Computational Efficiency for Calculating Determinants of Block Matrices

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Received 1 April 2022; Revised 26 December 2022; Accepted 9 February 2023

Abstract

In this paper, we study a method for calculating the determinant of a matrix divided into four submatrices using the formula of Schur. The purpose of this study is to analyze and provide the formula for the number of flops for matrix determinants consisting of submatrices of different sizes. The results are analyzed based on matrices with a submatrix on the first main diagonal of size 1×1. It shows that using the formula of Schur to calculate the determinant gives the number of flops close to calculating the determinant directly using the Gaussian elimination method. We also prove the relationship between the number of flops of determinant calculations by using the formula of Schur and direct determinant calculations using the Gaussian elimination method. Numerical experiments are presented, and the conclusions of the theoretical analysis are well supported.

Keywords: Determinant; Block matrix; Formula of Schur; Gaussian elimination method; Floating-point operations per second

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1. Introduction

The determinant of a matrix is one of the main contents in matrix theory or linear algebra. It is very useful in several areas of sciences and engineering; for example, solving a system of linear equations, finding the area and volume of an object, and finding integrals of a multivariable function. For a small size of a matrix, the determinant can be calculated by the Leibniz formula and Laplace expansion. However, there is some complexity to use the mentioned methods to calculate the large size of a matrix. In [1], the authors have listed methods that can be used to find the determinant of a matrix, for instance, the LU decomposition method, QR decomposition method and Cholesky decomposition method. Moreover, they also proposed a new method for determinant calculation that is faster than the existing method. However, it may occur to us how we might compare the performance of the procedure for computing the determinant. One way to evaluate the performance is counting a floating- point operations per second (flops).

Some studies have proposed a novel way for computing the determinant of a matrix with the idea of reducing the size of a matrix; for example, Chio method [2], Dodgson condensation [3] and Reduce order method [4]. A block matrix is a matrix that has been partitioned into submatrices called blocks. Partitioning will reduce the size of the original matrix to small submatrices. We may consider matrix partitions to be the idea of reducing the size of a matrix. There are many methods for finding a determinant of block matrices. The formula of Schur, [5], [6], is one of the popular methods to evaluate the determinant of a block matrix. It is an essential technique in numerical analysis, statistics, and matrix analysis. The general idea of this method is to first partition a matrix into four blocks where the first main diagonal is a square and nonsingular matrix. Then, the determinant of the original matrix is the product of the determinants of submatrices described in the formula of Schur. So, this method allows us to calculate the determinant of small matrices instead of computing a large size of the original matrix. The formula of Schur and the original method should give the same results for calculating the determinant. However, we may want to know that which method has better performance.

In this paper, we present the exact formula of the number of flops to compute the determinant based on the formula of Schur. In determinant computation, we also examine the size of the diagonal submatrices in relation to the number of flops. Moreover, we compare and analyze the performance of determinant calculations resulting in the formula of Schur to the Gaussian elimination method.

The rest of the paper is organized as follows: we first discuss related definitions and theorems that we will use throughout the paper in Section 2. In the results section, we demonstrate the main theorems showing the formula of flops for computing the determinant of a block matrix by the formula of Schur, including the comparison efficiency of the methods. The summary of this work will be discussed in the discussion and conclusion sections.

2. Research Methodology

In this section, we focus on an $m \times m$ block matrix, M, partitioned into four submatrix blocks as follows

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (1)$$

where A, B, C and D are block matrices having sizes $n \times n$, $n \times (m-n)$, $(m-n) \times n$, $(m-n) \times (m-n)$, respectively.

Theorem 1 (Schur complement) [5], [6], [7] If the matrix A is invertible, then Schur complement of a block matrix (1) is defined by

$$M / A = D - CA^{-1}B . \qquad (2)$$

In particular, one of the applications of the Schur complement is to compute the determinant of block matrices and allow us to derive useful formulae for the factorization of a block matrix [8], [9]. The Schur complement was presented by Schur in 1918 [5].

Theorem 2 (Formula of Schur) [5], [7] If the matrix A is invertible, the determinant of a block matrix (1) is given by

$$\det(M) = \det(A)\det(D - CA^{-1}B) . \qquad (3)$$

In the case of reading acquisition, several properties and its applications of the Schur complement are available in the literature [5], [6], [7], [9] and references therein.

Definition 3 (Floating-point operations per second (flops)) A flops is a norm to measure the performance of a computer operating calculation in a mathematical floating point.

In other words, flops tells us the number of operations to reach the command in the program. The fewer flops, the better computer performance.

The principle for counting flops for an operator in mathematics; summation, subtraction, multiplication and division, is counting one operator to be one flops. For example, $2+3$ requires 1 flops, $4 \times (5+2)$ requires 2 flops. In a matrix algebra, each operation requires more flops than a single number because one matrix has more than one entry to operate. **Table 1** shows examples of flops involving vector and matrix operations. More examples can be found in [10].

Define k is a real number, C is a matrix of size $n \times n$, A is a matrix of size $m \times n$, B is a matrix of size $n \times p$, u and v are column vectors of size n and w is a row vector of size m.

Table 1 The number of flops for computing some vector and matrix operations

	Product	Summation	Flops
kv	n	-	n
kA	nm	-	nm
$v^T u$	n	$n-1$	$2n-1$
vw	nm	-	nm
Au	nm	$m(n-1)$	$2nm-m$
AB	mnp	$mnp(n-1)$	$2mnp-mp$

Theorem 4 (Flops of Matrix Inversion) [11] If an invertible square matrix is of size $m \times m$, then the required number of operations to find its inverse by adapted Gaussian elimination method is

$$3m^3 - m \text{ when } m > 1.$$

Theorem 5 (Flops of Determinant by Gaussian Elimination Method) [12] Suppose that a square matrix M of size $m \times m$, defined in (1), when $m > 1$. The required number of operations to find its determinant by Gaussian elimination method is

$$\frac{1}{6}\left(-6+5m-3m^2+4m^3\right).$$

3. Results

In this section, we focus on an $m \times m$ block matrix, M, defined in (1), where $m > 1$. We first analyze the relationship between the partition of block matrices and the number of flops in the determinant calculation. Here we partition the matrix M into four submatrix blocks as given in (1). Without loss of generality, we may assume that the submatrix block A is a square matrix whose size is $n \times n$ where $1 \le n \le m-1$. Moreover, a flops of calculating the determinants will be counted based on the Gaussian elimination method while we use the conventional approach in **Theorem 4** to calculate the number of flops of the matrix inversion. In this paper, the number of flops will be studied in calculating the determinant using and not using the formula of Schur. In other words, we apply the Gaussian elimination method to calculate the determinant of the original matrix, M, comparing to the determinant of such matrix by using the formula of Schur. Our results are shown as below:

In the following, we present a computer algorithm of determinants calculation by the formula of Schur.

Algorithm : Method of the formula of Schur	
Step 1	Insert order of matrix M and order of the submatrix A defined in (1)
Step 2	Create a loop for calculating A^{-1} by **Theorem 4**
Step 3	Create a loop for calculating $A^{-1}B$
Step 4	Create a loop for calculating $CA^{-1}B$
Step 5	Create a loop for calculating $D-CA^{-1}B$
Step 6	Create a loop for calculating $\det(D-CA^{-1}B)$ and $\det(A)$ by **Theorem 5**
Step 7	Calculate the determinant $\det(M)$ from the result in Step 6 by **Theorem 2**

Theorem 6 Suppose a square matrix M of size $m \times m$, is partitioned as a block matrix (1). If the matrix A of size $n \times n$ is invertible, then the number of flops of the matrix M using the formula of Schur is given by the function flop as

$$\text{flop}(n) = \begin{cases} \frac{1}{6}\beta & \text{for} \quad n=1 \\ \frac{1}{6}\left(-6+\alpha+\beta\right) & \text{for } 2 \le n \le m \end{cases}$$

where $\alpha = 18n^3 - 6n$ and

$\beta = 5m - 3m^2 + 4m^3$.

Proof: As A is an $n \times n$ invertible matrix, the number of flops to calculate the inverse of A

$$\begin{cases} 1 & \text{for} \quad n=1 \\ 3n^3 - n & \text{for } 2 \le n \le m \end{cases} \tag{3}$$

Next, to calculate the number of flops for a matrix $D - CA^{-1}B$, it consists of the following 3 steps: First, calculate flops of $A^{-1}B$. Second, calculate flops of $C(A^{-1}B)$. Third, calculate flops of $D - CA^{-1}B$. Therefore, the number of flops used to find the matrix $D - CA^{-1}B$ is equal to

$$n(m-n)(2m-1). \tag{4}$$

Moreover, the flops count for computing the determinant of the matrix $D - CA^{-1}B$ requires

$$\frac{1}{6}\left(-6 + 5(m-n) - 3(m-n)^2 + 4(m-n)^3\right)$$

flops. (5)

Finally, the number of flops for the multiplication between $\det(A)$ and $\det(D - CA^{-1}B)$ requires

$$1 \text{ flops} \tag{6}$$

Adding (3) - (6), a total of flops for computing the determinant when the size of the matrix A is 1×1

$$\frac{1}{6}\left(5m - 3m^2 + 4m^3\right),$$

otherwise it requires total of flops

$$\frac{1}{6}\left(-6 - 6n + 18n^3 + 5m - 3m^2 + 4m^3\right).$$

Proposition 7 Given M is a square matrix whose submatrix block A has a size of 1×1, defined in equation (1). The difference of the number of flops in calculating the determinant of the matrix M by using the Gaussian elimination method and the formula of Schur is 1.

Proof: Let M be a square matrix of size 1×1, whose submatrix block A has a size of 1×1. Without loss of generality, we may assume that $m > 1$. By **Theorem 5**, the number of flops for computing the determinant of the matrix M by Gaussian elimination method is $\frac{1}{6}\left(-6+\beta\right)$. By **Theorem 6**, the number of flops for computing the determinant by formula of Schur is $\frac{\beta}{6}$. Consider the difference of the number of flops of both methods is 1, as desired.

Theorem 6 gives the formula of flops measurements for calculating the determinant of a block matrix in the formula of Schur. Because determinant calculations involve basic algebraic operations, the size of the matrix affects the number of algebraic operations. The following experiments will demonstrate the number of flops and the partitioning of a matrix using the formula of Schur.

Table 2 The number of flops for computing the determinant of a 7×7 matrix versus with several sizes of the block matrix A

Sizes of the submatrix block A	The number of flops
1	210
2	231
3	287
4	397
5	579
6	851

Table 2 shows the relationship between the number of flops for computing the determinant of a 7×7 matrix M and the partition of the matrix M having four submatrix blocks. We perform our experiment using the submatrix block A of size from 1×1 to 6×6 and count the number of flops via the formula in **Theorem 6.** This shows that the operation counts increase dramatically as the size of the submatrix block A increases. In contrast to the number of flops to calculate the determinant of the matrix M directly based on the Gaussian elimination method requires only 209 flops. **Figure 1** also shows a semilogy plot of the relationship between the size of the submatrix and the number of flops for calculating the determinant of matrices. In this experiment, we fix the original matrix M of several sizes 7×7 and 20×20 matrices. The obtained results are consistent with what we discussed earlier. That is, the size of the submatrix block is directly related to the number of flops in the determinant calculation.

Moreover, we see that the number of flops when the submatrix block A of size 1×1 is close to the number of flops when calculating the determinant directly using the Gaussian elimination method.

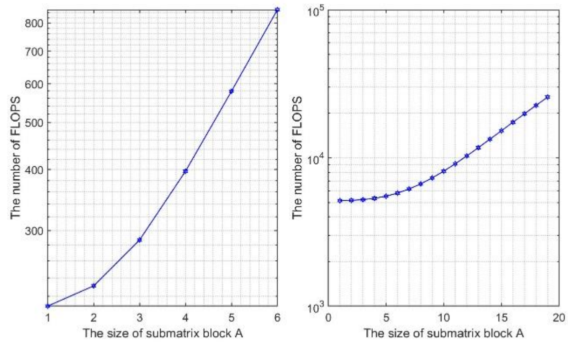

Figure 1 The number of flops for computing the determinant of matrices versus with several sizes of the block matrix A: Left, M is a 7×7 matrix, Right, M is a 20×20 matrix.

That raises the question of the number of flops will be when the matrix M is larger and the submatrix block A has a size of 1×1. Further details are described in **Table 3**.

Table 3 The number of flops for computing the determinant of matrices with different dimensions where the size of the block matrix A is 1×1

Sizes of matrix M	The number of flops: With Gaussian elimination method	The number of flops: With Formula of Schur
2×2	4	5
3×3	15	16
4×4	37	38
5×5	74	75
6×6	130	131
7×7	203	210

Sizes of matrix M	The number of flops	
	With Gaussian elimination method	With Formula of Schur
8×8	315	316
9×9	452	453
10×10	624	625

Table 3 shows the number of flops for computing the determinant of the matrix M of several sizes by using the formula of Schur and by using the Gaussian elimination method computing the determinant directly. In the experiment, we consider the matrix M partitioned into four submatrix blocks given in equation (1) of size from 1×1 where its submatrix block A has a size of 1×1. **Figure 2** demonstrates a semilogy plot of the comparison the number of flops for calculating the determinant of matrices between using and not using the formula of Schur where the submatrix block A is of size 1×1. In this experiment, we use the original matrix M of several sizes from 2×2 to 20×20.

From **Table 3** and **Figure 2**, we see that the number of flops of both methods are similar to each other and have the same effect in the same direction. In other words, the number of flops between using and not using the formula of Schur is increasing when the size of the matrix M is increasing, and the difference in the number of flops for both methods is 1 regardless of the size of the matrix M, which confirms our result in **Proposition 7**. Moreover, if the size of the matrix increases, then the basic manipulations can be applied to the determinant of the matrix. This results in an increase in the number of flops in both methods, which is shown in **Table 3**.

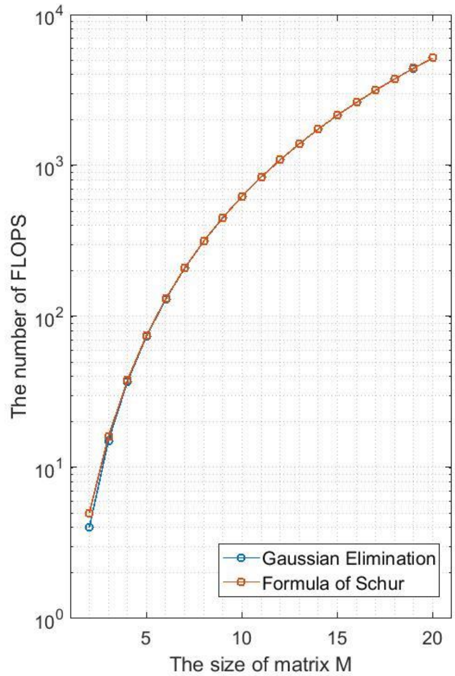

Figure 2 The number of flops for computing the determinant of matrices with different dimensions where the size of the block matrix A is 1×1 for both methods.

4. Discussion

The Schur complement is a method to find the determinant of a matrix by using submatrix blocks. This method should give a smaller number of flops than the Gaussian elimination method of calculating the determinant directly without dividing the matrix into submatrices. In the formula of Schur, although calculating the determinant of submatrices gives smaller numbers of flops than the original matrix, calculating the inverse of the submatrices and the multiplication operations of the submatrices in the formula will produce more flops. The result is that the increasing number of flops compensates for the decrease in the number of flops from computing determinants of submatrices. That is why the number of flops for both methods is not significantly different.

In addition, any square matrices can be broken into four submatrix blocks. In this work, we consider submatrices in a partitioned matrix having the main diagonal blocks square matrices. These submatrices identified in (1) can be of various sizes depending on a partition of the original matrix. We found that calculating the determinant of matrices using the formula of Schur gives the best efficiency in terms of the number of flops when the submatrix block A is of size 1.

To develop strategies to lessen the amount of flops than the existing approach, one can find an inversion matrix that produces less flops than the technique employed in this study. Another strategy that we are researching is the pattern of splitting the matrix into more than 4 blocks.

Remark. Moreover, there is another way to compare the performance of the methods by using the time execution. However, we focus on the operation count or flops for calculating the determinant of a matrix. Since the time execution depends on coding techniques, we may omit this approach to compare the performance of the calculation.

5. Conclusion

In this paper, the authors study a method for determining the determinant by the formula of Schur of a matrix written as a 2×2 block matrix. We provide the exact formula for the number of operations (flops) for calculating the determinant of a matrix by the formula of Schur. This formula is used to determine the efficiency of determinant calculations on each partition of a block matrix. The results show that the partitioning of matrix blocks directly affects the number of flops in the determinant calculation by using the formula of Schur. In addition, the matrix M with a submatrix on the first main-diagonal blocks of size 1×1 yields the smallest number of flops relative to the different sizes. The authors further investigate the efficiency of the determinant calculation of block matrices with such properties when the initial matrix size is larger. In this case, we calculate the determinant of the matrix by Gaussian elimination method and the formula of Schur and compare the number of flops for each method. The study found that the number of flops obtained by the formula of Schur whose the first sub-matrix on the main diagonal is of size 1×1 is as good as the Gaussian elimination method. All experiment results illustrated confirm our theoretical results.

6. Acknowledgement

The authors would like to thank the referee for carefully reading our manuscript and providing such constructive comments and suggestions that improved the paper's quality into its current form.

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