The Tangent Stiffness Uncertainty in Strain-hardening and Strain-softening Soil

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ABSTRACT: Load distribution in piles can be evaluated from strain-gage records by applying the tangent modulus method to the measured strains. The method requires tests in soil exhibiting a plastic response to the relative movement between the pile and the soil. Where the response instead is hardening or softening with increasing movement, the evaluated pile material stiffness becomes larger and smaller than true, respectively. This is demonstrated by analysis of a hypothetical test on a pile with a constant stiffness (EA/L) tested in ideally plastic, strain-hardening, and strain-softening soil.

KEYWORDS: Pile Stiffness, Secant Method, Tangent Method, Plastic Response, Hardening, Softening

1. INTRODUCTION

A conventional static head-down pile loading test provides information on the load-movement response of the pile head, but the response further down the pile, notably the pile toe, must be inferred from information beyond the loading test records. For a short pile, the basic pile-head response is often sufficient for completing a pile foundation design. For a longer pile, but yet a pile of moderate length, performing a bidirectional test will provide the needed information on the pile-toe response. But, a long test pile will have to be instrumented. The most common instrumentation consists of placing a pair or pairs of strain gages at certain levels to obtain the average strain over the pile cross section and converting the strains to axial load in the pile.

Where the measured strains are unaffected by shaft resistance, the conversion applies the method of direct secant stiffness. For gage levels affected by shaft resistance, the tangent stiffness method is used. Both methods of determining the stiffness of the pile material are considered straight-forward and rely on using the measured strains to determine the pile stiffness, *EA/L*, where E = Young modulus of the pile material, A = pile cross sectional area, and L = 1 m length of pile (Fellenius 2019).

The E-modulus of steel is known accurately as it is a constant value (29.5 x 106 ksi or 205 GPa). In contrast, not only does the concrete modulus range widely, it is also often not a constant but reduces with increasing strain. This means that when load is applied to a pile or a column, the load-movement may be in the shape of a curve rather than a straight line. The tangent stiffness method enables establishing the pile stiffness as a function of the imposed strain.

Lately, I have seen analysis results showing inconsistent relations of stiffness as a function of strain for tests in strain-hardening and strain-softening soil. To illustrate the issue, this paper presents a summary of the analytical principles of determining load from measured strain with examples and, then, shows results of a hypothetical pile tested in three soils with different shaft resistance response, one fully plastic, one strain-hardening, and a strain-softening.

2. THE DIRECT SECANT METHOD

It is common to calculate E-modulus of concrete as a the relation between the modulus and the cylinder strength, as proposed by the American Concrete Institute ACI 318-14 Manual: $E_{CONCRTETE}$ = 57,000$\sqrt{f'_c}$ (psi) or $E_{concrete}$ = 5,000$\sqrt{\sigma_{strength}}$ (MPa). However, the relation is not particularly reliable and it is usually better to determine the modulus from the actual load-strain measurements, as follows.

For records from loading a free-standing pile (like a column), a plot of the slope of a plot of load versus strain would indeed represent the axial stiffness, *EA/L*, of the pile. In contrast to a column, however, the axial load in a pile is not constant, but, due to shaft resistance, it diminishes proportionally with the distance from the load application (at the pile head or at the bidirectional cell). Therefore, before the shaft resistance is fully mobilized, the slope of the load-versus-strain curve is steeper than that of its equivalent column, i.e., the apparent stiffness is larger than the true stiffness of the pile. Once the shaft resistance is fully mobilized, it is usually taken as implicit that the continued soil response is plastic and, therefore, the slope of the load-strain curve represents the true stiffness of the pile.

The pile stiffness is best determined from a gage level that is unaffected by shaft resistance, which means that the records should be from a gage level near the pile head, or sufficiently near to have only negligible influence from shaft resistance between the load at the pile head and the gage level. Similar condition applies to gage levels near the bidirectional cell level, though these may be rendered less suitable due to presence of residual force in the pile at the gage level.

The stress-strain curve can be assumed to follow a second-degree line: $y = ax2 + bx + c$, where y is stress and x is strain (Fellenius 1989). The constants a and b (the constant c is zero) can be determined from analysis of the records themselves.

For a concreted pipe pile or for a concrete pile—driven or bored—the load-strain relation is normally linear, but, as mentioned, concrete is sometimes strain-dependent, as illustrated in Figure 1, showing the near-pile-head gage level records of a head-down test on a 600-mm diameter spun pile driven in Pusan, Korea (Kim et al. 2011). The load-strain line is not linear, but slightly curved, that is, the stiffness, *EA/L*, of the pile is strain-dependent and diminishes with increasing strain. The actual stiffness at a specific load-strain point is difficult to discern. Plotting the data as shown in Figure 2, i.e., in a "direct secant" plot, the load divided by the measured strain (Q/ϵ) vs. strain (ϵ), increases the resolution of the stiffness vs. strain of the pile. The secant stiffness is E_sA/L (L = unit length; one metre), where E_s is the secant modulus. Thus, the stiffness is $E_sA = a\epsilon + b$, i.e., it is a function of strain, with "a" being the slope of the line and "b" the ordinate intercept.

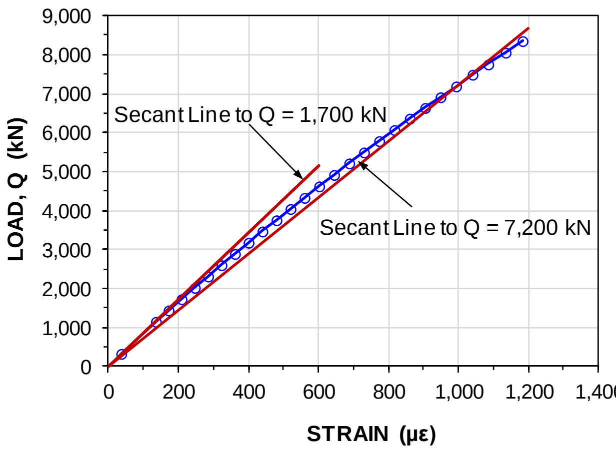

Figure 1 Load vs. strain for gage records close to the pile head

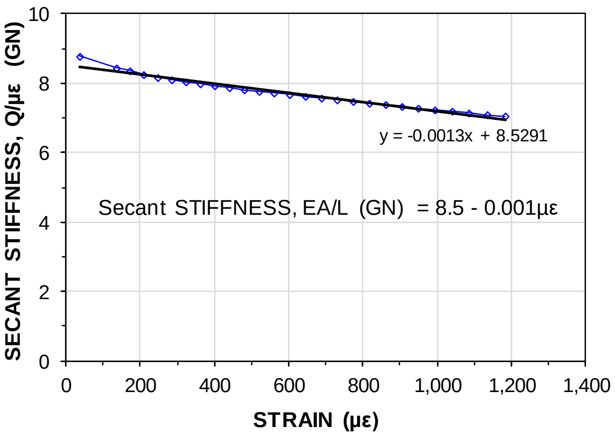

Figure 2 Near pile-head gage level secant stiffness vs. measured strain for the spun-pile

A linear regression of the straight line plot of the records from the nearest gage provides the equation constants and the stiffness relation $EA/L = 8.5 - 0.001\epsilon$. Thus, at small strain, stiffness of the curve is about 8.5 GN/m and at 1,000 με strain, the stiffness is 7.5 GN/m.

Note, an important condition for the direct secant method to work is that the pile has negligible locked-in (residual) strains and shaft resistance between the jack or bidirectional cell and the gage level. Equally important is that the test is performed with equal size load increments and equal load-holding durations and no unloading-reloading sequences.

The straight-line response is not always immediately apparent because the "zero"-reference of the records may not have been accurately known. This is illustrated in Figure 3, which is from a head-down static loading test on a 900-mm diameter bored pile installed in Jakarta, Indonesia. The gage record was from the gage level nearest the pile head, about 1.5 m below the ground surface. The secant stiffness trend was not fully established for the first couple of values. This could be because in the beginning of the test, the zero reference for strain might have been influenced by random effects such as bending and sideways movements introducing a more or less constant error to the readings. The effect of such error diminishes with increasing load and imposed strain. For the shown case, a "correction" of a mere 8 $\mu\epsilon$ added to all strain records removed the initially curved portion of the secant line and established the secant line.

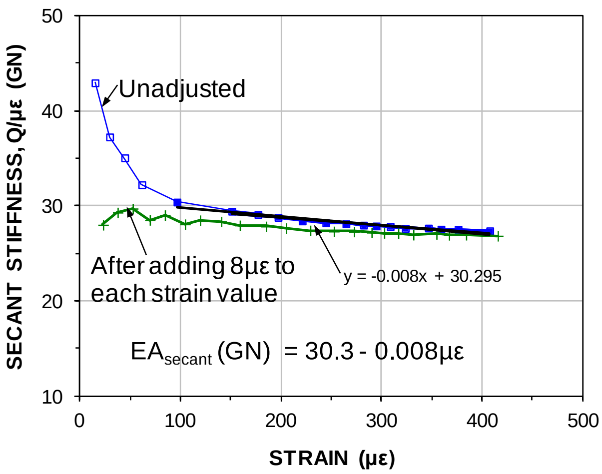

Figure 3 Secant stiffness (Q/με) vs. measured strain for a 900-mm bored pile (data from GeoOptima 2011)

3. THE TANGENT STIFFNESS METHOD

The need for knowing the initial (the "zero") reference of strain when applying the direct secant method can be removed by instead determining the tangent stiffness (incremental stiffness), which is does not require knowing the true zero value. The construction of the tangent stiffness (change of load over change of strain vs. strain) is similar to that of the secant stiffness (change of load over strain vs. strain). The tangent modulus of the composite material is a straight line, that can be used to establish the expression for the secant elastic modulus line allowing for converting every measured strain value to stress and load via its corresponding strain-dependent secant modulus. For a pile taken as a free-standing column (case of no shaft resistance), the tangent stiffness of the composite material (with reducing *E*-modulus) is a straight line with a slight slope from larger to a smaller. Every measured strain value can then be converted to stress via its corresponding strain-dependent secant stiffness. The test procedure was carried out in two phases. Phase 1 comprised four load increments applied every one hour up to the desired working load (6,154 kN), which was held for six hours, whereupon four additional load increments were applied to twice the working load, which was held for 36 hours. About 48 hours after start of test, the pile was unloaded, then, Phase 2 started by applying four increments to the working load, which was held for 16.5 hours. The pile was then given six additional increments to a 14,750-kN maximum test load, held for 16 hours, whereafter the pile was unloaded. The total test duration was 120 hours.

To numerically convert a tangent stiffness relation to a secant stiffness relation is simple. Eqs. 1 - 3 show the interrelations of Et and Es. (The following presents the mathematics without the pile cross section area, A).

The equation for the tangent modulus, E_t:

(1) $$E_t = \left(\frac{d\sigma}{d\varepsilon}\right) = a\varepsilon + b$$

which can be integrated to provide a relation for stress as a function of the strain:

(2) $$\sigma = \left(\frac{a}{2}\right)\varepsilon^2 + b\varepsilon$$

Combining Eqs. 2 and 3

(4) $$\sigma \;=\; E_s\,\varepsilon \;=\; 0.5a\,\varepsilon^2 + b\,\varepsilon$$

where

(5) $$E_s \;=\; 0.5a\,\varepsilon + b$$

where E_t = tangent modulus of composite pile material.

[N.B., the proper term for the tangent modulus is really "chord" rather than "tangent". However, if the two points are very close, the chord and tangent moduli can be considered equal. In actual tests, they are not, but I keep using the term "tangent", because shifting to "chord" would be "over-academic"].

E_s	=	secant modulus of composite pile material
E_t	=	tangent modulus of composite pile material ($E_t = a\,\varepsilon + b$)
σ	=	stress (load divided by cross section area)
$d\sigma$	=	($\sigma_{n+1} - \sigma_n$) = change of stress from one load increment to the next
a	=	slope of the tangent modulus line
ε	=	measured strain (always measured in units of microstrain, $\mu\varepsilon$; $\mu = 10^{-6}$.
$d\varepsilon$	=	($\varepsilon_{n+1} - \varepsilon_n$) = change of strain from one load increment to the next
b	=	y-intercept of the tangent modulus line (i.e., initial tangent modulus)

For a gage located near the pile head (in particular, if above the ground surface, the tangent modulus calculated for each increment is unaffected by shaft resistance and it is the true modulus. For gage records from further down the pile, the first load increments reaching the gage levels are substantially reduced by shaft resistance along the pile above the gage location and the induced strain does not permit determining the secant modulus. However, in contrast to the direct secant method, the tangent stiffness method is applicable also to the records affected by shaft resistance between the applied load (jack on the pile head or bidirectional cell). Initially, therefore, the tangent modulus values will be large. However, as the shaft resistance is being mobilized down the pile, the strain increments become larger and, therefore, the calculated modulus values become smaller. When all shaft resistance above a gage level is mobilized, the calculated modulus values for the subsequent increases in load at that gage location are the tangent modulus values of the pile cross section.

Figure 4 shows a tangent stiffness (incremental stiffness for an one-unit length element) plot of strain-gage records from the same (c.f., Figure. 3) head-down static loading test records as used for the secant modulus plot (gage level is close -to the pile head, and, therefore, unaffected by shaft resistance). The linear regression of the values shown in the figure is E_tA = 29.2 - 0.012$\mu\varepsilon$, which, per Eqs. 1 - - 3, gave essentially the same E_sA relation (E_sA = 30.3 - 0.008$\mu\varepsilon$) as the direct secant method.

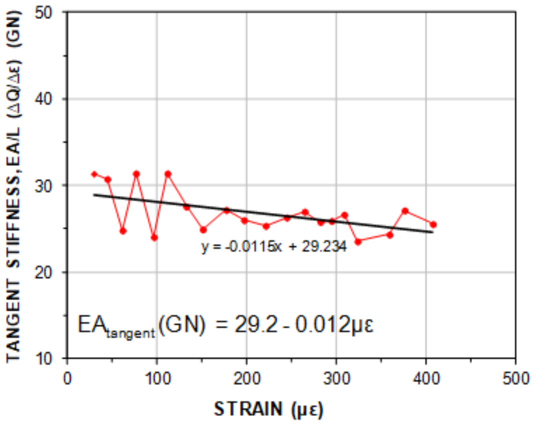

Figure 4 Tangent stiffness determined from the strain records unaffected by shaft resistance, L = 1.0 m

The tangent stiffness plot (also called the "incremental stiffness method") eliminates the uncertainty of the "zero"-reading. However, because differentiation will exaggerate small variations in the data, the tangent plot shows more scatter than found in the direct secant method. The secant stiffness plot is less sensitive to such variations and produces a smoother curve, but requires a well-established zero-reference.

Note, also the tangent stiffness method requires that the test data are from a properly performed test where all increments are equal and held for equal length of time, and where no unloading/reloading cycles have been included. If not, the gage evaluation will be adversely affected, possibly show to be useless without significant wishful guesswork.

Theoretically, the knowledge of the strain-dependent, composite, secant modulus relation, the measured strain values are converted to the stress in the pile at the gage location. The load at the gage is then obtained by multiplying the stress by the pile cross sectional area. However, other than for a premanufactured piles, such as a precast concrete pile or a steel pile, the pile size is not known accurately. But it does not have to be known, because, the evaluation of axial load in a pile does not require accurate knowledge of the pile cross section area, *A*, if instead of thinking E-modulus, the analysis is made for the pile stiffness, *AE*, directly, as used in Figures. 2 and 3. The load at the gage is then obtained by multiplying the measured strains with the evaluated stiffness.

4. LIMITATION OF THE TANGENT METHOD

The tangent stiffness method presumes a plastic response to movement of the pile in relation to the soil. I have previously thought that the stiffness determined by the tangent method would be negligibly affected by soil exhibiting moderate strain-hardening or strain-softening. Lately, however, I have found that the evaluated stiffness of the pile can indeed be quite different from the actual axial stiffness of the pile. The following fictional example of results of a static loading test on an instrumented pile illustrates the response in a non-plastic soil. The example pertains to a 650-mm diameter, 25 m long pile in a soil with a 2,000 kg/m^3 density and a pore pressure that is hydrostatically distributed from a groundwater table at 1.0-m depth. The beta-coefficient is 0.30 throughout the soil profile as mobilized at 5-mm movement for all pile elements. The unit pile toe stress is 5 MPa as mobilized at 5 mm movement. The pile material is reinforced concrete with a 2,400 kg/m^3 density and an E-modulus of 30 GPa which is constant across the full strain or stress range of the test.

To calculate the results of the virtual static loading test on the pile based on the foregoing values, the only additional information needed is the soil load-movement response to applied load, i.e., t-z and q-z functions. Three alternative t-z assumptions are now introduced, as illustrated in Figure 5. First t-z alternative is response according to the Van der Veen function (Fellenius 2019) with a function coefficient, *b*, of 1.00, modeling a soil response that initially is more or less linearly elastic becoming plastic at a 5 mm movement. Second alternative is a Chin-Kondner hyperbolic function (Fellenius 2019) with a function coefficient, C_1, of 0.0093, modeling a strain-hardening soil for which the load-movement shape for the first 5 mm movement response is more or less equal to that of the first alternative, then, for movement continuing beyond 5 mm, the resistance increases becoming 120 % of that at 5 mm at 400 mm movement. Third alternative is a Zhang function (Fellenius 2019) with a function coefficient, *a*, of 0.0090 modeling a strain-softening soil that reaches a peak at 5 mm movement and softening beyond this to 80% of that at 5 mm at 40 mm movement. For all three alternatives, the toe response, q-z, is set to a Gwizdala function (Fellenius 2019) with a function coefficient, θ, of 0.50, and a 5-MPa target unit toe resistance, r_t.

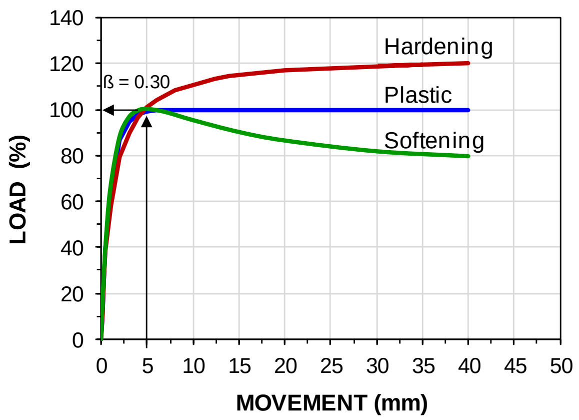

Figure 5 Three alternative t-z functions

The pile and soil information was input to UniPile5 (Goudreault and Fellenius 2013) to simulate a static loading test with four strain-gage levels at 4, 12, 18, and 23 m below the ground surface (and pile head). As shown in Figure 6, the simulation produced precise 'measurements' of load, strains, and movements at pile head, gage levels, and pile toe for each of the three alternative soil responses whose only difference is in regard to the t z functions. The pile toe response (q-z) is the same for all three alternatives. Note, the input of a constant E-modulus (30 GPa), means that the axial stiffness, *EA/L*, is 10 GN/m. Thus, a 'measured' strain value, *s* (με), converts to a load, Q = 10*s* (kN).

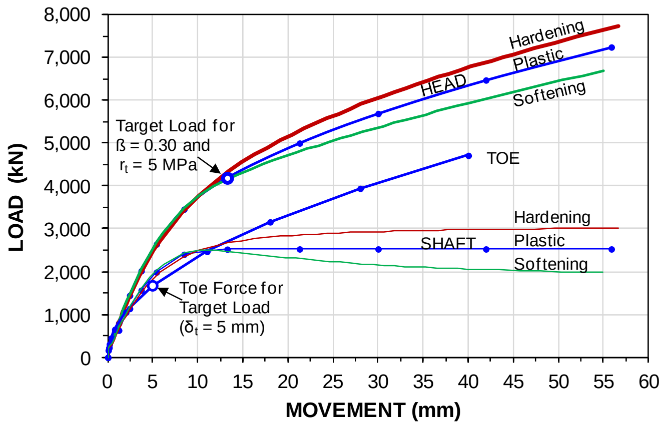

Figure 6 The load-movement results of the three simulated tests

Figure 7 shows the load distributions for the pile subjected to plastic shaft response. At the 6,000-kN applied load, the load distribution is indicated for all three piles. The distribution for the assumed shaft resistance beta-coefficient at each pile element and the toe resistance for 5-mm movement, respectively—the Target Load—is indicated by the red curve, which distribution is the same for all three alternatives.

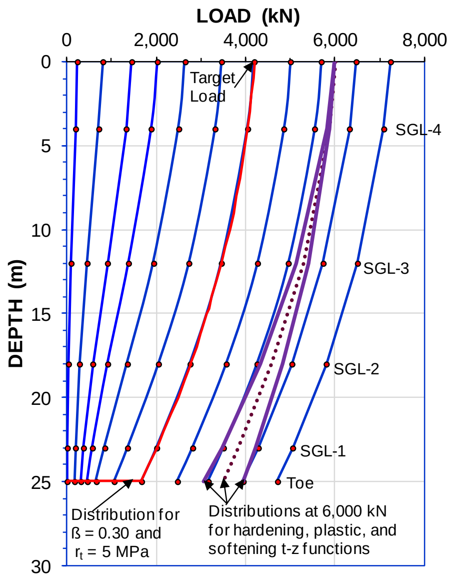

Figure 7 The load distribution for the pile subjected to plastic t-z response

The three alternative load-movement results allow for a back-calculation of the "test results" as if they were from actual tests, in regard to determining the tangent modulus relations for the gage levels, which is the very purpose of simulating the static loading tests. Figure 8 shows the tangent stiffness for the alternative of plastic soil response at the four gage levels. As no surprise, beyond the loads affected by shaft resistance above the gage level, the stiffness is a constant value and the same 10 GN/m as that used for determining the loads in the simulation. Gage level SGL-4 is 4.0 m below the pile head and its records include the effect of shaft resistance between the pile head and the gage level. Therefore, the stiffness determined by the secant stiffness method applied to the SGL-4 records is to some small degree affected by shaft resistance between the pile head and the gage level. However, although not shown, by subtracting 10 με from each strain value, a straight-line relation can be obtained that indicates a 10.0 GN/m direct secant stiffness, E_sA. Thus, the back-calculation results for plastic response verify the pile stiffness—of course.

The tangent stiffness curves and the SGL-4 secant stiffness for the alternative of hardening t-z response are shown in Figure 9. The tangent stiffness evaluated from the uppermost gage level, SGL-4, (blue line) shows an evaluated axial stiffness, EtA = 10 GN/m, that is constant after the first about 200 με, which is close to the actual value. However, the stiffness values of the gages further down (SGL-1 through SGL-3) do not imply a horizontal line anywhere close to the true 10-GN value. The response of SGL-3 at 12 m depth implies a stiffness relation, indicated by the dashed line, that would be interpreted to a tangent stiffness reducing with increasing strain from an about 12 GN/m initial value to less than 10 GN/m at large strain. The E_sA would change correspondingly with increasing strain. The plot of the two deeper gage levels show even larger stiffness reduction for increasing strain. It is obvious that the strain-hardening soil response falsely indicates a pile material stiffness that reduces with increasing strain. Had the pile material also exhibited reduction of concrete stiffness with increasing strain, the stiffness reduction would have been larger.

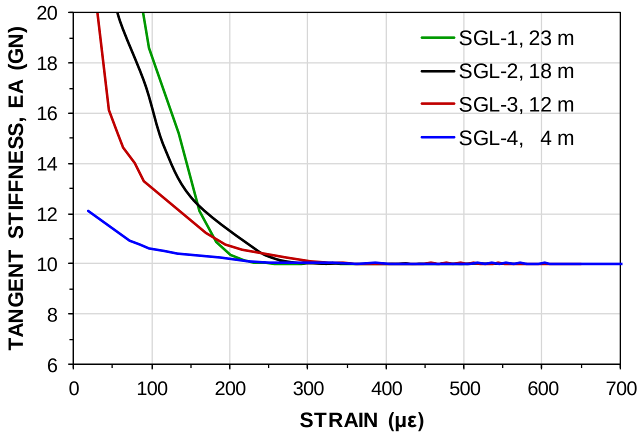

Figure 8 Tangent stiffness for to plastic t-z response

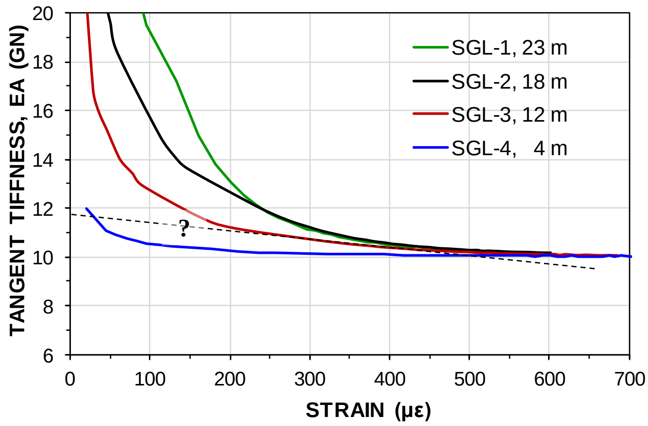

Figure 9 Tangent stiffness for hardening t-z response

Figure 10 shows the stiffness for the alternative of softening t-z response. Again, the records from the shallow gage level, SGL 4, indicated the correct pile stiffness. However, for the deeper gage levels, there was little agreement between the calculated tangent stiffness and actual stiffness until very large strain and large movement had developed (where the t-z curve shows little change with increasing movement, c.f. Figure 5, and the response is essentially plastic).

Repeating the simulations for case with t-z functions of different movement before the 100-% resistance and/or different ratio between shaft resistance and toe resistance results in quantitatively different E_sA-relations for the hardening, and softening tangent stiffness analyses. However, all show that for non-plastic t-z response hardening and softening t-z response of the soil above a gage level, a strain-the hardening soil will tend to indicate an average stiffness that initially is larger than the true value of a stiffness and then that reduces with increasing force, i.e., exaggerating a real tendency showing for a stiffness that starts out to large and reduces with increasing strain. In case of a softening response, the tangent method will indicate a non-linear smaller than true stiffness relation that only at large strain approaches something close to the true value.

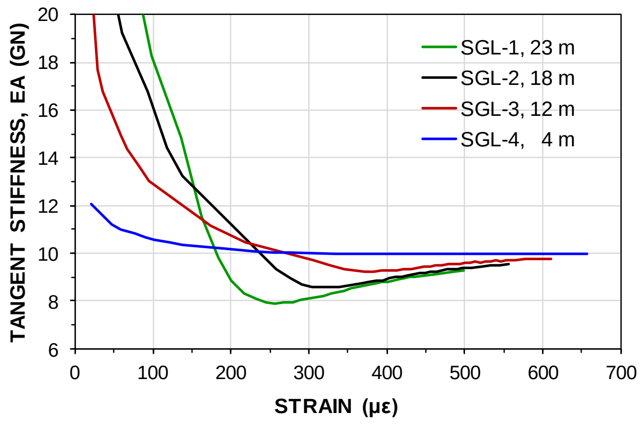

Figure 10 Tangent stiffness for softening t-z response

5. CONCLUSIONS

The foregoing indicates a limitation of the analysis of the strain records according to the tangent modulus method in strain-hardening or strain-softening soil. This significantly affects the reliability and use of not just the method, but of strain-gage instrumentation. Therefore, unless the pile axial stiffness is determined from gage records more or less unaffected by the soil resistance, a non-constant axial stiffness determined from strain-gage evaluation must be considered vague and be treated as approximate.

In case of a bidirectional test, a strain-gage level is often located near the bidirectional cell level and its can then be suitable for assessing the pile stiffness by the secant method. Note, however, that those gage levels must be close enough to the cell level to only include a small influence of shaft resistance between the cell and gage level, but sufficiently away from the cell for the pile cross section to have developed a uniform stress across the pile.

A bidirectional test provides a load at the cell location that is independent of modulus uncertainty, residual load, and cross section variations. Therefore, the bidirectional test is significantly more suitable for assessing the load distribution of a pile than a strain-gage instrumented head-down test.

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