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The Bivariate Length Biased - Power Garima Distribution Derived from Copula: Properties and Application to Environmental Data

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ABSTRACT

Bivariate distributions calculate the probabilities for simultaneous outcomes of two random variables. They are essential for understanding the relationship between two variables. This study proposed a new bivariate distribution called the bivariate length-biased power Garima (BLBPG) distribution, created by combining the Farlie-Gumbel-Morgenstern (FGM) copula with a length-biased power Garima distribution. The BLBPG distribution describes lifetime bivariate data with a weak correlation between variables as a flexible alternative to bivariate lifetime distributions for modeling real-valued data in applications. The proposed distribution yielded various properties including joint conditional probability functions, reliability (survival) and hazard functions, and generating a random variable. The maximum likelihood estimation was presented to estimate the parameters of the proposed distribution and a Monte Carlo simulation study was conducted to evaluate the performance of the estimators. A practical application of the proposed bivariate distribution to analyze environmental data was also demonstrated.

Keywords: Bivariate distribution; FGM copula; Dependent variables; Numerical simulation.

1. Introduction

Probability distributions are necessary statistical tools used to represent the various characteristics of data sets. Hence, it is crucial to investigate the selection of an appropriate probability distribution for each dataset. Two distinct categories of data exist as discrete or continuous variables. It is imperative to determine the suitable distribution for each type of data to enhance the precision of data

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modeling. The concept of length-biased distribution is used to develop models to find probabilistic distributions suitable for data analysis [1]. One length-biased distribution, called the length-biased power Garima distribution, has been introduced for lifetime data analysis. A variety of real-world data sets including Kiama Blowhole, times between failures, waiting time, breaking stress of carbon fibers, time to failure of turbochargers, and strength of glass fibers can be modeled by comparing data fitting to the length-biased Garima, power Garima, and Garima distributions. The distribution is interesting due to its flexible distribution, and the probability density function (pdf) has a unimodal distribution that has several shapes, such as left-skewed, right-skewed, and almost symmetrical [2].

Bivariate distributions are essential tools in statistics and data analysis to provide a foundational understanding of how two variables interact and to understand the relationships between two random variables. There are various concepts, including copulas, to derive bivariate distributions in statistical theory [3, 4]. Several kinds of copulas including the Plackett, Gumbel, Clayton, Frank, and Farlie-Gumbel-Morgenstern (FGM) copulas with different properties and features are discussed in the literature. Copulas are an invaluable instrument for explaining a distribution that involves two variables. According to Nelsen, a copula is a mathematical function that connects bivariate distribution functions having uniform (0, 1) margins [4], while Sklar presented the joint pdf and joint cumulative distribution function (cdf) for two marginal univariate distributions [4-6].

Copulas are mathematical functions that provide a connection between the marginal distributions of variables and their joint distribution, hence enabling flexible modeling of dependence. The FGM is widely recognized as one of the most prevalent bivariate distributions. The FGM copula is a statistical tool commonly employed in bivariate or multivariate analysis to represent the dependence structure among random variables. One of its main advantages is its simplicity and ease of practical use. Morgenstern first introduced the FGM copula [7]. Many bivariate distributions are derived from the FGM copula such as the bivariate gamma [8], bivariate Pareto [9], bivariate generalized exponential [10], bivariate modified Weibull [11], bivariate Weibull [12], bivariate Lomax-Claim [13], bivariate Lomax [14], and bivariate inverse Lindley [15] distributions.

This study applied the FGM copula to determine the correlation and dependencies between two variables in the length-biased power Garima (LBPG) distribution, called the bivariate length-biased power Garima distribution. The unknown parameters were estimated using the maximum likelihood (ML) approach. Numerical simulation and its practical implementation using real data are studied.

2. Materials and Methods

2.1 The FGM copula

Let X and Y be random variables with marginal distribution functions $G(x)$ and $G(y)$, respectively. If F is a bivariate distribution function with margins $G(x)$ and $G(y)$, there must exist a copula C such that

$$F(x,y) = C(G(x),G(y);\theta) = C(u,v;\theta), \quad (2.1)$$

where θ is introduced as a dependence parameter. A copula is defined as a function $C:[0,1]^2 \to [0,1]$, if the following two conditions are satisfied [15, 16]; (i) C satisfies, for all $u,v \in [0,1]$, the boundary conditions: $C(u,0) = C(0,v) = 0$, and $C(u,1) = u$, $C(1,v) = v$; (ii) C is two-increasing, i.e., for all $u_1,u_2,v_1,v_2 \in [0,1]$ satisfying $u_1 \le u_2$ and $v_1 \le v_2$, the following inequality holds: $C(u_1,v_2) + C(u_1,v_1) - C(u_2,\ v_1) - C(u_1,v_2) > 0$. If $C(u,v) = C(v,u)$ then C is symmetric for every $(u,v) \in [0,1]^2$, otherwise, C is asymmetric [17].

According to the FGM copula, the joint distribution function can be expressed as follows [7, 15, 16, 18]:

$$C(u,v) = uv[1+\theta(1-u)(1-v)], \quad (2.2)$$

where $u = G(x)$ and $v = G(y)$ and $-1 \le \theta \le 1$. is a dependence parameter. Its corresponding pdf is

$$c(u,v) = 1+\theta(1-2u)(1-2v). \quad (2.3)$$

Based on condition (ii), the property of two-increasing can be substituted with the condition

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v} \ge 0, \quad (2.4)$$

where $c(u,v)$ is the copula density. The associated joint pdf is

$$f(x,y) = c(G(x),G(y);\theta)g(x)g(y), \quad (2.5)$$

where $g(x)$ and $g(y)$ are the marginal pdf of X and Y respectively.

2.2 The LBPG distribution

Let X be a random variable distributed in the LBPG distribution with shape parameter λ and scale parameter β. The pdf and cdf can be expressed as

$$g(x) = \frac{\lambda\beta^{1+1/\lambda}(1+\beta+\beta x^{\lambda})x^{\lambda}e^{-\beta x^{\lambda}}}{(2+\frac{1}{\lambda}+\beta)\Gamma(1+\frac{1}{\lambda})}, \quad (2.6)$$

$$G(x) = 1 - \frac{(1+\beta)\gamma(1+\frac{1}{\lambda},\beta x^{\lambda}) + \gamma(2+\frac{1}{\lambda},\beta x^{\lambda})}{(2+\frac{1}{\lambda}+\beta)\Gamma(1+\frac{1}{\lambda})}, \quad (2.7)$$

for $x, \lambda, \beta > 0$, where $\Gamma(t) = \int_0^{\infty} s^{t-1}e^{-s}ds$ and $\gamma(t,x) = \int_0^{x} s^{t-1}e^{-s}ds$ are gamma and lower incomplete gamma functions, respectively for $t > 0$. Its corresponding survival function is

$$S(x) = \frac{(1+\beta)\gamma(1+\frac{1}{\lambda},\beta x^{\lambda}) + \gamma(2+\frac{1}{\lambda},\beta x^{\lambda})}{(2+\frac{1}{\lambda}+\beta)\Gamma(1+\frac{1}{\lambda})}. \quad (2.8)$$

According to the LBPG distribution by [19], it can be reduced to the length-biased Garima distribution when λ is equal to 1.

2.3 Model comparison criteria

The Akaike information criterion (AIC) and Bayesian information criterion (BIC) are statistical criteria used to compare models to find the best fit among several alternative models. The present study utilized the AIC and the BIC to choose the most suitable model for fitting the data. The optimal model is characterized by lower values of AIC and BIC. The formulas for the AIC and BIC are as follows:

$$\text{AIC} = -2\ell(\hat{\mathbf{\Omega}}) + 2q, \quad (2.9)$$

$$\text{BIC} = -2\ell(\hat{\mathbf{\Omega}}) + q\log n, \quad (2.10)$$

where q is the number of estimators in the model, n is the number of observations, and $\ell(\hat{\mathbf{\Omega}})$ is the value of the log-likelihood function at the model's estimated parameter vector $\hat{\mathbf{\Omega}}$.

3. Results and Discussion

3.1 A new bivariate distribution

A developed bivariate distribution, named as the bivariate length-biased power Garima (BLBPG) distribution, is introduced. This distribution is obtained using the copula concept of Sklar and the FGM copula.

Let (X,Y) be a bivariate random variable distributed in the BLBPG distribution, then its joint distribution function (see in Appendix) is

$$\begin{aligned} F(x,y) = &\left(1-\left[(1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right]\right) \\ &\times\left(1-\left[(1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right]\right) \\ &\times\left\{1+\theta\left[(1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right]\right. \\ &\left.\times\left[(1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right]\right\}, \end{aligned} \quad (3.1)$$

where $\delta_{j,k}(z)=\dfrac{\gamma(j+1/\lambda_k,\beta_k z^{\lambda_k})}{(2+1/\lambda_k+\beta_k)\Gamma(1+1/\lambda_k)}$ for $j,k=1,2$. The joint density function for Eq. (3.1) is denoted by

$$
\begin{aligned}
f(x,y)=&\frac{\lambda_1\beta_1^{1+1/\lambda_1}(1+\beta_1+\beta_1x^{\lambda_1})x^{\lambda_1}e^{-\beta_1x^{\lambda_1}}}{(2+1/\lambda_1+\beta_1)\Gamma(1+1/\lambda_1)}\\
&\times\frac{\lambda_2\beta_2^{1+1/\lambda_2}(1+\beta_2+\beta_2y^{\lambda_2})y^{\lambda_2}e^{-\beta_2y^{\lambda_2}}}{(2+1/\lambda_2+\beta_2)\Gamma(1+1/\lambda_2)}\\
&\times\Big\{1+\theta\big[2(1+\beta_1)\delta_{1,1}(x)+2\delta_{2,1}(x)-1\big]\\
&\times\big[2(1+\beta_2)\delta_{1,2}(y)+2\delta_{2,2}(y)-1\big]\Big\},
\end{aligned}
\tag{3.2}
$$

where $\lambda_1,\beta_1,\lambda_2,\beta_2>0$ and $-1\le\theta\le 1$.

Graphs for joint pdf and cdf plots of the BLBPG distribution are shown in Fig. 1. The shape of the BLBPG distribution is unimodal and the shapes of the plots in Fig. 1 (a1) and Fig. 1(b1) do not change. The only change is the increase in dispersion when theta has a positive value ($\theta=0.5$). When comparing the plots of Fig. 1 (b1) and Fig. 1 (c1) for fixed $\theta=0.5$, the shape does not change. The only change is a decrease in dispersion when λ_2 and β_2 values change. If $\lambda_1=\lambda_2=1$ then the BLBPG distribution reduces to the bivariate length-biased Garima (BLBG) distribution. Its joint distribution and density functions respectively are

$$
\begin{aligned}
F_1(x,y)=&\left(1-\frac{(1+\beta_1)\gamma(2,\beta_1x)+\gamma(3,\beta_1x)}{3+\beta_1}\right)\\
&\times\left(1-\frac{(1+\beta_2)\gamma(2,\beta_2y)+\gamma(3,\beta_2y)}{3+\beta_2}\right)\\
&\times\left\{1+\theta\left[\frac{(1+\beta_1)\gamma(2,\beta_1x)+\gamma(3,\beta_1x)}{3+\beta_1}\right]\right.\\
&\times\left\{1+\theta\left[\frac{(1+\beta_1)\gamma(2,\beta_1x)+\gamma(3,\beta_1x)}{3+\beta_1}\right]\right.\\
&\left.\times\left[\frac{(1+\beta_2)\gamma(2,\beta_2y)+\gamma(3,\beta_2y)}{3+\beta_2}\right]\right\},
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
f_1(x,y)=&\frac{(1+\beta_1+\beta_1x)xye^{-\beta_1x-\beta_2y}(1+\beta_2+\beta_2y)}{(3+\beta_1)(3+\beta_2)}\\
&\times\beta_1^2\beta_2^2\left\{1+\theta\left(\frac{2}{3+\beta_1}[(1+\beta_1)\gamma(2,\beta_1x)\right.\right.\\
&\left.+\gamma(3,\beta_1x)]-1\right)\left(\frac{2}{3+\beta_{21}}[\gamma(3,\beta_2y)\right.\\
&\left.\left.+(1+\beta_2)\gamma(2,\beta_2y)]-1\right)\right\}.
\end{aligned}
\tag{3.4}
$$

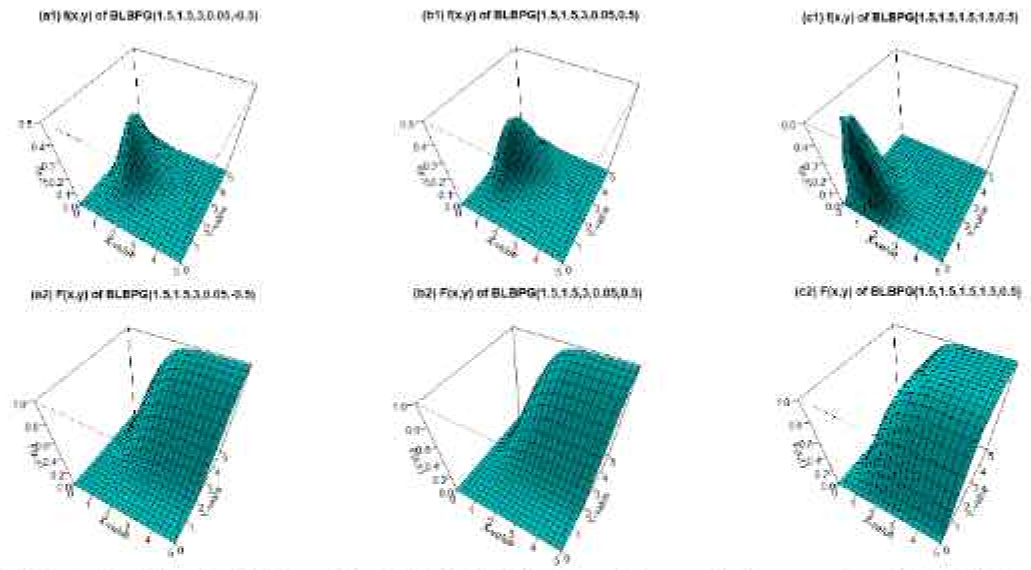

Fig. 1. The joint pdf and cdf plots of the BLBPG distribution with specified parameters of $(\lambda_1,\beta_1,\lambda_2,\beta_2,\theta)$.

3.2 Some statistical properties

Several statistical characteristics of the BLBPG distribution are presented, such as conditional probability functions, reliability and hazard functions, and variable generation.

3.2.1 Marginal and conditional distributions

Let $(X,Y)\sim$ BLBPG $(\lambda_1,\beta_1,\lambda_2,\beta_2,\theta)$ with the joint pdf as Eq. (3.2). The marginal distributions of X and Y are provided in Eq. (2.6). Some probability functions can be easily derived:

(i) The conditional density function of X given Y is

$$f(x\mid y)=\frac{f(x,y;\lambda_1,\beta_1,\lambda_2,\beta_2,\theta)}{g(y;\lambda_2,\beta_2)}$$
$$=\left\{1+\theta\left[2(1+\beta_1)\delta_{1,1}(x)+2\delta_{2,1}(x)-1\right]\right.$$
$$\left.\times\left[2(1+\beta_2)\delta_{1,2}(y)+2\delta_{2,2}(y)-1\right]\right\}\lambda_1 x^{\lambda_1}$$
$$\times\frac{\beta_1^{1+1/\lambda_1}(1+\beta_1+\beta_1 x^{\lambda_1})e^{-\beta_1 x^{\lambda_1}}}{(2+1/\lambda_1+\beta_1)\Gamma(1+1/\lambda_1)}. \tag{3.5}$$

(ii) The conditional distribution function of X given Y is

$$F(x\mid y)=G(x;\lambda_1,\beta_1)\{1+\theta[1-G(x;\lambda_1,\beta_1)]$$
$$\times[1-2G(y;\lambda_2,\beta_2)]\}$$
$$=\left(1-\left[(1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right]\right)$$
$$\times\left\{1+\theta\left[(1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right]\right.$$
$$\left.\times\left[(1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right]\right\}. \tag{3.6}$$

(iii) The conditional reliability (survival) distribution of X given Y is

$$S(x\mid y)=S(x;\lambda_1,\beta_1)\{1+\theta[1-S(x;\lambda_1,\beta_1)]$$
$$\times[1-2S(y;\lambda_2,\beta_2)]\}$$
$$=1-\left(1-\left[(1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right]\right)$$
$$\times\left\{1+\theta\left[(1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right]\right.$$
$$\left.\times\left[(1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right]\right\}. \tag{3.7}$$

3.2.2 Survival and hazard functions

Let X and Y be random variables distributed in the LBPG distribution with the survival function as Eq. (2.8). The survival (reliability) function of the bivariate distribution obtained using the copula concept is $S(x,y)=C(S(x),S(y))$ [4, 20]. From the FGM copula in Eq. (2.2), the survival function of the BLBPG distribution can be expressed as

$$S(x,y)=S(x)S(y)\left[1+\theta(1-S(x))(1-S(y))\right]$$
$$=\left((1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right)$$
$$\times\left((1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right)$$
$$\times\left\{1+\theta\left[1-\left((1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right)\right]\right.. \tag{3.8}$$

Its corresponding bivariate hazard (failure) function of the BLBPG distribution can be defined as

$$h(x,y)=\frac{f(x,y;\lambda_1,\beta_1,\lambda_2,\beta_2,\theta)}{S(x,y;\lambda_1,\beta_1,\lambda_2,\beta_2,\theta)}$$
$$=\frac{\lambda_1\beta_1^{1+1/\lambda_1}(1+\beta_1+\beta_1 x^{\lambda_1})x^{\lambda_1}e^{-\beta_1 x^{\lambda_1}}}{(2+1/\lambda_1+\beta_1)\Gamma(1+1/\lambda_1)}$$
$$\times\frac{\lambda_2\beta_2^{1+1/\lambda_2}(1+\beta_2+\beta_2 y^{\lambda_2})y^{\lambda_2}e^{-\beta_2 y^{\lambda_2}}}{(2+1/\lambda_2+\beta_2)\Gamma(1+1/\lambda_2)}$$
$$\times\left\{1+\theta\left[2(1+\beta_1)\delta_{1,1}(x)+2\delta_{2,1}(x)-1\right]\right.$$
$$\left.\times\left[2(1+\beta_2)\delta_{1,2}(y)+2\delta_{2,2}(y)-1\right]\right\}$$
$$\times\left\{1+\theta\left[1-\left((1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right)\right]\right.$$
$$\left.\times\left[1-\left((1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right)\right]\right\}^{-1}$$
$$\times\left\{\left((1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right)\right.$$
$$\times\left((1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right). \tag{3.9}$$

3.2.3 Generating BLBPG random variables

This proposed bivariate random variables by generating random samples using the conditional copula distribution function from $f(x,y)=f(y)f(x\mid y)$ [4]. The literature analysis conducted by [15] also documented the application of this approach for generating a bivariate inverse Lindley random variable.

The generation of BLBPG random samples can be carried out through the utilization of the conditional technique. The random variables X and Y are represented by the joint distribution function in Eq. (3.1). From Eq. (2.2), the FGM copula yields the conditional function $c_u(v)=\frac{d}{du}C(u,v)$.

Therefore, the technique can be employed to create the random numbers (x_i, y_i) as follows:

(1) Generate u and t from the uniform distribution on the interval (0, 1).

(2) Set $t=v[1+\theta(1-v)(1-2u)]$ and solve for v.

(3) Compute $x=G^{-1}(u;\lambda_1,\beta_1)$ and $y=G^{-1}(v;\lambda_2,\beta_2)$ where $G^{-1}(\cdot)$ is the inverse cdf of the LBPG distribution.

(4) Finally, set (X,Y) where $X=x$ and $Y=y$.

The inverse of the LBPG cdf in step (3) can be generated using the LBPG package with the rLBPG function [21] in R [22].

3.2.4 The r-th moment about the origin of BLBPG random variables

Let X and Y be random variables that follow the LBPG distribution, with the pdf and cdf represented by Eq. (2.6) and Eq. (2.7), respectively. The r-th moments regarding the origin of BLBPG random variables, as represented by Eq. (3.2), can be found in the Appendix section.

$$E[(XY)^r]=\theta\left\{\left[E[X^r]-2\tau_{r,x}\right]\left[E[Y^r]-2\tau_{r,y}\right]\right\}+E[X^r]E[Y^r],$$

where $E[X^r]$ and $E[Y^r]$ are the r-th moment about the origin of X and Y (see [2]),

$$E[X^r]=\frac{(1+\beta_1)\Gamma\left(1+\frac{r+1}{\lambda_1}\right)+\Gamma\left(2+\frac{r+1}{\lambda_1}\right)}{\beta^{r/\lambda_1}\left(2+\frac{1}{\lambda_1}+\beta_1\right)\Gamma\left(1+\frac{r+1}{\lambda_1}\right)},$$

$$E[Y^r]=\frac{(1+\beta_2)\Gamma\left(1+\frac{r+1}{\lambda_2}\right)+\Gamma\left(2+\frac{r+1}{\lambda_2}\right)}{\beta^{r/\lambda_2}\left(2+\frac{1}{\lambda_2}+\beta_2\right)\Gamma\left(1+\frac{r+1}{\lambda_2}\right)},$$

$$\tau_{r,x}=\int_0^\infty x^r g(x;\lambda_1,\beta_1)G(x;\lambda_1,\beta_1)dx, \text{ and}$$

$$\tau_{r,y}=\int_0^\infty y^r g(y;\lambda_2,\beta_2)G(y;\lambda_2,\beta_2)dy.$$

For $r=1$, we have the expectation (mean) and covariance of X and Y are respectively

$$E[XY]=\left(\frac{(1+\beta_1)\Gamma\left(1+\frac{2}{\lambda_1}\right)+\Gamma\left(2+\frac{2}{\lambda_1}\right)}{\beta^{1/\lambda_1}\left(2+\frac{1}{\lambda_1}+\beta_1\right)\Gamma\left(1+\frac{2}{\lambda_1}\right)}\right)\times\left(\frac{(1+\beta_2)\Gamma\left(1+\frac{2}{\lambda_2}\right)+\Gamma\left(2+\frac{2}{\lambda_2}\right)}{\beta^{1/\lambda_2}\left(2+\frac{1}{\lambda_2}+\beta_2\right)\Gamma\left(1+\frac{2}{\lambda_2}\right)}\right)$$
$$+\theta\left\{\left[\frac{(1+\beta_1)\Gamma\left(1+\frac{2}{\lambda_1}\right)+\Gamma\left(2+\frac{2}{\lambda_1}\right)}{\beta^{1/\lambda_1}\left(2+\frac{1}{\lambda_1}+\beta_1\right)\Gamma\left(1+\frac{2}{\lambda_1}\right)}-2\tau_{1,x}\right]\times\left[\frac{(1+\beta_2)\Gamma\left(1+\frac{2}{\lambda_2}\right)+\Gamma\left(2+\frac{2}{\lambda_2}\right)}{\beta^{1/\lambda_2}\left(2+\frac{1}{\lambda_2}+\beta_2\right)\Gamma\left(1+\frac{2}{\lambda_2}\right)}-2\tau_{1,y}\right]\right\},$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$
$$= \theta\left\{\left[E[X] - 2\tau_{1,x}\right]\left[E[Y] - 2\tau_{1,y}\right]\right\}$$
$$+E[X]E[Y] - E[X]E[Y]$$
$$= \theta\left\{\left[\frac{(1+\beta_1)\Gamma\left(1+\frac{2}{\lambda_1}\right)+\Gamma\left(2+\frac{2}{\lambda_1}\right)}{\beta^{1/\lambda_1}\left(2+\frac{1}{\lambda_1}+\beta_1\right)\Gamma\left(1+\frac{2}{\lambda_1}\right)} - 2\tau_{1,x}\right]\right.$$
$$\left.\times\left[\frac{(1+\beta_2)\Gamma\left(1+\frac{2}{\lambda_2}\right)+\Gamma\left(2+\frac{2}{\lambda_2}\right)}{\beta^{1/\lambda_2}\left(2+\frac{1}{\lambda_2}+\beta_2\right)\Gamma\left(1+\frac{2}{\lambda_2}\right)} - 2\tau_{1,y}\right]\right\}.$$

3.3 Parameter estimation

Let $X_1,\ldots,X_n$ be a random sample of size n from the BLBPG distribution with the joint pdf in Eq. (3.2). Then, its log-likelihood function is

$$\ell(\mathbf{\Theta}) = \log\prod_{i=1}^{n}\left\{\frac{\lambda_1\beta_1^{1+1/\lambda_1}(1+\beta_1+\beta_1 x_i^{\lambda_1})x_i^{\lambda_1}e^{-\beta_1 x_i^{\lambda_1}}}{(2+1/\lambda_1+\beta_1)\Gamma(1+1/\lambda_1)}\right.$$
$$\times\frac{\lambda_2\beta_2^{1+1/\lambda_2}(1+\beta_2+\beta_2 y_i^{\lambda_2})y_i^{\lambda_2}e^{-\beta_2 y_i^{\lambda_2}}}{(2+1/\lambda_2+\beta_2)\Gamma(1+1/\lambda_2)}$$
$$\times\left\{1+\theta\left[2(1+\beta_1)\delta_{1,1}(x_i)+2\delta_{2,1}(x_i)-1\right]\right.$$
$$\left.\left.\times\left[2(1+\beta_2)\delta_{1,2}(y_i)+2\delta_{2,2}(y_i)-1\right]\right\}\right\}, \quad (3.10)$$

where $\mathbf{\Theta}$ is a vector of $\lambda_1, \beta_1, \lambda_2, \beta_2$, and θ. The ML estimators $(\hat{\lambda}_1, \hat{\beta}_1, \hat{\lambda}_2, \hat{\beta}_2, \hat{\theta})$ can be obtained by solving Eq. (3.10) simultaneously, the estimates of each estimator are

$$\left.\frac{\partial\ell(\mathbf{\Theta})}{\partial\lambda_1}\right|_{\lambda_1=\hat{\lambda}_1} = 0, \left.\frac{\partial\ell(\mathbf{\Theta})}{\partial\beta_1}\right|_{\beta_1=\hat{\beta}_1} = 0, \left.\frac{\partial\ell(\mathbf{\Theta})}{\partial\lambda_2}\right|_{\lambda_2=\hat{\lambda}_2} = 0,$$
$$\left.\frac{\partial\ell(\mathbf{\Theta})}{\partial\beta_2}\right|_{\beta_2=\hat{\beta}_2} = 0, \left.\frac{\partial\ell(\mathbf{\Theta})}{\partial\theta}\right|_{\theta=\hat{\theta}} = 0.$$

Since these equations are no closed-form expressions for $(\hat{\lambda}_1, \hat{\beta}_1, \hat{\lambda}_2, \hat{\beta}_2, \hat{\theta})$ we used the numerical technique to calculate the ML estimates through statistical software. This study employed the nlm function in R software [22] version 4.3.1 to find the ML estimates of the parameters of the BLBPG distribution.

3.4 Monte Carlo simulation study

The Monte Carlo simulation study was conducted to measure the performance of the ML estimators for the BLBPG distribution with four cases of different parameter sets $(\lambda_1, \beta_1, \lambda_2, \beta_2, \theta)$. The first three cases, case 1: (1.5, 1.5, 3, 0.05, -0.5), case 2: (1.5, 1.5, 3, 0.05, 0.5), and case 3: (1.5, 1.5, 1.5, 1.5, 0.5) following Fig. 1, are studied. Three sets of parameters were selected to study different θ values while fixing other parameters (cases 1 and 2) and different λ_2 and β_2 for other fixed parameters. The situation mimicking the case study in Section 3.5 (case 4: set of parameters (0.4804, 1.7789, 0.3360, 2.5843, 0.3804) was added to see the simulation results. The study employed Monte Carlo simulation using R [22] version 4.3.1 and was carried out in the following steps:

1) Generate random samples (x_i, y_i) from BLBPG$(\lambda_1, \beta_1, \lambda_2, \beta_2, \theta)$ with different sizes (n) 10, 30, 50, 100, 200, and 500 reflecting small to large samples.

2) Estimate the value of $\lambda_1, \beta_1, \lambda_2, \beta_2$, and θ via the ML method.

3) 1,000 repetitions $(M = 1{,}000)$ were made to calculate the average of ML estimates (Est) and the MSE (mean square error) of each estimator.

Table 1 displays the Est and MSE values of each ML estimator, as obtained from the simulation results. The ML estimators are biased. However, when the sample sizes are large, these estimators give a value that closely approximates the true values. As the sample size increased, the MSE values of all parameters decreased. Moreover, Figs. 2–5 illustrate the box plots of the estimators for each parameter in each case. The findings indicated that the distribution of each ML estimator had a symmetrical distribution when considering large sample sizes. As the sample size increased, the value between the lower and upper estimates decreased.

3.5 Application study

We examined several real data sets to test the effectiveness of the BLBPG distribution. The study focused on the drought data set for Nebraska's Panhandle climate division. The actual drought data set revealed a total of 83 drought episodes within this division. The data included in this study were obtained from [9], which provided the monthly modified Palmer Drought Severity Index (PDSI) spanning from January 1895 to December 2004. The bivariate data sets x and y represent the duration of drought and non-drought, respectively. The determination of the BLBPG distribution, along with its sub-model known as the BLBG distribution, involved fitting the model using ML estimates to the observed data. The proposed distribution was compared with some bivariate distributions that also had five parameters such as the bivariate gamma (BG) distribution [8], the bivariate generalized exponential (BGE) distribution [10], and the bivariate Weibull (BW) distribution [12]. The joint pdf values of each bivariate distribution are shown below.

1) The joint pdf of the BG distribution:

$$f_2(x,y)=\left(\frac{x^{k_1-1}e^{-x/\eta_1}}{\eta_1^{k_1}\Gamma(k_1)}\right)\left(\frac{y^{k_2-1}e^{-y/\eta_2}}{\eta_2^{k_2}\Gamma(k_2)}\right)\left\{1+\theta\left[1-\frac{2\gamma(k_1,x/\eta_1)}{\Gamma(k_1)}\right]\left[1-\frac{2\gamma(k_2,y/\eta_2)}{\Gamma(k_2)}\right]\right\}, \quad (3.14)$$

where $x,y>0$ and the parameters $k_1,\eta_1,k_2,\eta_2>0$ and $-1<\theta<1$.

2) The joint pdf of the BGE distribution:

$$f_3(x,y)=a_1b_1e^{-b_1x}(1-e^{-b_1x})^{a_1-1}(1-e^{-b_2y})^{a_2-1}\times a_2b_2e^{-b_2y}\left\{1+\theta\left[1-2(1-e^{-b_1x})^{a_1}\right]\times\left[1-2(1-e^{-b_2y})^{a_2}\right]\right\}, \quad (3.15)$$

where $x,y>0$ and the parameters $a_1,b_1,a_2,b_2>0$ and $-1<\theta<1$.

3) The joint pdf of BW distribution:

$$f_4(x,y)=\frac{\alpha_1}{\omega_1}\left(\frac{\alpha_1}{\omega_1}\right)^{-\alpha_1x}e^{-(x/\omega_1)^{\alpha_1}}\frac{\alpha_2}{\omega_2}\left(\frac{\alpha_2}{\omega_2}\right)^{-\alpha_2y}\times e^{-(y/\omega_2)^{\alpha_2}}\left\{1+\theta\left[1-2\left(1-e^{-(x/\omega_1)^{\alpha_1}}\right)\right]\times\left[1-2(1-e^{-b_2y})^{a_2}\right]\right\}, \quad (3.16)$$

where $x,y>0$ and the parameters $\alpha_1,\omega_1,\alpha_2,\omega_2>0$ and $-1<\theta<1$.

To derive the ML estimates for each distribution, the nlm function in R version 4.3.1 [22] was utilized. Various models were evaluated for their fit using the AIC and BIC metrics. The data shown in Table 2 indicates that the BLBPG distribution had the lowest values for both AIC and BIC compared to the values of the other distributions. The BLBPG distribution exhibited a superior level of fit for the observed dataset in comparison to the alternative distributions.

The probability function plots for the datasets are shown in Fig. 6. In the pdf plots, the function has its maximum, which is equal to 0.0436984 (see Fig. 6(a)), at point (x, y) = (0, 0). Its value is nearly zero for large values of x (or y). From the plot of the survival function $S(x,y)=P(X>x,Y>y)$ in Fig. 6(b), the function has its maximum, which is equal to 1, at point $(x,y)=(0,0)$. Its value is nearly zero for large values of the duration of drought x (or the duration of non-drought y). For the duration of drought (x) and non-drought (y), the hazard function $h(x,y)$ is expected to be high at first and then gradually decreasing in the end (see Fig. 6(c)), reflecting a lower hazard for drought severity.

Table 1. Simulation results for the parameter estimates of the BLBPG distribution.

Parameter	n	Case 1		Case 2		Case 3		Case 4	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
λ_1	10	1.8294	0.4298	1.8439	0.4662	1.8216	0.4539	0.5910	0.0554
	30	1.6003	0.0852	1.5939	0.0874	1.5951	0.0890	0.5766	0.0390
	50	1.5460	0.0388	1.5739	0.0557	1.5518	0.0458	0.5436	0.0227
	100	1.5336	0.0204	1.5360	0.0236	1.5425	0.0292	0.5130	0.0100
	200	1.5138	0.0108	1.5219	0.0145	1.5223	0.0166	0.4981	0.0063
	500	1.5086	0.0072	1.5082	0.0068	1.5048	0.0043	0.4938	0.0033
β_1	10	1.4364	0.2582	1.4277	0.2864	1.4527	0.2673	1.6108	0.6487
	30	1.4690	0.0863	1.4774	0.0881	1.4763	0.0905	1.6118	0.5254
	50	1.4870	0.0458	1.4752	0.0584	1.4874	0.0487	1.6749	0.4089
	100	1.4870	0.0263	1.4903	0.0286	1.4913	0.0334	1.7299	0.2641
	200	1.4939	0.0152	1.4927	0.0173	1.4915	0.0190	1.7754	0.1867
	500	1.5000	0.0089	1.4991	0.0082	1.4979	0.0055	1.7612	0.0943
λ_2	10	3.5421	1.2686	3.5214	1.4227	1.8246	0.7260	0.4118	0.0261
	30	3.1560	0.3020	3.2009	0.3447	1.5874	0.0879	0.3968	0.0173
	50	3.0903	0.1444	3.1044	0.1733	1.5515	0.0443	0.3648	0.0080
	100	3.0706	0.0823	3.0532	0.0814	1.5402	0.0328	0.3496	0.0044
	200	3.0266	0.0443	3.0327	0.0502	1.5238	0.0164	0.3453	0.0026
	500	3.0148	0.0180	3.0084	0.0214	1.5037	0.0042	0.3393	0.0014
β_2	10	0.0446	0.0017	0.0522	0.0027	1.4095	0.2578	2.3286	1.3419
	30	0.0510	0.0011	0.0492	0.0010	1.4805	0.0922	2.3791	1.3419
	50	0.0494	0.0005	0.0498	0.0006	1.4955	0.0523	2.5309	0.7276
	100	0.0485	0.0003	0.0502	0.0004	1.4950	0.0351	2.5910	0.4931
	200	0.0501	0.0002	0.0500	0.0002	1.4901	0.0190	2.5832	0.2831
	500	0.0498	0.0001	0.0503	0.0001	1.4977	0.0054	2.5934	0.1496
θ	10	-0.1129	0.4160	0.1264	0.3924	0.1314	0.3716	0.2007	0.2764
	30	-0.3012	0.2231	0.3105	0.2046	0.3032	0.2072	0.2270	0.2238
	50	-0.4286	0.1106	0.4264	0.1170	0.3863	0.1404	0.2639	0.1651
	100	-0.4742	0.0685	0.4727	0.0718	0.4338	0.0996	0.3091	0.1108
	200	-0.4938	0.0475	0.4978	0.0457	0.4824	0.0562	0.3486	0.0584
	500	-0.4965	0.0246	0.5010	0.0257	0.4951	0.0163	0.3775	0.0416

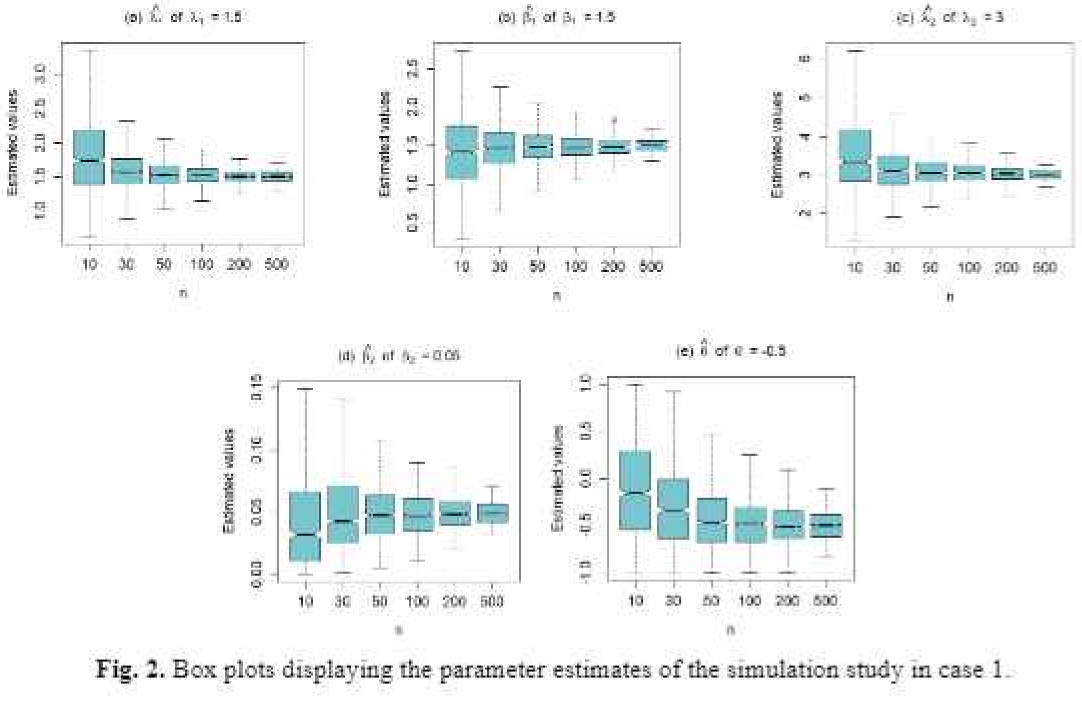

Fig. 2. Box plots displaying the parameter estimates of the simulation study in case 1.

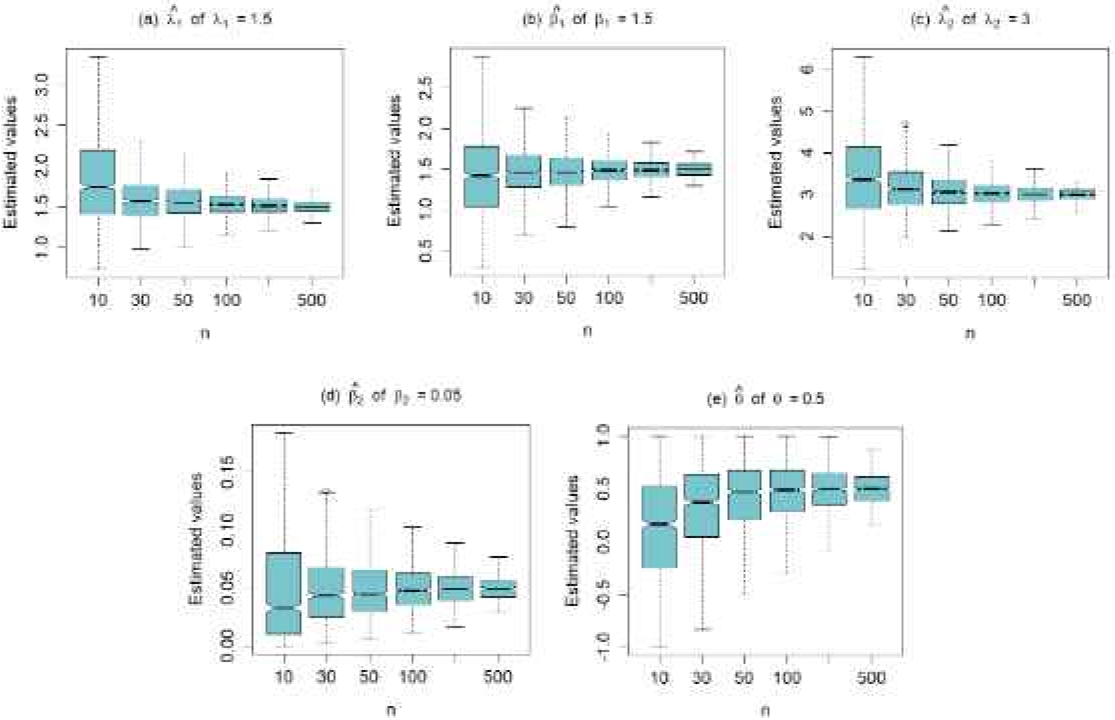

Fig. 3. Box plots displaying the parameter estimates the simulation study in case 2.

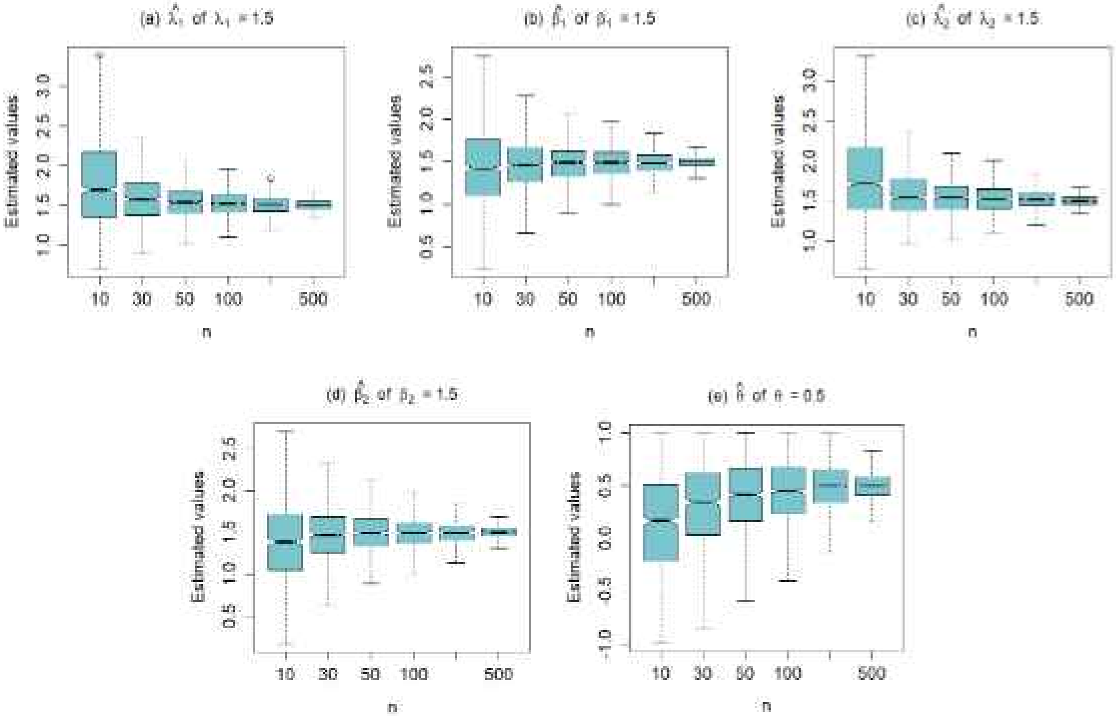

Fig. 4. Box plots displaying the parameter estimates the simulation study in case 3.

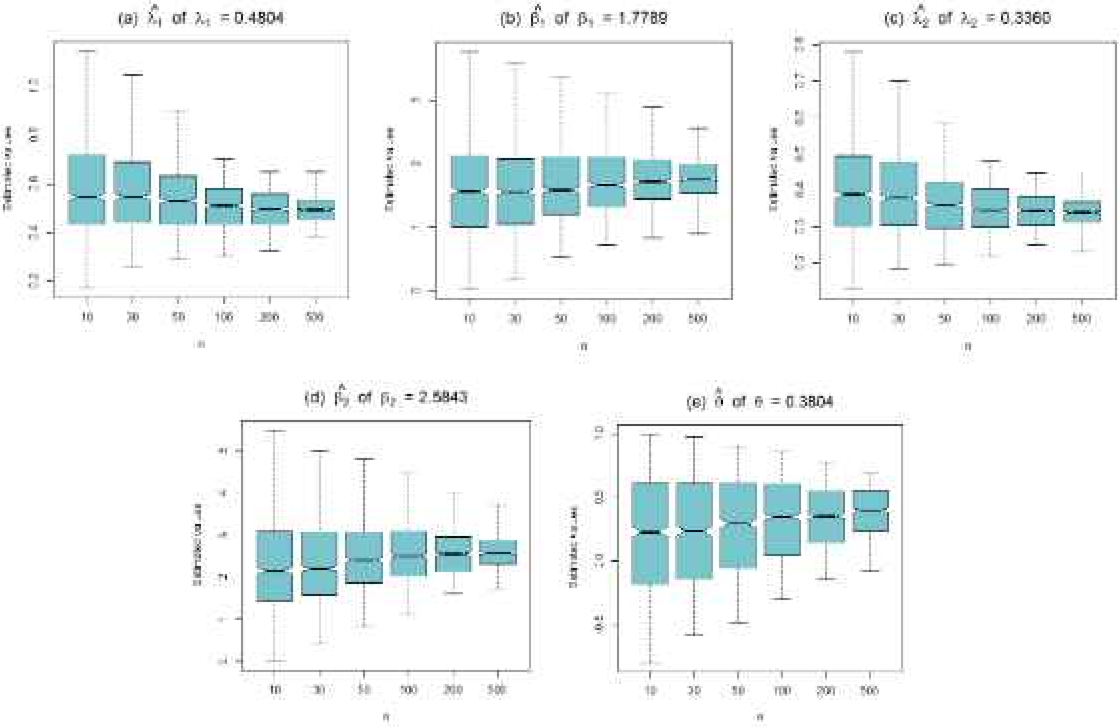

Fig. 5. Box plots displaying the parameter estimates the simulation study in case 4.

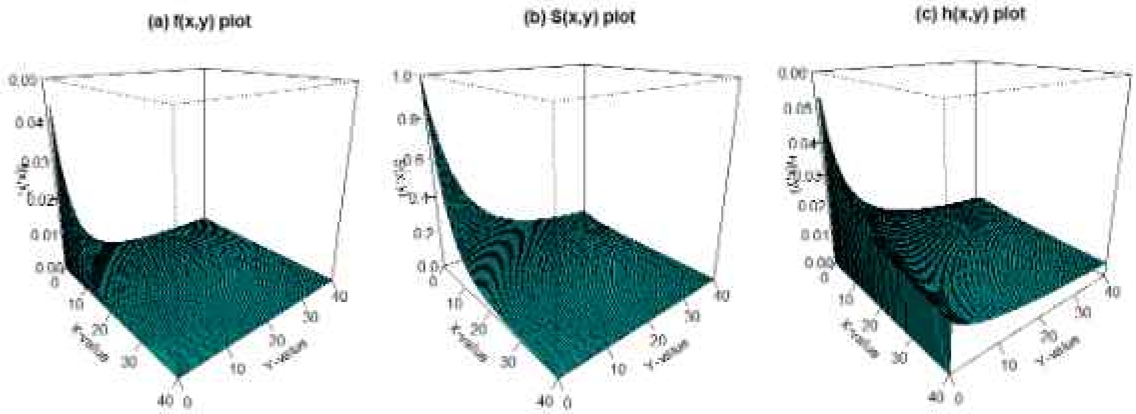

Fig. 6. Plots of the pdf, survival function, and hazard function for (X,Y) ~ BLBPG $(0.4804, 1.7789, 0.3360, 2.5843, 0.3804)$.

Table 2. The log-likelihood (LL), AIC, and BIC values for drought data were estimated using the ML method [9].

ML estimates	-LL	AIC	BIC
BG distribution $\hat{k}_1 = 0.9611, \hat{\eta}_1 = 0.1573,$ $\hat{k}_2 = 0.6373, \hat{\eta}_2 = 0.0631,$ $\hat{\theta} = 0.4535$	496.46	1,002.9	1,015.0
BGE distribution $\hat{\alpha}_1 = 0.9946, \hat{b}_1 = 0.1628,$ $\hat{\alpha}_2 = 0.6327, \hat{b}_2 = 0.0710,$ $\hat{\theta} = 0.4615$	497.05	1,004.1	1,016.2
BW distribution $\hat{\alpha}_1 = 0.9034, \hat{\omega}_1 = 5.7017,$ $\hat{\alpha}_2 = 0.7134, \hat{\omega}_2 = 7.6768,$ $\hat{\theta} = 0.4208$	491.97	993.9	1,006.0
BLBG distribution $\hat{\beta}_1 = 0.4055, \hat{\beta}_2 = 0.2249,$ $\hat{\theta} = 0.4657$	601.05	1,208.1	1,215.4
BLBPG distribution $\hat{\lambda}_1 = 0.4804, \hat{\beta}_1 = 1.7789,$ $\hat{\lambda}_2 = 0.3360, \hat{\beta}_2 = 2.5843,$ $\hat{\theta} = 0.3804$	481.01	972.0	984.1

Fig. 6 shows the probability function plots for the datasets. In the pdf plots, the function has its maximum, which is equal to 0.0436984 (see Fig. 6(a)), at point (x, y) = (0, 0). Its value is nearly zero for large values of x (or y). From the plot of the survival function $S(x,y) = P(X > x, Y > y)$ in Fig. 6(b), the function has its maximum, which is equal to 1, at point $(x,y) = (0,0)$. Its value is nearly zero for large values of the duration of drought x (or the duration of non-drought y). For the duration of drought (x) and non-drought (y), the hazard function $h(x,y)$ is expected to be high at first and then gradually decreasing in the end (see Fig. 6(c)), reflecting a lower hazard for drought severity.

4. Conclusion

The FGM copula serves as the basis for the present study's novel bivariate length-biased power Garima (BLBPG) distribution. The ML approach was used to estimate the parameters of the BLBPG distribution for drought data [9]. The characteristics of the bivariate distribution were introduced. The simulation study was conducted to estimate the parameters of the BLBPG distribution and evaluate the efficacy of the maximum likelihood method. The ML estimator of each parameter for the BLBPG distribution are biased. However, the ML estimators of each parameter give the values close to the true parameters when the sample sizes were large. As the sample sizes increased, the MSE values of all parameters decreased. The BLBPG distribution, which was generated from the FGM copula, demonstrated a good fit for the real data application. Furthermore, the proposed bivariate distribution exhibited greater efficiency compared to alternative distributions.

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References

[1] Gupta RC, Akman HO. On the reliability studies of a weighted inverse Gaussian model. Journal of Statistical Planning and Inference 1995; 48(1): 69-83.

[2] Klinjan K, Aryuyuen S. The length-biased power Garima distribution and its application to model lifetime data. Songklanakarin Journal of Science & Technology, 2021; 43(3): 667-76.

[3] Lai CD. Constructions of continuous bivariate distributions. Journal of the Indian Society for Probability and Statistics 2004; 8: 21-43.

[4] Nelsen RB. An introduction to copulas, 2nd ed., Springer, New York, NY.; 2006.

[5] Sklar M. Fonctions de répartition à n dimensions et leurs marges. In Annales de l'ISUP 1959; 8: 229-231.

[6] Sklar A. Random variables, joint distribution functions, and copulas. Kybernetika 1973; 9(6): 449-60.

[7] Morgenstern D. Einfache Beispiele zweidimensionaler Verteilungen. Mitteilingsblatt für Mathematische Statistik, 1956; 8: 234-5.

[8] Kotz S, Balakrishnan N, Johnson NL. Continuous multivariate distributions, Volume 1: Models and applications, volume 1, John Wiley & Sons; 2004.

[9] Nadarajah S. A bivariate Pareto model for drought. Stochastic Environmental Research and Risk Assessment 2009; 23: 811-22.

[10] Abd Elaal MK, Jarwan RS. Inference of bivariate generalized exponential distribution based on copula functions. Applied Mathematical Sciences 2017; 11(24): 1155-86.

[11] Peres MVDO, Achcar JA, Martinez EZ. Bivariate modified Weibull distribution derived from Farlie-Gumbel-Morgenstern copula: a simulation study. Electronic Journal of Applied Statistical Analysis 2018; 11(2): 463-88.

[12] Almetwally EM, Muhammed HZ, El-Sherpieny ESA. Bivariate Weibull distribution: properties and different methods of estimation. Annals of Data Science 2020; 7: 163-93.

[13] Zhao J, Faqiri H, Ahmad Z, Emam W, Yusuf M, Sharawy A. The Lomax-claim model: bivariate extension and applications to financial data. Complexity 2021; 2021: 1-17.

[14] Hamed MS, Cordeiro GM, Yousof HM. A new compound Lomax model: Properties, copulas, modeling and risk analysis utilizing the negatively skewed insurance claims data. Pakistan Journal of Statistics and Operation Research 2022; 18(3): 601-31.

[15] Abulebda M, Pandey A, Tyagi S. On bivariate inverse Lindley distribution derived from copula. Thailand Statistician 2023; 21(2): 291-304.

[16] Saminger-Platz S, Kolesárová A, Šeliga A, Mesiar R, Klement EP. The impact on the properties of the EFGM copulas when extending this family. Fuzzy Sets and Systems 2021; 415: 1-26.

[17] Bekrizadeh H. Generalized family of copulas: definition and properties. Thailand Statistician 2021; 19(1): 162-77.

[18] Aryuyuen, S, Panphut W, Pudprommarat, C. Bivariate Sushila distribution based on copulas: properties, simulations, and applications. Lobachevskii Journal of Mathematics 2024; 44(11): 4592-609.

[19] Shenbagaraja R, Rather AA, Subramanian C. On some aspects of length biased technique with real life data. Science, technology & Development 2019; 8(9): 326-35.

[20] Basu AP. Bivariate failure rate. Journal of the American Statistical Association 1971; 66(333): 103-4.

[21] Klinjan K, Aryuyuen S. R package for the length-biased power Garima distribution to analyze lifetime data. International Journal of Mathematics and Computer Science 2022; 17(3): 1351-62.

[22] R Core Team et al. R: A language and environment for statistical computing. [cited 2023 November 1]. Available from: https://www.R-project.org/.

Appendix.

1. The joint distribution function of the BLBPG distribution

From the copula concept of Eq. (2.1) and Eq. (2.2), we have

$$F(x,y) = C(u,v;\theta) = uv[1+\theta(1-u)(1-v)],$$

where $u = G(x;\lambda_1,\beta_1)$ and $v = G(y;\lambda_2,\beta_2)$ is the cdf of the LBPG distribution as in Eq. (2.7). Then the joint distribution function of the BLBPG distribution is

$$
\begin{aligned}
F(x,y) &= \left\{1-\frac{(1+\beta_1)\gamma(1+\frac{1}{\lambda_1},\beta_1 x^{\lambda_1})+\gamma(2+\frac{1}{\lambda_1},\beta_1 x^{\lambda_1})}{(2+\frac{1}{\lambda_1}+\beta_1)\Gamma(1+\frac{1}{\lambda_1})}\right\}\\
&\times\left\{1-\frac{(1+\beta_2)\gamma(1+\frac{1}{\lambda_2},\beta_2 y^{\lambda_2})+\gamma(2+\frac{1}{\lambda_2},\beta_2 y^{\lambda_2})}{(2+\frac{1}{\lambda_2}+\beta_2)\Gamma(1+\frac{1}{\lambda_2})}\right\}\\
&\times\left\{1+\theta\left[\frac{(1+\beta_1)\gamma(1+\frac{1}{\lambda_1},\beta_1 x^{\lambda_1})+\gamma(2+\frac{1}{\lambda_1},\beta_1 x^{\lambda_1})}{(2+\frac{1}{\lambda_1}+\beta_1)\Gamma(1+\frac{1}{\lambda_1})}\right]\right.\\
&\left.\times\left[\frac{(1+\beta_2)\gamma(1+\frac{1}{\lambda_2},\beta_2 y^{\lambda_2})+\gamma(2+\frac{1}{\lambda_2},\beta_2 y^{\lambda_2})}{(2+\frac{1}{\lambda_2}+\beta_2)\Gamma(1+\frac{1}{\lambda_2})}\right]\right\}\\
&=\left(1-\left[(1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right]\right)\\
&\times\left(1-\left[(1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right]\right)\\
&\times\left\{1+\theta\left[(1+\beta_1)\delta_{1,1}(x)+\delta_{2,1}(x)\right]\right.\\
&\left.\times\left[(1+\beta_2)\delta_{1,2}(y)+\delta_{2,2}(y)\right]\right\},
\end{aligned}
$$

where $\delta_{j,k}(z)=\dfrac{\gamma(j+1/\lambda_k,\beta_k z^{\lambda_k})}{(2+1/\lambda_k+\beta_k)\Gamma(1+1/\lambda_k)}$ for $j,k=1,2$

2. The joint density function of the BLBPG distribution

The copula density of Eq. (2.3) is

$$c(u,v)=1+\theta(1-2u)(1-2v)$$

and the joint pdf of Eq. (2.5) is

$$
\begin{aligned}
f(x,y) &= c(G(x),G(y);\theta)g(x)g(y)\\
&=\{1+\theta[1-2G(x;\lambda_1,\beta_1)][1-2G(y;\lambda_2,\beta_2)]\}\\
&\quad\times g(x;\lambda_1,\beta_1)g(y;\lambda_2,\beta_2)
\end{aligned}
$$

where $g(x;\lambda_1,\beta_1)$ and $g(y;\lambda_2,\beta_2)$ are the marginal pdf as shown in Eq. (2.6). The joint density function of the BLBPG distribution is

$$
\begin{aligned}
f(x,y) &= \left(\frac{\lambda_1\beta_1^{1+1/\lambda_1}(1+\beta_1+\beta_1 x^{\lambda_1})x^{\lambda_1}e^{-\beta_1 x^{\lambda_1}}}{(2+\frac{1}{\lambda_1}+\beta_1)\Gamma(1+\frac{1}{\lambda_1})}\right)\\
&\times\left(\frac{\lambda_2\beta_2^{1+1/\lambda_2}(1+\beta_2+\beta_2 x^{\lambda_2})y^{\lambda_2}e^{-\beta_2 y^{\lambda_2}}}{(2+\frac{1}{\lambda_2}+\beta_2)\Gamma(1+\frac{1}{\lambda_2})}\right)\\
&\times\left\{1+\theta\left[2\left(\frac{(1+\beta_1)\gamma(1+\frac{1}{\lambda_1},\beta_1 x^{\lambda_1})+\gamma(2+\frac{1}{\lambda_1},\beta_1 x^{\lambda_1})}{(2+\frac{1}{\lambda_1}+\beta_1)\Gamma(1+\frac{1}{\lambda_1})}\right)-1\right]\right.\\
&\left.\times\left[2\left(\frac{(1+\beta_2)\gamma(1+\frac{1}{\lambda_2},\beta_2 y^{\lambda_2})+\gamma(2+\frac{1}{\lambda_2},\beta_2 y^{\lambda_2})}{(2+\frac{1}{\lambda_2}+\beta_2)\Gamma(1+\frac{1}{\lambda_2})}\right)-1\right]\right\}\\
&=\frac{\lambda_1\beta_1^{1+1/\lambda_1}(1+\beta_1+\beta_1 x^{\lambda_1})x^{\lambda_1}e^{-\beta_1 x^{\lambda_1}}}{(2+1/\lambda_1+\beta_1)\Gamma(1+1/\lambda_1)}\\
&\times\frac{\lambda_2\beta_2^{1+1/\lambda_2}(1+\beta_2+\beta_2 y^{\lambda_2})y^{\lambda_2}e^{-\beta_2 y^{\lambda_2}}}{(2+1/\lambda_2+\beta_2)\Gamma(1+1/\lambda_2)}\\
&\times\left\{1+\theta\left[2(1+\beta_1)\delta_{1,1}(x)+2\delta_{2,1}(x)-1\right]\right.\\
&\left.\times\left[2(1+\beta_2)\delta_{1,2}(y)+2\delta_{2,2}(y)-1\right]\right\}.
\end{aligned}
$$

3. If $\lambda_1=\lambda_2=1$ the BLBPG distribution reduces to the BLBG distribution, with joint distribution and density functions, respectively as

$$
\begin{aligned}
F_1(x,y) &= \left(1-\frac{(1+\beta_1)\gamma(2,\beta_1 x)+\gamma(3,\beta_1 x)}{3+\beta_1}\right)\\
&\times\left(1-\frac{(1+\beta_2)\gamma(2,\beta_2 y)+\gamma(3,\beta_2 y)}{3+\beta_2}\right)\\
&\times\left\{1+\theta\left[\frac{(1+\beta_1)\gamma(2,\beta_1 x)+\gamma(3,\beta_1 x)}{3+\beta_1}\right]\right.\\
&\left.\times\left[\frac{(1+\beta_2)\gamma(2,\beta_2 y)+\gamma(3,\beta_2 y)}{3+\beta_2}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
f_1(x,y) &= \frac{(1+\beta_1+\beta_1 x)xye^{-\beta_1 x}e^{-\beta_2 y}(1+\beta_2+\beta_2 y)}{(3+\beta_1)(3+\beta_2)}\\
&\times\beta_1^2\beta_2^2\left\{1+\theta\left(\frac{2}{3+\beta_1}[(1+\beta_1)\gamma(2,\beta_1 x)\right.\right.\\
&\left.+\gamma(3,\beta_1 x)]-1\right)\left(\frac{2}{3+\beta_{21}}[\gamma(3,\beta_2 y)\right.\\
&\left.\left.+(1+\beta_2)\gamma(2,\beta_2 y)]-1\right)\right\}.
\end{aligned}
$$

4. The r-th moment about the origin of BLBPG random variable

$$E[(XY)^r] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^r y^r)c\left(G(x),G(y);\theta\right)g(x)g(y)dxdy$$

$$= \int_0^{\infty}\int_0^{\infty}(x^r y^r)\Big[\big\{1+\theta[1-2G(x;\lambda_1,\beta_1)][1-2G(y;\lambda_2,\beta_2)]\big\}$$

$$\times g(x;\lambda_1,\beta_1)g(y;\lambda_2,\beta_2)\Big]dxdy$$

$$= \int_0^{\infty}x^r g(x;\lambda_1,\beta_1)dx\int_0^{\infty}y^r g(y;\lambda_2,\beta_2)dy$$

$$+\theta\left\{\int_0^{\infty}x^r g(x;\lambda_1,\beta_1)[1-2G(x;\lambda_1,\beta_1)]dx\right.$$

$$\left.\times\int_0^{\infty}y^r g(y;\lambda_2,\beta_2)[1-2G(y;\lambda_2,\beta_2)]dy\right\}$$

$$= E(X^r)\cdot E(Y^r) +$$

$$+\theta\left\{\left[E(X^r)-2\int_0^{\infty}x^r g(x;\lambda_1,\beta_1)G(x;\lambda_1,\beta_1)dx\right]\right.$$

$$\left.\times\left[E(Y^r)-2\int_0^{\infty}x^r g(x;\lambda_1,\beta_1)G(x;\lambda_1,\beta_1)dx\right]\right\}$$

$$= \theta\left\{\left[E[X^r]-2\tau_{r,x}\right]\left[E[Y^r]-2\tau_{r,y}\right]\right\}$$

$$+E[X^r]E[Y^r],$$

where

$$\tau_{r,x} = \int_0^{\infty}x^r g(x;\lambda_1,\beta_1)G(x;\lambda_1,\beta_1)dx,$$

$$\tau_{r,y} = \int_0^{\infty}y^r g(y;\lambda_2,\beta_2)G(y;\lambda_2,\beta_2)dy.$$