



# Some New Results on Fixed Points for $\varpi$ -Distances in Complex-Valued Metric Spaces

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## ABSTRACT

In this paper, we introduce the notion of a  $\varpi$ -distance in complete complex-valued metric spaces and prove some fixed point theorems for mappings satisfying some appropriate inequalities in complete complex-valued metric spaces. Moreover, we deduce new fixed point results in complete complex-valued metric spaces and provide some examples to illustrate the usability of the obtained results.

**Keywords:**  $\varpi$ -distance;  $\Omega$ -distance;  $c$ -distance; Complex-valued metric spaces; Fixed point; Generalized  $c$ -distance;  $w\tau$ -distance;  $w$ -distance

## 1. Introduction

The Banach's contraction mapping principle is widely recognized as the source of a metric fixed point theory. The existence and uniqueness of a fixed point of operators or mappings has been a subject of a great interest since the work of a Banach in 1922 [1]. The concept of nonexpansive mappings has also been widely studied in the following works [2, 3]. This prin-

ciple has been applied in different spaces by mathematicians, for example D-metric spaces, quasimetric spaces, quasi b-metric spaces, b-metric-like spaces, Dislocated quasi-b-metric spaces, and G-metric spaces (see [4-8]) have already been obtained. A new space called the complex valued metric space which is more general than well-know metric spaces has been introduced by Azam et al. [9]. Naturally, this new idea

can be utilized to define complex valued normed spaces and complex valued inner product spaces which, in turn, offer a lot of scope for further investigations. Many authors have studied a fixed point theory in complex valued metric space (see [10-16]).

On the other hand, in 1996, Takahashi et al.[17] introduced the notion of a  $w$ -distance on a metric space and proved a nonconvex minimization theorem which generalizes Caristi’s fixed-point results and the  $\epsilon$ -variational principle. After that in 2011, Cho et al. [18] introduced the concept of a  $c$ -distance in a cone metric space. For more details about  $c$ -distance (see [19, 20] and the references contained therein). The concept of a  $\Omega$ -distance of  $G$ -metric spaces and constructed some fixed point theorems in  $G$ -metric spaces by using the notion of a  $\Omega$ -distance introduced by Saadati et al. [21]. Moreover, the results of Saadati et al. was made clear to shoot up by Shatanawi and Pitea [22]. Recently, a new concept of a  $w$  $t$ -distance on  $b$ -metric spaces, which is a  $b$ -metric version of the  $w$ -distance of Takahashi et al. [17] was introduced by Saadati et al. [23] and proved some fixed point results in a partially ordered  $b$ -metric space.

Moreover, Mohanta [24] has generalized the results of Saadati et al. [23]. The concept of a generalized  $c$ -distance on a cone  $b$ -metric space was introduced by Xu et al. [25] which is a generalization of  $c$ -distance of Cho et al. [18] and proved some fixed and common fixed point results in ordered cone  $b$ -metric spaces using this distance . For more details about generalized  $c$ -distance see [26].

The above concept, we introduce the notion of a  $\varpi$ -distance in complete complex-valued metric spaces and prove some fixed point theorems for mappings satisfying some appropriate inequalities in complete complex-valued metric spaces.

Moreover, we deduce new fixed point results in complete complex-valued metric spaces and provide some examples to illustrate the usability of the obtained results.

## 2. Preliminaries

Throughout this paper, we will write  $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$  and  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Let  $\mathbb{C}$  be the set of complex numbers and  $\sigma_1, \sigma_2 \in \mathbb{C}$ , we define a partial order  $<$  and  $\lesssim$  on  $\mathbb{C}$  as follows:

- (i)  $\sigma_1 < \sigma_2$  if and only if  $\text{Re}(\sigma_1) < \text{Re}(\sigma_2)$  and  $\text{Im}(\sigma_1) < \text{Im}(\sigma_2)$
- (ii)  $\sigma_1 \lesssim \sigma_2$  if and only if  $\text{Re}(\sigma_1) \leq \text{Re}(\sigma_2)$  and  $\text{Im}(\sigma_1) \leq \text{Im}(\sigma_2)$ .

Now, we recall some property of a complex valued metric space.

**Definition 2.1** ([9]). Let  $X$  be a nonempty set. Suppose that the mapping  $\Gamma : X \times X \rightarrow \mathbb{C}$  satisfies the following conditions:

- ( $\Gamma_1$ )  $0 \lesssim \Gamma(\zeta, \eta)$ , for all  $\zeta, \eta \in X$ ;
- ( $\Gamma_2$ )  $\Gamma(\zeta, \eta) = 0$  if and only if  $\zeta = \eta$  for all  $\zeta, \eta \in X$ ;
- ( $\Gamma_3$ )  $\Gamma(\zeta, \eta) = \Gamma(\eta, \zeta)$  for all  $\zeta, \eta \in X$ ;
- ( $\Gamma_4$ )  $\Gamma(\zeta, \eta) \lesssim \Gamma(\zeta, \sigma) + \Gamma(\sigma, \eta)$ , for all  $\zeta, \eta, \sigma \in X$ .

Then  $\Gamma$  is called a *complex valued metric* on  $X$  and  $(X, \Gamma)$  is called a *complex valued metric space*.

**Example 2.2** ([9]). Defined  $\Gamma : X \times X \rightarrow \mathbb{C}$  as follows:

$$\Gamma(\sigma_1, \sigma_2) = \begin{cases} \frac{2}{3}|\zeta_1 - \zeta_2| + \frac{i}{2}|\zeta_1 - \zeta_2| \\ \text{if } \sigma_1, \sigma_2 \in X_1, \\ \frac{1}{2}|\eta_1 - \eta_2| + \frac{i}{3}|\eta_1 - \eta_2| \\ \text{if } \sigma_1, \sigma_2 \in X_2, \\ (\frac{2}{3}\zeta_1 + \frac{1}{2}\eta_2) + i(\frac{1}{2}\zeta_1 + \frac{1}{3}\eta_2) \\ \text{if } \sigma_1 \in X_1, \sigma_2 \in X_2, \\ (\frac{1}{2}\eta_1 + \frac{2}{3}\zeta_2) + i(\frac{1}{3}\eta_1 + \frac{1}{2}\zeta_2) \\ \text{if } \sigma_1 \in X_2, \sigma_2 \in X_1, \end{cases}$$

Let  $X_1 = \{\sigma \in \mathbb{C} : 0 \leq \text{Re}(\sigma) \leq 1, \text{Im}(\sigma) = 0\}$ ,  $X_2 = \{\sigma \in \mathbb{C} : 0 \leq \text{Im}(\sigma) \leq 1, \text{Re}(\sigma) = 0\}$  and let  $X = X_1 \cup X_2$ . Let  $\sigma_1 = \zeta_1 + i\eta_1$ ,  $\sigma_2 = \zeta_2 + i\eta_2 \in X$ . Then  $(X, \Gamma)$  is a complete complex valued metric space.

**Example 2.3** ([9]). Let  $X = C([1, 3], \mathbb{R})$ ,  $a > 0$  and for every  $\zeta, \eta \in X$  let  $M_{\zeta\eta} = \max_{t \in [1, 3]} |\zeta(t) - \eta(t)|$ ,  $\Gamma(\zeta, \eta) = M_{\zeta\eta} \sqrt{1 + a^2} e^{i \tan^{-1} a}$ . Then  $(X, \Gamma)$  is a complex valued metric space.

**Definition 2.4** ([9]). Let  $(X, \Gamma)$  be a complex valued metric space.

(i) A point  $\zeta \in X$  is called *interior point* of a set  $B \subseteq X$  whenever there exists  $0 < r \in \mathbb{C}$  such that  $N(\zeta, r) := \{\eta \in X : \Gamma(\zeta, \eta) < r\} \subseteq B$ .

(ii) A point  $\zeta \in X$  is called *limit point* of a set  $B \subseteq X$  whenever for every  $0 < r \in \mathbb{C}$  such that  $N(\zeta, r) \cap (X - B) \neq \emptyset$ .

(iii) A subset  $B \subseteq X$  is called *open* whenever each element of  $B$  is an interior point of  $B$ .

(iv) A subset  $B \subseteq X$  is called *closed* whenever each limit point of  $B$  belongs to  $B$ .

(v) The family  $F = \{N(\zeta, r) : \zeta \in X, 0 < r\}$  is a sub-basis for a topology on  $X$ . we denote this complex topology  $\tau_c$ . Indeed, the topology  $\tau_c$  is Hausdorff.

**Definition 2.5** ([9]). Let  $(X, \Gamma)$  be a complex valued metric space and  $\{\zeta_n\}$  be a sequence in  $X$  and  $\zeta \in X$ .

(i) If for every  $c \in \mathbb{C}$ , with  $0 < c$  there is  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $\Gamma(\zeta_n, \zeta) < c$ , then  $\{\zeta_n\}$  is said to be *convergent*,  $\{\zeta_n\}$  converges to  $\zeta$  and  $\zeta$  is limit point of  $\{\zeta_n\}$ . We denote this by  $\zeta_n \rightarrow \zeta$  as  $n \rightarrow \infty$  or  $\lim_{n \rightarrow \infty} \zeta_n = \zeta$ .

(ii) If for every  $c \in \mathbb{C}$ , with  $0 < c$  there is  $N \in \mathbb{N}$  such that for all  $n >$

$N$ ,  $\Gamma(\zeta_n, \zeta_{n+m}) < c$ , where  $m \in \mathbb{N}$ , then  $\{\zeta_n\}$  is said to be *Cauchy sequence*.

(iii) If every Cauchy sequence in  $X$  is convergent, then  $(X, \Gamma)$  is said to be a *complete complex valued metric space*.

**Lemma 2.6** ([9]). Let  $(X, \Gamma)$  be a complex valued metric space and let  $\{\zeta_n\}$  be a sequence in  $X$ . Then  $\{\zeta_n\}$  converges to  $\zeta$  if and only if  $|\Gamma(\zeta_n, \zeta)| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 2.7** ([9]). Let  $(X, \Gamma)$  be a complex valued metric space and let  $\{\zeta_n\}$  be a sequence in  $X$ . Then  $\{\zeta_n\}$  is a Cauchy sequence if and only if  $|\Gamma(\zeta_n, \zeta_{n+m})| \rightarrow 0$  as  $n \rightarrow \infty$ , where  $m \in \mathbb{N}$ .

**Definition 2.8.** Let  $\mathbb{C}$  be the set of complex number and  $M \subseteq \mathbb{C}$ .

(i)  $f$  is continuous at  $\sigma_0 \in \mathbb{C}$  if and only if for all  $\epsilon > 0$ , there exists  $\delta > 0$ , if  $|\sigma - \sigma_0| < \delta$ , then  $|f(\sigma) - f(\sigma_0)| < \epsilon$  for all  $\sigma \in \mathbb{C}$ .

(ii)  $f$  is continuous on  $M$  if and only if  $f$  is continuous at  $\sigma_0$  for all  $\sigma_0 \in M$ .

**Remark 2.9** ([14]). We obtained that following statements hold:

(i) If  $\sigma_1 \lesssim \sigma_2$  and  $\sigma_2 \lesssim \sigma_3$ , then  $\sigma_1 \lesssim \sigma_3$ .

(ii) If  $\sigma \in \mathbb{C}$ ,  $a, b \in \mathbb{R}$  and  $a \leq b$ , then  $a\sigma \lesssim b\sigma$ .

(iii) If  $0 \lesssim \sigma_1 \lesssim \sigma_2$ , then  $|\sigma_1| \leq |\sigma_2|$ .

### 3. $\varpi$ -distance

In this section, we introduce the notion of  $\varpi$ -distance in complete complex-valued metric spaces and prove some lemma in such a space.

Let us recall that a complex-valued function  $f$  defined on a complex-valued metric space  $X$  is said to be lower semi-continuous at a point  $\sigma$  in  $X$  if either  $\liminf_{\zeta_n \rightarrow \sigma} f(\zeta_n) = \infty$  or  $f(\sigma) \leq$

$\liminf_{\zeta_n \rightarrow \sigma} f(\zeta_n)$ , whenever  $\zeta_n \in X$  for each  $n \in \mathbb{N}$  and  $\zeta_n \rightarrow \sigma$

**Definition 3.1.** Let  $(X, \Gamma)$  be a complex valued metric space. Then a function  $p : X \times X \rightarrow \mathbb{C}$  is called a  $\varpi$ -distance on  $X$  if the following are satisfied:

- (1)  $p(\zeta, \eta) \geq 0$  for all  $\zeta, \eta \in X$ ;
- (2)  $p(\zeta, \eta) \leq p(\zeta, \sigma) + p(\sigma, \eta)$  for all  $\zeta, \eta, \sigma \in X$ ;
- (3) for all  $\zeta \in X$ ,  $p(\zeta, \cdot) : X \rightarrow \mathbb{C}$  is lower semicontinuous i.e., if  $\zeta \in X$ ,  $\eta_n \rightarrow \eta \in X$  then  $p(\zeta, \eta) \leq \liminf_{n \rightarrow \infty} p(\zeta, \eta_n)$ ;
- (4) for all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that  $p(\zeta, \sigma) \leq \delta$  and  $p(\sigma, \eta) \leq \delta$  imply  $p(\zeta, \eta) \leq \epsilon$ , where  $\epsilon, \delta \in \mathbb{C}$ .

Now, we provide examples of  $\varpi$ -distance in complex-valued metric space.

**Example 3.2.** Let  $(X, \Gamma)$  be a complex valued metric space. Then  $p = \Gamma$  is a  $\varpi$ -distance on  $X$

*Proof.* (1), (2) and (3) are clearly. To show (4). Let  $\epsilon > 0$ . Setting  $\delta = \frac{\epsilon}{2}$ . Then, we have  $\Gamma(\zeta, \eta) \leq \Gamma(\zeta, \sigma) + \Gamma(\sigma, \eta) = p(\zeta, \sigma) + p(\sigma, \eta) \leq \epsilon$ , where  $p(\zeta, \sigma) \leq \delta$  and  $p(\sigma, \eta) \leq \delta$ .  $\square$

**Example 3.3.** Let  $(X, \Gamma)$  be a complex valued metric space. Then a function  $p : X \times X \rightarrow \mathbb{C}$  defined by  $p(\zeta, \eta) = c$  for every  $\zeta, \eta \in X$ ,  $p$  is a  $\varpi$ -distance on  $X$ , where  $c \in \mathbb{C}$  and  $c > 0$ .

*Proof.* (1), (2) and (3) clear. To show (4), let  $\epsilon > 0$ . Setting  $\delta = \frac{\epsilon}{2}$ . Then,  $p(\zeta, \sigma) \leq \delta$  and  $p(\sigma, \eta) \leq \delta$  imply  $\Gamma(\zeta, \eta) \leq \epsilon$ .  $\square$

**Example 3.4.** Let  $X$  be a normed linear space with  $\|\cdot\|$  and let  $\Gamma(\zeta, \eta) = \|\zeta - \eta\| + i\|\zeta - \eta\|$ , for all  $\zeta, \eta \in X$ . Then a function  $p : X \times X \rightarrow \mathbb{C}$  defined by  $p(\zeta, \eta) = (\|\zeta\| + \|\eta\|) + (\|\zeta\| + \|\eta\|)i$  for every  $\zeta, \eta \in X$ ,  $p$  is a  $\varpi$ -distance on  $X$

*Proof.* Let  $\zeta, \eta, \sigma \in X$ . Then,

$$p(\zeta, \eta) = (\|\zeta\| + \|\eta\|) + (\|\zeta\| + \|\eta\|)i \leq p(\zeta, \sigma) + p(\sigma, \eta).$$

(1), (2) and (3) clear. To show (4), let  $\epsilon > 0$ . Setting  $\delta = \frac{\epsilon}{2}$ . Then,

$$\Gamma(\zeta, \eta) = \|\zeta - \eta\| + i\|\zeta - \eta\| \leq (\|\zeta\| + \|\eta\|) + (\|\zeta\| + \|\eta\|)i \leq p(\zeta, \sigma) + p(\sigma, \eta) \leq \epsilon. \quad \square$$

**Example 3.5.** Let  $(X, \Gamma)$  be a complex valued metric space, and let  $\mathfrak{I}$  be continuous mapping from  $X$  into itself. Then a function  $p : X \times X \rightarrow \mathbb{C}$  defined by  $p(\zeta, \eta) = \max\{\Gamma(\mathfrak{I}\zeta, \eta), \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta)\}$  for every  $\zeta, \eta \in X$ ,  $p$  is a  $\varpi$ -distance on  $X$ .

*Proof.* We see that (1) holds. We show (2). Let  $\zeta, \eta, \sigma \in X$ . If  $\Gamma(\mathfrak{I}\zeta, \sigma) \geq \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\sigma)$ , then

$$p(\zeta, \sigma) \leq \Gamma(\mathfrak{I}\zeta, \sigma) \leq \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta) + \Gamma(\mathfrak{I}\eta, \sigma) \leq \max\{\Gamma(\mathfrak{I}\zeta, \eta), \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta)\} + \max\{\Gamma(\mathfrak{I}\eta, \sigma), \Gamma(\mathfrak{I}\eta, \mathfrak{I}\sigma)\} = p(\zeta, \eta) + p(\eta, \sigma).$$

On the other hand, we have

$$p(\zeta, \sigma) \leq \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\sigma) \leq \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta) + \Gamma(\mathfrak{I}\eta, \mathfrak{I}\sigma) \leq \max\{\Gamma(\mathfrak{I}\zeta, \eta), \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta)\} + \max\{\Gamma(\mathfrak{I}\eta, \sigma), \Gamma(\mathfrak{I}\eta, \mathfrak{I}\sigma)\} = p(\zeta, \eta) + p(\eta, \sigma).$$

Hence (2) holds. Since  $\mathfrak{I}$  is continuous, we have for all  $\zeta \in X$ ,  $p(\zeta, \cdot) : X \rightarrow \mathbb{C}$  is lower semicontinuous. Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{2}$ . Therefore, if  $p(\zeta, \sigma) \leq \delta$  and  $p(\sigma, \eta) \leq \delta$ , then  $p(\zeta, \mathfrak{I}\sigma) \leq \delta$  and  $p(\mathfrak{I}\sigma, \eta) \leq \delta$ . Hence  $\Gamma(\zeta, \eta) \leq \Gamma(\mathfrak{I}\sigma, \zeta) + \Gamma(\mathfrak{I}\sigma, \eta) \leq \epsilon$ .  $\square$

Next, we prove some basis lemma for  $\varpi$ -distance in complex-valued metric space.

**Lemma 3.6.** Let  $(X, \Gamma)$  be a complex valued metric spaces, and let  $p$  be an  $\varpi$ -distance on  $X$ . Let  $\{\zeta_n\}, \{\eta_n\}$  be sequences in  $X$ ,  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $\mathbb{C}$  with  $\alpha_n \geq 0$  and  $\beta_n \geq 0$ ,  $\{\alpha_n\}$  and  $\{\beta_n\}$  convergent to zero and let  $\zeta, \eta, \sigma, a \in X$ . Then we have the following:

(i) if  $p(\zeta_n, \eta_n) \leq \alpha_n$  and  $p(\zeta_n, \sigma) \leq \beta_n$  for all  $n \in \mathbb{N}$ , then  $\eta_n \rightarrow \sigma$ ;

(ii) if  $p(\zeta_n, \eta) \leq \alpha_n$  and  $p(\zeta_n, \sigma) \leq \beta_n$  for all  $n \in \mathbb{N}$ , then  $\eta = \sigma$ .

In particular, if  $p(\zeta, \eta) = 0$  and  $p(\zeta, \sigma) = 0$ , then  $\eta = \sigma$ ;

(iii) if  $p(\zeta_n, \zeta_m) \leq \alpha_n$  for all  $n, m \in \mathbb{N}$  with  $m \geq n$ , then  $\{\zeta_n\}$  is a Cauchy sequence;

(iv) if  $p(\eta, \zeta_n) \leq \alpha_n$  for all  $n \in \mathbb{N}$ , then  $\{\zeta_n\}$  is a Cauchy sequence.

*Proof.* First, we show (i). Let  $\epsilon > 0$ . From the definition of  $\varpi$ -distance, there exists a  $\delta > 0$  such that  $p(\mu, \nu) \leq \delta$  and  $p(\mu, \sigma) \leq \delta$  imply  $d(\nu, \sigma) \leq \epsilon$ . Since  $\{\alpha_n\}$  and  $\{\beta_n\}$  are converging to zero, we have  $n_0 \in \mathbb{N}$  such that  $\alpha_n \leq \delta$  and  $\beta_n \leq \delta$  for all  $n \geq n_0$ . Then we get, for any  $n \geq n_0$ ,

$$p(\zeta_n, \eta_n) \leq \alpha_n \leq \delta,$$

$$p(\zeta_n, \sigma) \leq \beta_n \leq \delta.$$

So,  $d(\eta_n, \sigma) \leq \epsilon$ , and then  $\{\eta_n\}$  converges to  $\sigma$ . (ii) holds, because using (i).

Now, to show that (iii) is true. Let  $\epsilon > 0$ . As in the proof of (ii), choose  $\delta > 0$ , and then  $n_0 \in \mathbb{N}$ . Then, for any  $m, n \geq n_0 + 1$

$$p(\zeta_{n_0}, \zeta_n) \leq \alpha_{n_0} \leq \delta,$$

$$p(\zeta_{n_0}, \zeta_m) \leq \alpha_{n_0} \leq \delta.$$

So,  $\Gamma(\zeta_n, \zeta_m) \leq \epsilon$ . Therefore  $\zeta_n$  is a Cauchy sequence. As in proof of (iii), we can prove (iv).  $\square$

## 4. Fixed point theorems

In this section, we will prove some fixed point theorems for mappings satisfying some appropriate inequalities in complete complex-valued metric spaces. Moreover, we deduce new fixed point results in complete complex-valued metric spaces and provide some examples to illustrate the usability of the obtained results. We suppose  $(X, \leq)$  is a partially ordered set and  $\mathfrak{I}$  is a mapping of  $X$  into itself. We say that  $\mathfrak{I}$  is non-decreasing if for  $\zeta, \eta \in X$ ,  $\zeta \leq \eta$  implies  $\mathfrak{I}\zeta \leq \mathfrak{I}\eta$ .

**Theorem 4.1.** Let  $(X, \Gamma)$  be a complete complex-valued metric spaces. Assume that  $\mathfrak{I}$  is a mapping from  $X$  into itself and a function  $p : X \times X \rightarrow [0, \infty)$  is  $\varpi$ -distance on  $X$ . Suppose that the following conditions are satisfied.

(i)  $(X, \leq)$  is a partially ordered set and  $\mathfrak{I}$  non-decreasing mapping;

(ii) for any fixed  $\zeta \in X$  with  $\zeta \leq \mathfrak{I}\zeta$

$$\inf_{\eta \in X} \{p(\zeta, \eta) + p(\zeta, \mathfrak{I}\zeta)\} > 0, \quad (4.1)$$

with  $\eta \neq \mathfrak{I}\eta$ ;

(iii) there exists an  $\zeta_0 \in X$  with  $\zeta_0 \leq \mathfrak{I}\zeta_0$ ;

(iv) there exists  $\alpha, \beta \in [0, \frac{1}{2})$  such that

$$p(\mathfrak{I}\zeta, \mathfrak{I}^2\eta) \lesssim \alpha p(\zeta, \mathfrak{I}\eta) + \beta \frac{p(\zeta, \mathfrak{I}\zeta)p(\mathfrak{I}\eta, \mathfrak{I}^2\eta)}{1 + p(\zeta, \mathfrak{I}\eta)}, \quad (4.2)$$

for all  $\zeta \leq \mathfrak{I}\zeta$  and any  $\eta \in X$  and  $\alpha + \beta < 1$ .

Then  $\mathfrak{I}$  has a fixed point. Moreover, if  $\nu = \mathfrak{I}\nu$ , then  $p(\nu, \nu) = 0$ .

*Proof.* Let  $\zeta_0$  be an arbitrary point in  $X$ , we define  $\zeta_{n+1} = \mathfrak{I}\zeta_n$ , for all  $n \in \mathbb{N}_0$ . If  $\mathfrak{I}\zeta_0 = \zeta_0$ , then the proof is complete. On the other

hand, suppose that  $\mathfrak{I}\zeta_0 \neq \zeta_0$ . Since  $\zeta_0 \leq \mathfrak{I}\zeta_0$  and  $\mathfrak{I}$  is non-decreasing, we have

$$\zeta_0 \leq \zeta_1 \leq \zeta_2 \leq \zeta_3 \leq \dots \leq \zeta_n \leq \zeta_{n+1} \leq \dots$$

Using the inequality (4.2) and  $p(\zeta_n, \mathfrak{I}\zeta_n) < 1 + p(\zeta_n, \mathfrak{I}\zeta_n)$ , we obtain that

$$\begin{aligned} p(\zeta_{n+1}, \zeta_{n+2}) &= p(\mathfrak{I}\zeta_n, \mathfrak{I}^2\zeta_n) \\ &\leq \alpha p(\zeta_n, \mathfrak{I}\zeta_n) + \\ &\quad \beta \frac{p(\zeta_n, \mathfrak{I}\zeta_n)p(\mathfrak{I}\zeta_n, \mathfrak{I}^2\zeta_n)}{1 + p(\zeta_n, \mathfrak{I}\zeta_n)} \\ &\leq \alpha p(\zeta_n, \zeta_{n+1}) + \\ &\quad \beta p(\zeta_{n+1}, \zeta_{n+2}). \end{aligned} \tag{4.3}$$

Thus,

$$\begin{aligned} p(\zeta_{n+1}, \zeta_{n+2}) &\leq \gamma p(\zeta_n, \zeta_{n+1}) \\ &\leq \gamma^2 p(\zeta_{n-1}, \zeta_n) \\ &\leq \vdots \\ &\leq \gamma^{n+1} p(\zeta_0, \zeta_1), \end{aligned} \tag{4.4}$$

where  $\gamma = \frac{\alpha}{1-\beta}$ . Then, for any  $n \in \mathbb{N}_0$  with  $m > n$ , using the triangle inequality, we obtain that

$$\begin{aligned} p(\zeta_n, \zeta_m) &\leq p(\zeta_n, \zeta_{n+1}) + p(\zeta_{n+1}, \zeta_{n+2}) + \\ &\quad \dots + p(\zeta_{m-1}, \zeta_m) \\ &\leq \gamma^n p(\zeta_0, \zeta_1) + \gamma^{n+1} p(\zeta_0, \zeta_1) + \\ &\quad \dots + \gamma^{m-1} p(\zeta_0, \zeta_1) \\ &\leq (\gamma^n + \gamma^{n+1} + \dots + \gamma^{m-1}) p(\zeta_0, \zeta_1) \\ &\leq \left(\frac{\gamma^n}{1-\gamma}\right) p(\zeta_0, \zeta_1). \end{aligned}$$

By using Lemma 3.6, we obtain that  $\{\zeta_n\}$  is a Cauchy sequence in  $X$ . Since  $X$  is complete, there exists  $\eta \in X$  such that  $\zeta_n \rightarrow \eta$ . For any  $n \in \mathbb{N}_0$ . Then since  $\{\zeta_n\}$  converges to  $\eta$  in  $(X, \Gamma)$  and  $p(\zeta_n, \cdot)$  is lower semi-continuous, we obtain that  $p(\zeta_n, \eta) \leq \liminf_{m \rightarrow \infty} p(\zeta_n, \zeta_m) \leq \frac{\gamma^n}{1-\gamma} p(\zeta_0, \zeta_1)$ . Now, we show that  $\eta$  is a fixed point of  $\mathfrak{I}$ ,

(i.e.  $\mathfrak{I}\eta = \eta$ ). If  $\mathfrak{I}\eta \neq \eta$ , then by using condition (4.1) and  $\zeta_n \leq \zeta_{n+1}$ , we get

$$\begin{aligned} 0 &< \inf_{\eta \in X} \{p(\zeta_n, \eta) + p(\zeta_n, \zeta_{n+1})\} \\ &\leq \inf_{\eta \in X} \left\{ \frac{\gamma^n}{1-\gamma} p(\zeta_0, \zeta_1) + \gamma^n p(\zeta_0, \zeta_1) \right\}. \end{aligned}$$

It implies that  $0 < 0$ , which is a contradiction. Therefore  $\eta = \mathfrak{I}\eta$ . Moreover, by the inequality (4.2), we have  $p(\eta, \eta) = 0$ .  $\square$

Now, we will prove the unique fixed point as follows:

**Corollary 4.2.** Let  $(X, \Gamma)$  be a complete complex-valued metric space. Assume that the function  $p : X \times X \rightarrow [0, \infty)$  is  $\varpi$ -distance on  $X$  and  $\mathfrak{I}$  is a mapping from  $X$  into itself are satisfying the conditions (i) – (iv) in Theorem 4.1 and  $X \neq F(\mathfrak{I})$ .

Then  $\mathfrak{I}$  has a unique fixed point. Moreover, if  $\nu = \mathfrak{I}\nu$ , then  $p(\nu, \nu) = 0$ .

*Proof.* We will show that  $\mathfrak{I}$  has a unique fixed point. Assume that  $\sigma, \eta$  in  $X$  are a fixed point of  $\mathfrak{I}$ . If  $\sigma \neq \eta$ , then by assumption and using condition (4.1), there exists  $\varsigma \in X$  such that  $\mathfrak{I}\varsigma \neq \varsigma$ , we get

$$0 < \inf_{\varsigma \in X} \{p(\eta, \varsigma) + p(\eta, \mathfrak{I}\eta)\}$$

and

$$0 < \inf_{\varsigma \in X} \{p(\sigma, \varsigma) + p(\sigma, \mathfrak{I}\sigma)\}.$$

Thus, we obtain that

$$\begin{aligned} 0 &< \inf_{\varsigma \in X} \{p(\eta, \varsigma) + p(\eta, \mathfrak{I}\eta)\} \\ &\quad + \inf_{\varsigma \in X} \{p(\sigma, \varsigma) + p(\sigma, \mathfrak{I}\sigma)\} \\ &\leq p(\eta, \mathfrak{I}\eta) + p(\sigma, \mathfrak{I}\sigma) = 0, \end{aligned} \tag{4.5}$$

which a contradiction. Hence  $\sigma = \eta$ . Therefore  $\mathfrak{I}$  has a unique fixed point.  $\square$

**Corollary 4.3.** Let  $(X, \Gamma)$  be a complete complex-valued metric spaces. Assume that the function  $p : X \times X \rightarrow [0, \infty)$  is  $\varpi$ -distance on  $X$  and  $\mathfrak{F}$  is a mapping from  $X$  into itself are satisfied the conditions (i)–(iv) in Theorem 4.1 and, if  $p(\zeta, \eta) = 0$ , then  $\zeta = \eta$  for all  $\zeta, \eta \in X$ . Then  $\mathfrak{F}$  has a unique fixed point. Moreover, if  $v = \mathfrak{F}v$ , then  $p(v, v) = 0$ .

*Proof.* We will to show that  $\mathfrak{F}$  has unique fixed point. Assume that  $\sigma, \eta$  in  $X$  are a fixed point of  $\mathfrak{F}$ . If  $\sigma \neq \eta$ , then using inequality (4.2), we have

$$\begin{aligned} p(\eta, \sigma) &= p(\mathfrak{F}\eta, \mathfrak{F}^2\sigma) \\ &\leq \alpha p(\eta, \mathfrak{F}\sigma) \\ &\quad + \beta \frac{p(\eta, \mathfrak{F}\eta)p(\mathfrak{F}\sigma, \mathfrak{F}^2\sigma)}{1 + p(\eta, \mathfrak{F}\sigma)} \\ &= \alpha p(\eta, \sigma). \end{aligned}$$

Since  $0 \leq \alpha < 1$ , we have  $\sigma = \eta$ . Therefore  $\mathfrak{F}$  has unique fixed point.  $\square$

**Example 4.4.** Let  $X = \mathbb{C}$ , and let  $|\cdot|$  be an absolute value of  $\mathbb{R}$ . Defined

$$p(\zeta, \eta) = |Re(\eta)| + i|Im(\eta)|,$$

for all  $\zeta, \eta \in X$ . Then  $p$  is  $\varpi$ -distance on  $X$ , by Example 3.3. Consider the function  $\mathfrak{F} : X \rightarrow X$  defined by  $\mathfrak{F}\zeta = \frac{\zeta}{2}$ . Thus,

$$\begin{aligned} p(\mathfrak{F}\zeta, \mathfrak{F}^2\eta) &= |\mathfrak{F}^2\eta| = |Re(\frac{\eta}{4})| + i|Im(\frac{\eta}{4})| \\ &= \frac{1}{2}(|Re(\frac{\eta}{2})| + i|Im(\frac{\eta}{2})|) \\ &\leq \frac{1}{2}(|Re(\frac{\eta}{2})| + i|Im(\frac{\eta}{2})|) + \\ &\frac{1}{3} \frac{(|Re(\frac{\zeta}{2})| + i|Im(\frac{\zeta}{2})|)(|Re(\frac{\eta}{4})| + i|Im(\frac{\eta}{4})|)}{1 + (|Re(\frac{\eta}{2})| + i|Im(\frac{\eta}{2})|)} \\ &= \alpha p(\zeta, \mathfrak{F}\eta) + \beta \frac{p(\zeta, \mathfrak{F}\zeta)p(\mathfrak{F}\eta, \mathfrak{F}^2\eta)}{1 + p(\zeta, \mathfrak{F}\eta)}, \end{aligned}$$

where  $\alpha = \frac{1}{2}$  and  $\beta = \frac{1}{3}$ . Then the conditions of Theorem 4.1 hold and the fixed

point of  $\mathfrak{F}$  is  $0 + 0i$  and  $P(0, 0) = 0$ . Moreover,  $\mathfrak{F}$  has a unique fixed point, because  $X \neq F(\mathfrak{F})$ .

Next, we will replace the inequality (4.2) by the inequality (4.7), which have a same the result of theorem 4.1 and prove the fixed point theorem as follows:

**Theorem 4.5.** Let  $(X, \Gamma)$  be a complete complex-valued metric spaces. Assume that  $\mathfrak{F}$  is a mapping from  $X$  into itself and a function  $p : X \times X \rightarrow [0, \infty)$  is  $\varpi$ -distance on  $X$ . Suppose that the following conditions are satisfied.

- (i)  $(X, \leq)$  is a partially ordered set and  $\mathfrak{F}$  non-decreasing mapping;
- (ii) for any fixed  $\zeta \in X$  with  $\zeta \leq \mathfrak{F}\zeta$

$$\inf_{\eta \in X} \{p(\zeta, \eta) + p(\zeta, \mathfrak{F}\zeta)\} > 0, \quad (4.6)$$

with  $\eta \neq \mathfrak{F}\eta$ ;

- (iii) there exists an  $\zeta_0 \in X$  with  $\zeta_0 \leq \mathfrak{F}\zeta_0$ ;

(iv) there exists  $\alpha, \beta \in [0, \frac{1}{2})$  such that

$$p(\mathfrak{F}\zeta, \mathfrak{F}^2\eta) \leq \alpha p(\zeta, \mathfrak{F}\eta) + \beta \frac{p(\zeta, \mathfrak{F}\zeta)p(\eta, \mathfrak{F}^2\eta)}{1 + p(\zeta, \mathfrak{F}\eta)}, \quad (4.7)$$

for all  $\zeta \leq \mathfrak{F}\zeta$  and any  $\eta \in X$  and  $\alpha + 2\beta < 1$ . Then  $\mathfrak{F}$  has a fixed point. Moreover, if  $v = \mathfrak{F}v$ , then  $p(v, v) = 0$ .

*Proof.* Let  $\zeta_0$  be an arbitrary point in  $X$ , we define  $\zeta_{n+1} = \mathfrak{F}\zeta_n$ , for all  $n \in \mathbb{N}_0$ . If  $\mathfrak{F}\zeta_0 = \zeta_0$ , then the proof is complete. On the other hand, suppose that  $\mathfrak{F}\zeta_0 \neq \zeta_0$ . Since  $\zeta_0 \leq \mathfrak{F}\zeta_0$  and  $\mathfrak{F}$  is non-decreasing, we have

$$\zeta_0 \leq \zeta_1 \leq \zeta_2 \leq \zeta_3 \leq \dots \leq \zeta_n \leq \zeta_{n+1} \leq \dots$$

Using the inequality (4.7) and  $p(\zeta_n, \mathfrak{F}\zeta_n) < 1 + p(\zeta_n, \mathfrak{F}\zeta_n)$ , we obtain that

$$p(\zeta_{n+1}, \zeta_{n+2}) = p(\mathfrak{F}\zeta_n, \mathfrak{F}^2\zeta_n)$$

$$\begin{aligned}
 &\lesssim \alpha p(\zeta_n, \mathfrak{I}\zeta_n) + \beta \frac{p(\zeta_n, \mathfrak{I}\zeta_n)p(\zeta_n, \mathfrak{I}^2\zeta_n)}{1 + p(\zeta_n, \mathfrak{I}\zeta_n)} \\
 &\leq \alpha p(\zeta_n, \zeta_{n+1}) + \beta p(\zeta_n, \zeta_{n+2}), \\
 &\leq \alpha p(\zeta_n, \zeta_{n+1}) + \beta p(\zeta_n, \zeta_{n+1}) + \\
 &\beta p(\zeta_{n+1}, \zeta_{n+2}). \tag{4.8}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 p(\zeta_{n+1}, \zeta_{n+2}) &\lesssim \gamma p(\zeta_n, \zeta_{n+1}) \\
 &\lesssim \gamma^2 p(\zeta_{n-1}, \zeta_n) \\
 &\lesssim \vdots \\
 &\lesssim \gamma^{n+1} p(\zeta_0, \zeta_1), \tag{4.9}
 \end{aligned}$$

where  $\gamma = \frac{\alpha+\beta}{1-\beta}$ . Then, for any  $n \in \mathbb{N}_0$  with  $m > n$ , using the triangle inequality, we obtain that

$$\begin{aligned}
 p(\zeta_n, \zeta_m) &\lesssim p(\zeta_n, \zeta_{n+1}) + p(\zeta_{n+1}, \zeta_{n+2}) + \\
 &\dots + p(\zeta_{m-1}, \zeta_m) \\
 &\lesssim \gamma^n p(\zeta_0, \zeta_1) + \gamma^{n+1} p(\zeta_0, \zeta_1) + \\
 &\dots + \gamma^{m-1} p(\zeta_0, \zeta_1) \\
 &\lesssim (\gamma^n + \gamma^{n+1} + \dots + \gamma^{m-1}) p(x_0, x_1) \\
 &\lesssim \left(\frac{\gamma^n}{1-\gamma}\right) p(\zeta_0, \zeta_1). \tag{4.10}
 \end{aligned}$$

By using Lemma 3.6, we obtain that  $\{\zeta_n\}$  is a Cauchy sequence in  $X$ . As in proof of Theorem 4.1, we obtain that  $\mathfrak{I}$  has a fixed point. Moreover, if  $\nu = \mathfrak{I}\nu$ , then  $p(\nu, \nu) = 0$ .  $\square$

**Corollary 4.6.** Let  $(X, \Gamma)$  be a complete complex-valued metric space. Assume that the function  $p : X \times X \rightarrow [0, \infty)$  is  $\varpi$ -distance on  $X$  and  $\mathfrak{I}$  is a mapping from  $X$  into itself are satisfied conditions (i) – (iv) in Theorem 4.5 and  $X \neq F(\mathfrak{I})$ .

Then  $\mathfrak{I}$  has a unique fixed point. Moreover, if  $\nu = \mathfrak{I}\nu$ , then  $p(\nu, \nu) = 0$ .

*Proof.* The same proof of Corollary 4.2.  $\square$

**Corollary 4.7.** Let  $(X, \Gamma)$  be a complete complex-valued metric space. Assume that the function  $p : X \times X \rightarrow [0, \infty)$  is  $\varpi$ -distance on  $X$  and  $\mathfrak{I}$  is a mapping from  $X$  into itself are satisfied conditions (i) – (iv) in Theorem 4.5 and, if  $p(\zeta, \eta) = 0$ , then  $\zeta = \eta$  for all  $\zeta, \eta \in X$ .

Then  $\mathfrak{I}$  has a unique fixed point. Moreover, if  $\nu = \mathfrak{I}\nu$ , then  $p(\nu, \nu) = 0$ .

*Proof.* We will show that  $\mathfrak{I}$  has unique fixed point. Assume that  $\sigma, \eta$  in  $X$  are a fixed point of  $\mathfrak{I}$ . If  $\sigma \neq \eta$ , then using inequality (4.7), we have

$$\begin{aligned}
 p(\eta, \sigma) &= p(\mathfrak{I}\eta, \mathfrak{I}^2\sigma) \\
 &\leq \alpha p(\eta, \mathfrak{I}\sigma) + \beta \frac{p(\eta, \mathfrak{I}\eta)p(\sigma, \mathfrak{I}^2\sigma)}{1 + p(\eta, \mathfrak{I}\sigma)} \\
 &= \alpha p(\eta, \sigma).
 \end{aligned}$$

Since  $0 \leq \alpha < 1$ , we have  $\sigma = \eta$ . Therefore  $\mathfrak{I}$  has unique fixed point.  $\square$

**Example 4.8.** Let  $X = \mathbb{C}$ , and define a mapping  $\Gamma : X \times X \rightarrow \mathbb{C}$  by  $|\zeta - \eta|$  for all  $\zeta, \eta \in X$ , then  $(X, \Gamma)$  can be easily verified as a complete complex-valued metric spaces. Defined  $p(\zeta, \eta) = |Re(\eta)| + i|Im(\eta)|$ , for all  $\zeta, \eta \in X$ . Then  $p$  is  $\varpi$ -distance on  $X$ , consider the function  $\mathfrak{I} : X \rightarrow X$  defined by

$$\mathfrak{I}(\zeta + i\eta) = \begin{cases} 0, & \zeta, \eta \in Q \\ 3 + 3i, & \zeta, \eta \in Q^c \\ 3, & \zeta \in Q^c, \eta \in Q \\ 3i, & \zeta \in Q, \eta \in Q^c. \end{cases} \tag{4.11}$$

Now for  $\zeta = \frac{1}{\sqrt{3}}$  and  $\eta = 0$  we get  $\Gamma(\mathfrak{I}(\frac{1}{\sqrt{3}}), \mathfrak{I}(0)) = \Gamma(3, 0) = 3 \lesssim \lambda \Gamma(\frac{1}{\sqrt{3}}, 0) = \lambda \frac{1}{\sqrt{3}}$ . Thus  $\lambda \geq 3\sqrt{3}$ , which is a contradiction as  $0 \leq \lambda < 1$ . However, notice that  $\mathfrak{I}^2\sigma = 0$ , so that  $0 = \Gamma(\mathfrak{I}^2\sigma_1, \mathfrak{I}^2\sigma_2) \lesssim \lambda \Gamma(\sigma_1, \sigma_2)$ , which shows that  $\mathfrak{I}^2$  satisfies the requirement of Bryant Theorem and  $\sigma = 0$  is the unique fixed point of  $\mathfrak{I}$ .

$$\begin{aligned}
 p(\mathfrak{I}\zeta, \mathfrak{I}^2\eta) &= |\mathfrak{I}^2\eta| = |Re(\frac{\eta}{9})| + i|Im(\frac{\eta}{9})| \\
 &= \frac{1}{3}(|Re(\frac{\eta}{3})| + i|Im(\frac{\eta}{3})|) \\
 &\approx \frac{1}{3}(|Re(\frac{\eta}{3})| + i|Im(\frac{\eta}{3})|) + \\
 &\frac{1}{4} \frac{(|Re(\frac{\zeta}{3})| + i|Im(\frac{\zeta}{3})|)(|Re(\frac{\eta}{9})| + i|Im(\frac{\eta}{9})|)}{1 + (|Re(\frac{\eta}{3})| + i|Im(\frac{\eta}{3})|)} \\
 &= \alpha p(\zeta, \mathfrak{I}\eta) + \beta \frac{p(\zeta, \mathfrak{I}\zeta)p(\eta, \mathfrak{I}^2\eta)}{1 + p(\zeta, \mathfrak{I}\eta)},
 \end{aligned}$$

where  $\alpha = \frac{1}{3}$  and  $\beta = \frac{1}{4}$ . Then the conditions of Theorem 4.5 hold and the fixed point of  $\mathfrak{I}$  is  $0 + 0i$  and  $p(0, 0) = 0$ . Moreover,  $\mathfrak{I}$  has a unique fixed point, because  $X \neq F(\mathfrak{I})$ , ( $F(\mathfrak{I})$  is means that the set of fixed point of  $\mathfrak{I}$ ).

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