilsrakoo and Saejung [8].
- (3) If $T := T_1 = T_2 = T_3$ and one of the conditions (C1), (C2), (C4) is satisfied and $\gamma_n \equiv 0$, then we obtain the results of Suantai [16].
- (4) If $T := T_1 = T_2 = T_3$ with condition (C1) is satisfied and $c_n = \beta_n = \gamma_n \equiv 0$, then we obtain the results of Xu and Noor [18].
- (5) If the condition (C5) is satisfied and $a_n = b_n = c_n \equiv 0$, then the sequence $\{x_n\}$ defined as in (3) converges strongly to a common fixed point of T_1, T_2 and T_3 .

The mapping $T : C \rightarrow X$ with $F(T) \neq \emptyset$ is said to satisfy Condition A [14] if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf \{\|x - q\| : q \in F(T)\}$. As Tan and Xu [17] pointed out, the Condition A is weaker than the compactness of C .

The following result gives a strong convergence theorem for asymptotically quasi-nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Condition A.

Theorem 2.5. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \rightarrow X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1, l_n \geq 1, m_n \geq 1, \sum_{n=1}^{\infty} (k_n - 1) < \infty, \sum_{n=1}^{\infty} (l_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n, b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C . Suppose T_1 satisfies Condition A and T_2, T_3 are uniformly L-Lipschitzian and one of the conditions (C1)-(C5) in Theorem 2.3 is satisfied. Then the sequence $\{x_n\}$ defined as in (1) converges strongly to a common fixed point of T_1, T_2 and T_3 .*

Proof. Let $q \in F$. By Lemma 2.1, $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists. Thus $\{x_n - q\}$ is bounded. Then there is a constant H such that $\|x_n - q\| \leq H$ for all $n \geq 1$. This together with (6), we have

$$(21) \quad \|x_{n+1} - q\| \leq \|x_n - q\| + D_n,$$

where $D_n = KH((k_n - 1) + (l_n - 1) + (m_n - 1)) + K(\delta_n + \sigma_n + \rho_n) < \infty$ for all $n \geq 1$. By Lemma 2.2, we have $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ ($i = 1, 2, 3$). Since T_1 satisfies Condition A, we obtain $\lim_{n \rightarrow \infty} d(x_n, F(T_1)) = 0$. Next, we want to show $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n \rightarrow \infty} d(x_n, F(T_1)) = 0$ and $\sum_{n=1}^{\infty} D_n < \infty$, for any $\epsilon > 0$, there exists a positive integer n_0 such that $d(x_n, F(T_1)) < \epsilon/4$ and $\sum_{k=n_0}^n D_k < \epsilon/2$ for all $n \geq n_0$. Now, let $n \in \mathbb{N}$ be such that $n \geq n_0$. Then we can find $q^* \in F$ such that $\|x_n - q^*\| < \epsilon/4$. This implies by (21) that for $m \geq 1$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q^*\| + \|x_n - q^*\| \\ &\leq 2\|x_n - q^*\| + \sum_{k=n}^{n+m-1} D_k \\ &= 2\|x_n - q^*\| + \sum_{k=n_0}^{n+m-1} D_k < 2\left(\frac{\epsilon}{4}\right) + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so it is convergent. Let $\lim_{n \rightarrow \infty} x_n = p$. Since $d(x_n, F(T_1)) \rightarrow 0$ as $n \rightarrow \infty$, it follows that $d(p, F(T_1)) = 0$ and hence $p \in F(T_1)$. Next, we want to show $p \in F(T_2) \cap F(T_3)$. Since T_2, T_3 are uniformly L-Lipschitzian and by Lemma 2.2, we obtain

$$\begin{aligned} \|T_i p - p\| &\leq \|T_i x_n - T_i p\| + \|T_i x_n - x_n\| + \|x_n - p\| \\ &\leq L\|x_n - p\| + \|T_i x_n - x_n\| + \|x_n - p\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus $T_i p = p$ ($i = 2, 3$). Therefore $p \in F$. □

In the next result, we prove weak convergence for the iterative scheme (1) for asymptotically quasi-nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \rightarrow X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1, l_n \geq 1, m_n \geq 1, \sum_{n=1}^{\infty} (k_n - 1) < \infty, \sum_{n=1}^{\infty} (l_n - 1) < \infty$ and*

$\sum_{n=1}^{\infty} (m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n, b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \delta_n <$

$\infty, \sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C . Suppose T_1, T_2, T_3 are uniformly L-Lipschitzian and $I - T_i$ ($i = 1, 2, 3$) is demiclosed at 0. If one of the following conditions (C1)-(C5) in Theorem 2.3 is satisfied, then the sequence $\{x_n\}$ defined as in (1) converges weakly to a common fixed point of T_1, T_2 and T_3 .

Proof. Assume one of the conditions (C1)-(C5) is satisfied. By Lemma 2.1 and Lemma 2.2, we have $\lim_{n \rightarrow \infty} \|T_i x_n - x_n\| = 0$ ($i = 1, 2, 3$). Since X is uniformly convex and $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_n \rightarrow u$ weakly as $n \rightarrow \infty$. Since $I - T_i$ is demiclosed at 0, we obtain $u \in F$. Suppose subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v , respectively. Also, since $I - T_i$ ($i = 1, 2, 3$) is demiclosed at 0, we have u and $v \in F$. By Lemma 2.1, we obtain $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. It follows from Lemma 1.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3 . □

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