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An Appreciation of Modified Cam Clay

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ABSTRACT: This paper presents an appreciation of the importance of the paper "On the generalised stress-strain behaviour of 'wet' clay" by Roscoe and Burland (1968), which sets out the Modified Cam Clay (MCC) framework for understanding the behavior of clays. In the last 50 years MCC has become one of the most important theoretical tools for understanding and modelling soil behaviour. Here I present a number of different aspects of the influence of MCC, as a simple conceptual model which allows hand calculations, as the basis of numerical modelling software, as the starting point for some of the most sophisticated of constitutive models developed today, and as a vehicle for discussion of fundamental features of soil behavior as a scientific topic.

KEYWORDS: Clay, Modified Cam Clay, Plasticity Theory

1. INTRODUCTION

In March 1968 the paper "On the generalised stress-strain behaviour of 'wet' clay", by Roscoe and Burland, was published as part of the proceedings of a conference on "Engineering Plasticity" held at Cambridge. This paper (here referred to as RB68) is generally regarded as the source for what has become known as the "Modified Cam Clay" model for soil behaviour. According to Google Scholar (January 2020) it has been cited 2881 times, and is the most cited paper by either of its two authors. This essay is an appreciation of the role and importance of RB68 and of Modified Cam Clay (MCC). It is neither a comprehensive review nor a detailed historical report, but simply represents a personal perspective on what is undoubtedly one of the key milestones in theoretical soil mechanics.

The late 1950's and the 1960's were intellectually exciting times in soil mechanics. In Cambridge, the group led by Roscoe was bringing together experimental data and theoretical concepts to devise a framework for understanding the behaviour of soils, later to become known as Critical State Soil Mechanics (CSSM). They drew on pre-war work by Casagrande (1936), Hvorslev (1936) and Rendulic (1936) and were strongly influenced by the textbook by Taylor (1948), and by the work of Drucker in the USA (*e.g.* Drucker (1956), Drucker, Gibson and Henkel (1957)). Key researchers in Roscoe's soil mechanics group included Wroth, Schofield, Poorooshasb, Thurairajah, Burland and Balasubramaniam. Others on the fringes of the group (but with primary interests in structures) who made key contributions included Calladine and Palmer. There was, by all accounts, a fierce rivalry with the Imperial College group, led by Skempton and with Bishop, Gibson, Parry and Henkel as key players, and with Rowe at Manchester. I cannot comment on these interactions and personal rivalries, as I was not a witness: I first heard of "soil mechanics" as an undergraduate in 1974, six years after the publication of RB68.

By the mid-1960's the key elements of CSSM were in place, and were set out in the textbook "Critical State Soil Mechanics" by Schofield and Wroth, also published in 1968 (and hereinafter referred to as SW68). One of the central concepts set out in the book was a model for the behaviour for soft clay. Cleverly, rather than referring to the model by some abstract list of the assumptions embodied in it, Schofield and Wroth gave their model a name: "Cam-clay", after the river that flows through Cambridge. This was a masterstroke, in that it gave a simple and memorable title by which the model could become known. It has to be said though that it also caused some confusion. In spite of Schofield and Wroth's assertion that "*The intention is to … remind our students that these are conceptual models – not real soil*", some students have certainly been confused about the reality of Cam-clay.

The ideas being developed in Cambridge were not simply crystallised in the SW68 book. Other concepts were being developed too. Burland, and others, were clearly troubled by the fact that the Cam-clay model exhibited what they considered to be unrealistic behaviour at very small shear stresses, and were working on ideas that remedied this. One of the central assumptions in the Cam-clay model is the hypothesis that the "rate of plastic work" is given by the expression:

$$\dot{W}_p = Mp\left|\dot{\varepsilon}_s^p\right| \tag{1}$$

(Throughout this paper we use the Cambridge "triaxial" parameters: definitions are given in the Notation at the end of the paper.)

The problem is that this leads to a curious "bullet-shaped" plastic potential, and hence also the same shape of yield surface when "associated flow" is assumed. This results in predicting unrealistic shear strains at small shear stresses. A suitable modification that dealt with the problem at small shear stresses was to change this hypothesis to:

$$\dot{W}_p = p\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2} \tag{2}$$

which appears as equation (33) in RB68.

The result is that the plastic potential (and yield surface) now becomes elliptical, providing a more satisfactory description of the behaviour at low shear stress. Although RB68 is now usually used as the key reference, the ideas had been published earlier. RB68 draws very heavily on Burland's thesis (Burland, 1967), which sets out most of the important results in detail. The vitally important Eq. (2) above had already been published in a letter to Géotechnique (Burland, 1965), including also the equation for the elliptical "yield locus":

$$p_c = p\left\{\frac{(q/p)^2 + M^2}{M^2}\right\} \tag{3}$$

Burland (1965) refers in turn to a letter to Géotechnique by Calladine (1963), which includes figures that appear to depict elliptical yield surfaces, and includes the comment "*If the yield curves are nearly elliptical, as they appear to be…*", but does not go so far as giving an analytical expression.

Why is this seemingly innocuous change to the detail of the model so important? The reason is that the elliptical yield surface is much more tractable mathematically, and opened up the possibility of not just using MCC as a conceptual model, but also as a practical tool for calculations of soil behaviour in real boundary value problems. This development depended on bringing together the soil model with the Finite Element Method, which was itself in a state of rapid

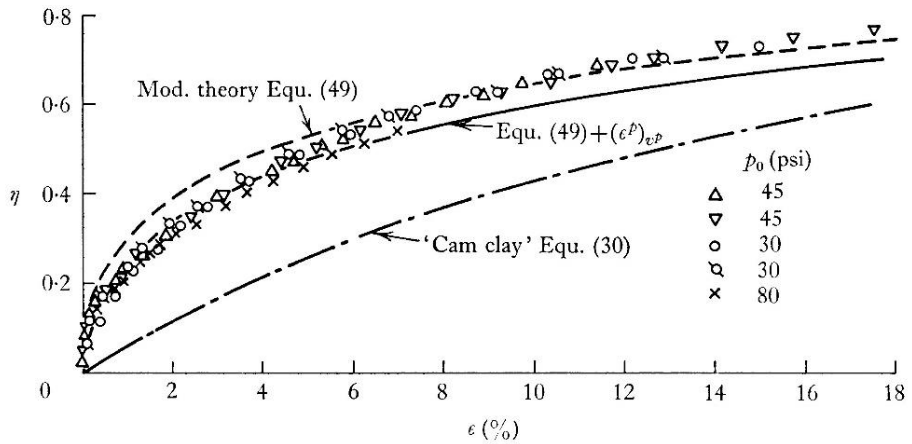


Figure 1 Observed and predicted shear stress ratio *v.* shear strain behaviour of normally consolidated Kaolin in drained triaxial tests (from Walker (1965)). Eq. (49) refers to the MCC model and Eq. (49) + $\left(\varepsilon^{p}\right)_{v^{p}}$ the "Revised" model. From Figure 10(a) of RB68.

development: an important early publication was by Clough (1960), and the widely used textbook by Zienkiewicz was first published in 1967.

2. RB68 AND MODIFIED CAM CLAY

Before moving on to discuss the impact of MCC it is worth highlighting some key elements of the RB68 paper. It is a long paper, and I do not attempt a comprehensive summary here. It begins with a general introduction to the ideas being developed in Cambridge at the time, followed by an exposition of what is now termed Modified Cam Clay. In modern terminology this would be recognised as an elastic-plastic model which employs a single isotropically hardening yield surface. It would be usually be presented in terms of:

1. The elastic response within the yield surface,
2. The equation for the yield surface (and by implication the plastic potential as "associated flow" is assumed),
3. The hardening law.

The presentation in RB68, however, does not follow this pattern, and to a modern reader it is harder to follow, but it must be remembered that the application of plasticity theory in soil mechanics was at the time in its infancy, and authors were still exploring how best to understand the concepts and explain them to others.

In the first version of the model presented in RB68 it is assumed that there is no recoverable shear strain $\varepsilon_s^e = 0$. This is the basis of the model now recognised as MCC, although elastic shear strains (more of which below) are almost always included. RB68 never calls this model explicitly "*Modified Cam Clay*", although they refer to it as "*the modified theory*".

Recognising (from experimental data) that the assumption of zero shear strains within their elliptical yield surface was unrealistic, RB68 then present a second version of the model. In this they include a second yield surface – geometrically just a straight line parallel to the *p*-axis, and introduce a rather complex hardening rule for the relationship between the position of this surface and the corresponding shear strains. This has become known as "*Revised Modified Cam Clay*", but again RB68 does not use this explicit terminology, although they refer to it as "*the revised modified theory*".

In RB68 the theories are compared with data, and Figure 1 reproduces Figure 10(a) of the original paper in which they demonstrate that MCC predicts shear strains much better than the original Cam-clay. The revised theory does a little better, but the improvement is not substantial for this normally consolidated clay.

In the event, the "*Revised*" model has not, to the author's knowledge, been widely adopted, largely because it would be much more complex to implement. Instead, as mentioned above, elastic shear strains are usually introduced. The commonest approach uses one of two methods. In the first a constant shear modulus is specified. This is simple theoretically, but does not represent the real behaviour of clays very well. In the second approach the bulk modulus is first calculated (from the slopes of the "swelling lines" in a consolidation plot) and a constant Poisson's ratio is assumed. This results in both bulk and shear moduli proportional to pressure (and also mildly dependent on specific volume). Whilst this represents the real behaviour of clays better, it is straightforward to show that it is theoretically unacceptable on thermodynamic grounds. This was clearly demonstrated by Zytynski *et al.* (1978). The author published a resolution of problem (Houlsby, 1985) by deriving pressure-dependent bulk and shear moduli from an elastic potential, but the uptake of those ideas has been disappointingly slow – too many constitutive modellers seem unconcerned by the fact that their model disobeys thermodynamic principles.

RB68 go on to discuss other improvements to MCC, including for instance adopting different critical state stress ratios in compression and extension, and adding a Mohr-Coulomb rupture condition on the "dry" side of critical.

Part II of the paper moves on to the modelling of clay under plane strain conditions, and discusses the generalisation of the shape of the yield surface in three dimensional principal stress space. Various sources of data are discussed and compared to MCC predictions with the data; a variety of simplifying assumptions are discussed.

3. MCC AS A MODERN PLASTICITY THEORY

It is useful at this stage to set out the model in modern elastic-plastic terminology, although following the notation used in RB68 as closely as is reasonable.

First we divide the strains into elastic (fully recoverable) and plastic components:

$$\varepsilon_v = \varepsilon_v^e + \varepsilon_v^p \,, \;\; \varepsilon_s = \varepsilon_s^e + \varepsilon_s^p \tag{4}$$

The elastic strain increments are given by the expressions:

$$\dot{\varepsilon}_v^e = \frac{\kappa}{pv}\dot{p} \,, \;\; \dot{\varepsilon}_s^e = \frac{\dot{q}}{3G} \tag{5}$$

Where the above describes the (slightly unrealistic but theoretically acceptable) version with a constant shear modulus. Note that the "bulk modulus" $K = pv/\kappa$ is proportional to the pressure and also to the specific volume *v*, although in most practical cases the variation of the latter is small.

The yield surface, *f*, which is the same as the plastic potential, *g*, is an ellipse of the form:

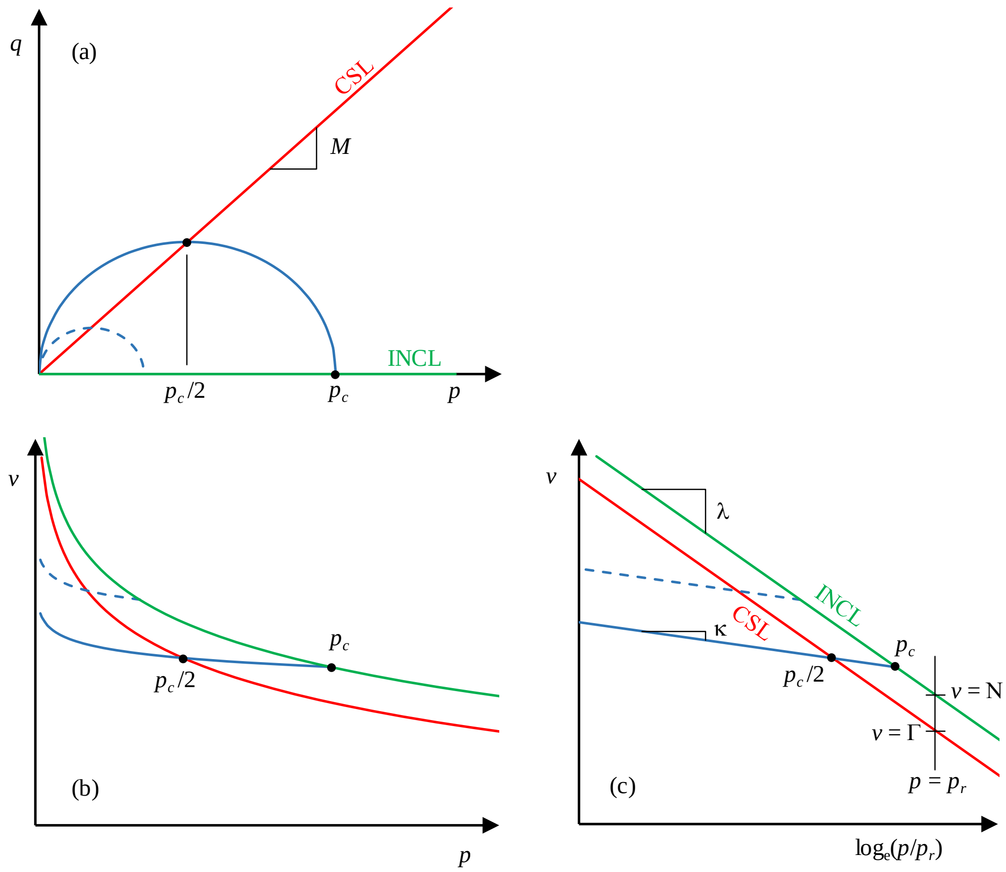


Figure 2 The Modified Cam Clay model in terms of triaxial parameters

$$f\left(p,q,p_c\right)=g\left(p,q,p_c\right)=p\left(p-p_c\right)+\frac{q^2}{M^2}=0 \qquad (6)$$

where p_c is the intercept on the *p*-axis (the isotropic preconsolidation pressure). Within the yield surface, *i.e.* for $f<0$, the plastic strain rates are zero, $\dot{\varepsilon}_v^p=0$, $\dot{\varepsilon}_s^p=0$.

If the stress point lies on the yield surface, $f=0$, then the ratios between the plastic strain rates are given by the differential of the plastic potential:

$$\dot{\varepsilon}_v^p=\Lambda\frac{\partial g}{\partial p}=\Lambda\left(2p-p_c\right),\ \dot{\varepsilon}_s^p=\Lambda\frac{\partial g}{\partial q}=\Lambda\frac{2q}{M^2} \qquad (7)$$

where the multiplier Λ can be determined by the "consistency condition" that during yield the stress point stays on the yield surface, so that $\dot{f}=\frac{\partial f}{\partial p}\dot{p}+\frac{\partial f}{\partial q}\dot{q}+\frac{\partial f}{\partial p_c}\dot{p}_c=\left(2p-p_c\right)\dot{p}+\frac{2q}{M^2}\dot{q}-p\dot{p}_c=0$, together with a "hardening rule" that links the size of the yield surface, specified by p_c, with the volumetric plastic strain ε_v^p . This is conveniently written in incremental form as $\dot{\varepsilon}_v^p=\frac{\lambda-\kappa}{p_c v}\dot{p}_c$.

Manipulation of the above equations allows the incremental compliance matrix to be written for both the elastic case (either $f<0$, or $\dot{f}<0$):

$$\begin{bmatrix}\dot{\varepsilon}_v\\ \dot{\varepsilon}_s\end{bmatrix}=\begin{bmatrix}\kappa/pv & 0\\ 0 & 1/3G\end{bmatrix}\begin{bmatrix}\dot{p}\\ \dot{q}\end{bmatrix} \qquad (8)$$

and the elastic-plastic case ($f=0$ and $\dot{f}=0$):

$$\begin{bmatrix}\dot{\varepsilon}_v\\ \dot{\varepsilon}_s\end{bmatrix}=\begin{bmatrix}\frac{\kappa}{pv}+\left(M^2-\eta^2\right)A & 2\eta A\\ 2\eta A & \frac{1}{3G}+\frac{4\eta^2}{\left(M^2-\eta^2\right)}A\end{bmatrix}\begin{bmatrix}\dot{p}\\ \dot{q}\end{bmatrix} \qquad (9)$$

where $\eta=q/p$ and $A=\frac{1}{\left(M^2+\eta^2\right)}\frac{\left(\lambda-\kappa\right)}{pv}$.

The model is usually illustrated as shown in Figure 2. In Figure 2(a) the elliptical yield surface in the $\left(p,q\right)$ plot is shown (blue). The Critical State Line (CSL, red) passes through the apex of the ellipse and has slope *M*. The Isotropic Normal Consolidation Line (INCL, green) coincides with the *p*-axis. The intersection of the yield surface with the *p*-axis is at $p=p_c$ and with the CSL at $p=p_c/2$. Figure 2(b) shows the corresponding consolidation plot in $\left(p,v\right)$ space. The curve INCL indicates the isotropic compression of a normally consolidated specimen. If unloading occurs from $p=p_c$, this is along the swelling curve (blue).

The consolidation behaviour is replotted in Figure 2(c) in $\left(\log_e\left(p\right),v\right)$ space, in which the consolidation and swelling lines are straight. Figure 2 also shows (dotted blue) a swelling line for unloading of a normally consolidated specimen at a lower pressure, and the corresponding smaller yield surface.

4. INTERPRETATION

4.1 Critical State Line

Noting that, from Eq. (7) and the formula for the yield surface we can obtain $\dot{\varepsilon}_v^p = \Lambda p\left(1-\frac{\eta^2}{M^2}\right)$, it follows that at $\eta = \frac{q}{p} = M$, $\dot{\varepsilon}_v^p = 0$, but $\dot{\varepsilon}_s^p = \Lambda\frac{2q}{M^2}$ can take an arbitrary value if the stress point is on the yield surface. Physically this means that if yield occurs on the Critical State Line, the clay can shear continuously with no change of stress or of volume. This was a generalisation of the concept of critical voids ratio, introduced by Casagrande (1936), and is the central and defining feature of Critical State Soil Mechanics (CSSM).

4.2 State Boundary Surface

Noting that $\dot{\varepsilon}_v = \dot{\varepsilon}_v^e + \dot{\varepsilon}_v^p = \frac{\kappa}{pv}\dot{p} + \frac{\lambda-\kappa}{p_c v}\dot{p}_c$ we obtain the result that $\dot{v} = -v\dot{\varepsilon}_v = -\kappa\frac{\dot{p}}{p} - (\lambda-\kappa)\frac{\dot{p}_c}{p_c}$, which integrates to give $v = \mathrm{N} - \kappa\log_e\left(\frac{p}{p_r}\right) - (\lambda-\kappa)\log_e\left(\frac{p_c}{p_r}\right)$, where p_r is a reference pressure (often taken as atmospheric pressure). For any stress point on the yield surface we can write $p_c = p + \frac{q^2}{M^2 p} = p\left(1+\frac{\eta^2}{M^2}\right)$, so for points on the yield surface we obtain:

$$\begin{aligned} v &= \mathrm{N} - \kappa\log_e\left(\frac{p}{p_r}\right) - (\lambda-\kappa)\log_e\left(\frac{p}{p_r}\left(1+\frac{\eta^2}{M^2}\right)\right) \\ &= \mathrm{N} - \lambda\log_e\left(\frac{p}{p_r}\right) - (\lambda-\kappa)\log_e\left(1+\frac{\eta^2}{M^2}\right) \end{aligned} \tag{10}$$

This is a unique surface in (p,q,v) space which is none other than the "State Boundary Surface", a concept first identified by Rendulic (1936), and later embedded as a central feature of CSSM. Of course in RB68 this result was presented the other way round – the existence of the state boundary surface was used to derive the hardening law.

4.3 Compression lines

Note that the form of Eq. (10) indicates that compression at any constant η value would be at slope $-\lambda$ in $(\log_e(p), v)$ space. Specifically for compression on the isotropic normal consolidation line (INCL) we obtain simply that $v = \mathrm{N} - \lambda\log_e\left(\frac{p}{p_r}\right)$.

On the Critical State Line $\eta = M$, and we obtain $v = \Gamma - \lambda\log_e\left(\frac{p}{p_r}\right)$, where $\Gamma = \mathrm{N} - (\lambda-\kappa)\log_e(2)$, so in the $(\log_e(p), v)$ INCL and CSL are parallel lines.

4.4 Swelling lines

As noted above, the specific volume is given by $v = \mathrm{N} - \kappa\log_e\left(\frac{p}{p_r}\right) - (\lambda-\kappa)\log_e\left(\frac{p_c}{p_r}\right)$, so that at constant p_c (*i.e.* any point within the yield surface) this relationship gives a straight line of slope $-\kappa$ in the $(\log_e(p), v)$ plot.

4.5 Plastic work rate

From Eq. (7) we can obtain that $\dot{W}_p = p\dot{\varepsilon}_v^p + q\dot{\varepsilon}_s^p = \Lambda p(2p - p_c) + 2\Lambda\frac{q^2}{M^2}$, and substituting the expression for the yield surface we obtain $\dot{W}_p = \Lambda p p_c$. On the other hand we can also derive $p\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2} = p\Lambda\sqrt{(2p-p_c)^2 + \frac{4q^2}{M^2}}$, and again substituting the expression for the yield surface this reduces to $p\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2} = \Lambda p p_c$, so we have demonstrated that $\dot{W}_p = p\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}$, the central assumption in RB68.

5. MCC AND THERMODYNAMICS

In 1978, as a research student under Peter Wroth's supervision, I was embarking on a study of constitutive modelling of soils. Aware of the Cambridge work on storage and dissipation of energy in soils, and the obvious link to the Laws of Thermodynamics, I was attempting to apply thermodynamic principles to soils. Wroth urged me to study thermodynamics more thoroughly, and by chance I came across the monograph by Ziegler (1977) in which he sets out an approach to thermodynamics with remarkable clarity. This book has had a major influence on my research.

Ziegler set out an approach in which the entire constitutive response of a material could be deduced from statements of just two scalar functions: one defining stored energy and the other the dissipation rate. Not only did this have an elegant mathematical simplicity, but the approach chimed well with the notions already adopted in Cambridge. A key difference was that Ziegler's formalism allowed a more rigorous application of thermodynamic principles.

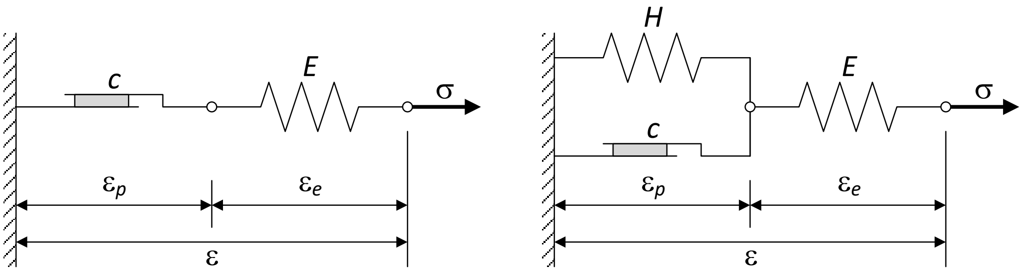
After some false starts, to my delight I was able to formulate Modified Cam Clay within Ziegler's approach (Houlsby, 1981), and I give a simplified version of the theory below. The necessary functions are the Helmholtz free energy f (closely related to the internal energy, but do not confuse this with f previously used for the yield function), and the dissipation function d,:

$$\begin{aligned} f &= f\left(\varepsilon_v, \varepsilon_s, \varepsilon_v^p, \varepsilon_s^p\right) \\ &= \kappa^* p_r \exp\left(\frac{\varepsilon_v - \varepsilon_v^p}{\kappa^*}\right) + \frac{3G}{2}\left(\varepsilon_s - \varepsilon_s^p\right)^2 \\ &\quad + (\lambda^* - \kappa^*) p_r \exp\left(\frac{\log_e(\Gamma/v_o) + \varepsilon_v^p}{\lambda^* - \kappa^*}\right) \end{aligned} \tag{11}$$

$$d = d\left(\varepsilon_v, \varepsilon_s, \varepsilon_v^p, \varepsilon_s^p, \dot{\varepsilon}_v^p, \dot{\varepsilon}_s^p\right) = \frac{p_c}{2}\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2} \tag{12}$$

where the definition $\frac{p_c}{2} = p_r \exp\left(\frac{\log_e(\Gamma/v_o) + \varepsilon_v^p}{\lambda^* - \kappa^*}\right)$ is introduced.

The model described by the thermodynamic functions is identical to the one described in the previous section using conventional plasticity theory, except that the negative slopes of the consolidation and swelling lines are λ^* and κ^* in a $(\log_e(p), \log_e(v))$ plot, rather than λ and κ in a $(\log_e(p), v)$ plot. This modified form was advocated by Butterfield (1979). The original form can be formulated within the thermodynamic approach, but the necessary expressions



(a) (b)

Figure 3 (a) simple plasticity model; (b) model with "frozen" energy

are slightly more complex. At a particular specific volume the parameters are related by $\lambda^* = \lambda/v$, $\kappa^* = \kappa/v$.

Ziegler's formulation requires no further assumptions. The stresses are calculated by differentiation:

$$p = \frac{\partial f}{\partial \varepsilon_v} = p_r \exp\left(\frac{\varepsilon_v - \varepsilon_v^p}{\kappa^*}\right) \tag{13}$$

$$q = \frac{\partial f}{\partial \varepsilon_s} = 3G\left(\varepsilon_s - \varepsilon_s^p\right) \tag{14}$$

and the formulation also requires that the following Eq. (15), (16) are obeyed. They embody Ziegler's "orthogonality principle", which is essentially a slightly stronger statement than the Second Law of Thermodynamics.

$$\begin{aligned} 0 &= \frac{\partial f}{\partial \varepsilon_v^p} + \frac{\partial d}{\partial \dot{\varepsilon}_v^p} \\ &= -p_r \exp\left(\frac{\varepsilon_v - \varepsilon_v^p}{\kappa^*}\right) + \frac{p_c}{2} + \frac{p_c}{2}\frac{\dot{\varepsilon}_v^p}{\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}} \end{aligned} \tag{15}$$

$$\begin{aligned} 0 &= \frac{\partial f}{\partial \varepsilon_s^p} + \frac{\partial d}{\partial \dot{\varepsilon}_s^p} \\ &= -3G\left(\varepsilon_s - \varepsilon_s^p\right) + \frac{p_c}{2}\frac{M^2\dot{\varepsilon}_s^p}{\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}} \end{aligned} \tag{16}$$

These are readily rearranged to give:

$$p - \frac{p_c}{2} = \frac{p_c}{2}\frac{\dot{\varepsilon}_v^p}{\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}} \tag{17}$$

$$\frac{q}{M} = \frac{p_c}{2}\frac{M\dot{\varepsilon}_s^p}{\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}} \tag{18}$$

Squaring and adding these equations, the right hand side reduces (for non-zero plastic strains) simply to $\left(p_c/2\right)^2$, and we deduce that whenever the plastic strain rates are non-zero the stresses must satisfy:

$$\left(p - \frac{p_c}{2}\right)^2 + \left(\frac{q}{M}\right)^2 = \left(\frac{p_c}{2}\right)^2 \tag{19}$$

This is, of course, the elliptical yield locus of MCC, but the result is remarkable, as at no stage in Ziegler's approach has the existence of a yield surface been postulated: instead its existence emerges as a direct consequence of the mathematical form of the dissipation function.

There is a parallel with the original development of MCC, in which the shape of the yield surface is deduced by assuming: (a) a function for the rate of plastic work (akin to the dissipation, but not identical to it), from which a "flow rule" and hence a plastic potential is deduced, and (b) Drucker's normality, which allows the plastic potential to be identified with the yield surface.

6. DISSIPATION AND THE RATE OF PLASTIC WORK IN MCC

This section is aimed at readers with an interest in some of the more subtle aspects of constitutive modelling and thermodynamics, and can be omitted by the more general reader.

There is a key difference between the expression for rate of plastic work $\dot{W}_p = p\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}$ in RB68, and the rate of dissipation $d = \frac{p_c}{2}\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}$. The first involves the multiplier p and the second the multiplier $p_c/2$. In most of the Cambridge work in the 1960's it was implicitly assumed that the rate of plastic work and the rate of dissipation were the same – in other words all work done by the plastic strains was dissipated. However, Palmer (1967) questioned this, and suggested that plastic straining could involve a combination of dissipation and storage of energy. The concepts are illustrated in Figure 3 for a simple one-dimensional model. On the left the plastic strains are entirely associated with dissipation in a sliding element with strength c, and $d = \dot{W}_p = \sigma\dot{\varepsilon}_p = c\left|\dot{\varepsilon}_p\right|$. On the right there is a spring of stiffness H in parallel with the slider. Whilst we still have $\dot{W}_p = \sigma\dot{\varepsilon}_p$, and the dissipation is still $d = c\left|\dot{\varepsilon}_p\right|$, the two are not the same as some plastic work is stored in the spring. It is easy to show that $\dot{W}_p = \left(H\varepsilon_p\right)\dot{\varepsilon}_p + d = \left(H\varepsilon_p\right)\dot{\varepsilon}_p + c\left|\dot{\varepsilon}_p\right|$, where $H\varepsilon_p$ is the current force in the spring H.

Such concepts are now widely recognised, and the storage of energy during plastic deformation underpins the development of "kinematic hardening" plasticity models. It has sometimes been termed "hidden" or "frozen" energy.

The MCC model described above is conceptually of the form in Figure 3(b), and the final term in the expression for the Helmholtz free energy, Eq. (11), represents the "frozen" energy. The model is remarkably close in concept to that described by Palmer (1967) in which some storage of energy is attributed to the consolidation process.

So which model is "correct" – RB68's model with no frozen energy and with rate of plastic work proportional to p or Houlsby

(1981) which includes a frozen energy term and employs dissipation proportional to $p_c/2$? A discussion in Houlsby (2000) sheds some further light on this issue. There I demonstrate that identically the same response to that described above can be obtained within Ziegler's formulation from the following expressions:

$$f = \kappa^* p_r \exp\left(\frac{\varepsilon_v - \varepsilon_v^p}{\kappa^*}\right) + \frac{3G}{2}\left(\varepsilon_s - \varepsilon_s^p\right)^2 \tag{20}$$

$$d = \frac{p_c}{2}\left(\dot{\varepsilon}_v^p + \sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}\right) \tag{21}$$

where one should note (a) the absence of the frozen energy term, and (b) a change to the dissipation function. Because identically the same constitutive response can be obtained from two different pairs of Helmholtz free energy and dissipation, it means that, conversely, observations of the constitutive response (*i.e.* stress-strain behaviour) cannot be used to deduce uniquely the dissipation or Helmholtz free energy. These functions are therefore strictly "unobservable" – a disappointing but inevitable conclusion. One can hypothesise, for instance, a particular dissipation function, and test the predictions against observations, but no matter how good the fit, one cannot prove that the dissipation function is "right". Another dissipation function might produce identical results.

In the second case (because there is no frozen energy) the dissipation is equal to the rate of plastic work. It is straightforward to show (once the formulae for the yield surface and the flow rule are taken into account) that $d = \frac{p_c}{2}\left(\dot{\varepsilon}_v^p + \sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2}\right) = p\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2} = \dot{W}_p$.

In that case one must question why one does not simply replace Eq. (21) by:

$$d = p\sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M\dot{\varepsilon}_s^p\right)^2} \tag{22}$$

and proceed with Ziegler's usual formulation. But if we do so we obtain a totally different result. Why?

Collins and Houlsby (1997) demonstrate that, if the dissipation depends explicitly on the stresses (as Eq. (22) does) then it is an inevitable result that on yielding the plastic strains will exhibit "non-associated" flow. In other words, for any "frictional" material (dissipation proportional to stress), it would be inconsistent to assume associated flow. If one follows through Ziegler's derivation with Eq. (22) rather than Eq. (21), then no useful model emerges.

This observation reveals an important, but subtle, feature of the formulation in which the constitutive response is derived from thermodynamic functions. The model derived depends on the functional form of the relevant functions, not merely their numerical values. The right hand sides of Eq. (21) and Eq. (22) always take the same numerical value, but the functional forms are clearly different, and the constitutive model derived depends on the functional forms, not the numerical values.

Collins and Houlsby's observation raises, however, a fundamental problem about the Cam Clay (and Modified Cam Clay) models. Both are based on the basic assumptions that (a) dissipation (or at least rate of plastic work) is proportional to mean stress, *i.e.* they are "frictional" and (b) Drucker's normality can be assumed. Collins and Houlsby show that (accepting Ziegler's orthogonality) these two assumptions are incompatible. Instead of Eq. (22) we must use Eq. (21): the dissipation is proportional to the preconsolidation stress, not the mean stress. I would term this family of models "quasi-frictional". This distinction points to a fundamental difference between the observed behaviour of sands and (normally consolidated and lightly overconsolidated) clays. Clays are "quasi-frictional"; they have a dissipation function proportional to preconsolidation pressure and exhibit, to a first approximation, "associated flow". Sands are truly "frictional"; they have a dissipation function proportional to true stress magnitude, and exhibiting a strongly "non-associated" response.

7. MCC AS A CONCEPTUAL MODEL

MCC can be used as a conceptual model at a purely qualitative level: for instance it can be used to sketch the shapes of undrained stress-paths for normally and lightly overconsolidated clays (and hence the pore pressures developed during undrained shearing), and these closely resemble those observed in triaxial tests. However, it can also be used to make some simple quantitative calculations about soil properties, and we pursue a couple of examples below.

(a) Earth pressure coefficient: The coefficient of earth pressure at rest for a normally consolidated clay can be estimated from the MCC model. Assuming that plastic strains dominate during consolidation, one can show that if $\dot{\varepsilon}_r = 0$, then $\frac{2}{3} = \frac{\dot{\varepsilon}_s}{\dot{\varepsilon}_v} \approx \frac{\dot{\varepsilon}_s^p}{\dot{\varepsilon}_v^p} = \frac{2q/M^2}{p\left(1-\eta^2/M^2\right)} = \frac{2\eta}{M^2-\eta^2}$. This gives the solution $\eta = \frac{-3+\sqrt{9+16M^2}}{4}$. Given that $K_o = \frac{p-q/3}{p+2q/3} = \frac{3-\eta}{3+2\eta}$, we obtain $K_o = \frac{15-\sqrt{9+16M^2}}{6+2\sqrt{9+16M^2}}$, and for $M = 1$ (which corresponds to $\phi' = 25.4^o$) this gives $K_o = 5/8 \approx 0.63$, whilst Jaky's approximate formula gives $K_o = 1-\sin\phi' = 4/7 \approx 0.57$ for this case, only 10% lower.

(b) Undrained strength: The undrained strength of a normally consolidated or lightly overconsolidated clay can be estimated by the following procedure. In an undrained test with initial mean stress p_i the specific volume is $v = \mathrm{N} - \kappa\log_e\left(\frac{p_i}{p_r}\right) - (\lambda-\kappa)\log_e\left(\frac{p_c}{p_r}\right)$. At large shear strain the state will approach the critical state line, for which $v = \Gamma - \lambda\log_e\left(\frac{p_f}{p_r}\right) = \mathrm{N} - (\lambda-\kappa)\log_e(2) - \lambda\log_e\left(\frac{p_f}{p_r}\right)$, where p_f is the mean effective stress at failure. But (because the test is undrained) this is the same as the initial specific volume, so we can deduce that $\mathrm{N} - \kappa\log_e\left(\frac{p_i}{p_r}\right) - (\lambda-\kappa)\log_e\left(\frac{p_c}{p_r}\right) = \mathrm{N} - (\lambda-\kappa)\log_e(2) - \lambda\log_e\left(\frac{p_f}{p_r}\right)$, which re-arranges to give $\log_e\left(\frac{p_f}{p_i}\right) = \frac{\lambda-\kappa}{\lambda}\log_e\left(\frac{p_c}{2p_i}\right)$ or $p_f = p_i\left(\frac{R_p}{2}\right)^{\Lambda}$, where $\Lambda = \frac{\lambda-\kappa}{\lambda}$, and $R_p = \frac{p_c}{p_i}$ is the initial overconsolidation ratio in terms of mean effective stress.

But the undrained strength is just equal to half the deviator stress at failure (*i.e.* at critical state), $s_u = \frac{q_f}{2} = \frac{Mp_f}{2} = \frac{Mp_i}{2}\left(\frac{R_p}{2}\right)^{\Lambda}$, or $\frac{s_u}{p_i} = \frac{M}{2^{1+\Lambda}}R_p^{\Lambda}$. In other words, the ratio of undrained strength to initial mean effective stress is given by a factor

$\left(s_u/p_i\right)_{NC} = M/2^{1+\Lambda}$ multiplied by the overconsolidation ratio to power Λ. Typical values $M \approx 1$ and $\Lambda \approx 0.8$ give $\left(s_u/p_i\right)_{NC} \approx 0.29$. This pattern of behaviour represents the undrained strength of soft clays very well.

There are a number of other simple relationships that can be deduced from the model that fit the behaviour of soft clays, and these have value for checking that data conform to expected patterns.

8. MCC AND NUMERICAL ANALYSIS

Perhaps a key advantage of MCC is that it can be fairly readily incorporated into a non-linear finite code. Such codes are now widespread and fairly routine, but Simpson (1973) was one of the first to work in this area. He was interested in problems in plane strain, and rather than formulating the MCC model for general stresses, and then extracting the plane strain solution (the approach that would most commonly be used today) he adopted a pragmatic approach and formulated a direct analogy of MCC, but expressed in the (s,t) variables convenient for plane strain problems rather than the (p,q) triaxial variables. He implemented this model (and a number of others) in a Finite Element program.

The MIT prediction symposium in 1974 was a key moment in the acceptance of critical state concepts in general, and the MCC model in particular. A road embankment on Boston Blue Clay had been partially constructed, but was then abandoned when plans changed. Charles "Chuck" Ladd (a Professor at MIT) had the bold idea of deliberately failing the embankment as an experiment. He invited leading experts to predict a number of quantities – pore pressures at certain locations after further fill had been added, final failure height *etc.*). The embankment was then failed by adding further fill, and a symposium held in which the predictions were presented and the actual results revealed.

Peter Wroth was one of the predictors for the MIT embankment, and he, and his students, used Simpson's plane strain version of the MCC model to make most of his predictions (Wroth, Thompson and Hughes, 1975). Because the exercise involved predicting 14 numbers, there was no single clear "winner" amongst the 10 predictors, but overall Wroth's predictions came out very well – he was closest on predicting the height at failure, and did well on predicting pore pressures. He tended to over-predict displacements. The result though was that, in a very public forum amongst the experts in the field, it was demonstrated that numerical modelling using Finite Element Analysis (FEA) could realistically predict soil behaviour for a "real" problem, and at the heart of the FEA was the Modified Cam-Clay model. The credibility of MCC as a practical engineering tool was firmly established.

Moving forward more than 40 years, many well-established Finite Element packages now offer MCC as one of their standard features, meaning that it is available for use by geotechnical engineers, without them needing to have specialized knowledge on programming techniques for constitutive modelling. For instance ABAQUS offers a version of the model, including some additional features that allow soft clay behaviour to be modelled with slightly more precision than in the original model. For instance, different M values can be employed for triaxial compression and extension (as anticipated in RB68), with a simple interpolation method for intermediate cases. There is also an option to adjust the shape of the yield surface on the "wet" side of critical. Such additional modifications are fairly typical of a host of variants that have been suggested.

PLAXIS, another code used widely in geomechanics, offers a very straightforward implementation of MCC without additional features. It employs the commonly used (but theoretically incorrect) constant Poisson's ratio. PLAXIS also offers other models, for instance the Ohta-Sekiguchi model, which is a later variant based on MCC.

9. MCC AND ADVANCED CONSTITUTIVE MODELS

One of the main aims of the development of MCC was to unify the treatment of deformation problems (mainly treated previously by purely elastic analyses) and capacity or failure (mainly treated by a separate set of calculations, usually employing limiting equilibrium methods). It successfully achieved this with the use of strain-hardening plasticity, but a problem remained that in reality some plastic strains occur "within" the yield surface. This was partly resolved in RB68 by the introduction of the "Revised" model, but the solution was not entirely satisfactory.

The recognition of the importance of non-linearity of soils at relatively small strains became a central theme of soil testing from the late 1970's, with important work throughout the 1980's and 1990's. It is now recognised widely that when secant stiffness is measured, the familiar "S-shaped curve" is obtained in a plot of secant shear stiffness against logarithm of strain amplitude. The increase of energy loss factor (often loosely referred to as damping), and the dependence of the tangent stiffness on immediate past history are allied phenomena. However, in many ways the theoretical modelling of these phenomena has lagged behind the appreciation of the phenomena from the experimental evidence.

The S-shaped curve can only be modelled if significant changes are made to elastic-plastic models (such as MCC) with single yield surfaces. Two main frameworks have been used to model small strain stiffness. The first are the multi-surface models: based on the work of either Iwan (1967) or Mroz (1967), they were first employed in soil mechanics by Prevost (1978). Several approaches have been made which use the MCC yield surface as a template for a series of surfaces of different sizes. The inner surfaces move within the outer surface, subject to kinematic hardening rules. Examples of this approach include a three-surface model by Stallebrass and Taylor (1997) and more general multisurface models for instance by Einav and Puzrin (2004) or Likitlersuang and Houlsby (2006).

The second framework is that of "bounding surface" models. These were introduced by Dafalias and Popov (1975). Perhaps the best-known model within this family is the "MIT E3" model described by Whittle and Kavvadas (1994). The "SANICLAY" model by Dafalias, Manzari and Papadimitriou (2006) is a further development.

In a different line of development, the MCC model has been applied to the modelling of unsaturated clays, requiring the consideration of "suction" in addition to other variables, see for instance Wheeler and Sivakumar (1995).

The above examples illustrate how MCC has been the starting point for some of the most sophisticated constitutive models used in modern soil mechanics.

10. POSTSCRIPT

There is no doubt that the paper RB68, which set out the central features on the Modified Cam Clay model, has been one of the most influential in the development of constitutive modelling in soil mechanics. Here the author has tried to demonstrate how multi-faceted the influence of MCC has been in the last 50 years. On the one hand MCC provides the conceptual framework that allows students to understand the fundamentals of the response of a soft clay, and allows simple "back-of-the-envelope" calculations of essential features such as undrained strength. At the opposite extreme it forms the basis of some of the most sophisticated constitutive models employed for detailed numerical analysis of geotechnical problems. As a scientific model it permits discussion about fundamental issues such as the thermodynamics of soils and the mechanisms of energy storage and dissipation.

Modified Cam-Clay has not just played an important role in the development of our subject over the last 50 years, it is as relevant today as when it was first published, and continues to enlighten our understanding of soils.

11. NOTATION

As far as possible this paper uses a notation close to that used in Roscoe and Burland (1968) (RB68), but with some minor departures to conform with more modern practice.

d	Dissipation function.
f	(1) Yield function, (2) Helmholtz free energy.
g	Plastic potential function.
G	Shear modulus.
K	Bulk modulus.
M	Slope of Critical State Line in (p,q) plot.
p	Mean effective stress. In triaxial test $\frac{1}{3}(\sigma'_a + 2\sigma'_r)$.
p_c	Isotropic preconsolidation pressure. In RB68 p_o.
p_r	Reference pressure. Often taken as atmospheric pressure. In RB68 implicitly taken as 1.0 psi.
q	Deviator stress. In triaxial test $\sigma'_a - \sigma'_r$.
s	Normal effective stress for plane strain, $\frac{1}{2}(\sigma'_1 + \sigma'_3)$.
t	Shear stress for plane strain, $\frac{1}{2}(\sigma'_1 - \sigma'_3)$.
v	Specific volume. RB68 uses void ratio $e = v - 1$.
v_o	Specific volume at zero strain.
$\dot{W}_p$	Rate of plastic work. In triaxial test $p\dot{\varepsilon}_v^p + q\dot{\varepsilon}_s^p$.
Γ	Value of specific volume on Critical State Line at $p = p_r$.
ε_s	Deviatoric strain. In triaxial test $\frac{2}{3}(\varepsilon_a - \varepsilon_r)$. In RB68 ε.
ε_v	Volumetric strain, equal to $-\log_e(v/v_o)$. In triaxial test $\varepsilon_a + 2\varepsilon_r$. In RB68 v.
$\varepsilon_v^e, \varepsilon_s^e$	Elastic strain components.
$\varepsilon_v^p, \varepsilon_s^p$	Plastic strain components.
η	Stress ratio q/p.
κ	Negative slope of swelling line in $(\log_e(p), v)$ plot.
κ^*	Negative slope of swelling line in $(\log_e(p), \log_e(v))$ plot.
λ	Negative slope of any constant η normal consolidation line in $(\log_e(p), v)$ plot.
λ^*	Negative slope of any constant η normal consolidation line in $(\log_e(p), \log_e(v))$ plot.
Λ	Plastic multiplier.
N	Value of specific volume on Isotropic Normal Consolidation Line at $p = p_r$.

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