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Power spectrum and minimum velocity threshold to generate Cherenkov radiation from the quantum field theory perspective

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Abstract

The theoretical study of Cherenkov radiation (CR) has been developing consistently from classical electrodynamics to quantum field theory. Electrodynamics theory has been popularly applied in the study of CR phenomena since the mid-20th century. The quantum field theory has been slightly utilized in CR, especially in the areas of power spectrum and velocity threshold. This paper aims to calculate the power spectrum of CR by applying quantum field theory and identifying the minimum velocity threshold necessary to generate CR in a water medium. The first step is to calculate the power spectrum of charged particles in the dielectric medium with quantum field theory perspectives by using the Poisson distribution and vacuum persistence probability. After deriving the power spectrum, the minimum velocity of CR can be directly calculated. The second step, the numerical calculation method has been performed to illustrate the minimum velocity threshold of the electron to produce CR in a water medium. The minimum velocity threshold required to generate CR is $\mathrm{v_{th}} \approx 0.12c$ (c is the speed of light), which equals values of the result from the electrodynamic theory. The last step, a comparison of the Cherenkov power spectrum as derived from electrodynamic theory and quantum field theory, found that both theories are similar results at low angular frequencies but inconsistent results at angular frequencies higher than $10^{15}\ rad/s$. This study demonstrates the potential of the quantum field theory for calculating the power spectrum of CR. The result may contribute to the knowledge of theoretical physics including future applications.

Keywords: Power spectrum, Velocity threshold, Cherenkov radiation, Quantum field theory

1. Introduction

Cherenkov radiation (CR) is the electromagnetic radiation generated by a charged particle moving through a transparent dielectric medium faster than the speed of light in the medium [1]. This radiation was discovered by Pavel Cherenkov in 1934 [2] and subsequently obtained the first theoretical interpretation by Frank and Tamm in 1937 [3]. Their work is fundamental knowledge applied experiments in physics, especially biomedical sciences. Particle physics uses CR principles to identify the energy and velocity of particles, for instance, the water Cherenkov detector which has an important role in discovering fundamental particles such as antiprotons and neutrinos [4,5]. In addition, astrophysics applies CR knowledge to determine the direction and energy of the cosmic rays and very high-energy gamma rays in the Earth's atmosphere [6]. Furthermore, recently biomedical sciences have used novel imaging techniques which as Cherenkov luminescence imaging for medical assessment through tissue including tumor diagnosis and treatment [7-9]. Therefore, considering the vast practical applications of CR, it is imperative to further enhance our understanding of this phenomenon.

In theoretical physics, the study of CR is constantly developing. Initially, theoretical developments of CR were based on classical electrodynamics theory, with Frank and Tamm pioneering the theoretical interpretation of the phenomenon. By applying electromagnetic theory, they successfully calculated the power radiation and determined the radiation characteristics with the Poynting vector [3]. Subsequently, this theory has been widely utilized by intellectuals to investigate the properties of CR, such as power spectrum, number of photons, velocity, and energy, in various media and conditions [10-12]. In 1976, Schwinger developed the source theory, which employs the quantum vacuum persistence amplitude to calculate the power spectrum of CR [13]. Thereafter in the 1990s, Manoukian applied the concept of source theory to investigate the CR and developed a mathematical model called field theory. He applied this method to calculate the properties of CR in isotropic media with different conducting plates [14,15]. Manoukian continued to advance the approach of field theory by incorporating quantum field theory, which applied the vacuum-to-vacuum transition probability and the Poisson distribution to calculate the power radiation until 2015 [16]. These CR theoretical developments reflect the process of validating the theory. Therefore, the verification of CR using modern theory, particularly quantum field theory is a significant subject of study.

Currently, the properties of CR such as power spectrum and velocity have been extensively studied in various media and under diverse conditions. However, a theoretical issue in the study of CR from the quantum field theory perspective including comparing the result between electrodynamics theory and quantum field theory has not yet been investigated. Additionally, the topic of the Cherenkov threshold is an essential factor occurrence of the CR phenomenon. Thus, this topic has been studied in various new materials to enhance the radiation efficiency of CR [17-19]. Studying and understanding the threshold of CR is tremendously important to developing theoretical particle physics [20]. Therefore, this paper purposes to calculate the power spectrum of CR by applying quantum field theory and finding the minimum velocity threshold required to generate the CR in a water medium. In the paper, the presentation is divided into three main topics, which are the calculation of the Cherenkov power spectrum by quantum field theory, the numerical calculations of the minimum velocity threshold by quantum field theory, and the comparison of the Cherenkov power spectrum by electrodynamics and quantum field theory.

2. Materials and methods

2.1 The calculation of the Cherenkov power spectrum by quantum field theory

This paper investigates the phenomenon of CR through the concepts of quantum field theory [21,22]. The paramount significant approaches in studying the Cherenkov phenomenon are the Poisson distribution and the vacuum-to-vacuum transition probability. The Poisson distribution, as described by Schwinger [23] and Manoukian [24], provides the probability distribution for the photon number N that is generated by the current is

$$\mathrm{Prob}[N = n] = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0, 1, \ldots, \text{and } \lambda = \langle N \rangle \tag{1}$$

where $\lambda = \langle N \rangle$ represents the average number of photons generated by the current source, and

$$\exp[-\langle N \rangle] = |\langle 0_+ | 0_- \rangle|^2 \tag{2}$$

The vacuum-to-vacuum transition amplitude is given by [25,26]

$$\langle 0_+ | 0_- \rangle = \exp\left[\frac{i}{2\hbar c^3} \int dx dx' J_\mu(x) D^{\mu\nu}(x, x') J_\nu(x')\right] \tag{3}$$

where $D^{\mu\nu}(x, x')$ is the photon propagator

the vacuum-to-vacuum transition probability can be obtained from Equation (3) is

$$|\langle 0_+ | 0_- \rangle|^2 = \exp\left[-\frac{1}{\hbar c^3} \int dx dx' J_\mu(x) (\mathrm{Im} D^{\mu\nu}(x, x')) J_\nu(x')\right] \tag{4}$$

where $dx = dx^1 dx^2 dx^3$, from Equation (2) the relation between the average number of photons and the vacuum-to-vacuum transition probability. The average number of photons emitted by the arbitrary current distribution to be

$$\langle N \rangle = \frac{1}{\hbar c^3} \int dx dx' J_\mu(x) \big(\mathrm{Im} D^{\mu\nu}(x, x')\big) J_\nu(x') \tag{5}$$

For the dielectric medium permittivity ε and permeability 1 were rewritten as:

$$\langle N \rangle = \frac{1}{\varepsilon \hbar c^3} \int dx dx' J_\mu(x) \big(\mathrm{Im} D^{\mu\nu}(x, x')\big) J_\nu(x') \tag{6}$$

The Equation (6) is valid for any current distribution. By considering a charged electron e moving in the medium along the x^1-axis with speed v. The associated current distribution can be expressed as follows:

$$J^i(x) = ev\delta^i \delta(x^2) \delta(x^3) \delta\left(x^1 - \frac{v}{c} x^0\right) \tag{7}$$

$$J^0(x) = ec\delta(x^2)\delta(x^3)\delta\left(x^1 - \frac{v}{c}x^0\right) \quad (8)$$

in our work conducted within the radiation gauge $A^0 = 0$, it is obtained that the photon propagator's components $D^{ij}(x, x'), i, j = 1,2,3$, satisfy with Lifshitz and Pitesti [27] ($\nu = 0,1,2,3$, and $i, j = 1,2,3$) is

$$\left[-\partial_\upsilon \partial^\nu \delta^{ij} + \partial^i \partial^j\right] D^{jk}(x, x') = \delta^{ik}\delta^{(4)}(x, x') \quad (9)$$

For an extension that is infinite, the 4D delta function $\delta^{(4)}(x, x') \equiv \delta^{(4)}(x - x')$ is in a straightforward given by

$$\delta^{(4)}(x - x') = \int \frac{(dQ)}{(2\pi)^4} e^{iQ(x-x')} \quad (10)$$

With moving along the x^1-axis, the relevant component of the propagator is $D^{11}(x - x')$ and can be expressed as follows:

$$D^{11}(x - x') = \int \frac{(dQ)}{(2\pi)^4} \frac{e^{iQ\cdot(x-x')} e^{-iQ^0\left(x^0 - x'^0\right)/\sqrt{\varepsilon}}}{\left[Q^2 - Q^{0^2} - i\delta\right]}, \quad \delta \to 0 \quad (11)$$

From the principles of the current conservation $\partial_\mu J^\mu(x) = 0$, we get for $\langle N \rangle$

$$\langle N \rangle = \frac{1}{\varepsilon^{1/2}\hbar c^3} \int (dx)(dx') \left(J(x) \cdot J(x') - \frac{1}{\varepsilon} J^0(x) J^0(x')\right) \times \int \frac{d^3Q}{(2\pi)^3 2|Q|} e^{iQ\cdot(x-x')} e^{-i|Q|\left(x^0 - x'^0\right)/\sqrt{\varepsilon}} \quad (12)$$

utilizing the principle of symmetry under the interchange in the process $(x - x') \leftrightarrow (x' - x)$ By insertion of the identity

$$1 = \int_0^\infty d\,\omega\,\delta\left(\omega - \frac{|Q|c}{\sqrt{\varepsilon}}\right) \quad (13)$$

in the integrand in Equation (12), we obtain $\langle N \rangle = \int_0^\infty d\,\omega N(\omega)$, with $Q = |Q|n$, $|Q| = \left((Q^1)^2 + |Q_{||}|^2\right)^{1/2}$, which from Equation (7) and (8) will obtain the expression for the photon number density to be

$$N(\omega) = \frac{e^2 v^2}{\varepsilon \hbar c^2}\left(1 - \frac{c^2}{\varepsilon v^2}\right) \int d\,x^0 dx'^0 \frac{dQ^1 \left(\pi d|Q_{||}|^2\right)}{16\pi^3 \omega} \times \delta\left(\omega - \frac{|Q|c}{\sqrt{\varepsilon}}\right) \exp\left[i\left(\frac{Q^1 v}{c} - \frac{\omega}{c}\right)\left(x^0 - x'^0\right)\right] \quad (14)$$

to acquire the total energy of radiation correlated with angular frequency, the Equation (14) must be multiplied by $\hbar\omega$. Ultimately, through the introduction of the variable $\tau = (x^0 - x'^0)/c$ and the execution of integrals over $Q^1, |Q_{||}|^2, \tau$, the expression for the Cherenkov power spectrum derived from quantum field theory is

$$P(\omega) = \frac{e^2}{4\pi} \frac{\omega v}{c^2}\left(1 - \frac{c^2}{\varepsilon v^2}\right) \quad (15)$$

to calulate the minimum threshold velocity to generate the Cherenkov radiation

$$P(\omega) > 0 \quad (16)$$

Therefore, imply that

$$\left(1 - \frac{c^2}{\varepsilon v^2}\right) > 0 \quad (17)$$

obtain that the minimum threshold velocity is

$$v_{th} > \frac{c}{\sqrt{\varepsilon}} \quad (18)$$

the Cherenkov power spectrum non-vanishing only for $v\sqrt{\varepsilon}/c > 1$, as a result of the delta function in Equation (14) which gives $|Q_{||}|^2 = \varepsilon\omega^2(1 - c^2/\varepsilon v^2)/c^2 > 0$, with $Q^1 = \omega/v$.

To verify the calculation results of the Cherenkov power spectrum derived from quantum field theory, the authors have compared the results with electrodynamic theory. From the Cherenkov power spectrum derived from electrodynamics theory [5,28] is given by:

$$P(\omega) = \frac{e^2\omega v}{c^2}\left(1 - \frac{c^2}{n^2 v^2}\right), \quad n > \frac{c}{v} \tag{19}$$

This study considers a charged electron moving in dielectric medium that permittivity ε and permeability $\mu = 1$ and $n = \sqrt{\mu\varepsilon}$. So, rewritten as

$$P(\omega) = \frac{e^2\omega v}{c^2}\left(1 - \frac{c^2}{\varepsilon v^2}\right) \tag{20}$$

and get the minimum threshold velocity is

$$v_{th} > \frac{c}{\sqrt{\varepsilon}} \tag{21}$$

From Equation (15) and (20), it can be seen that the result of the Cherenkov power spectrum from quantum field theory is similar to electrodynamics theory which has a difference of only $1/4\pi$. Likewise, Equation (18) and (21), show that the minimum threshold velocity derived from both theories is an equivalence.

3. Results and discussion

3.1 The numerical calculations of the minimum velocity threshold by quantum field theory

Numerical calculations are methods that rely on numerical values to model mathematical procedures, which reflect real-world circumstances [29]. This capability permits an improved comprehension of the Cherenkov phenomena. For the numerical computations, the result of the quantum field theory takes into account a charged electron traversing through water, considering the value of the dielectric permittivity of water $\varepsilon = 80$. The computation of the minimum velocity threshold for generating Cherenkov radiation (CR) (18) leads to the determination that the minimum threshold velocity of a moving charged electron in water is $v_{th} > 0.11c$. Thus, it can be inferred that the minimum velocity threshold required for generating CR in water must be at the minimum approximately 0.12c, where c represents the speed of light in a vacuum. In order to visualize data and functions in both 2D and 3D, we utilize Wolfram Mathematica for numerical calculations and plotting functions [30], resulting in Figure 1.

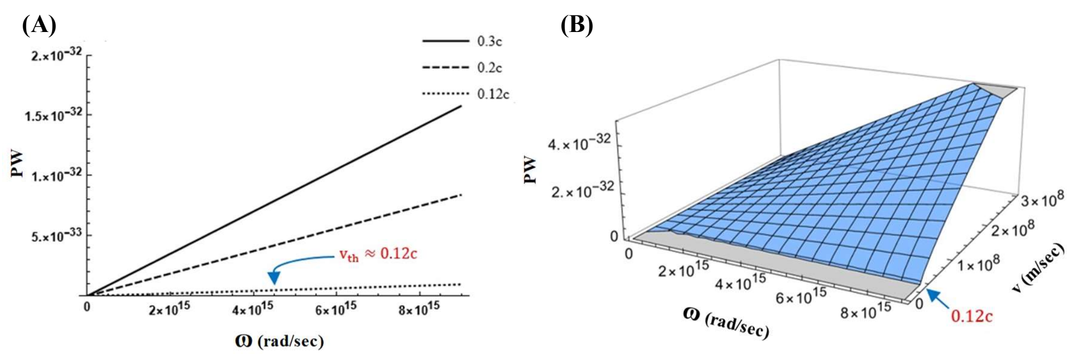

Figure 1 Cherenkov power spectrum for a moving electron in water derived from quantum field theory. (A) The correlation 2D of power spectrum P with angular frequency ω at velocities of v=0.12c, v=0.2c, and v=0.3c. (B) The correlation 3D of power spectrum P with angular frequency ω at velocities v=0 to c.

The depiction of the Cherenkov power spectrum (15) can be observed in Figure 1 (A& B), which showcases the correlation of the power spectrum at discrete velocities of three distinct values in a two-dimensional graph and the correlation of the power spectrum at continuous velocities from zero to the speed of light in a three-dimensional graph respectively. In Figure 1A, the correlation of power spectrum P with angular frequency ω for visible light in the range of 10^{15} rad/s at various velocities of v=0.12c, v=0.2c, and v=0.3c is illustrated in two dimensions, thereby signifying that the threshold velocity for CR generation in water is approximately 0.12c. This leads to the surpassing of the power spectrum beyond 0. It is notable that the aforementioned figure elucidates that the power spectrum is directly proportional to the angular frequency and velocity. Whereas Figure 1B portrays the correlation in three dimensions of the power spectrum P with angular frequency ω at velocities ranging from v=0 to c (3×10^8 m/s). In this regard, it can be inferred that the power spectrum remains below 0 at velocities less than 0.12c. This result highlights the significance of velocity in influencing the power spectrum of CR. These figures provide an understanding of the power spectrum of CR at different velocities and angular frequencies, thereby expanding the scope of its applications in particle physics and other fields.

In addition, an interesting point to consider is the result calculations of the minimum velocity threshold for CR generation between quantum field theory and electrodynamics theory. The calculations are derived from the principle of quantum field theory, as expressed in equation (18), as well as from the principle of electrodynamics theory, as represented in equation (21). It is important to note that equation (15) corresponds to the power spectrum of CR derived from quantum field theory, while equation (20) corresponds to the power spectrum derived from electrodynamics theory. Both equations (15) and (20) can be employed to calculate the minimum threshold velocity required to generate CR, which will obtain results as equations (18) and (21) respectively. Interestingly, it is observed that the value of the minimum velocity threshold in water is consistent between both theories, which yields approximately 0.12c. The calculation results support the notion that quantum field theory and electrodynamics theory yield corresponding results in terms of the minimum velocity threshold for CR generation.

3.2 The comparison of the Cherenkov power spectrum by electrodynamics and quantum field theory.

In order to verify the result of the Cherenkov power spectrum at a minimum velocity threshold as predicted by both quantum field theory (15) and electrodynamics theory (20). In Figures 2A and 2B presented the comparison of the correlation in two dimensions of the power spectrum P at range angular frequency ω lower and higher 10^{15} rad/s. Upon analyzing the graph illustrated in Figure 2A has been found that the power spectrum P, as derived by both electrodynamic theory and quantum field theory, demonstrates values that are either similar or comparable when the range of angular frequency ω is lower than 10^{15} rad/s. It is shown that the Cherenkov power spectrum of both theories has corresponding results at low angular frequencies. Conversely, the results obtained from Figure 2B reveal that the power spectrum P exhibits significant divergence in value when the range of angular frequency ω is higher than 10^{15} rad/s. Remarkable that the Cherenkov power spectrum of both theories began to have significantly inconsistent results at higher angular frequencies.

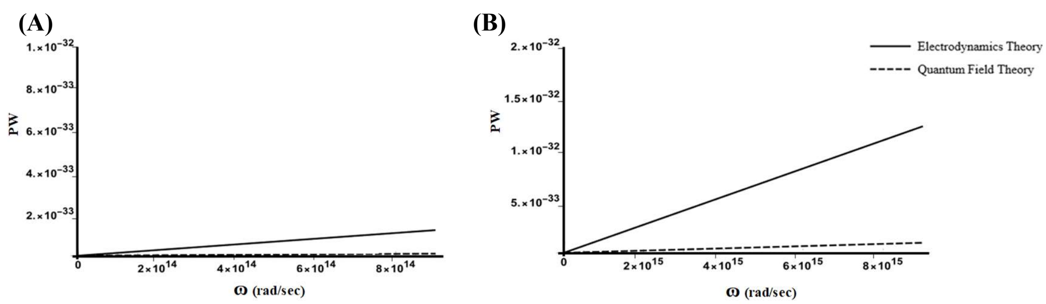

Figure 2 The comparison of the Cherenkov power spectrum for a moving electron in water at a minimum velocity threshold derived from electrodynamics theory and quantum field theory. (A) The correlation 2D of power spectrum P at range angular frequency ω lower 10^{15} rad/s. (B) The power spectrum P at range angular frequency ω higher 10^{15} rad/s.

The inconsistency of the Cherenkov power spectrum at angular frequencies higher than 10^{15} rad/s between electrodynamics theory and quantum field theory is because of the limitation and potentiality of each theory in describing the phenomena. Electrodynamics theory, which is rooted in the classical physics derived by Frank and Tamm [3], has limitations for describing the behavior of subatomic particles such as electrons and photons. Moreover, this theory neglects the concept of wave-particle duality [31]. On the other hand, quantum field theory, based on quantum physics, possesses the potential to describe the behavior of these particles including the quantum effect, and also uses the concept of wave-particle duality [32], which corresponds to the reality of the phenomena at the quantum level. Additionally, the CR phenomenon is characterized as a blue light caused by electrons moving faster than light in water, that associated with the quantum effect. Hence this phenomenon occurs at angular frequencies in the range of 10^{15} rad/s. As mentioned above, a consequence of the Cherenkov power spectrum results cannot be accurately predicted by electrodynamics theory [33], which demonstrates obvious differences to quantum field theory as shown in Figure 2B. Therefore, when studying CR or other radiation phenomena, the quantum field theory should be applied to those phenomena because this theory is a more accurate prediction.

4. Conclusion

This study presents a novel approach for calculating the power spectrum and the minimum velocity threshold of Cherenkov radiation (CR) in a water medium by applying quantum field theory. The initial step is the calculation of the power spectrum and the minimum velocity threshold to create CR in the dielectric medium from the quantum field theory perspective. The quantum field theory can calculate the power spectrum and the minimum velocity of CR, the results are similar to the electrodynamics theory derived by Frank and Tamm. The second step is the numerical calculation method by utilizing Wolfram Mathematica to indicate the minimum velocity threshold of the electron that produces CR in a water medium. The result found that the minimum velocity threshold required to generate CR is $v_{th} \approx 0.12c$, which equals the values of the result from the electrodynamic theory. The final step is a comparison of the Cherenkov power spectrum derived from electrodynamic theory and quantum field theory which both theories obtained similar results at low angular frequencies but inconsistent results at angular frequencies higher than $10^{15}\ rad/s$. Additionally, found that the power spectrum of CR depends on the velocity and angular frequency. These results demonstrate the potentiality of quantum field theory for calculating the power spectrum of CR. This study contributes to a comprehensive understanding of the power spectrum of CR at varying velocities and angular frequencies, thereby expanding the scope of its applications in various fields such as particle physics, astrophysics, and biomedical sciences.

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