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การออกแบบแบบแบ่งกลุ่มเป็น 2 คลาสขนาด 1 และ m โดยมี $\lambda_1 = 5,\ \lambda_2 = 1$

GDDs with two associate classes and with one group of size 1 and m groups of size n and $\lambda_1 = 5,\ \lambda_2 = 1$

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บทคัดย่อ

การออกแบบแบบแบ่งกลุ่ม $GDD(v = 1 + n + n + \cdots + n, 1 + m,\ 3,\ \lambda_1,\ \lambda_2)$ คือ คู่อันดับ (V, B) โดยที่ V เป็นเซตของสัญลักษณ์ขนาด $(1 + mn)$ และ B เป็นเซตของกลุ่มย่อยขนาด 3 (เรียกว่า บล็อก) ของ V ที่สอดคล้องกับเงื่อนไขต่อไปนี้ : เซต V ขนาด $(1 + mn)$ ถูกแบ่งออกเป็น 1 กลุ่มที่มีขนาด 1 และ m กลุ่มที่แต่ละกลุ่มมีขนาด n โดยที่ สมาชิกในแต่ละคู่ที่อยู่ในกลุ่มเดียวกันจะปรากฏร่วมกันในบล็อกของ B จำนวน λ_1 บล็อก และ สมาชิกในแต่ละคู่ที่จากต่างกลุ่มจะปรากฏร่วมกันในบล็อกของ B จำนวน λ_2 บล็อก ในผลงานวิจัยนี้ เราได้หาหลักเกณฑ์ที่จำเป็นและเพียงพอสำหรับการมีอยู่ของการออกแบบแบบแบ่งกลุ่ม ในรูปแบบ $GDD(v = 1 + n + n + \cdots + n, 1 + m,\ 3, 5, 1)$ โดยที่ $m,\ n\ \geq 3$

ABSTRACT

A group divisible design $GDD(v = 1 + n + n + \cdots + n, 1 + m,\ 3,\ \lambda_1,\ \lambda_2)$ is an ordered pair (V, B) where V is an $(1 + mn) -$ set of symbols and B is a collection of $3 -$ subsets (called blocks) of V satisfying the following properties: the $(1 + mn) -$ set is divided into 1 group of size 1 and m groups of size n; each pair of symbols from the same group occurs in exactly λ_1 blocks in B; and each pair of symbols from different groups occurs in exactly λ_2 blocks in B. In this paper, we find necessary and sufficient conditions for the existence of a group divisible design group divisible design $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$, where $m, n\ \geq 3$.

คำสำคัญ: การออกแบบแบ่งกลุ่มได้ การออกแบบบล็อกสมดุลแบบไม่สมบูรณ์ การแยกกราฟ

Keywords: Group divisible designs, Balanced incomplete block design, Graph decomposition

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INTRODUCTION

A *balanced incomplete block design* denoted by $BIBD(v, b, r, k, \lambda)$ is a pair (V, B) where V is a $v-$ set of symbols and B is a collection of b $k-$ subsets (called blocks) of V such that each element in V is contained in exactly r blocks and any $2-$ subsets of V is contained in exactly λ blocks. The numbers v, b, r, k and λ are parameters of $BIBD$.

Trivial necessary conditions for the existence of a $BIBD(v, b, r, k, \lambda)$ are

$$vr = bk \quad \text{and} \quad r(k-1) = \lambda(v-1).$$

With these conditions, a $BIBD(v, b, r, k, \lambda)$ is usually written as $BIBD(v, k, \lambda)$.

A group divisible design denoted by $GDD(v = v_1 + v_2 + \cdots + v_g,\ g,\ k,\ \lambda_1,\ \lambda_2)$, is an triple (V, G, B), where V is a $v-$ set of symbols, G is a partition of v into g sets of size $v_1,\ v_2,\ \cdots,\ v_g$ and B is a collection of $k-$ subsets (called blocks) of V such that each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks. Elements occurring together in the same group are called *first associates*, and elements occurring in different groups are called *second associates*. It is clear that if the indices λ_1 and λ_2 are equal, then the design is a balanced incomplete block design $BIBD$. The existence problem of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs. More recently, much work has been done on the existence of such designs when $\lambda_1 = 0$ (Colbourn and Dinitz, 2007) and Fu *et al.* (2000) called the designs when $\lambda_1 = 0$ as a partially balanced incomplete block designs ($PBIBDs$). The existence question for $k = 3$ was solved by Fu and Rodger (1998) and Fu *et al.,* (2000) when all groups are of the same size. Recently, the necessary and sufficient conditions for the existence of a $GDD(v = 1 + n + n, 3, 3, \lambda_1, \lambda_2)$ were found by Lapchinda, Punnim and Pabhapote (Lapchinda *et al.*, 2013; 2014).

In this paper, we consider the problem of determining necessary conditions for the existence of a $GDD(v = v_1 + v_2 + \cdots + v_g,\ g, k,\ \lambda_1,\ \lambda_2)$, and prove that the conditions are sufficient. We will see that necessary conditions for the existence of a $GDD(v = 1 + n + n + \cdots + n, 1 + m,\ 3,\ \lambda_1,\ \lambda_2)$ can be easily obtained by describing it graphically as follows.

Let G and H be multigraphs. A *decomposition* of H is a set of subgraphs of H whose edge sets partition the edge set of H. A subgraph in a decomposition is a $G-$ *block* if it is isomorphic to G. A decomposition of H into $G-$blocks is a $G-$*decomposition* of H. We write $G|H$ if there exists a $G-$decomposition of H.

Let λK_v denote the multigraph on v vertices in which each pair of distinct vertices is joined by exactly λ edges. Let G_1 and G_2 be vertex disjoint graphs. Then $G_1 \vee_\lambda G_2$ is the graph obtained from the union of G_1 and G_2 and by joining each vertex in G_1 to each vertex in G_2 with λ edges.

PRELIMINARY RESULTS

In this section, we will review some known results concerning triple designs that will be used in the sequel, most of which are taken from Lindner and Rodger (2009).

The following results on the existence of $\lambda-$fold triple systems are well known (Lindner and Rodger, 2009).

Theorem 2.1 (Lindner and Rodger, 2009). Let n be a positive integer. Then a $BIBD(n,3,\lambda)$ exists if and only if λ and n are in one of the following cases:

1) $\lambda \equiv 0\ \ (mod\ 6)\ and\ n \neq 2$,
2) $\lambda \equiv 1\ or\ 5\ (mod\ 6)\ and\ n \equiv 1\ or\ 3\ (mod\ 6)$,
3) $\lambda \equiv 2\ or\ 4\ (mod\ 6) and\ n \equiv 0\ or\ 1\ (mod\ 3)$,
4) $\lambda \equiv 3\ (mod\ 6)\ and\ n\ is\ odd$

The following Results are found in handbook of combinatorial designs by Colbourn and Dinitz (2007).

Theorem 2.2 (Colbourn and Dinitz, 2007) The necessary and sufficient conditions for the existence of a $GDD(v = n + n + \cdots + n, m, 3,\ 0, \lambda)$ are:

1) $m \geq 3$,
2) $\lambda(m-1)n \equiv 0\ (mod\ 2)$,
3) $\lambda m(m-1)n^2 \equiv 0\ (mod\ 6)$.

Theorem 2.3 (Colbourn and Dinitz, 2007) The necessary and sufficient conditions for the existence of a $GDD(v = n + n + \cdots + n, m,\ 3,\ \lambda_1,\ \lambda_2)$ are:

1) $\lambda_1(n-1) + \lambda_2(m-1)n \equiv 0\ (mod\ 2)$ and
2) $\lambda_1 mn(n-1) + \lambda_2 m(m-1)n^2 \equiv 0\ (mod\ 3)$.

The following notations will be used throughout the paper for our constructions.

1) Let V be a $v-$set. We use $K(V)$ for the complete graph K_V on the vertex set V.

2) Let V be a $v-$set. $BIBD(v,3,\lambda)$ can be defined as

$$BIBD(v,3,\lambda) = \{B \mid (V,B)\ is\ a\ BIBD(v,3,\lambda\}.$$

3) We denote $(X,Y,Z;B)$ for a $GDD(v = n_1 + n_2 + n_3, 3, 3, \lambda_1, \lambda_2)$ if X, Y and Z are n_1-set, n_2-set, and n_3-set, respectively.

4) We denote $(X, Y_1, Y_2, \dots, Y_m; B)$ for a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, \lambda_1, \lambda_2)$ if $|Y_i| = n$.

5) We denote $(Y_1, Y_2, \dots, Y_m; B)$ for a $GDD(v = n + n + \cdots + n, m, 3, \lambda_1, \lambda_2)$ if $|Y_i| = n$.

6) Let X, Y and Z be disjoint sets of cardinality $n_1,\ n_2$ and n_3, respectively. We define $GDD(X,Y,Z;\lambda_1,\lambda_2)\ a\ GDD(X,Y,Z;\lambda_1,\ \lambda_2) = \{B \mid (X,Y,Z;B)$ is a $GDD(v = n_1 + n_2 + n_3, 3,3, \lambda_1, \lambda_2)\}$.

7) When we say that B is a collection of subsets (blocks) of a $v-$set V, B may contain repeated blocks. Thus " ∪ " in our construction will be used for the union of multisets.

In 2014, Lapchinda, Punnim and Pabhapote (Lapchinda *et al.*, 2014) has found the necessary and sufficient conditions for the existence of a $GDD(v = 1 + n + n, 3, 3, \lambda_1, \lambda_2)$.

Theorem 2.4 (Lapchinda *et al.*, 2014) Let n be an integer, $n \geq 3$ and $\lambda_1 > \lambda_2$. Then $GDD(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ exists if and only if

$$\lambda_1 n(n-1) + \lambda_2 n(n+2) \equiv 0 \ (mod\ 3)\ and\ \lambda_1(n-1) + \lambda_2(n+1) \equiv 0 \ (mod\ 2).$$

In 2019, Lapchinda (2019) has found the necessary and sufficient conditions for the existence of a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, \lambda_1, \lambda_2)$ where $\lambda_1 = 3,\ \lambda_2 = 1$ and $m, n \geq 3$.

Theorem 2.5 (Lapchinda, 2019) Let m, n and t be positive integers and $m, n \geq 3$. A $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 3, 1)$ exists if and only if

1) $m \equiv 0, 2, 3\ or\ 5\ (mod\ 6)\ and\ n \equiv 0\ or\ 4\ (mod\ 6)$,
2) $m \equiv 0\ (mod\ 6)\ and\ n \equiv 1\ or\ 3\ (mod\ 6)$,
3) $m \equiv 0, 2\ or\ 3\ (mod\ 6)\ and\ n \equiv 2\ (mod\ 6)$,
4) $m \equiv 0\ (mod\ 6)\ and\ n \equiv 5\ (mod\ 6)$,
5) $m \equiv 1\ (mod\ 6)\ and\ n \equiv 0\ (mod\ 6)$,
6) $m \equiv 4\ (mod\ 6)\ and\ n \equiv 0\ or\ 3\ (mod\ 6)$.

RESULTS AND DISCUSSION

We have the following theorem.

Theorem 3.1 Let m and n be positive integers and $m, n \geq 3$. If a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$ exists, then

1) $m \equiv 0\ or\ 2\ (mod\ 6)\ and\ n \equiv 0, 1, 2, 3, 4\ or\ 5\ (mod\ 6)$,
2) $m \equiv 1\ (mod\ 6)\ and\ n \equiv 0\ (mod\ 6)$,
3) $m \equiv 3\ (mod\ 6)\ and\ \equiv 0, 2\ or\ 4\ (mod\ 6)$,
4) $m \equiv 4\ (mod\ 6)\ and\ n \equiv 0\ or\ 3\ (mod\ 6)$,
5) $m \equiv 5\ (mod\ 6)\ and\ n \equiv 0, 2\ or\ 4\ (mod\ 6)$.

Proof. The existence of a $GDD(v = 1 + n + n + \cdots + n, 1 + m,\ 3, 5, 1)$ is easily seen to be equivalent to the existence of a K_3 −decomposition of $5K_n \vee 5K_n \vee \cdots \vee 5K_n$ (m copies of $5K_n$).

The graph $5K_n \vee 5K_n \vee \cdots \vee 5K_n$ is of order $1 + mn$ and size

$$5m\binom{n}{2} + \left[mn + \binom{m}{2}n^2\right]$$

It contains 1 vertex of degree mn and mn vertices of degree $5(n-1) + [1 + (m-1)n]$. Thus the degree of each vertex must be even and size of the graph must be divisible by 3. Then the existence of a $K_3 -$ decomposition of $5K_n \vee 5K_n \vee \cdots \vee 5K_n$ implies

$$3\left|\left[5m\binom{n}{2} + \left[mn + \binom{m}{2}n^2\right]\right], 2|mn\ and\ 2\right|\left[5(n-1) + [1 + (m-1)n]\right]$$ ■

We will prove that the necessary conditions in Theorem 3.1 are sufficient.

Lemma 3.1 Let m and n be positive integers and $m, n \geq 3$. If $m \equiv 0$ or $2\ (mod\ 6)$ and $n \equiv 0, 1, 3$ or $4\ (mod\ 6)$, then there exists a

$$GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1).$$

Proof. Let $m \equiv 0$ or $2\ (mod\ 6)$ and $n \equiv 0, 1, 3$ or $4\ (mod\ 6)$. Let X be a singleton set and $Y_1, Y_2, \ldots, Y_m$ be $n-$ sets. Since $mn + 1 \equiv 1$ or $3\ (mod\ 6)$, by Theorem 2.1 2), there exists a $BIBD(1 + mn, 3, 1)$. Let $(X \cup (\cup_{i=1}^{m} Y_i), B)$ be a $BIBD(X \cup (\cup_{i=1}^{m} Y_i), 3, 1)$. Since $n \equiv 0, 1, 3$ or $4\ (mod\ 6)$, by Theorem 2.1 3), there exists a $BIBD(n, 3, 4)$. Let (Y_i, B_i) be a $BIBD(Y_i, 3, 4)$, where $i = 1, 2, 3, \ldots, m$. Then $B = \cup (\cup_{i=1}^{m} B_i)$ forms a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$. ■

Using a similar construction as above, it is true for the case when $m \equiv 1\ (mod\ 6)$ and $n \equiv 0\ (mod\ 6)$, $m \equiv 3\ (mod\ 6)$ and $n \equiv 0$ or $4\ (mod\ 6)$, and $m \equiv 4\ (mod\ 6)$ and $n \equiv 0$ or $3\ (mod\ 6)$, and $m \equiv 5\ (mod\ 6)$ and $n \equiv 0$ or $4\ (mod\ 6)$. Thus we have the following lemmas.

Lemma 3.2 Let m and n be positive integers and $m, n \geq 3$. If $m \equiv 1\ (mod\ 6)$ and $n \equiv 0\ (mod\ 6)$, then there exists a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$.

Lemma 3.3 Let m and n be positive integers and $m, n \geq 3$. If $m \equiv 3\ (mod\ 6)$ and $n \equiv 0$ or $4\ (mod\ 6)$, then there exists a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$.

Lemma 3.4 Let m and n be positive integers and $m, n \geq 3$. If $m \equiv 4\ (mod\ 6)$ and $n \equiv 0$ or $3\ (mod\ 6)$, then there exists a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$.

Lemma 3.5 Let m and n be positive integers and $m, n \geq 3$. If $m \equiv 5\ (mod\ 6)$ and $n \equiv 0$ or $4\ (mod\ 6)$, then there exists a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$.

Lemma 3.6 Let m and n be positive integers and $m, n \geq 3$. If $m \equiv 3\ (mod\ 6)$ and $n \equiv 2\ (mod\ 6)$, then there exists a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$.

Proof. Let $m \equiv 3\ (mod\ 6)$ and $n \equiv 2\ (mod\ 6)$. Let X be a singleton set and $Y_1, Y_2, \ldots, Y_m$ be $n-$sets. Since $1 + n \equiv 3\ (mod\ 6)$, by Theorem 2.1 2), there exists a $BIBD(1 + n, 3, 1)$. Let $(X \cup Y_i, B_i)$ be a $BIBD(X \cup Y_i, 3, 1)$ for $i = 1, 2, 3, \ldots, m$. Since $m \equiv 3\ (mod\ 6)$ and $n \equiv 2\ (mod\ 6)$, by Theorem 2.3, there exists a $GDD(v = n + n + \cdots + n, 1 + m, 3, 4, 1)$. Let $(Y_1, Y_2, \ldots, Y_m; B_0)$ be a $GDD(v = n + n + \cdots + n, 1 + m, 3, 4, 1)$. Then $B = B_0 \cup (\cup_{i=1}^{m} B_i)$ forms a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$. ■

By the same construction, we have the following lemma.

Lemma 3.7 Let m and n be positive integers and $m, n \geq 3$. If $m \equiv 5\ (mod\ 6)$ and $n \equiv 2\ (mod\ 6)$, then there exists a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$.

Thus we have the following lemma.

Lemma 3.8 Let m and n be positive integers and $m, n \geq 3$. If $m \equiv 0$ or $2\ (mod\ 6)$ and $n \equiv 2$ or $5\ (mod\ 6)$, then there exists a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5, 1)$.

Proof. Let $m \equiv 0$ or $2\ (mod\ 6)$ and $n \equiv 2$ or $5\ (mod\ 6)$. Let X be a singleton set and $Y_1, Y_2, \dots, Y_m$ be $n-$sets. Since $n \equiv 2$ or $5\ (mod\ 6)$, by Theorem 2.4 there exists a $GDD(v = 1 + n + n, 3,3,5,1)$. For $i = 1, 3, 5, \dots, m-1$, let (X, Y_i, Y_{i+1}, B_i) be a $GDD(v = 1 + n + n, 3,3,5,1)$. Since $\frac{m}{2} \equiv 0$ or $1\ (mod\ 6)$ and $2n \equiv 0\ (mod\ 2)$, by Theorem 2.2 there exists a $GDD\left(v = 2n + 2n + \cdots + 2n, \frac{m}{2}, 3, 0,1\right)$. Let $(Y_1 \cup Y_2, Y_3 \cup Y_4, \dots, Y_{m-1} \cup Y_m, B_0)$ be a $GDD\left(v = 2n + 2n + \cdots + 2n, \frac{m}{2}, 3, 0,1\right)$. Then $B = B_0 \cup \left(\bigcup_{i \in \{1,3,5,\dots,m-1\}} B_i\right)$ forms a $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5,1)$. ■

Combining Theorem 3.1 and Lemmas 3.1 – 3.8, we have finally the following theorem.

Theorem 3.2 Let m, n be positive integers and $m, n \geq 3$. $GDD(v = 1 + n + n + \cdots + n, 1 + m, 3, 5,1)$ exists, if and only if

1) $m \equiv 0\ or\ 2\ (mod\ 6)\ and\ n \equiv 0, 1, 2, 3, 4\ or\ 5\ (mod\ 6)$,

2) $m \equiv 1\ (mod\ 6)\ and\ n \equiv 0\ (mod\ 6)$,

3) $m \equiv 3\ (mod\ 6)\ and\ n \equiv 0, 2\ or\ 4\ (mod\ 6)$,

4) $m \equiv 4\ (mod\ 6)\ and\ n \equiv 0\ or\ 3\ (mod\ 6)$,

5) $m \equiv 5\ (mod\ 6)\ and\ n \equiv 0, 2\ or\ 4\ (mod\ 6)$.

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