Interdisciplinary Research Review
Volume 19, Number 5, Pages 1 – 10

Weighted Likelihood Estimator for Exponentiated Weibull Distribution

Kanlaya Booblha[1*]

[1] Department of Mathematics, Faculty of Science, Naresuan University

Abstract

In this study, weighted likelihood estimator was applied to the Exponentiated Weibull distribution (EW) with contamination and the performance of the maximum likelihood estimator and the weighted likelihood estimator was compared. The central distribution was fixed to be the Exponentiated Weibull distribution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta})$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the shape parameters and $\boldsymbol{\theta}$ is the scale parameter with (2,1,1) and (2,2,2) and the contamination is Exponentiated Weibull distribution with parameter $\boldsymbol{\alpha_1} = \boldsymbol{\alpha}(\mathbf{1} + \Delta), \boldsymbol{\beta_1} = \boldsymbol{\beta}(\mathbf{1} + \Delta)$, $\boldsymbol{\theta_1} = \boldsymbol{\theta}(\mathbf{1} + \Delta)$ where $\boldsymbol{\Delta}$ = 1, 5 and the contamination proportion ($\boldsymbol{\varepsilon}$)= 0.01, 0.03, and 0.05 and the values of pre–assigned small probability k= 0.01, 0.03, 0.05 with shape parameter $\boldsymbol{\alpha}$ = 2. Monte Carlo simulation was performed to compare the performance of the maximum likelihood estimator and the weighted likelihood estimator for estimate $\boldsymbol{\beta}$ are the shape parameters and $\boldsymbol{\theta}$ is the scale parameter. The simulation results are based on the 10,000 replace. The efficiency of the maximum likelihood estimator and the weighted likelihood estimator are compared based on the bias values and the root mean square error (RMSE). The result shows that the sample size increases as the bias and root mean square error of the maximum likelihood estimator and the weighted likelihood estimator decrease in most of the cases. The weighted likelihood estimator method for $\boldsymbol{\theta}$ provides better than the maximum likelihood estimator, resulting in term of the bias and root mean square error when k is large for the estimator scale parameter $\boldsymbol{\theta}$. While the maximum likelihood estimator method for $\boldsymbol{\beta}$ provides better weighted likelihood estimator, resulting in term of the bias and root mean square error for the estimator shape parameter $\boldsymbol{\beta}$. A real dataset on breaking stress of carbon fibers was presented to show the performance of the proposed methodology. Therefore, in the presence of contamination in the data highlights that the weighted likelihood estimator does the better estimates for the parameters.

Keywords: Exponentiated Weibull, Outlier, Weighted likelihood estimator

Article history: Received 07 January 2024, Revised 10 February 2024, Accepted 02 August 2024

1. Introduction

The Exponentiated Weibull (EW) distribution was introduced by Mudholkar and Srivastava [1], this distribution is an extension of the well-known Weibull distribution by adding shape parameter. The Exponentiated Weibull distribution has a scale parameter and two shape parameters. The properties of the distribution were studied by Gupta and Kundu [2]. They observed that many properties of the Exponentiated Weibull distribution are like those of the Weibull or gamma family. The Exponentiated Weibull distribution as a failure model is more realistic than that of monotone failure rates and plays an important role in the analysis of many types of survival data [3]. The applications of the Exponentiated Weibull distribution are used for modeling of extreme value data using floods, firmware system failure, distribution for excess-of-loss insurance data, and software reliability data [4]. The Exponentiated Weibull distribution has become more appealing in reliability engineering, such as when the performance of airborne optical communications is evaluated by modeling the atmospheric turbulence [5]. The stress data of carbon fibers and life test data of ball bearings

*Corresponding author; e-mail: kanlayab@nu.ac.th

are also fitted by the Exponentiated Weibull model [6].

The estimation of the parameters for Exponentiated Weibull distribution using the life testing data is an important problem. In life testing, due to the short total time spent on the experiment and the limited number of used units, the experiment is often terminated before all the units fail, according to Manisha et. al [7].

The maximum likelihood (ML) method is an effective and important approach for parameter estimation. The method finds a value of the parameter that maximizes the likelihood function. The maximum likelihood estimation (MLE) method has many large sample properties that make it attractive. It is asymptotically consistent, which means that as the sample size gets larger, the estimates converge to the true values. It is asymptotically efficient, which means that for large samples, it produces the most precise estimates. It is asymptotically unbiased, which means that for large samples, one expects to get the true value on average. The estimates themselves are normally distributed if the sample is large enough. These are all excellent large sample properties. However, it is complicated to solve the maximum likelihood equations by conventional numerical methods.

Real-life data often contain contaminated data due to various reasons. Consequently, we assume any sample group is a mixture of good and bad observations. Contaminated data refers to data points that deviate significantly from the expected pattern or distribution, potentially affecting the accuracy of analysis. Numbers far beyond the normal range, such as a person's height of 10 feet. Data points that don't fit the overall pattern, like a negative age value. Incorrectly recorded information was for example a typo in a numerical value and inaccurate readings from equipment or instruments readings from equipment or instruments. However, when data is contaminated with outliers, maximum likelihood estimation (MLE) becomes highly unreliable [8]. Regarding the Exponentiated Weibull distribution parameter estimation, no single method consistently outperforms others due to estimator properties and data variability. Estimating the EW parameters in the presence of outliers is crucial for reliability applications. Their estimation methods assume, however, that the number of outliers and their distribution families are known. Ahmed, Volodin, and Hussein [9] proposed the weighted likelihood estimator (WLE) for robust estimation of exponential distribution parameters. The weighted likelihood method was introduced as a generalization of the local likelihood method and can be global, as demonstrated by one of the applications in Hu and Zidek [10].

In this paper, the weighted likelihood estimator method for the parameters of the Exponentiated Weibull distribution was developed and proposed to obtain the estimates of the Exponentiated Weibull parameters when the data set shows contamination. Finally, the simulation studies were extended to compare the maximum likelihood estimation and weighted likelihood estimator methods based on the bias and root mean square error of the parameter estimator.

2. Weighted Likelihood Estimators

The Exponential Weibull (EW) family contains distributions with non-monotone failure rates besides a broader class of monotone failure rates [3]. The Exponentiated Weibull distribution has a scale parameter and two shape parameters [11]. The cumulative distribution function (cdf) and the probability density function (pdf) of a random variable described by the EW distribution is given by:

$$F(x) = \left[1 - exp\left(-\frac{x^{\beta}}{\theta^{\beta}}\right)\right]^{\alpha} \tag{1}$$

and $$f(x) = \frac{\alpha\beta}{\theta^{\beta}} x^{\beta-1} exp\left(-\frac{x^{\beta}}{\theta^{\beta}}\right)\left[1 - exp\left(-\frac{x^{\beta}}{\theta^{\beta}}\right)\right]^{\alpha-1} \quad ; x > 0, \alpha > 0, \beta > 0, \theta > 0. \tag{2}$$

respectively, for $x > 0, \alpha > 0, \beta > 0$, and $\theta > 0$, where α and β are the shape parameters and θ is the scale parameter.

Let $X_1, X_2, \dots, X_n$ be a random sample of size n, drawn from a probability density function $f(x)$ where α, β, θ are an unknown parameter. Consider the Exponentiated Weibull distribution a probability density function given in (2), then the likelihood function will be:

$$L(\alpha,\beta,\theta) = \prod_{i=1}^{n} \frac{\alpha\beta}{\theta^\beta} x_i^{\beta-1} exp\left(-\frac{x_i^\beta}{\theta^\beta}\right)\left[1 - exp\left(-\frac{x_i^\beta}{\theta^\beta}\right)\right]^{\alpha-1} \quad (3)$$

Now the log likelihood function can be written as:

$$L(\alpha,\beta,\theta) = nln\,\alpha + nln\,\beta + (\beta-1)\sum_{i=1}^{n} ln\,x_i - n\beta\,ln\,\theta - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^\beta + (\alpha-1)\sum_{i=1}^{n} ln\left[1 - exp\left(-\left(\frac{x_i}{\theta}\right)^\beta\right)\right]. \quad (4)$$

Therefore, the maximum likelihood estimation of α, β, θ say $\hat{\alpha}, \hat{\beta}, \hat{\theta}$, respectively, where maximize must satisfy the normal equation given by

$$\frac{\partial}{\partial\alpha} L(\alpha,\beta,\theta) = \frac{n}{\alpha} + \sum_{i=1}^{n} ln\left[1 - exp\left(-\left(\frac{x_i}{\theta}\right)^\beta\right)\right] = 0 \quad (5)$$

$$\frac{\partial}{\partial\beta} L(\alpha,\beta,\theta) = \frac{n}{\beta} + \sum_{i=1}^{n} ln\,x_i - n\,ln\,\theta - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^\beta ln\left(\frac{x_i}{\theta}\right) + (\alpha-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^\beta ln\left(\frac{x_i}{\theta}\right) exp\left(-\left(\frac{x_i}{\theta}\right)^\beta\right)}{1-exp\left(-\left(\frac{x_i}{\theta}\right)^\beta\right)} = 0 \quad (6)$$

$$\frac{\partial}{\partial\theta} L(\alpha,\beta,\theta) = -\frac{n\beta}{\theta} + \frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^\beta - \frac{(\alpha-1)\beta}{\theta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^\beta exp\left(-\left(\frac{x_i}{\theta}\right)^\beta\right)}{1-exp\left(-\left(\frac{x_i}{\theta}\right)^\beta\right)} = 0 \quad (7)$$

Numerical computation during data analysis used standard iterative procedures such as Newton-Raphson method [12]. It is obtaining the maximum likelihood estimation by maximizing equation (4).

Let $x^{(n)} = \{X_1, X_2, \dots, X_n\}$ be a random sample from a distribution with a probability density function $f(x)$ where α, β, θ are an unknown parameter. The weighted likelihood estimators (WLE) of $\{\alpha, \beta, \theta\}$ are obtained by maximizing the weighted likelihood function:

$$L\left(\alpha,\beta,\theta \,|x^{(n)}\right) = \sum_{i=1}^{n} w_i\left(x^{(n)}\right) ln\left(f(x_i;\alpha,\beta,\theta)\right), \quad (8)$$

where $w_i\left(x^{(n)}\right), 1 \le i \le n$ are the weights which depend on the sample. Following the idea presented by Ahmed, Volodin, and Hussein [9], we let the weight w_i that corresponds to the i^{th} observation to be 1, if its estimated likelihood is sufficiently large, and 0 elsewhere. That is:

$$w_i = \begin{cases} 1 & \text{if} \quad f\left(x_i;\hat{\alpha},\hat{\theta},\hat{\beta}\right) > C \\ 0 & \quad\text{otherwise} \end{cases} \quad (9)$$

where $\hat{\alpha}, \hat{\beta}, \hat{\theta}$ are the maximum likelihood estimation of the parameter (α, θ, β). We consider,

$$f\left(x_i;\hat{\alpha},\hat{\theta},\hat{\beta}\right) > C \quad (10)$$

$$\frac{\hat{\alpha}\hat{\beta}}{\hat{\theta}^{\hat{\beta}}} x_i^{\hat{\beta}-1} exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\left[1 - exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\right]^{\hat{\alpha}-1} > C \quad (11)$$

$$x_i^{\hat{\beta}-1} exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\left[1 - exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\right]^{\hat{\alpha}-1} > C\,\frac{\hat{\theta}^{\hat{\beta}}}{\hat{\alpha}\hat{\beta}}. \quad (12)$$

We assume that $C = a\left(\frac{\hat{\alpha}\hat{\beta}}{\hat{\theta}^{\hat{\beta}}}\right)$ where assume that a is chosen from the condition of a small probability of rejection of an observation when we sample from the non-contamination of the Exponentiated Weibull distribution with cumulative distribution function [9]. We define C by the given pre-assigned small probability k as:

$$k = P\left[\max_{1\le i\le n} X_i > \hat{\theta}\left[-ln\left(C\frac{\hat{\theta}^{\hat{\beta}}}{\hat{\alpha}\hat{\beta}}\right) + \left(\hat{\beta}-1\right)\ln(x_i) + ln\left[1 - exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\right]^{\hat{\alpha}-1}\right]^{1/\hat{\beta}}\right] \quad (13)$$

$$k = 1 - \prod_{i=1}^{n} P\left[x_i \le \hat{\theta}\left[-ln\left(C\frac{\hat{\theta}^{\hat{\beta}}}{\hat{\alpha}\hat{\beta}}\right) + (\hat{\beta}-1)\,ln(x_i) + ln\left[1 - exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\right]^{\hat{\alpha}-1}\right]^{1/\hat{\beta}}\right]. \tag{14}$$

We get

$$\alpha \approx \frac{n}{k}\left(x_i^{\hat{\beta}-1}\right)\left[1 - exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\right]^{\hat{\alpha}-1} \tag{15}$$

$$x > \hat{\theta}\left[-ln\left(C\frac{\hat{\theta}^{\hat{\beta}}}{\hat{\alpha}\hat{\beta}}\right) + (\hat{\beta}-1)\,ln(x_i) + ln\left[1 - exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\right]^{\hat{\alpha}-1}\right]^{1/\hat{\beta}}. \tag{16}$$

So we reject an observation from the sample if:

$$x > \hat{\theta}\left[-ln\left(\frac{n}{k}\left(x_i^{\hat{\beta}-1}\right)\left[1 - exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\right]^{\hat{\alpha}-1}\right) + (\hat{\beta}-1)\,ln(x_i) + ln\left[1 - exp\left(-\frac{x_i^{\hat{\beta}}}{\hat{\theta}^{\hat{\beta}}}\right)\right]^{\hat{\alpha}-1}\right]^{1/\hat{\beta}} \tag{17}$$

$$x > \hat{\theta}\left[ln\left(\frac{n}{k}\right)\right]^{1/\hat{\beta}}. \tag{18}$$

Let the weighted likelihood estimator $(\tilde{\alpha},\tilde{\theta},\tilde{\beta})$ of the parameter (α,θ,β) be defined as the solution of the equation $\frac{\sum_{k=1}^{m}\partial f\left(x_{i_k};\alpha,\theta,\beta\right)}{\partial\alpha} = 0$, $\frac{\sum_{k=1}^{m}\partial f\left(x_{i_k};\alpha,\theta,\beta\right)}{\partial\theta}$ and $\frac{\sum_{k=1}^{m}\partial f\left(x_{i_k};\alpha,\theta,\beta\right)}{\partial\beta} = 0$, where $x_{i_1}, x_{i_2}, \dots, x_{i_m}$ are the remaining observations in the sample after the rejection method. In the case of the Exponentiated Weibull distribution:

$$L(\alpha,\beta,\theta) = \prod_{k=1}^{m}\frac{\alpha\beta}{\theta^{\beta}}x_{i_k}{}^{\beta-1}\,exp\left(-\frac{x_{i_k}{}^{\beta}}{\theta^{\beta}}\right)\left[1 - exp\left(-\frac{x_{i_k}{}^{\beta}}{\theta^{\beta}}\right)\right]^{\alpha-1}. \tag{19}$$

Now the log likelihood function can be written as:

$$L(\alpha,\beta,\theta) = mln\,\alpha + mln\beta + (\beta-1)\sum_{k=1}^{m} ln\,x_{i_k} - m\beta\,ln\,\theta - \sum_{k=1}^{m}\left(\frac{x_{i_k}}{\theta}\right)^{\beta} + (\alpha-1)\sum_{k=1}^{m} ln\left[1 - exp\left(-\left(\frac{x_{i_k}}{\theta}\right)^{\beta}\right)\right]. \tag{20}$$

Therefore, the weighted likelihood estimator of α,β,θ say $\tilde{\alpha},\tilde{\beta},\tilde{\theta}$, respectively, which maximize must satisfy the normal equation given by:

$$\frac{\partial}{\partial\alpha}L(\alpha,\beta,\theta) = \frac{m}{\alpha} + \sum_{k=1}^{m} ln\left[1 - exp\left(-\left(\frac{x_{i_k}}{\theta}\right)^{\beta}\right)\right] = 0 \tag{21}$$

$$\frac{\partial}{\partial\beta}L(\alpha,\beta,\theta) = \frac{m}{\beta} + \sum_{k=1}^{m} ln\,x_{i_k} - m\,ln\,\theta - \sum_{k=1}^{m}\left(\frac{x_{i_k}}{\theta}\right)^{\beta} ln\left(\frac{x_{i_k}}{\theta}\right) + (\alpha-1)\sum_{k=1}^{m}\frac{\left(\frac{x_{i_k}}{\theta}\right)^{\beta} ln\left(\frac{x_{i_k}}{\theta}\right) exp\left(-\left(\frac{x_{i_k}}{\theta}\right)^{\beta}\right)}{1-exp\left(-\left(\frac{x_{i_k}}{\theta}\right)^{\beta}\right)} = 0 \tag{22}$$

$$\frac{\partial}{\partial\theta}L(\alpha,\beta,\theta) = -\frac{m\beta}{\theta} + \frac{\beta}{\theta}\sum_{k=1}^{m}\left(\frac{x_{i_k}}{\theta}\right)^{\beta} - \frac{(\alpha-1)\beta}{\theta}\sum_{k=1}^{m}\frac{\left(\frac{x_{i_k}}{\theta}\right)^{\beta} exp\left(-\left(\frac{x_{i_k}}{\theta}\right)^{\beta}\right)}{1-exp\left(-\left(\frac{x_{i_k}}{\theta}\right)^{\beta}\right)} = 0. \tag{23}$$

It is obtaining the maximum likelihood estimation by maximizing equation (20) numerically. Numerical computation during data analysis by use of standard iterative procedures such as the Newton-Raphson method. The Newton-Raphson method can be applied to generate a sequence that converges to the weighted likelihood estimator.

3. Simulation Study

The comparison is based on the root mean square error as follows. Generate samples of 10,000 size (n) are 30, 50, and 100. We assume that the sample $(x_1, x_2, \dots, x_n)$ is taken from the distribution $G_{\varepsilon}(x)$, the ε–contamination model is defined as $G_{\varepsilon}(x) = (1-\varepsilon)F(x,\Theta) + \varepsilon F_1(x,\Theta_1)$. The central model $F(x,\Theta)$ is the Exponentiated Weibull distribution with parameter (α,β,θ). For the contamination $F_1(x,\Theta_1)$ is Exponentiated Weibull distribution with parameter $(\alpha_1, \beta_1, \theta_1)$ where $\alpha_1 = \alpha(1+\Delta), \beta_1 = \beta(1+\Delta)$, $\theta_1 = \theta(1+\Delta)$ and $\Delta > 0$. Let ε denoted the contamination proportion where $0 < \varepsilon < 1$. If a random variable U has the uniform distribution on the interval [0,1], it will have the Exponentiated Weibull distribution with parameter (α, β, θ). In this study, we consider the failure rate is increasing function when scale parameter α $\alpha > 1$, so we set $\alpha = 2$ and $\beta, \theta \geq 1$ so the central distribution to be the Exponentiated Weibull with (2,1,1) and (2,2,2). And the contamination is Exponentiated Weibull distribution with parameter $(\alpha_1, \beta_1, \theta_1)$ where $\Delta = 1$, 5 and $\varepsilon = 0.01$, 0.03, and 0.05 and the values of preassigned small probability $k = 0.01$, 0.03, 0.05. We performed a Monte Carlo simulation to compare the perform of maximum likelihood estimation and the weighted likelihood estimator for estimate when β are the shape parameters and θ is the scale parameter. The simulation results are based on the 10,000 replace and the simulation was done uses statistical software R version 4.1.3. The efficiency of the maximum likelihood estimation and the weighted likelihood estimator were compared based on the bias values and the root mean square error. The bias values were considered as a bias for comparison between estimator methods, which take the following form: $Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$ and $Bias(\hat{\beta}) = E(\hat{\beta}) - \beta$ when fix $\alpha = 2$. The root mean square error (RMSE) was considered as a bias for comparison between estimator methods, which take the following form: $RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{10,000}(\hat{\theta}-\theta)^2}{10,000}}$ and $RMSE(\hat{\beta}) = \sqrt{\frac{\sum_{i=1}^{10,000}(\hat{\beta}-\beta)^2}{10,000}}$ when fix α =2.

4. Result and discussion

In this section, we compared the performance of the maximum likelihood estimation and weighted likelihood estimator methods to estimate parameters Exponentiated Weibull with outlier through simulation.

Table 1 presents the results on bias and root mean square error of the maximum likelihood estimation and the weighted likelihood estimator when the central distribution is the Exponentiated Weibull distribution with parameter (2,1,1) and the contamination is Δ=1. The sample size increases as the bias and root mean square error of the maximum likelihood estimation and the weighted likelihood estimator decrease in most of the cases in term of parameters θ and β. For the maximum likelihood estimation and the weighted likelihood estimator method both of parameters θ and β, the ε increase as the bias and root mean square error increase. For the weighted likelihood estimator method, the magnitude of the bias and root mean square error decrease as k increases. For ε=0.03, the weighted likelihood estimator method provides better results in terms of bias and root mean square error. The weighted likelihood estimator method provides better results in terms of bias and root mean square error compared to the maximum likelihood estimation method.

Table 2 presents the bias and root mean square error for maximum likelihood estimation and weighted likelihood estimator when the central distribution is the Exponentiated Weibull distribution with parameter (2,1,1) and the contamination level is Δ=2. Generally, both maximum likelihood estimation and weighted likelihood estimator decrease bias and root mean square error for parameters θ and β as sample size increases. However, increasing contamination proportion (ε) leads to higher bias and root mean square error for both estimators. The weighted likelihood estimator demonstrates a reduction in bias and root mean square error with increasing weight parameter (k). Notably, the weighted likelihood estimator outperforms the maximum likelihood estimation in terms of bias and root mean square error for sample sizes of 30 and 50 when ε=0.05. Overall, the weighted likelihood estimator shows better performance compared to the maximum likelihood estimation in terms of the bias and root means square error.

Table 1. Bias and RMSE of the MLE and WLE for parameter θ and β when $(\alpha,\beta,\theta)=(2,1,1)$ and $\Delta=1$.

ε	method	$n=30$				$n=50$				$n=100$			
		Bias		RMSE		Bias		RMSE		Bias		RMSE	
		$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$
0.01	MLE	0.038	0.067	0.164	0.171	0.030	0.045	0.127	0.125	0.024	0.035	0.090	0.088
	WLE ($k=0.01$)	0.036	0.066	0.164	0.171	0.030	0.045	0.127	0.125	0.023	0.032	0.088	0.085
	WLE ($k=0.03$)	0.033	0.065	**0.161**	**0.170**	0.029	0.044	**0.126**	**0.123**	0.024	0.033	0.088	0.085
	WLE ($k=0.05$)	**0.031**	**0.063**	0.163	0.171	**0.029**	**0.044**	0.127	0.125	**0.022**	**0.030**	**0.086**	**0.084**
0.03	MLE	0.072	0.097	0.175	0.183	0.066	0.077	0.139	0.148	0.061	0.063	0.106	0.103
	WLE ($k=0.01$)	0.069	0.096	0.173	0.183	0.066	0.077	0.137	0.147	0.060	0.063	0.104	0.103
	WLE ($k=0.03$)	0.072	0.096	0.175	0.184	0.064	0.073	0.138	0.147	0.060	0.062	0.104	0.102
	WLE ($k=0.05$)	**0.069**	**0.095**	**0.171**	**0.182**	**0.063**	**0.072**	**0.136**	**0.146**	**0.060**	**0.060**	**0.104**	**0.101**
0.05	MLE	0.108	0.132	0.191	0.206	0.102	0.111	0.160	0.162	0.097	0.093	0.129	0.138
	WLE ($k=0.01$)	0.106	0.130	0.189	0.205	0.100	0.111	0.157	0.162	0.094	0.092	0.127	0.135
	WLE ($k=0.03$)	0.107	0.131	0.191	0.210	**0.099**	**0.110**	**0.156**	**0.161**	0.097	0.092	0.128	0.138
	WLE ($k=0.05$)	**0.103**	**0.130**	**0.188**	**0.204**	0.101	0.112	0.159	0.163	**0.094**	**0.090**	**0.127**	**0.130**

Table 2. Bias and RMSE of the MLE and WLE for parameter θ and β when $(\alpha,\beta,\theta)=(2,1,1)$ and $\Delta=5$.

ε	method	$n=30$				$n=50$				$n=100$			
		Bias		RMSE		Bias		RMSE		Bias		RMSE	
		$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$
0.01	MLE	0.096	0.131	0.185	0.211	0.089	0.108	0.150	0.162	0.084	0.090	0.120	0.122
	WLE ($k=0.01$)	0.096	0.127	0.185	0.207	0.089	0.103	0.150	0.159	0.084	0.085	0.119	0.117
	WLE ($k=0.03$)	0.095	0.125	0.184	0.197	0.087	0.103	0.149	0.152	**0.082**	**0.082**	**0.118**	**0.114**
	WLE ($k=0.05$)	**0.092**	**0.118**	**0.182**	**0.195**	**0.087**	**0.098**	**0.148**	**0.150**	0.084	0.084	0.119	0.114
0.03	MLE	0.242	0. 271	0.287	0.328	0.235	0.240	0.264	0.276	0.231	0.218	0.246	0.236
	WLE ($k=0.01$)	0.240	0.259	0.285	0.317	0.235	0.233	0.264	0.268	0.231	0.211	0.246	0.229
	WLE ($k=0.03$)	0.239	0.259	0.285	0.317	0.233	0.222	0.262	0.256	**0.229**	**0.203**	**0.243**	**0.220**
	WLE ($k=0.05$)	**0.235**	**0.245**	**0.281**	**0.299**	**0.232**	**0.218**	**0.260**	**0.252**	0.230	0.207	0.245	0.225
0.05	MLE	0.378	0. 398	0.407	0.447	0.373	0.364	0.391	0.393	0.369	0.338	0.378	0.353
	WLE ($k=0.01$)	0.378	0. 384	0.407	0.433	0.372	0.351	0.391	0.380	0.369	0.332	0.378	0.347
	WLE ($k=0.03$)	0.373	0. 366	0.404	0.412	0.370	0.343	0.388	0.372	0.368	0.322	0.377	0.336
	WLE ($k=0.05$)	**0.373**	**0. 360**	**0.403**	**0.407**	**0.368**	**0.334**	**0.386**	**0.362**	**0.366**	**0.314**	**0.376**	**0.328**

Table 3. Bias and RMSE of the MLE and WLE for parameter θ and β when $(\alpha,\beta,\theta)=(2,2,2)$ and $\Delta=1$.

ε	method	$n=30$				$n=50$				$n=100$			
		Bias		RMSE		Bias		RMSE		Bias		RMSE	
		$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$
0.01	MLE	0.035	0.144	0.115	0.252	0.033	0.103	0.090	0.183	0.029	0.064	0.065	0.121
	WLE ($k=0.01$)	0.034	0.135	0.115	0.247	0.033	0.096	0.090	0.181	0.029	0.061	0.065	0.121
	WLE ($k=0.03$)	0.035	0.122	0.114	0.234	0.032	0.085	0.089	0.174	0.029	0.057	0.065	0.118
	WLE ($k=0.05$)	**0.033**	**0.121**	**0.111**	**0.235**	**0.032**	**0.083**	**0.089**	**0.173**	**0.028**	**0.055**	**0.064**	**0.117**
0.03	MLE	0.091	0.208	0.129	0.280	0.085	0.157	0.104	0.203	0.082	0.122	0.084	0.145
	WLE ($k=0.01$)	0.089	0.199	0.127	0.273	0.084	0.148	0.104	0.200	0.082	0.119	0.084	0.143
	WLE ($k=0.03$)	0.090	0.186	0.129	0.256	0.084	0.141	0.104	0.194	0.081	0.116	0.084	0.140
	WLE ($k=0.05$)	**0.086**	**0.185**	**0.127**	**0.257**	**0.084**	**0.138**	**0.104**	**0.191**	**0.081**	**0.113**	**0.083**	**0.138**
0.05	MLE	0.141	0.267	0.148	0.305	0.138	0.217	0.130	0.234	0.135	0.183	0.113	0.176
	WLE ($k=0.01$)	0.141	0.255	0.148	0.296	0.137	0.208	0.129	0.229	0.135	0.176	0.113	0.171
	WLE ($k=0.03$)	0.140	0.240	0.148	0.283	0.136	0.200	0.129	0.221	0.134	0.169	0.113	0.168
	WLE ($k=0.05$)	**0.138**	**0.238**	**0.146**	**0.281**	**0.134**	**0.196**	**0.128**	**0.218**	**0.134**	**0.169**	**0.112**	**0.166**

Table 4. Bias and RMSE of the MLE and WLE for parameter θ and β when $(\alpha, \beta, \theta) = (2,2,2)$ and $\Delta = 5$.

ε	method	$n = 30$				$n = 50$				$n = 100$			
		Bias		RMSE		Bias		RMSE		Bias		RMSE	
		$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$
0.01	MLE	0.128	0.242	0.143	0.294	0.123	0.193	0.122	0.219	0.121	0.156	0.105	0.161
	WLE ($k = 0.01$)	0.126	0.226	0.143	0.283	0.123	0.185	0.122	0.215	0.120	0.151	0.104	0.157
	WLE ($k = 0.03$)	0.126	0.217	0.143	0.272	0.122	0.177	0.121	0.210	0.121	0.146	0.104	0.154
	WLE ($k = 0.05$)	**0.124**	**0.213**	**0.142**	**0.270**	**0.121**	**0.173**	**0.120**	**0.206**	**0.120**	**0.143**	**0.104**	**0.152**
0.03	MLE	0.362	0.491	0.277	0.438	0.357	0.434	0.266	0.363	0.353	0.389	0.256	0.304
	WLE ($k = 0.01$)	0.361	0.475	0.277	0.422	0.355	0.424	0.265	0.352	0.353	0.380	0.256	0.297
	WLE ($k = 0.03$)	0.359	0.462	0.277	0.411	0.356	0.414	0.265	0.346	0.351	0.377	0.255	0.295
	WLE ($k = 0.05$)	**0.357**	**0.457**	**0.275**	**0.405**	**0.354**	**0.407**	**0.264**	**0.341**	**0.350**	**0.372**	**0.254**	**0.291**
0.05	MLE	0.589	0.744	0.430	0.602	0.585	0.674	0.422	0.521	0.581	0.624	0.415	0.463
	WLE ($k = 0.01$)	0.589	0.716	0.430	0.579	0.585	0.666	0.422	0.515	0.580	0.618	0.414	0.459
	WLE ($k = 0.03$)	0.588	0.695	0.429	0.562	0.584	0.643	0.421	0.497	0.580	0.604	0.415	0.448
	WLE ($k = 0.05$)	**0.585**	**0.687**	**0.427**	**0.554**	**0.582**	**0.637**	**0.420**	**0.492**	**0.579**	**0.597**	**0.413**	**0.443**

Table 3 presents a comparison of the maximum likelihood estimation and the weighted likelihood estimator in terms of bias and root mean square error for the Exponentiated Weibull distribution with parameter (2,2,2) distribution with a contamination level of Δ=1. Results indicate that both maximum likelihood estimation and weighted likelihood estimator generally decrease bias and root mean square error for parameters θ and β as sample size increases. However, increasing contamination proportion (ε) leads to increases in bias and root mean square error for both estimators. The weighted likelihood estimator demonstrates a reduction in bias and root mean square error with increasing weight parameter (k). Notably, the weighted likelihood estimator outperforms the maximum likelihood estimation in terms of bias and root mean square error for all sample sizes when ε=0.05. Overall, the weighted likelihood estimator method provides better results in terms of bias and root mean square error compared to the maximum likelihood estimation method.

Table 4 presents the results on bias and root mean square error of the maximum likelihood estimation and weighted likelihood estimator when the central distribution is the Exponentiated Weibull distribution with parameter (2,2,2) and the contamination level is Δ=2. As the sample size increases, the bias and root mean square error of both maximum likelihood estimation and the weighted likelihood estimator generally decreases for parameters θ and β. Both estimation methods exhibit increased bias and root mean square error as the contamination proportion (ε) increases. For the weighted likelihood estimator, the magnitude of bias and root mean square error decreases as the weight parameter (k) increases. When ε=0.05, the weighted likelihood estimator outperforms the maximum likelihood estimation in terms of bias and root mean square error across all sample sizes. Overall, the weighted likelihood estimator provides better performance compared to the maximum likelihood estimation in terms of bias and root mean square error.

5. Real data Analysis

In this section, a real dataset is presented to show the performance of the proposed methodology. We consider a data set on breaking stress of carbon fibers (in Gba) from Nichols and Padgett [13] which includes the following values: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2 ,1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38 ,1.84, 0.39, 3.68 ,2.48, 0.85 ,1.61 ,2.79 ,4.7 ,2.03 ,1.8 ,1.57 ,1.08 ,2.03 ,1.61, 2.12, 1.89, 2.88, 2.82, 2.05 and 3.65. This suggests that the data set following the Exponentiated Weibull distribution were fitted by the method of maximum likelihood. The ML estimators of parameter (α, β, θ) is (1.317, 2.409, 2.682) as shown in Table 5. The goodness-of-fit test for the exponentiated Weibull distribution yielded a Kolmogorov-Smirnov statistic (KS) of 0.0064 with a p-value of 0.8014, indicating a good fit. To create a contamination of 1% (ε=0.01) into the data set, the last observation (3.65) was changed to 7.65. Applying the maximum likelihood approach to the contaminated data yielded the maximum likelihood estimation of (2.494, 1.628, 2.041) with a log-likelihood of -148.085. The weighted likelihood estimator approach produces estimates of (1.367, 2.353, 2.635) with a log-likelihood of -139.783. Therefore, the presence of contamination in the data highlights that the weighted likelihood estimator does the better estimates for the parameters.

Table 5. Parameters of the fitted distributions

Method	$\alpha\ (S.E.)$	$\beta(S.E.)$	$\theta(S.E.)$	-Log Likelihood	KS (p-value)
MLE (original data)	1.317 (0.0001)	2.409(0.0240)	2.682(0.0003)	-141.332	0.0064(0.8014)
MLE	2.494(0.0050)	1.628(0.0030)	2.041(0.0050)	-148.085	
WLE	1.367(0.0040)	2.353(0.0023)	2.635(0.0041)	-139.783	

6. Conclusion

This study conducted a comparative analysis of Maximum Likelihood Estimation and Weighted Likelihood Estimation methods for estimating parameters of the Exponentiated Weibull distribution in the presence of outliers. Simulation studies were conducted under various conditions of sample size, contamination level, and distribution parameters. The results consistently demonstrate the superiority of the weighted likelihood estimator method over the maximum likelihood estimation in terms of bias and root mean squared error for estimating the scale parameter (θ) of the Exponentiated Weibull distribution with parameter distribution, especially when the contamination level is significant. The results on bias and root mean square error of the maximum likelihood estimation and weighted likelihood estimator for the central distribution are the EW (2,1,1) and EW (2,2,2) when fix α=2 and the distribution of the contamination is Δ=1,5. The characteristics of the maximum likelihood estimation and weighted likelihood estimator for each situation are summarized as follows. The sample size increases as the bias and root mean square error of the maximum likelihood estimation and weighted likelihood estimator decrease in most of the cases. For the maximum likelihood estimation and the weighted likelihood estimator method both of parameters θ and β, the ε increases as the bias and root mean square error increases. For the maximum likelihood estimation method both of parameters θ and β, the ε increases as the bias and root mean square error increase. For the weighted likelihood estimator method for the scale parameter θ, the magnitude of the bias and root mean square error decreases as k increases. The bias and root mean square error when comparing values of the maximum likelihood estimation method are close to those of the weighted likelihood estimator method when k=0.01. On the other hand, the bias and root mean square error values of the weighted likelihood estimator are smaller than those of the maximum likelihood estimation for all cases. Hence, in the case when Δ=1,5 based on the bias and root mean square error of the weighted likelihood estimator method, it provides better results when comparing to the maximum likelihood estimation method when k is large. While both methods showed increased bias and root mean square error with increasing contamination, the weighted likelihood estimator exhibits a more robust performance, particularly for larger values of the shape parameter.

The findings of this study underscore the limitations of the maximum likelihood estimation method in handling outliers within the context of the Exponentiated Weibull distribution. The weighted likelihood estimator, on the other hand, emerges as a more reliable approach for estimating the scale parameter in the presence of contaminated data. The results of this study suggest that the weighted likelihood estimator is a viable alternative to the maximum likelihood estimation for analyzing data from the Exponentiated Weibull distribution when contamination is suspected. While this research focused on the scale parameter, future studies could explore the performance of the weighted likelihood estimator for estimating the shape parameter under different contamination scenarios. Additionally, investigating the sensitivity of the weighted likelihood estimator to different weight functions would provide further insights into its robustness. The successful application of the weighted likelihood estimator to the Exponentiated Weibull distribution opens avenues for extending this methodology to other complex distributions, such as the exponentiated generalized Weibull distribution. Moreover, incorporating censored observations into the weighted likelihood estimator framework could enhance its applicability to real-world datasets with incomplete information. Overall, the results of this study contribute to the development of robust estimation techniques for the Exponentiated Weibull distribution and provide valuable insights for researchers and practitioners dealing with contaminated data.

7. Acknowledgements

The authors gratefully acknowledge the financial support provided by the Faculty of Science, Naresuan University Research Fund under the Faculty of Science Research Scholar, Contract No. P2565C017.

References

[1] Mudholkar, Govind S., Deo Kumar Srivastava, and Marshall Freimer. "The exponentiated Weibull family: A reanalysis of the bus-motor-failure data." *Technometrics* 37.4 (1995): 436-445.

[2] Gupta, Rameshwar D., and Debasis Kundu. "Exponentiated exponential family: an alternative to gamma and Weibull distributions." *Biometrical Journal: Journal of Mathematical Methods in Biosciences* 43.1 (2001): 117-130.

[3] Nassar, Manal M., and Fathy H. Eissa. "On the exponentiated Weibull distribution." *Communications in Statistics-Theory and Methods* 32.7 (2003): 1317-1336.

[4] Sobhi, Mashail M. AL, and Ahmed A. Soliman. "Estimation for the exponentiated Weibull model with adaptive Type-II progressive censored schemes." *Applied Mathematical Modelling* 40.2 (2016): 1180-1192.

[5] Jing, Zhao, et al. "Performance analysis for mixed FSO/RF Nakagami-m and Exponentiated Weibull dual-hop airborne systems." *Optics Communications* 392 (2017): 294-299.

[6] Khan, R. U., Zaki Anwar, and Haseeb Athar. "Recurrence relations for single and product moments of dual generalized order statistics from exponentiated Weibull distribution." *Aligarh J. Statist* 28 (2008): 37-45.

[7] Pal, Manisha, M. Masoom Ali, and Jungsoo Woo. "Exponentiated weibull distribution." *Statistica* 66.2 (2006): 139-147.

[8] Boudt, Kris, Derya Caliskan, and Christophe Croux. "Robust explicit estimators of Weibull parameters." *Metrika* 73.2 (2011): 187-209.

[9] Ahmed, Ejaz S., Andrei I. Volodin, and Abdulkadir A. Hussein. "Robust weighted likelihood estimation of exponential parameters." *IEEE Transactions on reliability* 54.3 (2005): 389-395.

[10] Hu, Feifang, and James V. Zidek. "The relevance weighted likelihood with applications." *Empirical Bayes and Likelihood Inference*. Springer, New York, NY, 2001. 211-235.

[11] Datta, Debanshee, and D. Datta. "Comparison of Weibull distribution and exponentiated Weibull distribution based estimation of mean and variance of wind data." *International Journal of Energy, Information and Communications* 4.4 (2013): 1-12.

[12] Alizadeh, M., et al. "Estimates of the PDF and the CDF of the exponentiated Weibull distribution." *Brazilian Journal of Probability and Statistics* 29.3 (2015): 695-716.

[13] Nichols, Michele D., and W. J. Padgett. "A bootstrap control chart for Weibull

percentiles." *Quality and reliability engineering international* 22.2 (2006): 141-151.