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Ternary Semigroups Characterized by Spherical Fuzzy Bi-ideals

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ABSTRACT

The concept of spherical fuzzy sets, was introduced by Ashraf et al. in 2019, and is a generalization of the concepts of intuitionistic fuzzy sets, picture fuzzy sets and Pythagorean fuzzy sets. In this paper, we introduce the concepts of spherical fuzzy quasi-ideals and spherical fuzzy bi-ideals of ternary semigroups and investigate some of their properties. Then, we characterize regular ternary semigroups in terms of spherical fuzzy left (resp., lateral, right) ideals, spherical fuzzy quasi-ideals and spherical fuzzy bi-ideals of ternary semigroups. Moreover, some characterizations of weakly regular ternary semigroups by the concepts of many types of spherical fuzzy ideals in ternary semigroups are discussed.

Keywords: Regular ternary semigroup; Spherical fuzzy set; Spherical fuzzy ideal; Spherical fuzzy bi-ideal; Weakly regular ternary semigroup

1. Introduction

In 1965, Zadeh [1] presented the concept of fuzzy subsets or fuzzy sets as a function from a nonempty set X to the unit interval $[0, 1]$. The concept of fuzzy subgroups was first introduced by Rosenfeld [2], which acted as the inspiration for the research of fuzzy algebraic structures. Later, Kuroki [3] also introduced the concept of fuzzy subsemigroups. Additionally, he examined fuzzy generalized bi-ideals of semigroups and applied the properties of fuzzy left and fuzzy right ideals of semigroups to characterize some classes of semigroups, see, [4, 5]. Lehmer [6] gave the notion of ternary semigroups as a generalization of semigroups. Next, Shabir and Rehman [7] characterized regular ternary semigroups by the properties of anti fuzzy left (resp., lateral, right) ideals, anti fuzzy quasi-ideals

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and anti fuzzy (resp., generalized) bi-ideals of ternary semigroups. Afterwards, the regular and intra-regular ternary semigroups were characterized by using the concepts of fuzzy left (resp., lateral, right) ideals of ternary semigroups by Kar and Sarkar [8]. As an extension of the concept of fuzzy sets, Atanassov [9] introduced the concept of intuitionistic fuzzy sets. In addition, the fuzzy sets actually determine the degree of memberships of an element in a given set, whereas the intuitionistic fuzzy sets provide both membership and non-membership degrees. In 2012, Akram [10] and Lekkoksung [11] investigated the intuitionistic fuzzy sets in ternary semigroups. Later, the notions of intuitionistic fuzzy ideals and intuitionistic fuzzy filters in ternary semigroups were studied by Lalithamani et al. [12].

The concept of Pythagorean fuzzy sets, as the sum of the squares of membership and non-membership degrees belongs to the unit interval $[0, 1]$, was suggested by Yager and Abbasov [13] in 2013. This concept generalizes the intuitionistic fuzzy sets. In 2020, Chinram and Pantiyakul [14] considered the concept of rough Pythagorean fuzzy ideals in ternary semigroups. Another generalization of the intuitionistic fuzzy sets is the notion of picture fuzzy sets, was introduced by Cuong and Kreinovich [15]. In fact, the sum of positive membership, neutral membership and negative membership grades is greater than 1. Then, the semigroups was characterized by the properties of various types of picture fuzzy ideals of semigroups see, e.g., [16–18]. Furthermore, Nakkhasen [19] considered characterizations of (m, n)-regular, $(m, 0)$-regular and $(0, n)$-regular semigroups in terms of picture fuzzy (m, n)-ideals, picture fuzzy $(m, 0)$-ideals and picture fuzzy $(0, n)$-ideals of semigroups.

As a further generalization of the Pythagorean fuzzy sets and the picture fuzzy sets, Ashraf et al. [20] introduced the notion of spherical fuzzy sets. The concept of spherical fuzzy sets was surveyed in many algebraic structures, for instance, Veerappan and Venkatesan [21] discussed about the connection between bi-ideals and spherical interval-valued fuzzy bi-ideals in gamma near-rings, Subha et al. [22] characterized semi-rings by their rough spherical fuzzy ideals, and Chinnadurai et al. [23] studied the notion of spherical fuzzy ideals of semigroups and provided its characteristics using sufficient examples in 2022. Recently, Krailoet et al. [24] have applied the concept of spherical fuzzy sets to ternary semigroups and introduced the notions of spherical fuzzy ternary subsemigroups and spherical fuzzy (resp., left, lateral, right) ideals of ternary semigroups. They characterized the spherical fuzzy ternary subsemigroups and the spherical fuzzy left (resp., lateral, right) ideals of ternary semigroups, and obtained relationship between rough set theory and spherical fuzzy sets of ternary semigroups.

The purpose of this article is to introduce the concepts of spherical fuzzy quasi-ideals and spherical fuzzy bi-ideals of ternary semigroups. In Section 2, we review the concepts of spherical fuzzy ternary subsemigroups and spherical fuzzy (resp., left, lateral, right) ideals in ternary semigroups and introduce the notion of spherical fuzzy quasi-ideals of ternary semigroups. Later, we study the relationships between spherical fuzzy (resp., left, right, lateral) ideals and spherical fuzzy quasi-ideals of ternary semigroups. Next, we define the concept of spherical fuzzy bi-ideals of ternary semigroups and give some of their properties in Section 3. Then, in Section 4, we discuss the characterizations of regular ternary semigroups by means of spherical fuzzy left (resp., lateral, right) ideals, spherical fuzzy quasi-ideals and spherical fuzzy bi-ideals of ternary semigroups. Section 5 shows that characterizations of weakly regular ternary semigroups using the concepts of many types of spherical fuzzy ideals of ternary semigroups are investigated. Moreover, we suggest some characterization of both regular and weakly regular ternary semigroups by spherical fuzzy ideals. Finally, Section 6 concludes the article.

2. Preliminaries

In this section, we will review some of the fundamental definitions that will be utilized throughout the paper.

A nonempty set T is said to be a *ternary semigroup* [6] if there exists a ternary operation $[\ ] : T \times T \times T \to T$, written as $(a, b, c) \to [abc]$, such that it satisfies the following associative law hods: $[[abc]de] = [a[bcd]e] = [ab[cde]]$ for all $a, b, c, d, e \in T$. For the sake of simplicity, we will write abc instead of $[abc]$ for each $a, b, c \in T$. Let A, B and C be any nonempty subsets of a ternary semigroup T. Then, we denote $ABC = \{abc \mid a \in A, b \in B, c \in C\}$.

Now, we recall the notions of many types of ideals in ternary semigroups which occurred in [25] as follows. Let A be a nonempty subset of a ternary semigroup T. Then: (i) A is called a *ternary subsemigroup* of T if $AAA \subseteq A$; (ii) A is called a *left* (resp., *lateral, right*) *ideal* of T if $TTA \subseteq A$ (resp., $TAT \subseteq A$, $ATT \subseteq A$); (iii) if A is a left, lateral and right ideal of T, then it is called an *ideal* of T; (iv) if A is a left and right ideal of T, then it is called a *two-sided ideal* of T.

A nonempty subset Q of a ternary semigroup T is said to be a *quasi-ideal* [26] of T if $(QTT \cap TQT \cap TTQ) \subseteq Q$ and $(QTT \cap TTQTT \cap TTQ) \subseteq Q$. A ternary subsemigroup B of a ternary semigroup T is called a *bi-ideal* [27] of T if $BTBTB \subseteq B$.

A *fuzzy set* [1] μ of a nonempty set X is a mapping $\mu : X \to [0, 1]$. Let μ and η be any two fuzzy sets of a nonempty set X. For every $x \in X$, we denote

$$(\mu \cap \eta)(x) = \min\{\mu(x), \eta(x)\},$$
$$(\mu \cup \eta)(x) = \max\{\mu(x), \eta(x)\}.$$

Next, we review the concept of spherical fuzzy sets, which was introduced by Ashraf et al. [20] in 2019. A *spherical fuzzy set* (briefly, SFS) $\mathcal{A}$ on a nonempty set X is defined by the form

$$\mathcal{A} := \{\langle x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\rangle \mid x \in X\},$$

where $\mu_{\mathcal{A}} : X \to [0, 1]$, $\eta_{\mathcal{A}} : X \to [0, 1]$ and $\nu_{\mathcal{A}} : X \to [0, 1]$ denote the degree of membership, the degree of hesitancy and the degree of non-membership of each $x \in X$, respectively, with the condition $0 \leq (\mu_{\mathcal{A}}(x))^2 + (\eta_{\mathcal{A}}(x))^2 + (\nu_{\mathcal{A}}(x))^2 \leq 1$.

Throughout this paper, we shall use the symbol $\mathcal{A}$ instead of the SFS $\mathcal{A} := \{\langle x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\rangle \mid x \in X\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be any two SFSs on a universe set X. Then, we denote:

(i) $\mathcal{A} \subseteq \mathcal{B}$ iff $\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x), \eta_{\mathcal{A}}(x) \leq \eta_{\mathcal{B}}(x)$ and $\nu_{\mathcal{A}}(x) \geq \nu_{\mathcal{B}}(x)$ for all $x \in X$;

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$;

(iii) $\mathcal{A} \cap \mathcal{B} := \{\langle x, (\mu_{\mathcal{A}} \cap \mu_{\mathcal{B}})(x), (\eta_{\mathcal{A}} \cap \eta_{\mathcal{B}})(x), (\nu_{\mathcal{A}} \cup \nu_{\mathcal{B}})(x)\rangle \mid x \in X\}$;

(iv) $\mathcal{A} \cup \mathcal{B} := \{\langle x, (\mu_{\mathcal{A}} \cup \mu_{\mathcal{B}})(x), (\eta_{\mathcal{A}} \cap \eta_{\mathcal{B}})(x), (\nu_{\mathcal{A}} \cap \nu_{\mathcal{B}})(x)\rangle \mid x \in X\}$.

Next, we recall the concepts of spherical fuzzy ternary subsemigroups, spherical fuzzy (resp., left, lateral, right) ideals and the product of three spherical fuzzy sets of ternary semigroups which defined in [24] as follows.

Definition 2.1 ([24]). A SFS $\mathcal{A}$ on a ternary semigroup T is called a *spherical fuzzy ternary subsemigroup* (briefly, SFSub) of T, if for every $a, b, c \in T$:

(i) $\mu_{\mathcal{A}}(abc) \geq \min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{A}}(b), \mu_{\mathcal{A}}(c)\}$;

(ii) $\eta_{\mathcal{A}}(abc) \geq \min\{\eta_{\mathcal{A}}(a), \eta_{\mathcal{A}}(b), \eta_{\mathcal{A}}(c)\}$;

(iii) $\nu_{\mathcal{A}}(abc) \leq \max\{\nu_{\mathcal{A}}(a), \nu_{\mathcal{A}}(b), \nu_{\mathcal{A}}(c)\}$.

Definition 2.2 ([24]). A SFS $\mathcal{A}$ on a ternary semigroup T is called:

(i) a *spherical fuzzy left ideal* (briefly, SFL) of T, if for every $a, b, c \in T$, $\mu_{\mathcal{A}}(abc) \geq \mu_{\mathcal{A}}(c), \eta_{\mathcal{A}}(abc) \geq \eta_{\mathcal{A}}(c)$ and $\nu_{\mathcal{A}}(abc) \leq \nu_{\mathcal{A}}(c)$;

(ii) a *spherical fuzzy lateral ideal* (briefly, SFM) of T, if for every $a, b, c \in T$, $\mu_{\mathcal{A}}(abc) \geq \mu_{\mathcal{A}}(b), \eta_{\mathcal{A}}(abc) \geq \eta_{\mathcal{A}}(b)$ and $\nu_{\mathcal{A}}(abc) \leq \nu_{\mathcal{A}}(b)$;

(iii) a *spherical fuzzy right ideal* (briefly, SFR) of T, if for every $a, b, c \in T$, $\mu_{\mathcal{A}}(abc) \geq \mu_{\mathcal{A}}(a), \eta_{\mathcal{A}}(abc) \geq \eta_{\mathcal{A}}(a)$ and $\nu_{\mathcal{A}}(abc) \leq \nu_{\mathcal{A}}(a)$;

(iv) a *spherical fuzzy ideal* (briefly, SFI) of T, if for every $a, b, c \in T$,

$$\mu_{\mathcal{A}}(abc) \geq \max\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{A}}(b), \mu_{\mathcal{A}}(c)\},$$
$$\eta_{\mathcal{A}}(abc) \geq \max\{\eta_{\mathcal{A}}(a), \eta_{\mathcal{A}}(b), \eta_{\mathcal{A}}(c)\},$$
$$\nu_{\mathcal{A}}(abc) \leq \min\{\nu_{\mathcal{A}}(a), \nu_{\mathcal{A}}(b), \nu_{\mathcal{A}}(c)\}.$$

By a *spherical fuzzy two-sided ideal* (briefly, SFT) $\mathcal{A}$ of a ternary semigroup T, we means that $\mathcal{A}$ is both a SFL and a SFR of T.

Let $\mathcal{A}, \mathcal{B}$ and C be any three SFSs on a ternary semigroup T. The *product* $\mathcal{A} \circ \mathcal{B} \circ C$, in [24], of $\mathcal{A}, \mathcal{B}$ and C is defined by

$$\mathcal{A} \circ \mathcal{B} \circ C := \{\langle x, (\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}} \circ \mu_C)(x), (\eta_{\mathcal{A}} \circ \eta_{\mathcal{B}} \circ \eta_C)(x), (\nu_{\mathcal{A}} \circ \nu_{\mathcal{B}} \circ \nu_C)(x)\rangle \mid x \in T\},$$

where

$$(\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}} \circ \mu_C)(x) = \begin{cases} \sup\limits_{x=abc} \min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{B}}(b), \mu_C(c)\} & \text{if } x \in TTT; \\ 0 & \text{otherwise,} \end{cases}$$

$$(\eta_{\mathcal{A}} \circ \eta_{\mathcal{B}} \circ \eta_C)(x) = \begin{cases} \sup\limits_{x=abc} \min\{\eta_{\mathcal{A}}(a), \eta_{\mathcal{B}}(b), \eta_C(c)\} & \text{if } x \in TTT; \\ 0 & \text{otherwise,} \end{cases}$$

$$(\nu_{\mathcal{A}} \circ \nu_{\mathcal{B}} \circ \nu_C)(x) = \begin{cases} \inf\limits_{x=abc} \max\{\nu_{\mathcal{A}}(a), \nu_{\mathcal{B}}(b), \nu_C(c)\} & \text{if } x \in TTT; \\ 1 & \text{otherwise.} \end{cases}$$

Let $\mathcal{T} := \{\langle x, \mu_{\mathcal{T}}(x), \eta_{\mathcal{T}}(x), \nu_{\mathcal{T}}(x)\rangle \mid x \in T\}$, where $\mu_{\mathcal{T}}(x) = 1, \eta_{\mathcal{T}}(x) = 1$ and $\nu_{\mathcal{T}}(x) = 0$ for all $x \in T$, be a SFS of T. We observe that $\mathcal{A} \subseteq \mathcal{T}$ for every SFS $\mathcal{A}$ of T.

Let A be any subset of a ternary semigroup T. The *spherical characteristic function* of A is defined by $C_A := \{\langle x, \mu_{C_A}(x), \eta_{C_A}(x), \nu_{C_A}(x)\rangle \mid x \in T\}$, where

$$\mu_{C_A}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta_{C_A}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases}$$
$$\nu_{C_A}(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise.} \end{cases}$$

We note that if $A = T$, then $C_A = \mathcal{T}$.

The following lemmas can be proved straightforwardly.

Lemma 2.3. *Let C_A, C_B and C_C be SFSs on a ternary semigroup T, where A, B and C are any nonempty subsets of T. Then the following conditions hold:*

(i) $C_A \cap C_B = C_{A\cap B}$;

(ii) $C_A \circ C_B \circ C_C = C_{ABC}$;

(iii) $A \subseteq B$ *if and only if* $C_A \subseteq C_B$.

Lemma 2.4. *Let T be a ternary semigroup and $\mathcal{A}$ be any SFS of T. Then the following statements hold:*

(i) $\mathcal{A} \circ \mathcal{T} \circ \mathcal{T}$ *is a SFR of T;*

(ii) $\mathcal{T} \circ \mathcal{A} \circ \mathcal{T}$ *is a SFM of T;*

(iii) $\mathcal{T} \circ \mathcal{T} \circ \mathcal{A}$ *is a SFL of T.*

Lemma 2.5 ([24])**.** *Let $\mathcal{A}$ be a SFS on a ternary semigroup T. Then the following properties hold:*

(i) $\mathcal{A}$ *is a SFSub of T if and only if* $\mathcal{A} \circ \mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$;

(ii) $\mathcal{A}$ *is a SFL of T if and only if* $\mathcal{T} \circ \mathcal{T} \circ \mathcal{A} \subseteq \mathcal{A}$;

(iii) $\mathcal{A}$ *is a SFM of T if and only if* $\mathcal{T} \circ \mathcal{A} \circ \mathcal{T} \subseteq \mathcal{A}$;

(iv) $\mathcal{A}$ *is a SFR of T if and only if* $\mathcal{A} \circ \mathcal{T} \circ \mathcal{T} \subseteq \mathcal{A}$.

The following lemma is obtained by Lemma 2.5.

Lemma 2.6. *Let $\mathcal{A}$ be a SFS on a ternary semigroup T. Then, $\mathcal{A}$ is a SFI of T if and only if* $(\mathcal{A} \circ \mathcal{T} \circ \mathcal{T}) \cup (\mathcal{T} \circ \mathcal{A} \circ \mathcal{T}) \cup (\mathcal{T} \circ \mathcal{T} \circ \mathcal{A}) \subseteq \mathcal{A}$.

Next, we introduce the concept of spherical fuzzy quasi-ideals of ternary semigroups and give some properties and an example of this concept.

Definition 2.7. A SFS $\mathcal{Q}$ on a ternary semigroup T is called a *spherical fuzzy quasi-ideal* (briefly, SFQ) of T if $(\mathcal{Q}\circ\mathcal{T}\circ\mathcal{T}\cap((\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T})\cup(\mathcal{T}\circ\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}\circ\mathcal{T}))\cap\mathcal{T}\circ\mathcal{T}\circ\mathcal{Q}) \subseteq \mathcal{Q}$.

Proposition 2.8. *Every SFQ of a ternary semigroup T is a SFSub of T.*

Proof. Let $\mathcal{Q}$ be a SFQ of a ternary semigroup T. Then, $\mathcal{Q} \circ \mathcal{Q} \circ \mathcal{Q} \subseteq \mathcal{Q} \circ \mathcal{T} \circ \mathcal{T}$, $\mathcal{Q} \circ \mathcal{Q} \circ \mathcal{Q} \subseteq \mathcal{T} \circ \mathcal{T} \circ \mathcal{Q}$ and $\mathcal{Q} \circ \mathcal{Q} \circ \mathcal{Q} \subseteq \mathcal{T}\circ\mathcal{Q}\circ\mathcal{T} \subseteq (\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T})\cup(\mathcal{T}\circ\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}\circ\mathcal{T})$. Hence, $\mathcal{Q} \circ \mathcal{Q} \circ \mathcal{Q} \subseteq \mathcal{Q} \circ \mathcal{T} \circ \mathcal{T} \cap ((\mathcal{T} \circ \mathcal{Q} \circ \mathcal{T}) \cup (\mathcal{T} \circ \mathcal{T} \circ \mathcal{Q} \circ \mathcal{T} \circ \mathcal{T})) \cap \mathcal{T} \circ \mathcal{T} \circ \mathcal{Q} \subseteq \mathcal{Q}$. By Lemma 2.5(i), we have that $\mathcal{Q}$ is a SFSub of T. □

Proposition 2.9. *Every SFL (resp., SFM, SFR, SFI) of a ternary semigroup T is a SFQ of T.*

Proof. Let $\mathcal{L}$ be a SFL of a ternary semigroup T. By Lemma 2.5(ii), we get that $\mathcal{T} \circ \mathcal{T} \circ \mathcal{L} \subseteq \mathcal{L}$. Thus, we have $\mathcal{L} \circ \mathcal{T} \circ \mathcal{T} \cap ((\mathcal{T} \circ \mathcal{L} \circ \mathcal{T}) \cup (\mathcal{T} \circ \mathcal{T} \circ \mathcal{L} \circ \mathcal{T} \circ \mathcal{T})) \cap \mathcal{T} \circ \mathcal{T} \circ \mathcal{L} \subseteq (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \cap (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \cap \mathcal{L} \subseteq \mathcal{T} \cap \mathcal{T} \cap \mathcal{L} = \mathcal{L}$. It follows that $\mathcal{L}$ is a SFQ of T. Similarly, we can prove the other cases. □

The converse of Proposition 2.9 is not true in general. This can be shown by the following example.

Example 2.10. Let $T = \{a, b, c, d, e\}$ and define the ternary operation $\cdot$ on T as follows:

$\cdot$	a	b	c	d	e
aa	a	a	a	a	a
ab	a	a	a	a	a
ac	a	a	a	a	a
ad	a	a	a	a	a
ae	a	a	a	a	a

$\cdot$	a	b	c	d	e
ba	a	a	a	a	a
bb	a	b	c	a	a
bc	a	a	a	b	c
bd	a	a	a	a	a
be	a	a	a	a	a

$\cdot$	a	b	c	d	e
ca	a	a	a	a	a
cb	a	a	a	a	a
cc	a	a	a	a	a
cd	a	b	c	a	a
ce	a	a	a	b	c

$\cdot$	a	b	c	d	e
da	a	a	a	a	a
db	a	d	e	a	a
dc	a	a	a	d	e
dd	a	a	a	a	a
de	a	a	a	a	a

$\cdot$	a	b	c	d	e
ea	a	a	a	a	a
eb	a	a	a	a	a
ec	a	a	a	a	a
ed	a	d	e	a	a
ee	a	a	a	d	e

Then, $(T, \cdot)$ is a ternary semigroup. Now, we define a SFS $\mathcal{A}$ on T as follows:

$\mathcal{A}$	$\mu_{\mathcal{A}}$	$\eta_{\mathcal{A}}$	$\nu_{\mathcal{A}}$
a	0.7	0.6	0.2
b	0.5	0.3	0.4
c	0.7	0.6	0.2
d	0.5	0.3	0.4
e	0.5	0.3	0.4

By routine calculations, we obtain that $\mathcal{A}$ is a SFQ of T, but it is not a SFL of T, because

$$\mu_{\mathcal{A}}(edc) = \mu_{\mathcal{A}}(e) = 0.5 < 0.7 = \mu_{\mathcal{A}}(c),$$

$$\eta_{\mathcal{A}}(edc) = \eta_{\mathcal{A}}(e) = 0.3 < 0.6 = \eta_{\mathcal{A}}(c),$$

$$\nu_{\mathcal{A}}(edc) = \nu_{\mathcal{A}}(e) = 0.4 > 0.2 = \nu_{\mathcal{A}}(c).$$

Moreover, $\mathcal{A}$ is neither a SFM nor a SFR of T.

3. Spherical Fuzzy Bi-ideals

In this section, we may also define the notion of spherical fuzzy bi-ideals of ternary semigroups and create some properties of it with the sufficient example.

Definition 3.1. A SFSub $\mathcal{A}$ of a ternary semigroup T is called a *spherical fuzzy bi-ideal* (briefly, SFB) of T, if for all $m, n, x, y, z \in T$:

(i) $\mu_{\mathcal{A}}(xmynz) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y), \mu_{\mathcal{A}}(z)\}$;

(ii) $\eta_{\mathcal{A}}(xmynz) \geq \min\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y), \eta_{\mathcal{A}}(z)\}$;

(iii) $\nu_{\mathcal{A}}(xmynz) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(y), \nu_{\mathcal{A}}(z)\}$.

Proposition 3.2. *Every SFQ of a ternary semigroup T is a SFB of T.*

Proof. Let $\mathcal{Q}$ be a SFQ of a ternary semigroup T. By Proposition 2.8, $\mathcal{Q}$ is a SFSub of T. Then, $\mathcal{Q}\circ\mathcal{Q}\circ\mathcal{Q} \subseteq \mathcal{Q}$ by Lemma 2.5(i). Thus, we have $\mathcal{Q}\circ\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}\circ\mathcal{Q} \subseteq \mathcal{Q}\circ\mathcal{T}\circ\mathcal{T}$, $\mathcal{Q}\circ\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}\circ\mathcal{Q} \subseteq \mathcal{T}\circ\mathcal{T}\circ\mathcal{Q}$ and $\mathcal{Q}\circ\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}\circ\mathcal{Q} \subseteq \mathcal{T}\circ\mathcal{Q}\circ\mathcal{T} \subseteq (\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}) \cup (\mathcal{T}\circ\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}\circ\mathcal{T})$. Since $\mathcal{Q}$ is a SFQ of T, this implies that $\mathcal{Q}\circ\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}\circ\mathcal{Q} \subseteq \mathcal{Q}\circ\mathcal{T}\circ\mathcal{T} \cap ((\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}) \cup (\mathcal{T}\circ\mathcal{T}\circ\mathcal{Q}\circ\mathcal{T}\circ\mathcal{T})) \cap \mathcal{T}\circ\mathcal{T}\circ\mathcal{Q} \subseteq \mathcal{Q}$. Hence, $\mathcal{Q}$ is a SFB of T. □

In contrast, a SFB of a ternary semigroup T may not be a SFQ as shown by the following example.

Example 3.3. Let

$$T = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{Z}_0^- \right\},$$

where $\mathbb{Z}_0^-$ is the set of all non-positive integers. Then, T is a ternary semigroup under the usual matrix multiplication (see [28]). Now, we consider

$$B = \left\{ \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & m \\ 0 & 0 & 0 \end{pmatrix} \mid m \in \mathbb{Z}_0^- \right\}.$$

Define the SFS $\mathcal{B}$ on T as follows:

$$\mu_{\mathcal{B}}(x) = \begin{cases} 0.8 & \text{if } x \in B, \\ 0 & \text{otherwise,} \end{cases}$$
$$\eta_{\mathcal{B}}(x) = \begin{cases} 0.5 & \text{if } x \in B, \\ 0 & \text{otherwise,} \end{cases}$$
$$\nu_{\mathcal{B}}(x) = \begin{cases} 0 & \text{if } x \in B, \\ 0.9 & \text{otherwise.} \end{cases}$$

We can see that $\mathcal{B}$ is a SFB of T. Consider, $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \in B$ and $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T \setminus B$. It turns out that

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Letting $a = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. This implies that, $a \notin B$. We obtain that $(\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}})(a) > \mu_{\mathcal{B}}(a), (\mu_{\mathcal{T}} \circ \mu_{\mathcal{B}} \circ \mu_{\mathcal{T}})(a) > \mu_{\mathcal{B}}(a)$ and $(\mu_{\mathcal{B}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{T}})(a) > \mu_{\mathcal{B}}(a)$. Thus,

$$\min\{(\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}})(a), (\mu_{\mathcal{T}} \circ \mu_{\mathcal{B}} \circ \mu_{\mathcal{T}})(a), (\mu_{\mathcal{B}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{T}})(a)\} > \mu_{\mathcal{B}}(a).$$

Similarly, we have that $\min\{(\eta_{\mathcal{T}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}})(a), (\eta_{\mathcal{T}} \circ \eta_{\mathcal{B}} \circ \eta_{\mathcal{T}})(a), (\eta_{\mathcal{B}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{T}})(a)\} > \eta_{\mathcal{B}}(a)$ and $\max\{(\nu_{\mathcal{T}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}})(a), (\nu_{\mathcal{T}} \circ \nu_{\mathcal{B}} \circ \nu_{\mathcal{T}})(a), (\nu_{\mathcal{B}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{T}})(a)\} < \nu_{\mathcal{B}}(a)$. This means that

$$(\mathcal{B} \circ \mathcal{T} \circ \mathcal{T}) \cap (\mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \cup \mathcal{T} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{T}) \cap (\mathcal{T} \circ \mathcal{T} \circ \mathcal{B}) \nsubseteq \mathcal{B}.$$

This shows that $\mathcal{B}$ is not a SFQ of T.

Theorem 3.4. *Let $\mathcal{A}$ be a SFSub of a ternary semigroup T. Then, $\mathcal{A}$ is a SFB of T if and only if $\mathcal{A} \circ \mathcal{T} \circ \mathcal{A} \circ \mathcal{T} \circ \mathcal{A} \subseteq \mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is a SFB of T. Let $a \in T$. If a is not to be expressible $a = bpcqd$ for all $b, c, d, p, q \in T$, then it is well done. Suppose that there exist $m, n, x, y, z \in T$ such that $a = xmynz$. Take $k = xmy$. Then, we have

$$\begin{aligned}
&(\mu_{\mathcal{A}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{A}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{A}})(a) \\
&= \sup_{a=knz} \min\{(\mu_{\mathcal{A}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{A}})(k), \\
&\qquad \mu_{\mathcal{T}}(n), \mu_{\mathcal{A}}(z)\} \\
&= \sup_{a=knz} \min\{\sup_{k=xmy} \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{T}}(m), \\
&\qquad \mu_{\mathcal{A}}(y)\}, \mu_{\mathcal{T}}(n), \mu_{\mathcal{A}}(z)\} \\
&= \sup_{a=knz} \sup_{k=xmy} \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{T}}(m), \mu_{\mathcal{A}}(y), \\
&\qquad \mu_{\mathcal{T}}(n), \mu_{\mathcal{A}}(z)\} \\
&= \sup_{a=knz} \sup_{k=xmy} \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y), \mu_{\mathcal{A}}(z)\} \\
&\leq \sup_{a=knz} \sup_{k=xmy} \mu_{\mathcal{A}}(xmynz) \\
&= \sup_{a=knz} \mu_{\mathcal{A}}(knz) \\
&= \mu_{\mathcal{A}}(a),
\end{aligned}$$

$$\begin{aligned}
&(\eta_{\mathcal{A}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{A}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{A}})(a) \\
&= \sup_{a=knz} \min\{(\eta_{\mathcal{A}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{A}})(k), \\
&\qquad \eta_{\mathcal{T}}(n), \eta_{\mathcal{A}}(z)\} \\
&= \sup_{a=knz} \min\{\sup_{k=xmy} \min\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{T}}(m),
\end{aligned}$$

$$\eta_{\mathcal{A}}(y)\}, \eta_{\mathcal{T}}(n), \eta_{\mathcal{A}}(z)\}$$
$$= \sup_{a=knz} \sup_{k=xmy} \min\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{T}}(m), \eta_{\mathcal{A}}(y),$$
$$\eta_{\mathcal{T}}(n), \eta_{\mathcal{A}}(z)\}$$
$$= \sup_{a=knz} \sup_{k=xmy} \min\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y), \eta_{\mathcal{A}}(z)\}$$
$$\leq \sup_{a=knz} \sup_{k=xmy} \eta_{\mathcal{A}}(xmynz)$$
$$= \sup_{a=knz} \eta_{\mathcal{A}}(knz)$$
$$= \eta_{\mathcal{A}}(a),$$

and

$$(\nu_{\mathcal{A}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{A}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{A}})(a)$$
$$= \inf_{a=knz} \max\{(\nu_{\mathcal{A}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{A}})(k),$$
$$\nu_{\mathcal{T}}(n), \nu_{\mathcal{A}}(z)\}$$
$$= \inf_{a=knz} \max\{\inf_{k=xmy} \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{T}}(m),$$
$$\nu_{\mathcal{A}}(y)\}, \nu_{\mathcal{T}}(n), \nu_{\mathcal{A}}(z)\}$$
$$= \inf_{a=knz} \inf_{k=xmy} \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{T}}(m), \nu_{\mathcal{A}}(y),$$
$$\nu_{\mathcal{T}}(n), \nu_{\mathcal{A}}(z)\}$$
$$= \inf_{a=knz} \inf_{k=xmy} \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(y), \nu_{\mathcal{A}}(z)\}$$
$$\geq \inf_{a=knz} \inf_{k=xmy} \nu_{\mathcal{A}}(xmynz)$$
$$= \inf_{a=knz} \nu_{\mathcal{A}}(knz)$$
$$= \nu_{\mathcal{A}}(a).$$

Therefore, $\mathcal{A} \circ \mathcal{T} \circ \mathcal{A} \circ \mathcal{T} \circ \mathcal{A} \subseteq \mathcal{A}$.

Conversely, let $m, n, x, y, z \in T$. Put $a = xmynz$. By the given assumption, we have

$$\mu_{\mathcal{A}}(xmynz)$$
$$= \mu_{\mathcal{A}}(a)$$
$$\geq (\mu_{\mathcal{A}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{A}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{A}})(a)$$
$$= \sup_{a=kpq} \min\{(\mu_{\mathcal{A}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{A}})(k),$$
$$\mu_{\mathcal{T}}(p), \mu_{\mathcal{A}}(q)\}$$
$$\geq \min\{(\mu_{\mathcal{A}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{A}})(xmy),$$
$$\mu_{\mathcal{T}}(n), \mu_{\mathcal{A}}(z)\}$$
$$= \min\{\sup_{xmy=bcd} \min\{\mu_{\mathcal{A}}(b), \mu_{\mathcal{T}}(c), \mu_{\mathcal{A}}(d)\},$$
$$\mu_{\mathcal{T}}(n), \mu_{\mathcal{A}}(z)\}$$
$$\geq \min\{\min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{T}}(m), \mu_{\mathcal{A}}(y)\},$$
$$\mu_{\mathcal{T}}(n), \mu_{\mathcal{A}}(z)\}$$
$$= \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y), \mu_{\mathcal{A}}(z)\},$$

$$\eta_{\mathcal{A}}(xmynz)$$
$$= \eta_{\mathcal{A}}(a)$$
$$\geq (\eta_{\mathcal{A}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{A}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{A}})(a)$$
$$= \sup_{a=kpq} \min\{(\eta_{\mathcal{A}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{A}})(k),$$
$$\eta_{\mathcal{T}}(p), \eta_{\mathcal{A}}(q)\}$$
$$\geq \min\{(\eta_{\mathcal{A}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{A}})(xmy),$$
$$\eta_{\mathcal{T}}(n), \eta_{\mathcal{A}}(z)\}$$
$$= \min\{\sup_{xmy=bcd} \min\{\eta_{\mathcal{A}}(b), \eta_{\mathcal{T}}(c), \eta_{\mathcal{A}}(d)\},$$
$$\eta_{\mathcal{T}}(n), \eta_{\mathcal{A}}(z)\}$$
$$\geq \min\{\min\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{T}}(m), \eta_{\mathcal{A}}(y)\},$$
$$\eta_{\mathcal{T}}(n), \eta_{\mathcal{A}}(z)\}$$
$$= \min\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y), \eta_{\mathcal{A}}(z)\},$$

and

$$\nu_{\mathcal{A}}(xmynz)$$
$$= \nu_{\mathcal{A}}(a)$$
$$\leq (\nu_{\mathcal{A}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{A}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{A}})(a)$$
$$= \inf_{a=kpq} \max\{(\nu_{\mathcal{A}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{A}})(k),$$
$$\nu_{\mathcal{T}}(p), \nu_{\mathcal{A}}(q)\}$$
$$\leq \max\{(\nu_{\mathcal{A}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{A}})(xmy),$$
$$\nu_{\mathcal{T}}(n), \nu_{\mathcal{A}}(z)\}$$
$$= \max\{\inf_{xmy=bcd} \max\{\nu_{\mathcal{A}}(b), \nu_{\mathcal{T}}(c), \nu_{\mathcal{A}}(d)\},$$
$$\nu_{\mathcal{T}}(n), \nu_{\mathcal{A}}(z)\}$$
$$\leq \max\{\max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{T}}(m), \nu_{\mathcal{A}}(y)\},$$
$$\nu_{\mathcal{T}}(n), \nu_{\mathcal{A}}(z)\}$$
$$= \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(y), \nu_{\mathcal{A}}(z)\}.$$

Consequently, $\mathcal{A}$ is a SFB of T. □

Theorem 3.5. *Let A be a nonempty subset of a ternary semigroup T. Then the following statements hold:*

(i) A is a ternary subsemigroup of T if and only if C_A is a SFSub of T;

(ii) A is a left ideal of T if and only if C_A is a SFL of T;

(iii) A is a lateral ideal of T if and only if C_A is a SFM of T;

(iv) A is a right ideal of T if and only if C_A is a SFR of T;

(v) A is an ideal of T if and only if C_A is a SFI of T;

(vi) A is a two-sided ideal of T if and only if C_A is a SFT of T;

(vii) A is a quasi-ideal of T if and only if C_A is a SFQ of T;

(viii) A is a bi-ideal of T if and only if C_A is a SFB of T.

Proof. (i) Assume that A is a ternary subsemigroup of T. Suppose that $\mu_{C_A}(abc) < \min\{\mu_{C_A}(a), \mu_{C_A}(b), \mu_{C_A}(c)\}$ for some $a, b, c \in T$. Then, $\mu_{C_A}(abc) = 0$ and $\min\{\mu_{C_A}(a), \mu_{C_A}(b), \mu_{C_A}(c)\} = 1$. This implies that $abc \notin A$ and $a, b, c \in A$. By the hypothesis, we have that $abc \in A$, which is a contradiction. Thus,

$$\mu_{C_A}(xyz) \geq \min\{\mu_{C_A}(x), \mu_{C_A}(y), \mu_{C_A}(z)\}$$

for all $x, y, z \in T$. If there exist $a, b, c \in T$ such that $\eta_{C_A}(abc) < \min\{\eta_{C_A}(a), \eta_{C_A}(b), \eta_{C_A}(c)\}$, then $\eta_{C_A}(abc) = 0$ and $\min\{\eta_{C_A}(a), \eta_{C_A}(b), \eta_{C_A}(c)\} = 1$. It follows that $abc \notin A$ and $a, b, c \in A$. By assumption, $abc \in A$, which is a contradiction. Hence,

$$\eta_{C_A}(xyz) \geq \min\{\eta_{C_A}(x), \eta_{C_A}(y), \eta_{C_A}(z)\}$$

for all $x, y, z \in T$. If $\nu_{C_A}(abc) > \max\{\nu_{C_A}(a), \nu_{C_A}(b), \nu_{C_A}(c)\}$ for some $a, b, c \in T$, then $\nu_{C_A}(abc) = 1$ and $\max\{\nu_{C_A}(a), \nu_{C_A}(b), \nu_{C_A}(c)\} = 0$. Also, $abc \notin A$, while $a, b, c \in A$. By the given assumption, we have that $abc \in A$. This is a contradiction. So,

$$\nu_{C_A}(xyz) \leq \max\{\nu_{C_A}(x), \nu_{C_A}(y), \nu_{C_A}(z)\}$$

for all $x, y, z \in T$. Therefore, C_A is a SFSub of T.

Conversely, assume that C_A is a SFSub of T. Let $x, y, z \in A$. Then,

$$\begin{aligned}\mu_{C_A}(xyz) &\geq \min\{\mu_{C_A}(x), \mu_{C_A}(y), \mu_{C_A}(z)\}\\ &= 1.\end{aligned}$$

It turns out that $\mu_{C_A}(xyz) = 1$. We obtain that $xyz \in A$. Thus, $AAA \subseteq A$. Consequently, A is a ternary subsemigroup of T.

The other statements can be proved in a similar way. □

4. Regular Ternary Semigroups

In this section, we study the characterizations of regular ternary semigroups by the properties of spherical fuzzy left (resp., lateral, right) ideals, spherical fuzzy quasi-ideals and spherical fuzzy bi-ideals of ternary semigroups. Moreover, we show that every spherical fuzzy bi-ideal is also a spherical fuzzy quasi-ideal in a regular ternary semigroup.

Definition 4.1 (cf. [25]). Let T be a ternary semigroup. An element $a \in T$ is called *regular* if there exists an element $x \in T$ such that $a = axa$, that is, $a \in aTa$. A ternary semigroup T is called *regular* if all its elements are regular.

Lemma 4.2 ([25]). *Let T be a ternary semigroup. Then, T is regular if and only if $RML = R \cap M \cap L$, for every left ideal L, every lateral ideal M and every right ideal R of T.*

Theorem 4.3. *Let T be a ternary semigroup. Then, T is regular if and only if $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} = \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$, for every SFL $\mathcal{L}$, every SFM $\mathcal{M}$ and every SFR $\mathcal{R}$ of T.*

Proof. Assume that T is regular. Let $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{R}$ be a SFL, a SFM and a SMR of T, respectively. By Lemma 2.5, we have that $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$. On the other hand, let $a \in S$. Then, there exists $x \in T$ such that $a = axa = (axa)(xax)(axa)$. Thus, we have

$$\begin{aligned}
&(\mu_{\mathcal{R}} \circ \mu_{\mathcal{M}} \circ \mu_{\mathcal{L}})(a) \\
&= \sup_{a=kpq} \min\{\mu_{\mathcal{R}}(k), \mu_{\mathcal{M}}(p), \mu_{\mathcal{L}}(q)\} \\
&\geq \min\{\mu_{\mathcal{R}}(axa), \mu_{\mathcal{M}}(xax), \mu_{\mathcal{L}}(axa)\} \\
&= \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{M}}(a), \mu_{\mathcal{L}}(a)\} \\
&= (\mu_{\mathcal{R}} \cap \mu_{\mathcal{M}} \cap \mu_{\mathcal{L}})(a),
\end{aligned}$$

$$\begin{aligned}
&(\eta_{\mathcal{R}} \circ \eta_{\mathcal{M}} \circ \eta_{\mathcal{L}})(a) \\
&= \sup_{a=kpq} \min\{\eta_{\mathcal{R}}(k), \eta_{\mathcal{M}}(p), \eta_{\mathcal{L}}(q)\} \\
&\geq \min\{\eta_{\mathcal{R}}(axa), \eta_{\mathcal{M}}(xax), \eta_{\mathcal{L}}(axa)\} \\
&= \min\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{M}}(a), \eta_{\mathcal{L}}(a)\} \\
&= (\eta_{\mathcal{R}} \cap \eta_{\mathcal{M}} \cap \eta_{\mathcal{L}})(a)
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_{\mathcal{R}} \circ \nu_{\mathcal{M}} \circ \nu_{\mathcal{L}})(a) \\
&= \inf_{a=kpq} \max\{\nu_{\mathcal{R}}(k), \nu_{\mathcal{M}}(p), \nu_{\mathcal{L}}(q)\} \\
&\leq \max\{\nu_{\mathcal{R}}(axa), \nu_{\mathcal{M}}(xax), \nu_{\mathcal{L}}(axa)\} \\
&= \max\{\nu_{\mathcal{R}}(a), \nu_{\mathcal{M}}(a), \nu_{\mathcal{L}}(a)\} \\
&= (\nu_{\mathcal{R}} \cup \nu_{\mathcal{M}} \cup \nu_{\mathcal{L}})(a).
\end{aligned}$$

This shows that $\mathcal{R} \cap \mathcal{M} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{M} \circ \mathcal{L}$. Hence, $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} = \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$.

Conversely, let L, M and R be any left ideal, lateral ideal and right ideal of a ternary semigroup T, respectively. By Theorem 3.5, we have that C_L, C_M and C_R are a SFL, a SFM and a SFR of T, respectively. Then, using the given assumption and Lemma 2.3, it implies that $C_{RML} = C_R \circ C_M \circ C_L = C_R \cap C_M \cap C_L = C_{R \cap M \cap L}$, and so $RML = R \cap M \cap L$. By Lemma 4.2, T is regular. □

Theorem 4.4. *Let T be a ternary semigroup. Then the following conditions are equivalent:*

(i) T is regular;

(ii) $\mathcal{B} = \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}$, for every SFB $\mathcal{B}$ of T;

(iii) $\mathcal{Q} = \mathcal{Q} \circ \mathcal{T} \circ \mathcal{Q} \circ \mathcal{T} \circ \mathcal{Q}$, for every SQB $\mathcal{Q}$ of T.

Proof. (i) $\Rightarrow$ (ii) Assume that T is regular. Let $\mathcal{B}$ be a SFB of T. By Theorem 3.4, we have that $\mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \subseteq \mathcal{B}$. Let $a \in T$. Then, there exists $x \in T$ such that $a = axa$. So, $a = (axa)xa$. Thus, we have

$$\begin{aligned}
&(\mu_{\mathcal{B}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}})(a) \\
&= \sup_{a=bcd} \min\{(\mu_{\mathcal{B}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}})(b), \\
&\qquad \mu_{\mathcal{T}}(c), \mu_{\mathcal{B}}(d)\} \\
&\geq \min\{(\mu_{\mathcal{B}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}})(axa), \\
&\qquad \mu_{\mathcal{T}}(x), \mu_{\mathcal{B}}(a)\} \\
&= \min\{\sup_{axa=kpq} \min\{\mu_{\mathcal{B}}(k), \mu_{\mathcal{T}}(p), \mu_{\mathcal{B}}(q)\}, \\
&\qquad \mu_{\mathcal{T}}(x), \mu_{\mathcal{B}}(a)\} \\
&\geq \min\{\min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{T}}(x), \mu_{\mathcal{B}}(a)\}, \\
&\qquad \mu_{\mathcal{T}}(x), \mu_{\mathcal{B}}(a)\} \\
&= \mu_{\mathcal{B}}(a),
\end{aligned}$$

$$\begin{aligned}
&(\eta_{\mathcal{B}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}})(a) \\
&= \sup_{a=bcd} \min\{(\eta_{\mathcal{B}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}})(b), \\
&\qquad \eta_{\mathcal{T}}(c), \eta_{\mathcal{B}}(d)\} \\
&\geq \min\{(\eta_{\mathcal{B}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}})(axa), \\
&\qquad \eta_{\mathcal{T}}(x), \eta_{\mathcal{B}}(a)\} \\
&= \min\{\sup_{axa=kpq} \min\{\eta_{\mathcal{B}}(k), \eta_{\mathcal{T}}(p), \eta_{\mathcal{B}}(q)\},
\end{aligned}$$

$$\eta_{\mathcal{T}}(x), \eta_{\mathcal{B}}(a)\}$$
$$\geq \min\{\min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{T}}(x), \eta_{\mathcal{B}}(a)\}, \eta_{\mathcal{T}}(x), \eta_{\mathcal{B}}(a)\}$$
$$= \eta_{\mathcal{B}}(a)$$

and

$$\begin{aligned}
&(\nu_{\mathcal{B}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}})(a) \\
&= \inf_{a=bcd} \max\{(\nu_{\mathcal{B}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}})(b), \nu_{\mathcal{T}}(c), \nu_{\mathcal{B}}(d)\} \\
&\leq \max\{(\nu_{\mathcal{B}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}})(axa), \nu_{\mathcal{T}}(x), \nu_{\mathcal{B}}(a)\} \\
&= \max\{\inf_{axa=kpq} \max\{\nu_{\mathcal{B}}(k), \nu_{\mathcal{T}}(p), \nu_{\mathcal{B}}(q)\}, \nu_{\mathcal{T}}(x), \nu_{\mathcal{B}}(a)\} \\
&\leq \max\{\max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{T}}(x), \nu_{\mathcal{B}}(a)\}, \nu_{\mathcal{T}}(x), \nu_{\mathcal{B}}(a)\} \\
&= \nu_{\mathcal{B}}(a).
\end{aligned}$$

This shows that $\mathcal{B} \subseteq \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}$. Hence, $\mathcal{B} = \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}$.

(ii) ⇒ (iii) By Proposition 3.2, it follows that (iii) holds.

(iii) ⇒ (i) Let $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{R}$ be a SFL, a SFM and a SFR of T, respectively. Then, $\mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$ is a SFQ of T. Hence, by (iii), we have

$$\begin{aligned}
&\mathcal{R} \cap \mathcal{M} \cap \mathcal{L} \\
&= (\mathcal{R} \cap \mathcal{M} \cap \mathcal{L}) \circ \mathcal{T} \circ (\mathcal{R} \cap \mathcal{M} \cap \mathcal{L}) \circ \mathcal{T} \circ (\mathcal{R} \cap \mathcal{M} \cap \mathcal{L}) \\
&\subseteq \mathcal{R} \circ (\mathcal{T} \circ \mathcal{M} \circ \mathcal{T}) \circ \mathcal{L} \\
&\subseteq \mathcal{R} \circ \mathcal{M} \circ \mathcal{L}.
\end{aligned}$$

On the other hand, we obtain that $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{T} \circ \mathcal{T} \subseteq \mathcal{R}$, $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} \subseteq \mathcal{T} \circ \mathcal{M} \circ \mathcal{T} \subseteq \mathcal{M}$ and $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} \subseteq \mathcal{T} \circ \mathcal{T} \circ \mathcal{L} \subseteq \mathcal{L}$. Also, $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$. Hence, $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} = \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$. By Theorem 4.3, we conclude that T is regular. □

Theorem 4.5. *Let T be a ternary semigroup. Then the following statements are equivalent:*

(i) T is regular;

(ii) $\mathcal{B} = \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}$, for every SFB $\mathcal{B}$ of T;

(iii) $\mathcal{Q} = \mathcal{Q} \circ \mathcal{T} \circ \mathcal{Q}$, for every SQB $\mathcal{Q}$ of T.

Proof. (i) ⇒ (ii) Let $\mathcal{B}$ be a SFB of T. By Theorem 4.4, $\mathcal{B} = \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}$. We obtain that $\mathcal{B} = \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \subseteq \mathcal{B} \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \circ \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}$ and $\mathcal{B} \circ \mathcal{T} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{T} \circ (\mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}) \subseteq \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \circ \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} = \mathcal{B}$. This implies that $\mathcal{B} = \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}$.

(ii) ⇒ (iii) It is obvious.

(iii) ⇒ (i) Assume that (iii) holds. Let $\mathcal{Q}$ be a SFQ of T. By the hypothesis, $\mathcal{Q} = \mathcal{Q} \circ \mathcal{T} \circ \mathcal{Q}$. Then, $\mathcal{Q} = \mathcal{Q} \circ \mathcal{T} \circ \mathcal{Q} = \mathcal{Q} \circ \mathcal{T} \circ \mathcal{Q} \circ \mathcal{T} \circ \mathcal{Q}$. Therefore, T is regular by Theorem 4.4. □

Proposition 4.6. *In a regular ternary semigroup T, every SFB of T is also a SFQ of T.*

Proof. Let $\mathcal{B}$ be a SFB of T. By Theorem 4.5, $\mathcal{B} = \mathcal{B} \circ \mathcal{T} \circ \mathcal{B}$. Then, $\mathcal{B} \circ \mathcal{T} \circ \mathcal{T}$, $(\mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \cup \mathcal{T} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{T})$ and $\mathcal{T} \circ \mathcal{T} \circ \mathcal{B}$ are a SFR, a SFM and a SFL of T, respectively. Hence, using Theorem 4.3, we have

$$\begin{aligned}
&(\mathcal{B} \circ \mathcal{T} \circ \mathcal{T}) \cap ((\mathcal{T} \circ \mathcal{B} \circ \mathcal{T}) \cup (\mathcal{T} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{T})) \cap (\mathcal{T} \circ \mathcal{T} \circ \mathcal{B}) \\
&= (\mathcal{B} \circ \mathcal{T} \circ \mathcal{T}) \circ ((\mathcal{T} \circ \mathcal{B} \circ \mathcal{T}) \cup (\mathcal{T} \circ \mathcal{T} \circ \mathcal{B} \circ \mathcal{T} \circ \mathcal{T})) \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{B}) \\
&= (\mathcal{B} \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \circ \mathcal{B} \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \circ \mathcal{B}) \\
&\quad \cup (\mathcal{B} \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \circ (\mathcal{T} \circ \mathcal{B} \circ \mathcal{T}) \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \circ \mathcal{B}) \\
&\subseteq (\mathcal{B} \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \circ \mathcal{B}) \cup (\mathcal{B} \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}) \circ \mathcal{B}) \\
&\subseteq (\mathcal{B} \circ \mathcal{T} \circ \mathcal{B}) \cup (\mathcal{B} \circ \mathcal{T} \circ \mathcal{B}) \\
&= \mathcal{B} \circ \mathcal{T} \circ \mathcal{B} \\
&= \mathcal{B}.
\end{aligned}$$

Consequently, $\mathcal{B}$ is a SFQ of T. □

Theorem 4.7. *Let T be a ternary semigroup. Then the following conditions are equivalent:*

(i) T is regular;

(ii) $\mathcal{B} \cap \mathcal{M} = \mathcal{B} \circ \mathcal{M} \circ \mathcal{B} \circ \mathcal{M} \circ \mathcal{B}$, for every SFB $\mathcal{B}$ and every SFM $\mathcal{M}$ of T;

(iii) $\mathcal{Q} \cap \mathcal{M} = \mathcal{Q} \circ \mathcal{M} \circ \mathcal{Q} \circ \mathcal{M} \circ \mathcal{Q}$, for every SFQ $\mathcal{Q}$ and every SFM $\mathcal{M}$ of T.

Proof. (i) ⇒ (ii) Assume that T is regular. Let $\mathcal{B}$ and $\mathcal{M}$ be a SFB and a SFM of T, respectively. By Lemma 2.5 and Theorem 3.4, we have that $\mathcal{B} \circ \mathcal{M} \circ \mathcal{B} \circ \mathcal{M} \circ \mathcal{B} \subseteq \mathcal{B} \cap \mathcal{M}$. Let $a \in T$. Then, there exists $x \in T$ such that $a = axa = a(xax)a(xax)a$. Thus, we have

$$\begin{aligned}
&(\mu_{\mathcal{B}} \circ \mu_{\mathcal{M}} \circ \mu_{\mathcal{B}} \circ \mu_{\mathcal{M}} \circ \mu_{\mathcal{B}})(a)\\
&= \sup_{a=bcd} \min\{(\mu_{\mathcal{B}} \circ \mu_{\mathcal{M}} \circ \mu_{\mathcal{B}})(b), \mu_{\mathcal{M}}(c), \mu_{\mathcal{B}}(d)\}\\
&\geq \min\{(\mu_{\mathcal{B}} \circ \mu_{\mathcal{M}} \circ \mu_{\mathcal{B}})(a(xax)a), \mu_{\mathcal{M}}(xax), \mu_{\mathcal{B}}(a)\}\\
&= \min\{\sup_{a(xax)a=kpq} \min\{\mu_{\mathcal{B}}(k), \mu_{\mathcal{M}}(p), \mu_{\mathcal{B}}(q)\}, \mu_{\mathcal{M}}(xax), \mu_{\mathcal{B}}(a)\}\\
&\geq \min\{\min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{M}}(xax), \mu_{\mathcal{B}}(a)\}, \mu_{\mathcal{M}}(xax), \mu_{\mathcal{B}}(a)\}\\
&\geq \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{M}}(a), \mu_{\mathcal{B}}(a), \mu_{\mathcal{M}}(a), \mu_{\mathcal{B}}(a)\}\\
&= \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{M}}(a)\}\\
&= (\mu_{\mathcal{B}} \cap \mu_{\mathcal{M}})(a),
\end{aligned}$$

$$\begin{aligned}
&(\eta_{\mathcal{B}} \circ \eta_{\mathcal{M}} \circ \eta_{\mathcal{B}} \circ \eta_{\mathcal{M}} \circ \eta_{\mathcal{B}})(a)\\
&= \sup_{a=bcd} \min\{(\eta_{\mathcal{B}} \circ \eta_{\mathcal{M}} \circ \eta_{\mathcal{B}})(b), \eta_{\mathcal{M}}(c), \eta_{\mathcal{B}}(d)\}\\
&\geq \min\{(\eta_{\mathcal{B}} \circ \eta_{\mathcal{M}} \circ \eta_{\mathcal{B}})(a(xax)a), \eta_{\mathcal{M}}(xax), \eta_{\mathcal{B}}(a)\}\\
&= \min\{\sup_{a(xax)a=kpq} \min\{\eta_{\mathcal{B}}(k), \eta_{\mathcal{M}}(p), \eta_{\mathcal{B}}(q)\}, \eta_{\mathcal{M}}(xax), \eta_{\mathcal{B}}(a)\}\\
&\geq \min\{\min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{M}}(xax), \eta_{\mathcal{B}}(a)\}, \eta_{\mathcal{M}}(xax), \eta_{\mathcal{B}}(a)\}\\
&\geq \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{M}}(a), \eta_{\mathcal{B}}(a), \eta_{\mathcal{M}}(a), \eta_{\mathcal{B}}(a)\}\\
&= \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{M}}(a)\}\\
&= (\eta_{\mathcal{B}} \cap \eta_{\mathcal{M}})(a),
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_{\mathcal{B}} \circ \nu_{\mathcal{M}} \circ \nu_{\mathcal{B}} \circ \nu_{\mathcal{M}} \circ \nu_{\mathcal{B}})(a)\\
&= \inf_{a=bcd} \max\{(\nu_{\mathcal{B}} \circ \nu_{\mathcal{M}} \circ \nu_{\mathcal{B}})(b), \nu_{\mathcal{M}}(c), \nu_{\mathcal{B}}(d)\}\\
&\leq \max\{(\nu_{\mathcal{B}} \circ \nu_{\mathcal{M}} \circ \nu_{\mathcal{B}})(a(xax)a), \nu_{\mathcal{M}}(xax), \nu_{\mathcal{B}}(a)\}\\
&= \max\{\inf_{a(xax)a=kpq} \max\{\nu_{\mathcal{B}}(k), \nu_{\mathcal{M}}(p), \nu_{\mathcal{B}}(q)\}, \nu_{\mathcal{M}}(xax), \nu_{\mathcal{B}}(a)\}\\
&\leq \min\{\max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{M}}(xax), \nu_{\mathcal{B}}(a)\}, \nu_{\mathcal{M}}(xax), \nu_{\mathcal{B}}(a)\}\\
&\leq \min\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{M}}(a), \nu_{\mathcal{B}}(a), \nu_{\mathcal{M}}(a), \nu_{\mathcal{B}}(a)\}\\
&= \max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{M}}(a)\}\\
&= (\nu_{\mathcal{B}} \cup \nu_{\mathcal{M}})(a).
\end{aligned}$$

Hence, $\mathcal{B} \cap \mathcal{M} \subseteq \mathcal{B} \circ \mathcal{M} \circ \mathcal{B} \circ \mathcal{M} \circ \mathcal{B}$. It turns out that $\mathcal{B} \cap \mathcal{M} = \mathcal{B} \circ \mathcal{M} \circ \mathcal{B} \circ \mathcal{M} \circ \mathcal{B}$.

(ii) ⇒ (iii) Since every SFQ of T is also a SFB of T, it follows that (iii) holds.

(iii) ⇒ (i) Let $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{R}$ be a SFL, a SFM and a SFR of T, respectively. It is not difficult to verify that $\mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$ is a SFQ of T. By the hypothesis, we have that

$$\begin{aligned}
&\mathcal{R} \cap \mathcal{M} \cap \mathcal{L}\\
&= \mathcal{M} \cap (\mathcal{R} \cap \mathcal{M} \cap \mathcal{L})\\
&= (\mathcal{R} \cap \mathcal{M} \cap \mathcal{L}) \circ \mathcal{M} \circ (\mathcal{R} \cap \mathcal{M} \cap \mathcal{L}) \circ
\end{aligned}$$

$$\begin{aligned}&\mathcal{M} \circ (\mathcal{R} \cap \mathcal{M} \cap \mathcal{L})\\ &\subseteq \mathcal{R} \circ (\mathcal{T} \circ \mathcal{M} \circ \mathcal{T}) \circ \mathcal{L}\\ &\subseteq \mathcal{R} \circ \mathcal{M} \circ \mathcal{L}.\end{aligned}$$

On the other hand, $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$ always. Thus, $\mathcal{R} \circ \mathcal{M} \circ \mathcal{L} = \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$. By Theorem 4.3, T is regular. □

Lemma 4.8 ([29]). *Let T be a ternary semigroup. Then, T is regular if and only if $R \cap L = RTL$, for every right ideal R and every left ideal L of T.*

Theorem 4.9. *Let T be a ternary semigroup. Then, T is regular if and only if $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{T} \circ \mathcal{L}$, for each SFL $\mathcal{L}$ and each SFR $\mathcal{R}$ of T.*

Proof. Assume that T is regular. Let $\mathcal{L}$ and $\mathcal{R}$ be a SFL and a SFR of T, respectively. We note that $\mathcal{T}$ is a SFM of T. Thus, by Theorem 4.3, it follows that $\mathcal{R} \circ \mathcal{T} \circ \mathcal{L} = \mathcal{R} \cap \mathcal{T} \cap \mathcal{L} = \mathcal{R} \cap \mathcal{L}$. Conversely, let L and R be any left ideal and right ideal of T, respectively. By Theorem 3.5, C_L is a SFL and C_R is a SFR of T. Thus, by assumption, it follows that $C_R \cap C_L = C_R \circ C_T \circ C_L$. Also, $R \cap L = RTL$ by Lemma 2.3. By Lemma 4.8, T is regular.

□

Theorem 4.10. *Let T be a ternary semigroup. Then the following statements are equivalent:*

(i) T is regular;

(ii) $\mathcal{B} \cap \mathcal{L} \subseteq \mathcal{B} \circ \mathcal{T} \circ \mathcal{L}$, for every SFB $\mathcal{B}$ and every SFL $\mathcal{L}$ of T;

(iii) $\mathcal{Q} \cap \mathcal{L} \subseteq \mathcal{Q} \circ \mathcal{T} \circ \mathcal{L}$, for every SFQ $\mathcal{Q}$ and every SFL $\mathcal{L}$ of T;

(iv) $\mathcal{B} \cap \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{T} \circ \mathcal{B}$, for every SFB $\mathcal{B}$ and every SFR $\mathcal{R}$ of T;

(v) $\mathcal{Q} \cap \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{T} \circ \mathcal{Q}$, for every SFQ $\mathcal{Q}$ and every SFR $\mathcal{R}$ of T.

Proof. (i) ⇒ (ii) Let $\mathcal{L}$ and $\mathcal{B}$ be a SFL and a SFB of T, respectively. For any $a \in T$, there exists $x \in T$ such that $a = axa$. Thus, we have

$$\begin{aligned}&(\mu_{\mathcal{B}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{L}})(a)\\ &= \sup_{a=bcd} \min\{\mu_{\mathcal{B}}(b), \mu_{\mathcal{T}}(c), \mu_{\mathcal{L}}(d)\}\\ &\geq \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{T}}(x), \mu_{\mathcal{L}}(a)\}\\ &= \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{L}}(a)\}\\ &= (\mu_{\mathcal{B}} \cap \mu_{\mathcal{L}})(a),\end{aligned}$$

$$\begin{aligned}&(\eta_{\mathcal{B}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{L}})(a)\\ &= \sup_{a=bcd} \min\{\eta_{\mathcal{B}}(b), \eta_{\mathcal{T}}(c), \eta_{\mathcal{L}}(d)\}\\ &\geq \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{T}}(x), \eta_{\mathcal{L}}(a)\}\\ &= \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{L}}(a)\}\\ &= (\eta_{\mathcal{B}} \cap \eta_{\mathcal{L}})(a),\end{aligned}$$

and

$$\begin{aligned}&(\nu_{\mathcal{B}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{L}})(a)\\ &= \inf_{a=bcd} \max\{\nu_{\mathcal{B}}(b), \nu_{\mathcal{T}}(c), \nu_{\mathcal{L}}(d)\}\\ &\leq \max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{T}}(x), \nu_{\mathcal{L}}(a)\}\\ &= \max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{L}}(a)\}\\ &= (\nu_{\mathcal{B}} \cup \nu_{\mathcal{L}})(a).\end{aligned}$$

It follows that $\mathcal{B} \cap \mathcal{L} \subseteq \mathcal{B} \circ \mathcal{T} \circ \mathcal{L}$.

(ii) ⇒ (iii) Since every SFQ of T is also a SFB of T, we obtain that (iii) holds.

(iii) ⇒ (i) Let $\mathcal{L}$ and $\mathcal{R}$ be a SFL and a SFR of T, respectively. Since every SFR of T is a SFB of T, we obtain that $\mathcal{R}$ is a SFQ of T. By the given assumption, $\mathcal{R} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{T} \circ \mathcal{L}$. On the other hand, $\mathcal{R} \circ \mathcal{T} \circ \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}$ always. Hence, $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{T} \circ \mathcal{L}$. By Theorem 4.9, we conclude that T is regular.

Similarly, we can show that (i) ⇒ (iv) ⇒ (v) ⇒ (i) holds. □

Theorem 4.11. *Let T be a ternary semigroup. Then the following conditions are equivalent:*

(i) T is regular;

(ii) $\mathcal{B}_1 \cap \mathcal{B}_2 \subseteq (\mathcal{B}_1 \circ \mathcal{T} \circ \mathcal{B}_2) \cap (\mathcal{B}_2 \circ \mathcal{T} \circ \mathcal{B}_1)$, for any two SFBs $\mathcal{B}_1$ and $\mathcal{B}_2$ of T;

(iii) $\mathcal{Q}_1 \cap \mathcal{Q}_2 \subseteq (\mathcal{Q}_1 \circ \mathcal{T} \circ \mathcal{Q}_2) \cap (\mathcal{Q}_2 \circ \mathcal{T} \circ \mathcal{Q}_1)$, for any two SFQs $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of T;

(iv) $\mathcal{B} \cap \mathcal{L} \subseteq (\mathcal{B} \circ \mathcal{T} \circ \mathcal{L}) \cap (\mathcal{L} \circ \mathcal{T} \circ \mathcal{B})$, for every SFB $\mathcal{B}$ and every SFL $\mathcal{L}$ of T;

(v) $\mathcal{Q} \cap \mathcal{L} \subseteq (\mathcal{Q} \circ \mathcal{T} \circ \mathcal{L}) \cap (\mathcal{L} \circ \mathcal{T} \circ \mathcal{Q})$, for every SFQ $\mathcal{Q}$ and every SFL $\mathcal{L}$ of T;

(vi) $\mathcal{B} \cap \mathcal{R} \subseteq (\mathcal{R} \circ \mathcal{T} \circ \mathcal{B}) \cap (\mathcal{B} \circ \mathcal{T} \circ \mathcal{R})$, for every SFB $\mathcal{B}$ and every SFR $\mathcal{R}$ of T;

(vii) $\mathcal{Q} \cap \mathcal{R} \subseteq (\mathcal{R} \circ \mathcal{T} \circ \mathcal{Q}) \cap (\mathcal{Q} \circ \mathcal{T} \circ \mathcal{R})$, for every SFQ $\mathcal{Q}$ and every SFR $\mathcal{R}$ of T;

(viii) $\mathcal{R} \cap \mathcal{L} \subseteq (\mathcal{R} \circ \mathcal{T} \circ \mathcal{L}) \cap (\mathcal{L} \circ \mathcal{T} \circ \mathcal{R})$, for every SFL $\mathcal{L}$ and every SFR $\mathcal{R}$ of T.

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be SFBs of T, let $a \in T$. Then, there exists $x \in T$ such that $a = axa$. It turns out that

$$\begin{aligned}
&((\mu_{\mathcal{B}_1} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}_2}) \cap (\mu_{\mathcal{B}_2} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}_1}))(a)\\
&= \min\{(\mu_{\mathcal{B}_1} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}_2})(a),\\
&\quad (\mu_{\mathcal{B}_2} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{B}_1})(a)\}\\
&= \min\{\sup_{a=bcd} \min\{\mu_{\mathcal{B}_1}(b), \mu_{\mathcal{T}}(c), \mu_{\mathcal{B}_2}(d)\},\\
&\quad \sup_{a=kpq} \min\{\mu_{\mathcal{B}_2}(k), \mu_{\mathcal{T}}(p), \mu_{\mathcal{B}_1}(q)\}\}\\
&\geq \min\{\min\{\mu_{\mathcal{B}_1}(a), \mu_{\mathcal{T}}(x), \mu_{\mathcal{B}_2}(a)\},\\
&\quad \min\{\mu_{\mathcal{B}_2}(a), \mu_{\mathcal{T}}(x), \mu_{\mathcal{B}_1}(a)\}\}\\
&= \min\{\mu_{\mathcal{B}_1}(a), \mu_{\mathcal{B}_2}(a)\}\\
&= (\mu_{\mathcal{B}_1} \cap \mu_{\mathcal{B}_2})(a),
\end{aligned}$$

$$\begin{aligned}
&((\eta_{\mathcal{B}_1} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}_2}) \cap (\eta_{\mathcal{B}_2} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}_1}))(a)\\
&= \min\{(\eta_{\mathcal{B}_1} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}_2})(a),\\
&\quad (\eta_{\mathcal{B}_2} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{B}_1})(a)\}\\
&= \min\{\sup_{a=bcd} \min\{\eta_{\mathcal{B}_1}(b), \eta_{\mathcal{T}}(c), \eta_{\mathcal{B}_2}(d)\},\\
&\quad \sup_{a=kpq} \min\{\eta_{\mathcal{B}_2}(k), \eta_{\mathcal{T}}(p), \eta_{\mathcal{B}_1}(q)\}\}\\
&\geq \min\{\min\{\eta_{\mathcal{B}_1}(a), \eta_{\mathcal{T}}(x), \eta_{\mathcal{B}_2}(a)\},\\
&\quad \min\{\eta_{\mathcal{B}_2}(a), \eta_{\mathcal{T}}(x), \eta_{\mathcal{B}_1}(a)\}\}\\
&= \min\{\eta_{\mathcal{B}_1}(a), \eta_{\mathcal{B}_2}(a)\}\\
&= (\eta_{\mathcal{B}_1} \cap \eta_{\mathcal{B}_2})(a),
\end{aligned}$$

and

$$\begin{aligned}
&((\nu_{\mathcal{B}_1} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}_2}) \cap (\nu_{\mathcal{B}_2} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}_1}))(a)\\
&= \max\{(\nu_{\mathcal{B}_1} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}_2})(a),\\
&\quad (\nu_{\mathcal{B}_2} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{B}_1})(a)\}\\
&= \max\{\inf_{a=bcd} \max\{\nu_{\mathcal{B}_1}(b), \nu_{\mathcal{T}}(c), \nu_{\mathcal{B}_2}(d)\},\\
&\quad \inf_{a=kpq} \max\{\nu_{\mathcal{B}_2}(k), \nu_{\mathcal{T}}(p), \nu_{\mathcal{B}_1}(q)\}\}\\
&\leq \max\{\max\{\nu_{\mathcal{B}_1}(a), \nu_{\mathcal{T}}(x), \nu_{\mathcal{B}_2}(a)\},\\
&\quad \max\{\nu_{\mathcal{B}_2}(a), \nu_{\mathcal{T}}(x), \nu_{\mathcal{B}_1}(a)\}\}\\
&= \max\{\nu_{\mathcal{B}_1}(a), \nu_{\mathcal{B}_2}(a)\}\\
&= (\nu_{\mathcal{B}_1} \cup \nu_{\mathcal{B}_2})(a).
\end{aligned}$$

This implies that $\mathcal{B}_1 \cap \mathcal{B}_2 \subseteq (\mathcal{B}_1 \circ \mathcal{T} \circ \mathcal{B}_2) \cap (\mathcal{B}_2 \circ \mathcal{T} \circ \mathcal{B}_1)$.

(ii) $\Rightarrow$ (iii) Since every SFQ of T is a SFB of T, it follows that (iii) holds.

(iii) $\Rightarrow$ (i) Let $\mathcal{L}$ and $\mathcal{R}$ be a SFL and a SFR of T, respectively. It is not difficult to verify that $\mathcal{R} \cap \mathcal{L}$ is a SFQ of T. Then, by assumption, we have that

$$\begin{aligned}
&\mathcal{R} \cap \mathcal{L}\\
&= (\mathcal{R} \cap \mathcal{L}) \cap (\mathcal{R} \cap \mathcal{L})\\
&\subseteq ((\mathcal{R} \cap \mathcal{L}) \circ \mathcal{T} \circ (\mathcal{R} \cap \mathcal{L}))\\
&\quad \cap ((\mathcal{R} \cap \mathcal{L}) \circ \mathcal{T} \circ (\mathcal{R} \cap \mathcal{L}))
\end{aligned}$$

$$= (\mathcal{R} \cap \mathcal{L}) \circ \mathcal{T} \circ (\mathcal{R} \cap \mathcal{L})$$
$$\subseteq \mathcal{R} \circ \mathcal{T} \circ \mathcal{L}.$$

Otherwise, $\mathcal{R} \circ \mathcal{T} \circ \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}$. This shows that $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{T} \circ \mathcal{L}$. By Theorem 4.9, T is regular.

Similarly, we can prove that (i) ⇒ (iv) ⇒ (v) ⇒ (i), (i) ⇒ (vi) ⇒ (vii) ⇒ (i) and (i) ⇔ (viii) hold. □

Theorem 4.12. *Let T be a ternary semigroup. Then the following statements are equivalent:*

(i) T is regular;

(ii) $\mathcal{B} \cap \mathcal{I} \cap \mathcal{L} \subseteq \mathcal{B} \circ \mathcal{I} \circ \mathcal{L}$, for every SFB $\mathcal{B}$, every SFI $\mathcal{I}$ and every SFL $\mathcal{L}$ of T;

(iii) $\mathcal{Q} \cap \mathcal{I} \cap \mathcal{L} \subseteq \mathcal{Q} \circ \mathcal{I} \circ \mathcal{L}$, for every SFQ $\mathcal{Q}$, every SFI $\mathcal{I}$ and every SFL $\mathcal{L}$ of T;

(iv) $\mathcal{R} \cap \mathcal{I} \cap \mathcal{B} \subseteq \mathcal{R} \circ \mathcal{I} \circ \mathcal{B}$, for every SFB $\mathcal{B}$, every SFI $\mathcal{I}$ and every SFR $\mathcal{R}$ of T;

(v) $\mathcal{R} \cap \mathcal{I} \cap \mathcal{Q} \subseteq \mathcal{R} \circ \mathcal{I} \circ \mathcal{Q}$, for every SFQ $\mathcal{Q}$, every SFI $\mathcal{I}$ and every SFR $\mathcal{R}$ of T;

(vi) $\mathcal{B} \cap \mathcal{M} \cap \mathcal{L} \subseteq \mathcal{B} \circ \mathcal{M} \circ \mathcal{L}$, for every SFB $\mathcal{B}$, every SFM $\mathcal{M}$ and every SFL $\mathcal{L}$ of T;

(vii) $\mathcal{Q} \cap \mathcal{M} \cap \mathcal{L} \subseteq \mathcal{Q} \circ \mathcal{M} \circ \mathcal{L}$, for every SFQ $\mathcal{Q}$, every SFM $\mathcal{M}$ and every SFL $\mathcal{L}$ of T;

(viii) $\mathcal{R} \cap \mathcal{M} \cap \mathcal{B} \subseteq \mathcal{R} \circ \mathcal{M} \circ \mathcal{B}$, for every SFB $\mathcal{B}$, every SFM $\mathcal{M}$ and every SFR $\mathcal{R}$ of T.

(ix) $\mathcal{R} \cap \mathcal{M} \cap \mathcal{Q} \subseteq \mathcal{R} \circ \mathcal{M} \circ \mathcal{Q}$, for every SFQ $\mathcal{Q}$, every SFM $\mathcal{M}$ and every SFR $\mathcal{R}$ of T.

Proof. (i) ⇒ (ii) Let $\mathcal{L}$, $\mathcal{I}$ and $\mathcal{B}$ be a SFL, a SFI and a SFB of T, respectively, and let $a \in T$. So, $a = axa = a(xax)a$ for some $x \in T$. Thus, we have

$$\begin{aligned}
&(\mu_{\mathcal{B}} \circ \mu_{\mathcal{I}} \circ \mu_{\mathcal{L}})(a) \\
&= \sup_{a=bcd} \min\{\mu_{\mathcal{B}}(b), \mu_{\mathcal{I}}(c), \mu_{\mathcal{L}}(d)\} \\
&\geq \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{I}}(xax), \mu_{\mathcal{L}}(a)\} \\
&\geq \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{I}}(a), \mu_{\mathcal{L}}(a)\} \\
&= (\mu_{\mathcal{B}} \cap \mu_{\mathcal{I}} \cap \mu_{\mathcal{L}})(a),
\end{aligned}$$

$$\begin{aligned}
&(\eta_{\mathcal{B}} \circ \eta_{\mathcal{I}} \circ \eta_{\mathcal{L}})(a) \\
&= \sup_{a=bcd} \min\{\eta_{\mathcal{B}}(b), \eta_{\mathcal{I}}(c), \eta_{\mathcal{L}}(d)\} \\
&\geq \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{I}}(xax), \eta_{\mathcal{L}}(a)\} \\
&\geq \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{I}}(a), \eta_{\mathcal{L}}(a)\} \\
&= (\eta_{\mathcal{B}} \cap \eta_{\mathcal{I}} \cap \eta_{\mathcal{L}})(a)
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_{\mathcal{B}} \circ \nu_{\mathcal{I}} \circ \nu_{\mathcal{L}})(a) \\
&= \inf_{a=bcd} \max\{\nu_{\mathcal{B}}(b), \nu_{\mathcal{I}}(c), \nu_{\mathcal{L}}(d)\} \\
&\leq \max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{I}}(xax), \nu_{\mathcal{L}}(a)\} \\
&\leq \max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{I}}(a), \nu_{\mathcal{L}}(a)\} \\
&= (\nu_{\mathcal{B}} \cup \nu_{\mathcal{I}} \cup \nu_{\mathcal{L}})(a).
\end{aligned}$$

This implies that $\mathcal{B} \cap \mathcal{I} \cap \mathcal{L} \subseteq \mathcal{B} \circ \mathcal{I} \circ \mathcal{L}$.

(ii) ⇒ (iii) Since for any SFQ of T is also a SFB of T, this implies that (iii) holds.

(iii) ⇒ (i) Let $\mathcal{L}$ and $\mathcal{R}$ be a SFl and a SFR of T, respectively. Also, $\mathcal{R}$ is a SFQ of T. Moreover, the SFS $\mathcal{T}$ on T is a SFI of T. By the given assumption, we have that $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \cap \mathcal{T} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{T} \circ \mathcal{L}$. On the other hand, $\mathcal{R} \circ \mathcal{T} \circ \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}$. Hence, $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{T} \circ \mathcal{L}$. By Theorem 4.9, T is regular.

Similarly, we can prove that (i) ⇒ (iv) ⇒ (v) ⇒ (i), (i) ⇒ (vi) ⇒ (vii) ⇒ (i) and (i) ⇒ (viii) ⇒ (ix) ⇒ (i) hold. □

5. Weakly Regular Ternary Semigroups

In this section, we characterize (resp., right, left) weakly regular ternary semigroups using the concepts of various kinds of spherical fuzzy ideals in ternary semigroups. Finally, we give some characterization of both regular and weakly regular ternary semigroups in terms of their spherical fuzzy ideals.

Definition 5.1 (cf. [29])**.** A ternary semigroup T is called *right* (resp., *left*) *weakly regular*, if for each $a \in T$, $a \in (aTT)^3$ (resp., $a \in (TTa)^3$). If T is both right and left weakly regular, then it is called *weakly regular*.

It is known that every regular ternary semigroup is right (resp., left) weakly, but the converse is not always true in general.

Lemma 5.2 ([29])**.** *Let T be a ternary semigroup. Then, T is right weakly regular if and only if $R \cap W = RWW$, for every right ideal R and every two-sided ideal W of T.*

Theorem 5.3. *Let T be a ternary semigroup. Then, T is right weakly regular if and only if $\mathcal{R} \cap \mathcal{W} = \mathcal{R} \circ \mathcal{W} \circ \mathcal{W}$, for every SFR $\mathcal{R}$ and every SFT $\mathcal{W}$ of T.*

Proof. Assume that T is right weakly regular. Let $\mathcal{R}$ and $\mathcal{W}$ be a SFR and a SFT of T, respectively. Let $a \in T$. Then, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3)$. Thus ,we have

$$\begin{aligned}
&(\mu_{\mathcal{R}} \circ \mu_{\mathcal{W}} \circ \mu_{\mathcal{W}})(a)\\
&= \sup_{a=bcd} \min\{\mu_{\mathcal{R}}(b), \mu_{\mathcal{W}}(c), \mu_{\mathcal{W}}(d)\}\\
&\geq \min\{\mu_{\mathcal{R}}(as_1t_1), \mu_{\mathcal{W}}(as_2t_2), \mu_{\mathcal{W}}(as_3t_3)\}\\
&\geq \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{W}}(a), \mu_{\mathcal{W}}(a)\}\\
&= \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{W}}(a)\}\\
&= (\mu_{\mathcal{R}} \cap \mu_{\mathcal{W}})(a),
\end{aligned}$$

$$\begin{aligned}
&(\eta_{\mathcal{R}} \circ \eta_{\mathcal{W}} \circ \eta_{\mathcal{W}})(a)\\
&= \sup_{a=bcd} \min\{\eta_{\mathcal{R}}(b), \eta_{\mathcal{W}}(c), \eta_{\mathcal{W}}(d)\}\\
&\geq \min\{\eta_{\mathcal{R}}(as_1t_1), \eta_{\mathcal{W}}(as_2t_2), \eta_{\mathcal{W}}(as_3t_3)\}\\
&\geq \min\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{W}}(a), \eta_{\mathcal{W}}(a)\}\\
&= \min\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{W}}(a)\}\\
&= (\eta_{\mathcal{R}} \cap \eta_{\mathcal{W}})(a)
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_{\mathcal{R}} \circ \nu_{\mathcal{W}} \circ \nu_{\mathcal{W}})(a)\\
&= \inf_{a=bcd} \max\{\nu_{\mathcal{R}}(b), \nu_{\mathcal{W}}(c), \nu_{\mathcal{W}}(d)\}\\
&\leq \max\{\nu_{\mathcal{R}}(as_1t_1), \nu_{\mathcal{W}}(as_2t_2), \nu_{\mathcal{W}}(as_3t_3)\}\\
&\leq \max\{\nu_{\mathcal{R}}(a), \nu_{\mathcal{W}}(a), \nu_{\mathcal{W}}(a)\}\\
&= \max\{\nu_{\mathcal{R}}(a), \nu_{\mathcal{W}}(a)\}\\
&= (\nu_{\mathcal{R}} \cup \nu_{\mathcal{W}})(a).
\end{aligned}$$

It follows that $\mathcal{R} \cap \mathcal{W} \subseteq \mathcal{R} \circ \mathcal{W} \circ \mathcal{W}$. On the other hand, $\mathcal{R} \circ \mathcal{W} \circ \mathcal{W} \subseteq \mathcal{R} \cap \mathcal{W}$. Therefore, $\mathcal{R} \cap \mathcal{W} = \mathcal{R} \circ \mathcal{W} \circ \mathcal{W}$.

Conversely, let R be a right ideal and W be a two-sided ideal of T. Then, by Theorem 3.5, we get that C_R is a SFR and C_W is a SFT of T. By the hypothesis, $C_R \cap C_W = C_R \circ C_W \circ C_W$. Thus, using Lemma 2.3, it follows that $R \cap W = RWW$. Consequently, T is right weakly regular by Lemma 5.2. □

Theorem 5.4. *Let T be a ternary semigroup. Then the following conditions are equivalent:*

(i) T is right weakly regular;

(ii) $\mathcal{B} \cap \mathcal{W} \cap \mathcal{R} \subseteq \mathcal{B} \circ \mathcal{W} \circ \mathcal{R}$, for every SFB $\mathcal{B}$, every SFT $\mathcal{W}$ and every SFR $\mathcal{R}$ of T;

(iii) $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R} \subseteq \mathcal{Q} \circ \mathcal{W} \circ \mathcal{R}$, for every SFQ $\mathcal{Q}$, every SFT $\mathcal{W}$ and every SFR $\mathcal{R}$ of T.

Proof. (i) ⇒ (ii) Let $\mathcal{B}, \mathcal{W}$ and $\mathcal{R}$ be a SFB, a SFT and a SFR of T, respectively. Let $a \in$

T. Then, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in T$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$. So, consider

$$\begin{aligned}
&(\mu_{\mathcal{B}} \circ \mu_{\mathcal{W}} \circ \mu_{\mathcal{R}})(a)\\
&= \sup_{a=bcd} \min\{\mu_{\mathcal{B}}(b), \mu_{\mathcal{W}}(c), \mu_{\mathcal{R}}(d)\}\\
&\geq \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{W}}(s_1t_1as_2t_2), \mu_{\mathcal{R}}(as_3t_3)\}\\
&\geq \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{W}}(s_1t_1a), \mu_{\mathcal{R}}(a)\}\\
&\geq \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{W}}(a), \mu_{\mathcal{R}}(a)\}\\
&= (\mu_{\mathcal{B}} \cap \mu_{\mathcal{W}} \cap \mu_{\mathcal{R}})(a),
\end{aligned}$$

$$\begin{aligned}
&(\eta_{\mathcal{B}} \circ \eta_{\mathcal{W}} \circ \eta_{\mathcal{R}})(a)\\
&= \sup_{a=bcd} \min\{\eta_{\mathcal{B}}(b), \eta_{\mathcal{W}}(c), \eta_{\mathcal{R}}(d)\}\\
&\geq \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{W}}(s_1t_1as_2t_2), \eta_{\mathcal{R}}(as_3t_3)\}\\
&\geq \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{W}}(s_1t_1a), \eta_{\mathcal{R}}(a)\}\\
&\geq \min\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{W}}(a), \eta_{\mathcal{R}}(a)\}\\
&= (\eta_{\mathcal{B}} \cap \eta_{\mathcal{W}} \cap \eta_{\mathcal{R}})(a)
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_{\mathcal{B}} \circ \nu_{\mathcal{W}} \circ \nu_{\mathcal{R}})(a)\\
&= \inf_{a=bcd} \max\{\nu_{\mathcal{B}}(b), \nu_{\mathcal{W}}(c), \nu_{\mathcal{R}}(d)\}\\
&\leq \max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{W}}(s_1t_1as_2t_2), \nu_{\mathcal{R}}(as_3t_3)\}\\
&\leq \max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{W}}(s_1t_1a), \nu_{\mathcal{R}}(a)\}\\
&\leq \max\{\nu_{\mathcal{B}}(a), \nu_{\mathcal{W}}(a), \nu_{\mathcal{R}}(a)\}\\
&= (\nu_{\mathcal{B}} \cup \nu_{\mathcal{W}} \cup \nu_{\mathcal{R}})(a).
\end{aligned}$$

This shows that $\mathcal{B} \cap \mathcal{W} \cap \mathcal{R} \subseteq \mathcal{B} \circ \mathcal{W} \circ \mathcal{R}$.

(ii) ⇒ (iii) Since every SFQ of T is a SFB of T, this implies that (iii) holds.

(iii) ⇒ (i) Let $\mathcal{R}$ and $\mathcal{W}$ be a SFR and a SFT of T, respectively. Then, $\mathcal{R}$ is also a SFQ of T by Proposition 2.9. Thus, using the given assumption, we have that $\mathcal{R} \cap \mathcal{W} = \mathcal{R} \cap \mathcal{W} \cap \mathcal{W} \subseteq \mathcal{R} \circ \mathcal{W} \circ \mathcal{W}$. Otherwise, $\mathcal{R} \circ \mathcal{W} \circ \mathcal{W} \subseteq \mathcal{R} \cap \mathcal{W}$ always. Hence, $\mathcal{R} \circ \mathcal{W} \circ \mathcal{W} = \mathcal{R} \cap \mathcal{W}$. By Theorem 5.3, T is right weakly regular. □

The following theorem can be proved similar to Theorem 5.4.

Theorem 5.5. *Let T be a ternary semigroup. Then the following statements are equivalent:*

(i) T is right weakly regular;

(ii) $\mathcal{B} \cap \mathcal{W} \subseteq \mathcal{B} \circ \mathcal{W} \circ \mathcal{W}$, for every SFB $\mathcal{B}$ and every SFT $\mathcal{W}$ of T;

(iii) $\mathcal{Q} \cap \mathcal{W} \subseteq \mathcal{Q} \circ \mathcal{W} \circ \mathcal{W}$, for every SFQ $\mathcal{Q}$ and every SFT $\mathcal{W}$ of T.

Lemma 5.6 ([30]). *Let T be a ternary semigroup. Then, T is right (resp., left) weakly regular if and only if $A_1 \cap A_2 \cap A_3 \subseteq A_1A_2A_3$, for every right (resp., left) ideals A_1, A_2, A_3 of T.*

Theorem 5.7. *Let T be a ternary semigroup. Then the following conditions are equivalent:*

(i) T is right weakly regular;

(ii) $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \subseteq \mathcal{R}_1 \circ \mathcal{R}_2 \circ \mathcal{R}_3$, for every SFRs $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ of T.

Proof. (i) ⇒ (ii) Assume that T is right weakly regular. Let $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ be SFRs of T. Let $a \in T$. Then, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in T$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3)$. So, we have

$$\begin{aligned}
&(\mu_{\mathcal{R}_1} \circ \mu_{\mathcal{R}_2} \circ \mu_{\mathcal{R}_3})(a)\\
&= \sup_{a=bcd} \min\{\mu_{\mathcal{R}_1}(b), \mu_{\mathcal{R}_2}(c), \mu_{\mathcal{R}_3}(d)\}\\
&\geq \min\{\mu_{\mathcal{R}_1}(as_1t_1), \mu_{\mathcal{R}_2}(as_2t_2), \mu_{\mathcal{R}_3}(as_3t_3)\}\\
&\geq \min\{\mu_{\mathcal{R}_1}(a), \mu_{\mathcal{R}_2}(a), \mu_{\mathcal{R}_3}(a)\}\\
&= (\mu_{\mathcal{R}_1} \cap \mu_{\mathcal{R}_2} \cap \mu_{\mathcal{R}_3})(a),
\end{aligned}$$

$$\begin{aligned}
&(\eta_{\mathcal{R}_1} \circ \eta_{\mathcal{R}_2} \circ \eta_{\mathcal{R}_3})(a)\\
&= \sup_{a=bcd} \min\{\eta_{\mathcal{R}_1}(b), \eta_{\mathcal{R}_2}(c), \eta_{\mathcal{R}_3}(d)\}\\
&\geq \min\{\eta_{\mathcal{R}_1}(as_1t_1), \eta_{\mathcal{R}_2}(as_2t_2), \eta_{\mathcal{R}_3}(as_3t_3)\}\\
&\geq \min\{\eta_{\mathcal{R}_1}(a), \eta_{\mathcal{R}_2}(a), \eta_{\mathcal{R}_3}(a)\}\\
&= (\eta_{\mathcal{R}_1} \cap \eta_{\mathcal{R}_2} \cap \eta_{\mathcal{R}_3})(a)
\end{aligned}$$

and

$$\begin{aligned}&(\nu_{\mathcal{R}_1} \circ \nu_{\mathcal{R}_2} \circ \nu_{\mathcal{R}_3})(a)\\ &= \inf_{a=bcd} \max\{\nu_{\mathcal{R}_1}(b), \nu_{\mathcal{R}_2}(c), \nu_{\mathcal{R}_3}(d)\}\\ &\leq \max\{\nu_{\mathcal{R}_1}(as_1t_1), \nu_{\mathcal{R}_2}(as_2t_2), \nu_{\mathcal{R}_3}(as_3t_3)\}\\ &\leq \max\{\nu_{\mathcal{R}_1}(a), \nu_{\mathcal{R}_2}(a), \nu_{\mathcal{R}_3}(a)\}\\ &= (\nu_{\mathcal{R}_1} \cup \nu_{\mathcal{R}_2} \cup \nu_{\mathcal{R}_3})(a).\end{aligned}$$

Hence, $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \subseteq \mathcal{R}_1 \circ \mathcal{R}_2 \circ \mathcal{R}_3$.

(ii) $\Rightarrow$ (i) Let R_1, R_2 and R_3 be right ideals of T. By Theorem 3.5, we have that C_{R_1}, C_{R_2} and C_{R_3} are SFRs of T. Then, using the assumption, it follows that $C_{R_1} \cap C_{R_2} \cap C_{R_3} \subseteq C_{R_1} \circ C_{R_2} \circ C_{R_3}$. By Lemma 2.3, this implies that $R_1 \cap R_2 \cap R_3 \subseteq R_1R_2R_3$. Therefore, T is right weakly regular, by Lemma 5.6. □

The following theorem holds if T is a left weakly regular ternary semigroup.

Theorem 5.8. *Let T be a ternary semigroup. Then the following conditions are equivalent:*

(i) T is left weakly regular;

(ii) $\mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \subseteq \mathcal{L}_1 \circ \mathcal{L}_2 \circ \mathcal{L}_3$, for every SFLs $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ of T.

Now, we present that a SFS $\mathcal{A}$ on a ternary semigroup T is called *idempotent*, if $\mathcal{A} = \mathcal{A} \circ \mathcal{A} \circ \mathcal{A}$.

Theorem 5.9. *Let T be a ternary semigroup. Then, T is right (resp., left) weakly regular if and only if every SFR (resp., SFL) of T is idempotent.*

Proof. Assume that T is right weakly regular. Let $\mathcal{R}$ be a SFR of T. By Lemma 2.5, $\mathcal{R} \circ \mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{T} \circ \mathcal{T} \subseteq \mathcal{R}$. Otherwise, by Theorem 5.7, implies that $\mathcal{R} = \mathcal{R} \cap \mathcal{R} \cap \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{R} \circ \mathcal{R}$. Hence, $\mathcal{R} = \mathcal{R} \circ \mathcal{R} \circ \mathcal{R}$. Conversely, let $a \in T$. Consider $A = a \cup aSS$. It is easy to show that A is a right ideal of T containing a. By Theorem 3.5, C_A is a SFR of T. Then, by the hypothesis and Lemma 2.3, we have that $C_A = C_A \circ C_A \circ C_A = C_{AAA}$. Also, $A = AAA$. We obtain that $a \in AAA$. Hence, we have

$$\begin{aligned}a &\in (a \cup aSS)(a \cup aSS)(a \cup aSS)\\ &\subseteq aaa \cup aaaSS \cup aaSSa \cup aaSSaSS\\ &\quad \cup aSSaa \cup aSSaaSS \cup aSSaSSa\\ &\quad \cup aSSaSSaSS.\end{aligned}$$

For each case, it follows that $a \in (aSS)^3$. Therefore, T is right weakly regular.

For other cases, we can prove similarly. □

Theorem 5.10. *Let T be a weakly regular ternary semigroup and $\mathcal{Q}$ be a SFS of T. Then, $\mathcal{Q}$ is a SFQ of T if and only if $\mathcal{Q} = (\mathcal{T} \circ \mathcal{T} \circ \mathcal{Q}) \cap (\mathcal{Q} \circ \mathcal{T} \circ \mathcal{T})$.*

Proof. The sufficiency of the proof is obvious. Now, let $\mathcal{Q}$ be a SFQ of T. By Lemma 2.4, we obtain that $\mathcal{Q} \circ \mathcal{T} \circ \mathcal{T}$ is a SFR of T. Then, using Theorem 5.9, we get that $\mathcal{Q} \circ \mathcal{T} \circ \mathcal{T} = (\mathcal{Q} \circ \mathcal{T} \circ \mathcal{T}) \circ \mathcal{Q} \circ (\mathcal{T} \circ \mathcal{T} \circ \mathcal{Q} \circ \mathcal{T} \circ \mathcal{T}) \subseteq \mathcal{T} \circ \mathcal{Q} \circ \mathcal{T}$. It turns out that

$$\begin{aligned}&(\mathcal{T} \circ \mathcal{T} \circ \mathcal{Q}) \cap (\mathcal{Q} \circ \mathcal{T} \circ \mathcal{T})\\ &\subseteq (\mathcal{T} \circ \mathcal{T} \circ \mathcal{Q}) \cap (\mathcal{T} \circ \mathcal{Q} \circ \mathcal{T}) \cap (\mathcal{Q} \circ \mathcal{T} \circ \mathcal{T})\\ &\subseteq (\mathcal{T} \circ \mathcal{T} \circ \mathcal{Q}) \cap (\mathcal{T} \circ \mathcal{Q} \circ \mathcal{T}\\ &\quad \cup \mathcal{T} \circ \mathcal{T} \circ \mathcal{Q} \circ \mathcal{T} \circ \mathcal{T}) \cap (\mathcal{Q} \circ \mathcal{T} \circ \mathcal{T})\\ &\subseteq \mathcal{Q}.\end{aligned}$$

On the other hand, let $a \in T$. Then, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in T$ such that $a = (s_1t_1a)(s_2t_2as_3t_3)a$. So, we have

$$\begin{aligned}&(\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{Q}})(a)\\ &= \sup_{a=bcd} \min\{\mu_{\mathcal{T}}(b), \mu_{\mathcal{T}}(c), \mu_{\mathcal{Q}}(d)\}\\ &\geq \min\{\mu_{\mathcal{T}}(s_1t_1a), \mu_{\mathcal{T}}(s_2t_2as_3t_3), \mu_{\mathcal{Q}}(a)\}\\ &= \mu_{\mathcal{Q}}(a),\end{aligned}$$

$$
\begin{aligned}
&(\eta_{\mathcal{T}} \circ \eta_{\mathcal{T}} \circ \eta_{Q})(a) \\
&= \sup_{a=bcd} \min\{\eta_{\mathcal{T}}(b), \eta_{\mathcal{T}}(c), \eta_{Q}(d)\} \\
&\geq \min\{\eta_{\mathcal{T}}(s_1t_1a), \eta_{\mathcal{T}}(s_2t_2as_3t_3), \eta_{Q}(a)\} \\
&= \eta_{Q}(a)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\nu_{\mathcal{T}} \circ \nu_{\mathcal{T}} \circ \nu_{Q})(a) \\
&= \inf_{a=bcd} \max\{\nu_{\mathcal{T}}(b), \nu_{\mathcal{T}}(c), \nu_{Q}(d)\} \\
&\leq \max\{\nu_{\mathcal{T}}(s_1t_1a), \nu_{\mathcal{T}}(s_2t_2as_3t_3), \nu_{Q}(a)\} \\
&= \nu_{Q}(a).
\end{aligned}
$$

This means that $Q \subseteq \mathcal{T} \circ \mathcal{T} \circ Q$. Similarly, we can show that $Q \subseteq Q \circ \mathcal{T} \circ \mathcal{T}$. Hence, $Q \subseteq (\mathcal{T} \circ \mathcal{T} \circ Q) \cap (Q \circ \mathcal{T} \circ \mathcal{T})$. Consequently, $Q = (\mathcal{T} \circ \mathcal{T} \circ Q) \cap (Q \circ \mathcal{T} \circ \mathcal{T})$. □

Theorem 5.11. *Let T be a ternary semigroup. Then, T is weakly regular if and only if $Q = (\mathcal{T} \circ \mathcal{T} \circ Q)^3 \cap (Q \circ \mathcal{T} \circ \mathcal{T})^3$, for any SFQ Q of T.*

Proof. Assume that T is weakly regular. Let Q be a SFQ of T. By Lemma 2.4 and Theorem 5.9, we have that the SFL $\mathcal{T} \circ \mathcal{T} \circ Q$ and the SFR $Q \circ \mathcal{T} \circ \mathcal{T}$ of T are idempotent. By Theorem 5.10, it follows that $Q = (\mathcal{T} \circ \mathcal{T} \circ Q) \cap (Q \circ \mathcal{T} \circ \mathcal{T}) = (\mathcal{T} \circ \mathcal{T} \circ Q)^3 \cap (Q \circ \mathcal{T} \circ \mathcal{T})^3$. Conversely, let $\mathcal{R}$ be a SFR of T. Also, $\mathcal{R}$ is a SFQ of T by Proposition 2.9. Then, using the given assumption, we have $\mathcal{R} = (\mathcal{T} \circ \mathcal{T} \circ \mathcal{R})^3 \cap (\mathcal{R} \circ \mathcal{T} \circ \mathcal{T})^3 \subseteq (\mathcal{R} \circ \mathcal{T} \circ \mathcal{T})^3 \subseteq \mathcal{R} \circ \mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$. This shows that $\mathcal{R} = \mathcal{R} \circ \mathcal{R} \circ \mathcal{R}$, that is, $\mathcal{R}$ is idempotent. Hence, T is right weakly regular. Similarly, we obtain that T is left weakly regular by Theorem 5.9. Therefore, T is weakly regular. □

Lemma 5.12 ([31]). *Let T be a ternary semigroup. Then the following statements are equivalent:*

(i) T is both regular and weakly regular;

(ii) $a \in aSSaSSa$, for any $a \in T$;

(iii) $R \cap W \cap L \subseteq RWL$, for every right ideal R, every two-sided ideal W and every left ideal L of T;

(iv) $R \cap L \subseteq RLTRL$, for every right ideal R and every left ideal L of T.

Theorem 5.13. *Let T be a ternary semigroup. Then the following conditions are equivalent:*

(i) T is both regular and weakly regular;

(ii) $\mathcal{R} \cap \mathcal{W} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{W} \circ \mathcal{L}$, for every SFR $\mathcal{R}$, every SFT $\mathcal{W}$ and every SFL $\mathcal{L}$ of T;

(iii) $\mathcal{R} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{L} \circ \mathcal{T} \circ \mathcal{R} \circ \mathcal{L}$, for every SFR $\mathcal{R}$ and every SFL $\mathcal{L}$ of T.

Proof. (i) ⇒ (ii) Let $\mathcal{R}$, $\mathcal{W}$ and $\mathcal{L}$ be a SFR, a SFT and a SFL of T, respectively. Let $a \in T$. By Lemma 5.12, $a = astaxya$ for some $s, t, x, y \in T$. Then, we have

$$
\begin{aligned}
&(\mu_{\mathcal{R}} \circ \mu_{\mathcal{W}} \circ \mu_{\mathcal{L}})(a) \\
&= \sup_{a=bcd} \min\{\mu_{\mathcal{R}}(b), \mu_{\mathcal{W}}(c), \mu_{\mathcal{L}}(d)\} \\
&\geq \min\{\mu_{\mathcal{R}}(ast), \mu_{\mathcal{W}}(axy), \mu_{\mathcal{L}}(a)\} \\
&\geq \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{W}}(a), \mu_{\mathcal{L}}(a)\} \\
&= (\mu_{\mathcal{R}} \cap \mu_{\mathcal{W}} \cap \mu_{\mathcal{L}})(a),
\end{aligned}
$$

$$
\begin{aligned}
&(\eta_{\mathcal{R}} \circ \eta_{\mathcal{W}} \circ \eta_{\mathcal{L}})(a) \\
&= \sup_{a=bcd} \min\{\eta_{\mathcal{R}}(b), \eta_{\mathcal{W}}(c), \eta_{\mathcal{L}}(d)\} \\
&\geq \min\{\eta_{\mathcal{R}}(ast), \eta_{\mathcal{W}}(axy), \eta_{\mathcal{L}}(a)\} \\
&\geq \min\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{W}}(a), \eta_{\mathcal{L}}(a)\} \\
&= (\eta_{\mathcal{R}} \cap \eta_{\mathcal{W}} \cap \eta_{\mathcal{L}})(a)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\nu_{\mathcal{R}} \circ \nu_{\mathcal{W}} \circ \nu_{\mathcal{L}})(a) \\
&= \inf_{a=bcd} \max\{\nu_{\mathcal{R}}(b), \nu_{\mathcal{W}}(c), \nu_{\mathcal{L}}(d)\}
\end{aligned}
$$

$$\leq \max\{\nu_{\mathcal{R}}(ast), \nu_{\mathcal{W}}(axy), \nu_{\mathcal{L}}(a)\}$$
$$\leq \max\{\nu_{\mathcal{R}}(a), \nu_{\mathcal{W}}(a), \nu_{\mathcal{L}}(a)\}$$
$$= (\nu_{\mathcal{R}} \cup \nu_{\mathcal{W}} \cup \nu_{\mathcal{L}})(a).$$

This shows that $\mathcal{R} \cap \mathcal{W} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{W} \circ \mathcal{L}$.

(ii) $\Rightarrow$ (iii) Let $\mathcal{R}$ be a SFR and $\mathcal{L}$ be a SFL of T. For any $a \in T$, $a = astaxya$ for some $s, t, x, y \in T$ by Lemma 5.12. So, $a = a(sta)(xyast)(axy)a$. Thus, we have

$$(\mu_{\mathcal{R}} \circ \mu_{\mathcal{L}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{R}} \circ \mu_{\mathcal{L}})(a)$$
$$= \sup_{a=bcd} \min\{(\mu_{\mathcal{R}} \circ \mu_{\mathcal{L}} \circ \mu_{\mathcal{T}})(b), \mu_{\mathcal{R}}(c), \mu_{\mathcal{L}}(d)\}$$
$$\geq \min\{(\mu_{\mathcal{R}} \circ \mu_{\mathcal{L}} \circ \mu_{\mathcal{T}})(astaxyast), \mu_{\mathcal{R}}(axy), \mu_{\mathcal{L}}(a)\}$$
$$= \min\{\sup_{astaxyast=mnk} \min\{\mu_{\mathcal{R}}(m), \mu_{\mathcal{L}}(n), \mu_{\mathcal{T}}(k)\}, \mu_{\mathcal{R}}(axy), \mu_{\mathcal{L}}(a)\}$$
$$\geq \min\{\min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(sta), \mu_{\mathcal{T}}(xyast)\}, \mu_{\mathcal{R}}(axy), \mu_{\mathcal{L}}(a)\}$$
$$\geq \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(a), \mu_{\mathcal{T}}(a), \mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(a)\}$$
$$= \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(a)\}$$
$$= (\mu_{\mathcal{R}} \cap \mu_{\mathcal{L}})(a),$$

$$(\eta_{\mathcal{R}} \circ \eta_{\mathcal{L}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{R}} \circ \eta_{\mathcal{L}})(a)$$
$$= \sup_{a=bcd} \min\{(\eta_{\mathcal{R}} \circ \eta_{\mathcal{L}} \circ \eta_{\mathcal{T}})(b), \eta_{\mathcal{R}}(c), \eta_{\mathcal{L}}(d)\}$$
$$\geq \min\{(\eta_{\mathcal{R}} \circ \eta_{\mathcal{L}} \circ \eta_{\mathcal{T}})(astaxyast), \eta_{\mathcal{R}}(axy), \eta_{\mathcal{L}}(a)\}$$
$$= \min\{\sup_{astaxyast=mnk} \min\{\eta_{\mathcal{R}}(m), \eta_{\mathcal{L}}(n), \eta_{\mathcal{T}}(k)\}, \eta_{\mathcal{R}}(axy), \eta_{\mathcal{L}}(a)\}$$
$$\geq \min\{\min\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{L}}(sta), \eta_{\mathcal{T}}(xyast)\}, \eta_{\mathcal{R}}(axy), \eta_{\mathcal{L}}(a)\}$$
$$\geq \min\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{L}}(a), \eta_{\mathcal{T}}(a), \eta_{\mathcal{R}}(a), \eta_{\mathcal{L}}(a)\}$$
$$= \min\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{L}}(a)\}$$
$$= (\eta_{\mathcal{R}} \cap \eta_{\mathcal{L}})(a),$$

and

$$(\nu_{\mathcal{R}} \circ \nu_{\mathcal{L}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{R}} \circ \nu_{\mathcal{L}})(a)$$
$$= \inf_{a=bcd} \max\{(\nu_{\mathcal{R}} \circ \nu_{\mathcal{L}} \circ \nu_{\mathcal{T}})(b), \nu_{\mathcal{R}}(c), \nu_{\mathcal{L}}(d)\}$$
$$\leq \max\{(\nu_{\mathcal{R}} \circ \nu_{\mathcal{L}} \circ \nu_{\mathcal{T}})(astaxyast), \nu_{\mathcal{R}}(axy), \nu_{\mathcal{L}}(a)\}$$
$$= \max\{\inf_{astaxyast=mnk} \max\{\nu_{\mathcal{R}}(m), \nu_{\mathcal{L}}(n), \nu_{\mathcal{T}}(k)\}, \nu_{\mathcal{R}}(axy), \nu_{\mathcal{L}}(a)\}$$
$$\leq \max\{\max\{\nu_{\mathcal{R}}(a), \nu_{\mathcal{L}}(sta), \nu_{\mathcal{T}}(xyast)\}, \nu_{\mathcal{R}}(axy), \nu_{\mathcal{L}}(a)\}$$
$$\leq \max\{\nu_{\mathcal{R}}(a), \nu_{\mathcal{L}}(a), \nu_{\mathcal{T}}(a), \nu_{\mathcal{R}}(a), \nu_{\mathcal{L}}(a)\}$$
$$= \max\{\nu_{\mathcal{R}}(a), \nu_{\mathcal{L}}(a)\}$$
$$= (\nu_{\mathcal{R}} \cup \nu_{\mathcal{L}})(a).$$

Hence, $\mathcal{R} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{L} \circ \mathcal{T} \circ \mathcal{R} \circ \mathcal{L}$.

The proofs of (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) are obtained by considering the spherical characteristic functions. □

6. Conclusion

In 2022, Krailoet at el. [24] introduced the notions of spherical fuzzy ternary subsemigroups and spherical fuzzy (resp., left, lateral, right) ideals of ternary semigroups. In this article, we have introduced two additional concepts: spherical fuzzy quasi-ideals and spherical fuzzy bi-ideals of ternary semigroups. Then, we considered the relationships between various kinds of ideals and their spherical fuzzy ideals in ternary semigroups.

In Section 4, we characterized regular ternary semigroups in terms of spherical fuzzy left (resp., lateral, right) ideals, spherical fuzzy quasi-ideals and spherical fuzzy bi-ideals of ternary semigroups which accrued in Theorem 4.3, Theorem

4.4 and Theorem 4.9. Subsequently, it was shown in Proposition 4.6 that spherical fuzzy quasi-ideals and spherical fuzzy bi-ideals coincide in a regular ternary semigroup.

In Section 5, the characterizations of weakly regular ternary semigroups by the concepts of many types of spherical fuzzy ideals of ternary semigroups are verified as shown in Theorem 5.3, Theorem 5.7 and Theorem 5.9. Finally, in Theorem 5.13, we gave some characterization of both regular and weakly regular ternary semigroups using their spherical fuzzy ideals of ternary semigroups.

In our future work, it will be possible to characterize many classes of regularities of ternary semigroups or other algebraic structures by the properties of spherical fuzzy sets.

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References

[1] Zadeh LZ. Fuzzy sets. Information and Control. 1965;8:338-53.

[2] Rosenfeld A. Fuzzy groups. Journal of Mathematical Analysis and Applications. 1971;35:512-7.

[3] Kuroki N. On fuzzy ideals and fuzzy bi-ideals in semigroups. Fuzzy Sets and Systems. 1981;5:203-15.

[4] Kuroki N. On fuzzy semigroups. Information Sciences. 1991;53:203-36.

[5] Kuroki N. Fuzzy generalized bi-ideals in semigroups. Information Sciences. 1992;66:235-43.

[6] Lehmer DH. A ternary analogue of abelian groups. American Journal of Mathematics. 1932;54: 329-38.

[7] Shabir M, Rehman N. Characterizations of ternary semigroups by their anti fuzzy ideals. Annals of Fuzzy Mathematics and Informatics. 2011;2:227-38.

[8] Kar S, Sarkar P. Fuzzy ideals of ternary semigroups. Fuzzy Information and Engineering. 2012;2:181-93.

[9] Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets and Systems. 1986;20:87-96.

[10] Akram M. Intuitionistic fuzzy points and ideals of ternary semigroups. International Journal of Algebra and Statistics. 2012;1:74-82.

[11] Lekkoksung S. Intuitionistic fuzzy bi-ideals of ternary semigroups. International Mathematical Forum. 2012;7:385-9.

[12] Lalithamani N, Prabakaran K, Remesh R. Intuitionistic fuzzy ideals and intuitionistic fuzzy filters of ternary semigroups. Advances in Mathematics: Scientific Journal. 2020;9:9535-40.

[13] Yager RR, Abbasov AM. Pythagorean membership grades, complex numbers, and decision making. International Journal of Intelligent Systems. 2013;28:436-52.

[14] Chinram R, Paniyakul T. Rough Pythagorean fuzzy ideals in ternary semigroups. Journal of Mathematics and Computer Science. 2020;20:302-12.

[15] Cuong BC, Kreinovich V. Picture fuzzy sets - a new concept for computational intelligence problems. Proceedings of the Third World Congress on Information and Communication Technologies WI-ICT. 2013;1-6.

[16] Nakkhasen W. Characterizing regular and intra-regular semigroups in terms of picture fuzzy bi-ideals. International Journal of Innovative Computing, Information and Control. 2021;17:2115-35.

[17] Yiarayong P. Semigroups characterized by picture fuzzy sets. International Journal of Innovative Computing, Information and Control. 2020;16:2121-30.

[18] Yiarayong P. Characterizations of semigroups by the properties of their picture fuzzy bi-ideals. Journal of Control and Decision. 2022;9:111-6.

[19] Nakkhasen W. On picture fuzzy (m, n)-ideals of semigroups. IAENG International Journal of Applied Mathematics. 2022;52:4.

[20] Ashraf S, Abdullah S, Mahmood T, Gahni F, Mahmood T. Spherical fuzzy sets and their applications in multi-attribute decision making problems. Journal of Intelligent & Fuzzy Systems. 2019;36:2829-44.

[21] Veerappan C, Venkatesan S. Spherical interval-valued fuzzy bi-ideals of gamma near-rings. Journal of Fuzzy Extension and Applications. 2020;1:314-24.

[22] Subha VS, Lavanya S, Aswini CB. Characterizations of semi-ring by rough spherical fuzzy ideals. Turkish Online Journal of Qualitative Inquiry. 2021;12:6103-16.

[23] Chinnadurai V, Bobin A, Arulselvam A. A study on spherical fuzzy ideals of semigroups. TWMS Journal of Applied and Engineering Mathematics. 2022;12:1202-12.

[24] Krailoet W, Chinram R, Petapirak M, Iampan A. Application of spherical fuzzy sets in ternary semigroups. International Journal of Analysis and Applications. 2022;20:29.

[25] Dutta TK, Kar S, Maity BK. On ideals in regular ternary semigroups. Discussiones Mathematicae General Algebra and Applications. 2008;28:147-59.

[26] Sioson FM. Ideal theory in ternary semigroups. Mathematica Japonica. 1965;10:63-84.

[27] Dixit VN, Dewan S. A note on quasi and bi-ideals in ternary semigroups. International Journal of Mathematics and Mathematical Sciences. 1995;18:501-8.

[28] Kar S, Sarkar P. Fuzzy quasi-ideals and fuzzy bi-ideals of ternary semigroups. Annals of Fuzzy Mathematics and Informatics. 2012;4:407-23.

[29] Shabir M, Bano M. Prime bi-ideals in ternary semigroups. Quasigroups and Related Systems. 2008;16:239-56.

[30] Lekkoksung N, Jampachon P. On right weakly regular ordered ternary semigroups. Quasigroups and Related Systems. 2014;22:257-66.

[31] Bashir S, Du X. Intra-regular and weakly regular ordered ternary semigroups. Annals of Fuzzy Mathematics and Informatics. 2017;13:539-51.