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Original research article

Finding the Domination Number of Amalgamations of Paths and Cycles at Connected Subgraphs

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ABSTRACT

Let $\gamma(G)$ denote the *domination number* of a graph G. Let G_1, G_2 be disjoint graphs with two subgraphs H_1, H_2, respectively such that there is a graph isomorphism f from H_1 to H_2. The *amalgamation of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f is the graph $G = G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ obtained by forming the disjoint union of G_1 and G_2 and then identifying H_1 and H_2 with respect to f. A graph H is called a *clone* of G if $H \cong H_1 \cong H_2$. In this case, G is called an *amalgamation of* G_1 *and* G_2 *at* H. Denote P_r and C_t the path and cycle of order r and $t \geq 3$, respectively. In this research paper, our primary focus lies in investigating the domination number of the amalgamation $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$, with the condition that $H_1 \cong H_2 \cong P_s$. We approach this problem by employing congruence properties modulo 3. For cases where $s \in \{1, 2, \min\{r,t\}\}$, we utilize these congruence properties to identify a minimum dominating set and determine the domination number of $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$. However, for $s \in \{3, 4, \ldots, \min\{r,t\} - 1\}$, we take a different approach. We construct a graph denoted as $PC(\alpha, \beta, \rho, \lambda)$ using four paths, namely $P_\alpha, P_\beta, P_\rho, P_\lambda$. We then establish that $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t \cong PC(\alpha, \beta, \rho, \lambda)$ for some $\alpha \in \{0, 1, \ldots, r\}$, $\beta = s - 2$, $\rho = t - s$, and $\lambda = r - (\alpha + s)$. Having established this correspondence, we proceed to determine the domination number $\gamma(P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t)$ using the corresponding value of $\gamma(PC(\alpha, \beta, \rho, \lambda))$. This approach allows us to gain valuable insights into the domination number of the amalgamation and explore the intricate relationships between these graph structures.

Keywords: Amalgamation; Cycle; Domination Number; Path

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1. Introduction

The domination number is one of the most important parameters in Graph Theory. The concept of the domination number was first introduced in 1958 by Berge [1], as the "coefficient of external stability". In 1962, Ore [2] studied the same concept and used the name "dominating set" and "domination number" for a graph. In 1977, a survey of the results about dominating sets was introduced by Cockayane and Hedetneimi [3]. They used the notation $\gamma(G)$ for the domination number of a graph G. In 1998, Haynes, Hedetniemi, and Slater [4] introduced a text devoted to fundamentals of dominations in graphs.

By P_n we denote the *path of order* n such that $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{\{v_i, v_{i+1}\} | i = 1, 2, \ldots, n-1\}$. In this study, we allow the existence of the graph with an empty set of vertices and an empty set of edges, named as the empty graph with notation P_0. By C_n we denote the *cycle of order* n ($n \geq 3$) such that $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $E(C_n) = \{\{u_i, u_{i+1}\} | i = 1, 2, \ldots, n\}$ where + is the addition modulo n. It is clear that $\gamma(C_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$ where $\lceil x \rceil$ is the least integer greater than or equal to x. Numerous mathematical researchers have extensively examined the domination numbers of graph products, including the cartesian products of two cycles [5] and the cross products of two paths [6]. In some cases, the domination number of a graph product can be bounded by the domination numbers of its constituent graphs, providing both upper and lower bounds. Among the graph products, the amalgamation of two graphs holds particular significance. It has been observed that an upper bound for the domination number of the amalgamation is equal to the sum of the domination numbers of the original graphs. This property adds to the allure and relevance of studying amalgamations in the realm of graph theory.

Let G_1, G_2 be disjoint graphs with two subgraphs H_1, H_2, respectively such that $H_1 \cong H_2$. Let f be a graph isomorphism from H_1 to H_2. The *amalgamation of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f is the graph $G = G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ obtained by forming the disjoint union of G_1 and G_2 and then identifying H_1 and H_2 with respect to f. Equivalently, $G = G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ is the graph such that

$$
\begin{aligned}
V(G) =&(V(G_1) \setminus V(H_1)) \cup (V(G_2) \setminus V(H_2)) \\
&\cup \{(v, f(v)) | v \in V(H_1)\}, \text{ and} \\
E(G) =&E(G_1 \setminus V(H_1)) \cup E(G_2 \setminus V(H_2)) \\
&\cup \{\{u, (v, f(v))\} | \{u, v\} \in E(G_1)\} \\
&\cup \{\{u, (v, f(v))\} | \{u, f(v)\} \in E(G_2)\} \\
&\cup \{\{(u, f(u)), (v, f(v))\} | \{u, v\} \in E(G_1) \\
&\text{ or } \{f(u), f(v)\} \in E(G_2)\}.
\end{aligned}
$$

We see at once that if $\{u, v\} \in E(G_1)$ and $\{f(u), f(v)\} \in E(G_2)$, then $\{u, v\} \in E(H_1)$ and $\{f(u), f(v)\} \in E(H_2)$. A graph H is called a *clone* of $G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ if $H \cong H_1 \cong H_2$. As an example, Fig. 1 illustrates the amalgamation $G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ with respect to the graph isomorphism $f : H_1 \to H_2$ defined by $f(v_i) = u_i$ for all $i = 1, 2, 3, 4$. The amalgamation or the glued graph of two graphs was introduced by Uiyyasathian [7] for finding a solution of the maximal-clique partition problem. In 2006, C. Promsakon and C. Uiyyasathian [8] gave an upper bound for the chromatic number of amalgamations of graphs. Recently, Boonmee, Ma-In and Panma [9] introduced the exact value of the domination number of an amalgamation of two cycles at connected subgraphs. Their approach involved constructing an isomorphic graph of the amalgamation of two cycles and employing congruence properties modulo 3 to find the domination number. Inspired by their method, our current focus is on finding the domination number of an amalgamation of a path and a cycle at connected subgraphs. By applying similar techniques, we aim to unveil the domination properties of this specific type of graph amalgamations.

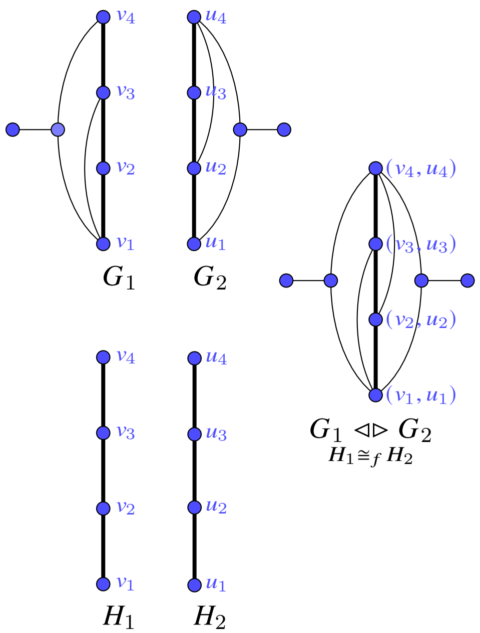

Fig. 1. An amalgamation of G_1 and G_2 at H_1 and H_2 with respect to f.

2. Basic Definitions and Results

Let G and H be two graphs. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we say that H is a *subgraph* of G or G *contains* H and denoted as $H \subseteq G$. When $H \subseteq G$ but $V(H) \neq V(G)$ or $E(H) \neq E(G)$, we say that H is a proper subgraph of G and denoted as $H \subset G$.

Let G be a graph. A vertex u is a *neighbor* of a vertex v in G if there is an edge $\{u, v\} \in E(G)$. Let us denote by $N[u]$ the set of all neighbors of u and u itself. A vertex u in G is said to *dominate* each of its neighbors and itself; that is, u dominates the vertices in $N[u]$. For a set S of vertices in G, let $N[S] = \cup_{u \in S} N[u]$ and S is said to *dominate* the vertices in $N[S]$. A set S of vertices in G is a *dominating set* of G if every vertex in $V(G) \setminus S$ is adjacent to some vertex in S, i.e., $N[S] = V(G)$. A *minimum dominating set* in G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the *domination number* of G, and is denoted by $\gamma(G)$. A dominating set of G with minimum cardinality is called a γ*−set* of G.

A mapping $f : V(G) \to V(H)$ is called a graph isomorphism if f is a one-to-one correspondence such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$. If such a function exists then we say that G and H are *isomorphic*, denoted by $G \cong H$. It is a well-known fact that if $G \cong H$ then $\gamma(G) = \gamma(H)$. A *graph automorphism* is simply a graph isomorphism from a graph to itself. The set of all graph isomorphisms from G to H will be denoted by $Iso(G, H)$ and the set of all graph automorphisms on G will be denoted by $Aut(G)$.

Note that if H is a connected subgraph of P_n, then $H \cong P_\alpha$ for some $\alpha \in \{1, 2, ..., n\}$ and if H is a connected subgraph of C_n, then $H \cong C_n$ or $H \cong P_\alpha$ for some $\alpha \in \{1, 2, ..., n\}$. It follows that if H_1 is a connected subgraph of P_r and H_2 is a connected subgraph of C_t such that $H_1 \cong H_2$, then H_1 and H_2 are paths of order s for some $s \leq \min\{r, t\}$. Moreover, $V(H_1) = \{v_{j+1}, v_{j+2}, \ldots, v_{j+s}\}$ and $V(H_2) = \{u_{k+1}, u_{k+2}, \ldots, u_{k+s}\}$ for some $j \in \{0, 1, \ldots, r\}$ and $k \in \{0, 1, \ldots, t\}$. This implies that if H is a connected clone of $G = P_r \underset{H_1 \cong_f H_2}{\lhd\!\rhd} C_t$, then $H \cong P_s$ for some $s \leq \min\{r, t\}$. Moreover, $Iso(H_1, H_2) = \{f_1, f_2\}$ such that $f_1(v_{j+q}) = u_{k+q}$ and

$f_2(v_{j+q}) = u_{k+(s-q)+1}$ for all $q = 1, 2, \ldots, s$. It follows easily that $Iso(H_2, H_1) = \{f_1^{-1}, f_2^{-1}\}$ and $P_r \underset{H_1 \cong_{f_i} H_2}{\lhd\rhd} C_t \cong C_t \underset{H_2 \cong_{f_i^{-1}} H_1}{\lhd\rhd} P_r$ for all $i = 1, 2$. Figs. 2-3 illustrate $G = P_r \underset{H_1 \cong_{f_1} H_2}{\lhd\rhd} C_t$ and $G' = P_r \underset{H_1 \cong_{f_2} H_2}{\lhd\rhd} C_t$, respectively. Define the mapping $\phi : V(G) \to V(G')$ by $\phi(v_l) = v_l$ for all $l = 1, 2, \ldots, \alpha, \alpha + s + 1, \alpha + s + 2, \ldots, r$, $\phi((v_{j+q}, u_{k+q})) = (v_{j+q}, u_{k+(s-q)+1})$ for all $q = 1, 2, \ldots, s$, and $\phi(u_{k+s+l}) = u_{k-l+1}$ for all $l = 1, \ldots, t-s$. It is easy to check that f is a graph isomorphism. This implies that if $H_1 \subseteq P_r$ and $H_2 \subseteq C_t$ are connected such that $H_1 \cong H_2$, then $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t \cong P_r \underset{H_1 \cong_g H_2}{\lhd\rhd} C_t$ for all $f, g \in Iso(H_1, H_2)$. Similarly, $C_t \underset{H_2 \cong_f H_1}{\lhd\rhd} P_r \cong C_t \underset{H_2 \cong_g H_1}{\lhd\rhd} P_r$ for all $f, g \in Iso(H_2, H_1)$, and so $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t \cong C_t \underset{H_2 \cong_g H_1}{\lhd\rhd} P_r$ for all $f \in Iso(H_1, H_2)$ and $g \in Iso(H_2, H_1)$. Moreover, for any connected subgraph $H_2' \subseteq C_t$ such that $H_2' \cong H_2$, we have $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t \cong P_r \underset{H_1 \cong_g H_2'}{\lhd\rhd} C_t$ for all $f \in Iso(H_1, H_2)$ and $g \in Iso(H_1, H_2')$. This implies the following lemma.

Lemma 2.1. *Let $H_1 \subseteq P_r$ and $H_2, H_2' \subseteq C_t$ be connected such that $H_1 \cong H_2 \cong H_2'$. Then*

1. $\gamma\left(P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t\right) = \gamma\left(P_r \underset{H_1 \cong_g H_2'}{\lhd\rhd} C_t\right)$ *for all $f \in Iso(H_1, H_2)$ and $g \in Iso(H_1, H_2')$,*

2. $\gamma\left(P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t\right) = \gamma\left(C_t \underset{H_2 \cong_g H_1}{\lhd\rhd} P_r\right)$ *for all $f \in Iso(H_1, H_2)$ and $g \in Iso(H_2, H_1)$.*

For simplicity of notation, we write G instead of $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$. Based on the result of Lemma 2.1, we will assume that $V(H_1) = \{v_{j+1}, v_{j+2}, \ldots, v_{j+s}\}$ for some $j \in \{0, 1, \ldots, r\}$ and $V(H_2) = \{u_1, u_2, ..., u_s\}$ and the isomorphism $f : H_1 \to H_2$ is defined by $f(v_{j+q}) = u_q$ for all $q = 1, 2, ..., s$. So $(v_{j+1}, u_1), (v_{j+2}, u_2), ..., (v_{j+s}, u_s) \in V(G)$ (see Fig. 4).

Our next goal is determine the domination number of six special cases of a graph G. Before determining, we define $v_{j+k} = (v_{j+k}, u_k)$ for $k = 1, 2$ and $V_a = \{v_i \in V(G) \mid i \equiv a \pmod 3, 1 \le i \le r\}$, $U_a = \{u_i \in V(G) \mid i \equiv a \pmod 3, 2 \le i \le t\}$, $U_a' = \{u_i \in V(G) \mid i \equiv a \pmod 3, 3 \le i \le t-1\}$ for all $a \in \{0, 1, 2\}$. Furthermore, $j \equiv f \pmod 3, m \equiv g \pmod 3$ and $t \equiv h \pmod 3$ where $f, g, h \in \{0, 1, 2\}$.

First of all, we consider the amalgamation G where $s = 1$(see Fig. 5).

Lemma 2.2. *If $s = 1$, then*

$$\gamma(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil + \left\lceil \frac{t-1}{3} \right\rceil & \textit{if } (j \equiv 2 \pmod 3 \\ & \textit{and } m \equiv 0 \pmod 3) \\ & \textit{or } (j \equiv 0 \pmod 3 \\ & \textit{and } m \equiv 2 \pmod 3) \\ & \textit{or } j \equiv m \equiv 2 \pmod 3, \\ \left\lceil \frac{r}{3} \right\rceil + \left\lceil \frac{t-3}{3} \right\rceil & \textit{otherwise.} \end{cases}$$

Proof. Let
$W_1 = \{(0,2,0), (2,0,0), (2,2,0), (0,2,1), (2,0,1), (2,2,1)\}$,
$W_2 = \{(0,2,2), (2,0,2), (2,2,2)\}$,
$W_3 = \{(0,0,0), (0,1,0), (0,0,2), (0,1,2)\}$,
$W_4 = \{(0,0,1), (0,1,1)\}$,
$W_5 = \{(1,0,0), (1,1,0), (1,0,2), (1,1,2)\}$,
$W_6 = \{(1,0,1), (1,1,1)\}$,
$W_7 = \{(1,2,0), (1,2,2)\}$,
$W_8 = \{(1,2,1)\}$,

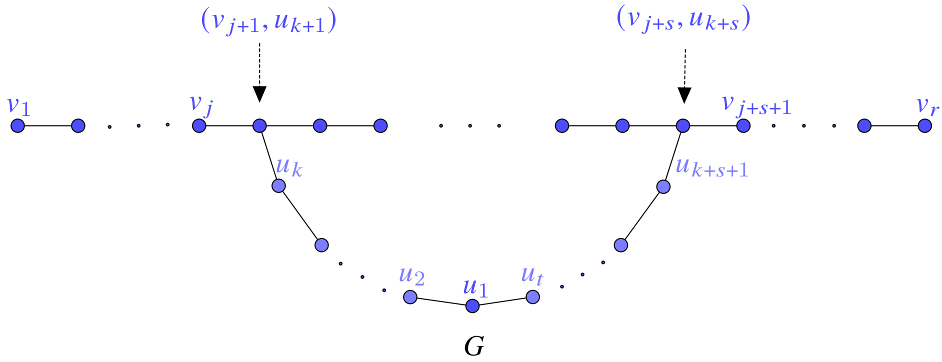

Fig. 2. The amalgamation of P_r and C_t at connected subgraphs H_1 and H_2 with respect to f_1.

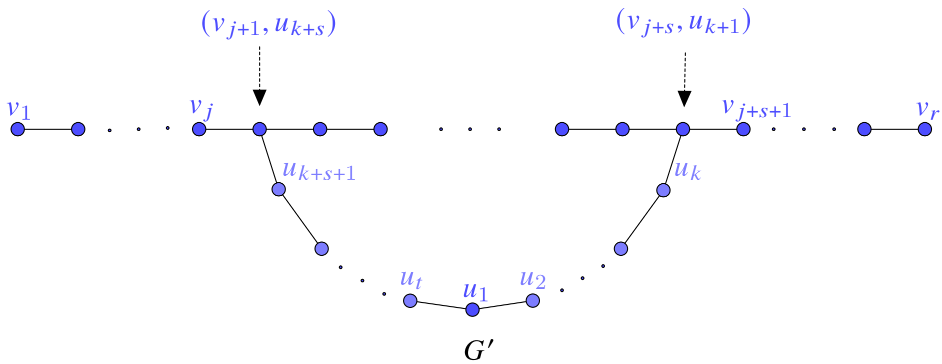

Fig. 3. The amalgamation of P_r and C_t at connected subgraphs H_1 and H_2 with respect to f_2.

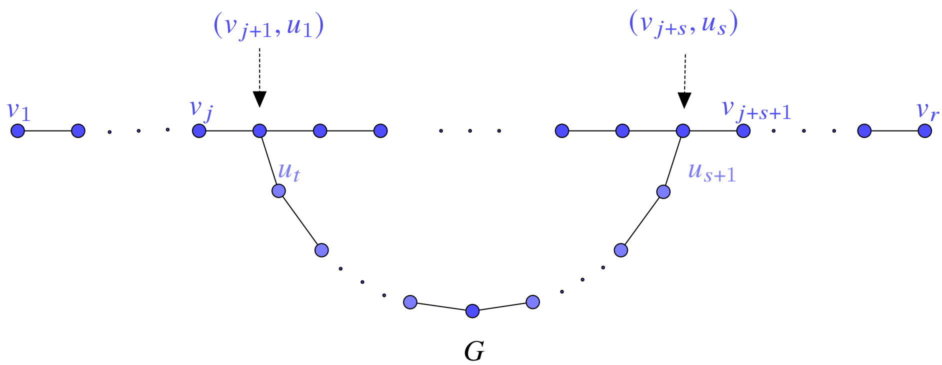

Fig. 4. The amalgamation of P_r and C_t at connected subgraphs H_1 and H_2 with respect to f.

$W_9 = \{(2,1,0),(2,1,2)\}$,
$W_{10} = \{(2,1,1)\}$,
$S_1 = V_2 \cup U_0, S_2 = V_2 \cup U_2, S_3 = V_1 \cup U_1'$,
$S_4 = V_1 \cup U_0', S_5 = V_2 \cup U_1', S_6 = V_2 \cup U_0'$,
$S_7 = V_2 \cup \{v_r\} \cup U_1', S_8 = V_2 \cup \{v_r\} \cup U_0'$,
$S_9 = V_0 \cup \{v_1\} \cup U_1', S_{10} = V_0 \cup \{v_1\} \cup U_0'$.

We can check at once that S_i is a minimum dominating set of G when $(f,g,h) \in W_i$ for all $i = 1,2,\dots,10$, respectively. Furthermore, $|S_i| = \lceil \frac{r}{3} \rceil + \lceil \frac{t-1}{3} \rceil$ for $i = 1,2$ and $|S_i| = \lceil \frac{r}{3} \rceil + \lceil \frac{t-3}{3} \rceil$ otherwise. □

After that, we will consider the dom-

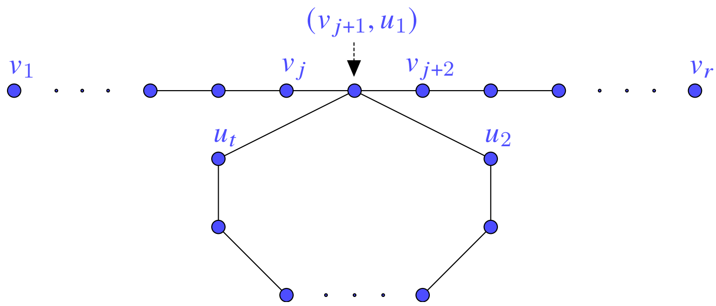

Fig. 5. The amalgamation of P_r and C_t at a connected clone P_1.

ination number of G where $s = 2$ and $t = 3$ (see Fig. 6).

Lemma 2.3. *If $s = 2$ and $t = 3$, then*

$$\gamma(G) = \begin{cases} \lceil \frac{r+1}{3} \rceil & \text{if } j \equiv m \equiv 2 \pmod 3, \\ \lceil \frac{r}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof. It is obvious that if $S_1 = V_2$, $S_2 = V_1, S_3 = V_0 \cup \{v_1\}, S_4 = V_2 \cup \{u_3\}$, $W_1 = \{(0,1),(0,0),(1,0),(1,2)\}$, $W_2 = \{(0,2),(2,0),(2,1)\}, W_3 = \{(1,1)\}$ and $W_4 = \{(2,2)\}$, then S_i is a minimum dominating set of G whenever $(f,g) \in W_i$ for $i = 1,2,3,4$, respectively. Moreover, $|S_1| = |S_2| = |S_3| = \lceil \frac{r}{3} \rceil$ and $|S_4| = \lceil \frac{r+1}{3} \rceil$. □

Next, we consider the domination number of G where $s = 2$ and $t \geq 4$ (see Fig. 7).

Lemma 2.4. *If $s = 2$ and $t \geq 4$, then*

$$\gamma(G) = \begin{cases} \lceil \frac{r}{3} \rceil + \lceil \frac{t-2}{3} \rceil, & \\ \text{if } j \equiv m \equiv 2 \pmod 3, & \\ \lceil \frac{r}{3} \rceil + \lceil \frac{t-4}{3} \rceil, & \\ \text{if } j \equiv m \equiv 1 \pmod 3 & \\ \text{or } j \equiv m+1 \equiv 0 \pmod 3 & \\ \text{or } j+1 \equiv m \equiv 0 \pmod 3, & \\ \lceil \frac{r}{3} \rceil + \lceil \frac{t-3}{3} \rceil, & \\ \text{otherwise.} & \end{cases}$$

Proof. Set $S_1 = V_2 \cup U_0, S_2 = V_2 \cup U_1 \setminus \{u_1\}$,
$S_3 = V_0 \cup \{v_1\} \cup U_2 \setminus \{u_2\}$,
$S_4 = V_2 \setminus \{v_{j+2}, ..., v_r\} \cup V_0 \setminus \{v_1, ..., v_{j+1}\} \cup U_2 \setminus \{u_2\}$,
$S_5 = V_1 \cup U_1 \setminus \{u_1\}$,
$S_6 = V_2 \setminus \{v_{j+1}, ..., v_r\} \cup V_0 \setminus \{v_1, ..., v_{j+2}\} \cup U_1 \setminus \{u_1\}$,
$S_7 = V_0 \cup \{v_1\} \cup U_1 \setminus \{u_1\}$,
$S_8 = V_2 \setminus \{v_{j+1}, ..., v_r\} \cup V_0 \setminus \{v_1, ..., v_{j+2}\} \cup U_0$,
$S_9 = V_2 \cup U_2 \setminus \{u_2\}, S_{10} = V_2 \cup U_1 \setminus \{u_1\}$,
$S_{11} = V_2 \cup U_1 \setminus \{u_1\}, S_{12} = V_2 \cup U_0$,
$S_{13} = V_1 \cup U_2 \setminus \{u_2\}, S_{14} = V_1 \cup U_1 \setminus \{u_1\}$,
$W_1 = \{(2,2,0)\}, W_2 = \{(2,2,1),(2,2,2)\}$,
$W_3 = \{(1,1,0),(1,1,2)\}, W_4 = \{(1,1,1)\}$,
$W_5 = \{(0,2,0),(0,2,2)\}, W_6 = \{(0,2,1)\}$,
$W_7 = \{(2,0,0),(2,0,2)\}, W_8 = \{(2,0,1)\}$,
$W_9 = \{(0,0,0),(0,1,0)\}$,
$W_{10} = \{(0,0,1),(0,0,2),(0,1,1),(0,1,2)\}$,
$W_{11} = \{(1,0,0),(1,0,2),(1,2,0)\}$,
$W_{12} = \{(1,0,1),(1,2,1),(1,2,2)\}$,
$W_{13} = \{(2,1,0)\}, W_{14} = \{(2,1,1),(2,1,2)\}$.

It is easy to see that S_i is a minimum dominating set of G whenever $(f,g,h) \in W_i$ for $i = 1,2,...,14$, respectively. Besides,
$|S_1| = |S_2| = \lceil \frac{r}{3} \rceil + \lceil \frac{t-2}{3} \rceil$, $|S_i| = \lceil \frac{r}{3} \rceil + \lceil \frac{t-4}{3} \rceil$ for $i = 3,4,...,8$ and $|S_i| = \lceil \frac{r}{3} \rceil + \lceil \frac{t-3}{3} \rceil$ for $i = 9,...,14$. □

In addition, we determine the graph

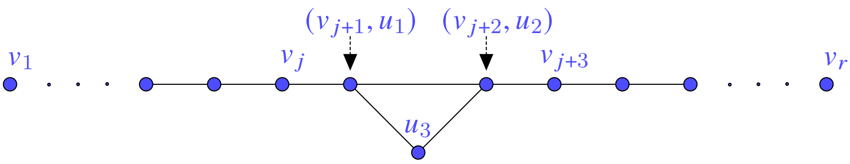

Fig. 6. The amalgamation of P_r and C_t at a connected clone P_2 where $t = 3$.

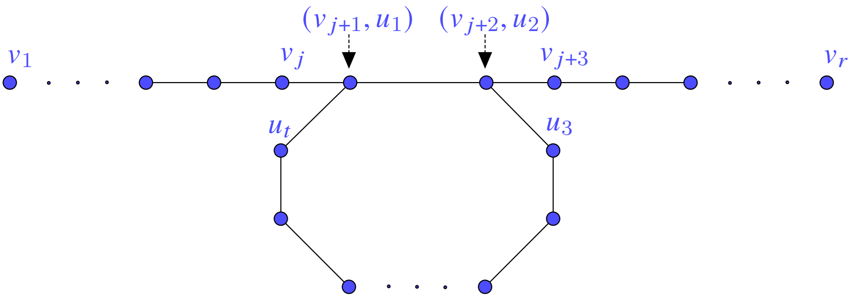

Fig. 7. The amalgamation of P_r and C_t at a connected clone P_2 where $t \geq 4$.

$G = P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_3$ such that $H_1 \cong P_3$ (see Fig. 8). It is easily seen that G is isomorphic to the amalgamation $G' = P_{r-1} \underset{H_1 \cong_f H_2}{\lhd\rhd} C_3$ such that $H_1 \cong P_2$. From the assumptions of Lemma 2.3 with r replaced by $r-1$, to obtain the following lemma.

Lemma 2.5. *If $s = t = 3$, then*

$$\gamma(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil, & \text{if } j \equiv m \equiv 2 \pmod 3, \\ \left\lceil \frac{r-1}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Let $t \geq 4$, we will consider the amalgamation $G = P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$ where $s = t = \min\{r, t\}$ (see Fig. 9). We see that G is isomorphic to the amalgamation $G' = P_{r-t+2} \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$ such that $H_1 \cong P_2$ and $t \geq 4$. Under the assumptions of Lemma 2.4 with r replaced by $r - t + 2$, we obtain the following lemma.

Lemma 2.6. *If $t \geq 4$ and $s = t = \min\{r, t\}$, then*

$$\gamma(G) = \begin{cases} \left\lceil \frac{r-t+2}{3} \right\rceil + \left\lceil \frac{t-2}{3} \right\rceil, \\ \text{if } j \equiv m \equiv 2 \pmod 3, \\ \\ \left\lceil \frac{r-t+2}{3} \right\rceil + \left\lceil \frac{t-4}{3} \right\rceil, \\ \text{if } j \equiv m \equiv 1 \pmod 3 \\ \text{or } j \equiv m+1 \equiv 0 \pmod 3 \\ \text{or } j+1 \equiv m \equiv 0 \pmod 3, \\ \\ \left\lceil \frac{r-t+2}{3} \right\rceil + \left\lceil \frac{t-3}{3} \right\rceil, \\ \text{otherwise.} \end{cases}$$

Note that if $H_1 \cong P_s$ and $s = r = \min\{r, t\}$, then $G \cong C_t$. We thus get the following lemma.

Lemma 2.7. *If $s = r = \min\{r, t\}$, then* $\gamma(G) = \left\lceil \frac{t}{3} \right\rceil$.

By Lemma 2.2-2.7, we know the domination number of G for all H_1 such that $H_1 \cong P_1, P_2, P_{\min\{r,t\}}$. We now

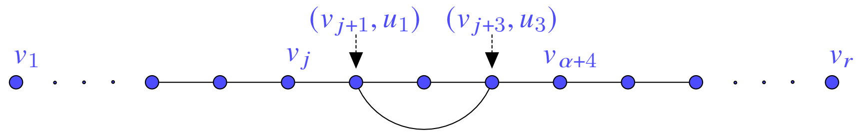

Fig. 8. The amalgamation of P_r and C_t at a connected clone P_s where $s = \min\{r, t\}$ and $t = 3$.

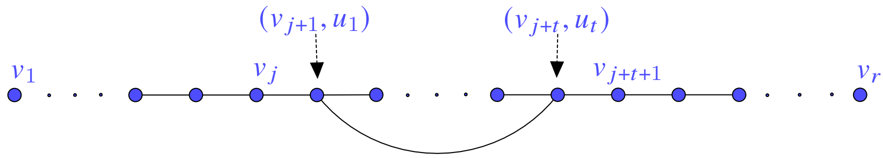

Fig. 9. The amalgamation of P_r and C_t at a connected clone P_t where $t \geq 4$.

consider the case $H_1 \cong P_s$ for all $s \in \{3, 4, ..., \min\{r, t\} - 1\}$.

Let $\alpha, \lambda \in \mathbb{N} \cup \{0\}$, $\beta, \rho \in \mathbb{N}$ and $A = \{\alpha, \beta, \rho, \lambda\}$. For $\alpha, \beta, \rho, \lambda \in \mathbb{N}$, we will denote by
P_α a path of order α with $V(P_\alpha) = \{1_1, 2_1, \ldots, \alpha_1\}$ and $E(P_\alpha) = \{\{i_1, (i+1)_1\} \mid i = 1, 2, \ldots, \alpha - 1\}$, P_β a path of order β with $V(P_\beta) = \{1_2, 2_2, \ldots, \beta_2\}$ and $E(P_\beta) = \{\{i_2, (i+1)_2\} \mid i = 1, 2, \ldots, \beta - 1\}$, P_ρ a path of order ρ with $V(P_\rho) = \{1_3, 2_3, \ldots, \rho_3\}$ and $E(P_\rho) = \{\{i_3, (i+1)_3\} \mid i = 1, 2, \ldots, \rho - 1\}$, P_λ a path of order λ with $V(P_\lambda) = \{1_4, 2_4, \ldots, \lambda_4\}$ and $E(P_\lambda) = \{\{i_4, (i+1)_4\} \mid i = 1, 2, \ldots, \lambda - 1\}$. Moreover, let us denote by $PC(\alpha, \beta, \rho, \lambda)$ the graph with $V(PC(\alpha, \beta, \rho, \lambda)) = \{x, y\} \cup \bigcup_{k \in A} V(P_k)$ and $E(PC(\alpha, \beta, \rho, \lambda)) =$

$$
\begin{cases}
\{\{x, 1_2\}, \{x, \rho_3\}, \{y, \beta_2\}, \{y, 1_3\}\} \cup \bigcup_{k \in A} E(P_k) & \text{if } \alpha = 0 \text{ and } \lambda = 0, \\
\{\{x, 1_2\}, \{x, \rho_3\}, \{y, \beta_2\}, \{y, 1_3\}, \{y, 1_4\}\} \cup \bigcup_{k \in A} E(P_k) & \text{if } \alpha = 0 \text{ and } \lambda \neq 0, \\
\{\{x, 1_1\}, \{x, 1_2\}, \{x, \rho_3\}, \{y, \beta_2\}, \{y, 1_3\}\} \cup \bigcup_{k \in A} E(P_k) & \text{if } \alpha \neq 0 \text{ and } \lambda = 0, \\
\{\{x, 1_1\}, \{x, 1_2\}, \{x, \rho_3\}, \{y, \beta_2\}, \{y, 1_3\}, \{y, 1_4\}\} \cup \bigcup_{k \in A} E(P_k) & \text{if } \alpha \neq 0 \text{ and } \lambda \neq 0.
\end{cases}
$$

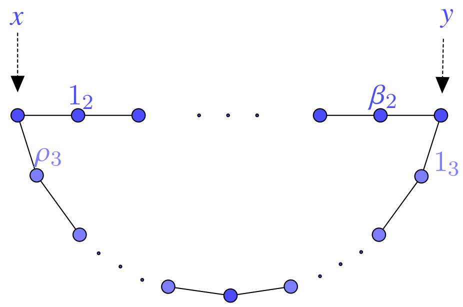

Fig. 10. $PC(\alpha, \beta, \rho, \lambda)$ with $\alpha = 0$ and $\lambda = 0$.

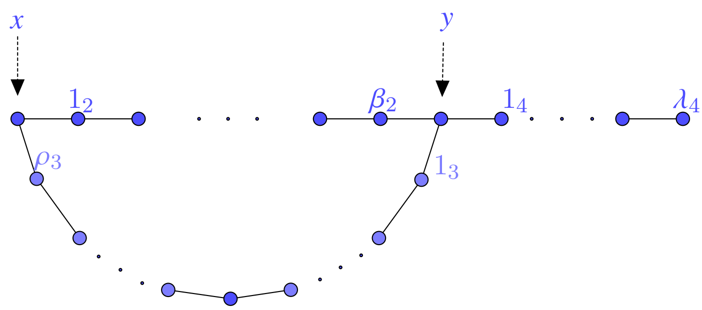

Fig. 11. $PC(\alpha, \beta, \rho, \lambda)$ with $\alpha = 0$ and $\lambda \neq 0$.

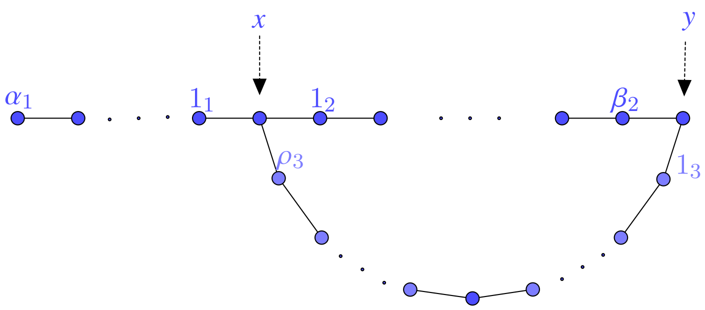

Fig. 12. $PC(\alpha, \beta, \rho, \lambda)$ with $\alpha \neq 0$ and $\lambda = 0$.

Figs. 10-13 illustrate $PC(\alpha, \beta, \rho, \lambda)$ with ($\alpha = 0$ and $\lambda = 0$), ($\alpha = 0$ and $\lambda \neq 0$), ($\alpha \neq 0$ and $\lambda = 0$) and ($\alpha \neq 0$ and $\lambda \neq 0$), respectively. Note that if $\alpha = 0$ and $\lambda = 0$ then $PC(\alpha, \beta, \rho, \lambda) \cong G \cong C_t$ such that $s = r = \min\{r, t\}$, and thus $\gamma(PC(\alpha, \beta, \rho, \lambda)) = \lceil \frac{t}{3} \rceil$.

Lemma 2.8. *Let* $G = P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$ *be such that* $V(H_1) = \{v_{j+1}, v_{j+2}, \ldots, v_{j+s}\}$ *for some* $j \in \{0, 1, \ldots, r\}$ *and* $s \in \{3, 4, ..., \min\{r, t\} - 1\}$. *Then the following conditions hold:*

1. *if* $j = 0$, *then* $G \cong PC(\alpha, \beta, \rho, \lambda)$ *where* $\alpha = 0$, $\beta = s - 2$, $\rho = t - s$ *and* $\lambda = r - s$;
2. *if* $j \neq 0$ *and* $s = r - j$, *then* $G \cong PC(\alpha, \beta, \rho, \lambda)$ *where* $\alpha = j$, $\beta = s - 2$, $\rho = t - s$ *and* $\lambda = 0$;

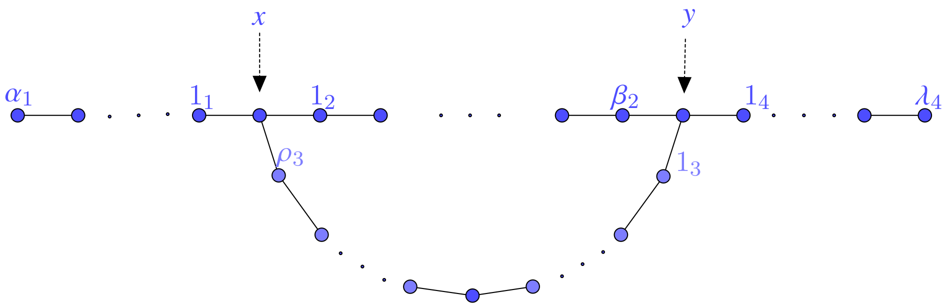

Fig. 13. $PC(\alpha, \beta, \rho, \lambda)$ with $\alpha \neq 0$ and $\lambda \neq 0$.

3. *if $j \neq 0$ and $s \neq r - j$, then $G \cong PC(\alpha, \beta, \rho, \lambda)$ where $\alpha = j$, $\beta = s - 2$, $\rho = t - s$ and $\lambda = r - (j + s)$.*

Proof. 1. Suppose that $j = 0$. Then $V(G) = \{(v_1, u_1), \ldots, (v_s, u_s)\} \cup \{u_{s+1}, \ldots, u_t\} \cup \{v_{s+1}, \ldots, v_r\}$. Since $s \in \{3, 4, \ldots, \min\{r, t\} - 1\}$, we get $s - 2, t - s, r - s \in \mathbb{N}$. Define $f : V(G) \to V(PC(0, s-2, t-s, r-s))$ by $f((v_1, u_1)) = x$, $f((v_a, u_a)) = (a-1)_2$ for all $a \in \{2, \ldots, s-1\}$, $f((v_s, u_s)) = y$, $f(u_{s+b}) = b_3$ for all $b \in \{1, \ldots, t-s\}$, and $f(v_{s+c}) = c_4$ for all $c \in \{1, \ldots, r-s\}$. Clearly, f is an isomorphism.

2. Suppose that $j \neq 0$ and $s = r - j$. Then $V(G) = \{v_1, \ldots, v_j\} \cup \{(v_{j+1}, u_1), \ldots, (v_{j+s}, u_s)\} \cup \{u_{s+1}, \ldots, u_t\}$. Define $f : V(G) \to V(PC(j, s-2, t-s, 0))$ by $f(v_a) = (j - (a-1))_1$ for all $a \in \{1, \ldots, j\}$, $f((v_{j+1}, u_1)) = x$, $f((v_{j+a}, u_a)) = (a-1)_2$ for all $a \in \{2, \ldots, s-1\}$, $f((v_{j+s}, u_s)) = y$, and $f(u_{s+b}) = b_3$ for all $b \in \{1, \ldots, t-s\}$. It is clear that f is an isomorphism.

3. Suppose that $j \neq 0$ and $s \neq r - j$. Then $r - (j+s) \in \mathbb{N}$ and $V(G) = \{v_1, \ldots, v_j\} \cup \{(v_{j+1}, u_1), \ldots, (v_{j+s}, u_s)\} \cup \{u_{s+1}, \ldots, u_t\} \cup \{v_{s+1}, \ldots, v_r\}$. Define $f : V(G) \to V(PC(j, s-2, t-s, r-(j+s)))$ by $f(v_a) = (j-(a-1))_1$ for all $a \in \{1, \ldots, j\}$, $f((v_{j+1}, u_1)) = x$, $f((v_{j+a}, u_a)) = (a-1)_2$ for all $a \in \{2, \ldots, s-1\}$, $f((v_{j+s}, u_s)) = y$, $f(u_{s+b}) = b_3$ for all $b \in \{1, \ldots, t-s\}$, and $f(v_{s+c}) = c_4$ for all $c \in \{1, \ldots, r-(j+s)\}$. It is also clear that f is an isomorphism.

□

3. Domination Numbers of Amalgamations of Paths and Cycles at Connected Subgraphs

In this section, we calculate the domination number of $PC(\alpha, \beta, \rho, \lambda)$ for all $\beta, \rho \in \mathbb{N}$ and for all $\alpha, \lambda \in \mathbb{N} \cup \{0\}$, not both zero. Then, by Lemma 2.8, we thus get the domination number $\gamma(G)$ for the case $H_1 \cong P_s$ for all $s \in \{3, 4, \ldots, \min\{r, t\} - 1\}$.

For $\alpha, \lambda \in \mathbb{N} \cup \{0\}$, $\beta, \rho \in \mathbb{N}$ and $a \in \{0, 1, 2\}$, let us denote by $S_{a\alpha} = \{j_1 | j \equiv a (\text{mod} 3) \text{ and } 1 \leq j \leq \alpha\}$, $S_{a\beta} = \{j_2 | j \equiv a (\text{mod} 3) \text{ and } 1 \leq j \leq \beta\}$, $S_{a\rho} = \{j_3 | j \equiv a (\text{mod} 3) \text{ and } 1 \leq j \leq \rho\}$ and $S_{a\lambda} = \{j_4 | j \equiv a (\text{mod} 3) \text{ and } 1 \leq j \leq \lambda\}$. Note that $S_{a0} = \emptyset$ for all $a \in \{0, 1, 2\}$. From now on, $\lceil x \rceil$ stands for the least integer greater than or equal to x. From basic properties of congruence modulo 3, we get the following remark.

Remark 3.1. *Let* $\alpha, \lambda \in \mathbb{N} \cup \{0\}$, $\beta, \rho \in \mathbb{N}$ *and* $x \in \{\alpha, \beta, \rho, \lambda\}$.

(1) *If* $x \equiv 0(\text{mod}3)$, *then* $|S_{0x}| = |S_{1x}| = |S_{2x}| = \lceil \frac{x}{3} \rceil = \lceil \frac{x-1}{3} \rceil = \lceil \frac{x-2}{3} \rceil$.

(2) *If* $x \equiv 1(\text{mod}3)$, *then* $|S_{0x}| = |S_{2x}| = \lceil \frac{x-1}{3} \rceil = \lceil \frac{x-2}{3} \rceil = \lceil \frac{x-3}{3} \rceil$ *and* $|S_{1x}| = \lceil \frac{x}{3} \rceil = \lceil \frac{x+1}{3} \rceil = \lceil \frac{x+2}{3} \rceil$.

(3) *If* $x \equiv 2(\text{mod}3)$, *then* $|S_{1x}| = |S_{2x}| = \lceil \frac{x-1}{3} \rceil = \lceil \frac{x}{3} \rceil = \lceil \frac{x+1}{3} \rceil$ *and* $|S_{0x}| = \lceil \frac{x-2}{3} \rceil = \lceil \frac{x-3}{3} \rceil = \lceil \frac{x-4}{3} \rceil$.

For simplicity of notation, we write PC instead of $PC(\alpha, \beta, \rho, \lambda)$ where $\beta, \rho \in \mathbb{N}$ and $\alpha, \lambda \in \mathbb{N} \cup \{0\}$, not both zero. Consider α, λ under congruence modulo 3, we get nine cases for α and λ:
$\alpha \equiv 0(\text{mod}3), \lambda \equiv 0(\text{mod}3)$;
$\alpha \equiv 0(\text{mod}3), \lambda \equiv 1(\text{mod}3)$;
$\vdots$
$\alpha \equiv 2(\text{mod}3), \lambda \equiv 2(\text{mod}3)$.
In each of these cases, we give upper bounds for $\gamma(PC)$ as follows.

Lemma 3.2. *Let* $\alpha \equiv 0(\text{mod}3)$ *and* $\lambda \equiv 0(\text{mod}3)$.

(1) *If* $\beta \equiv 2(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) *If* $\rho \equiv 2(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) *If* $\beta \equiv 0(\text{mod}3)$ *and* $\rho \not\equiv 2(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(4) *If* $\beta \equiv 1(\text{mod}3)$ *and* $\rho \equiv 0(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$.

(5) *If* $\beta \equiv 1(\text{mod}3)$ *and* $\rho \equiv 1(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$.

Proof. Set
$S_1 = S_{0\alpha} \cup S_{0\beta} \cup S_{0\rho} \cup S_{0\lambda} \cup \{x, y\}$,
$S_2 = S_{0\alpha} \cup S_{0\beta} \cup S_{0\rho} \cup S_{0\lambda} \cup \{x, y\}$,
$S_3 = S_{0\alpha} \cup S_{0\beta} \cup S_{2\rho} \cup S_{2\lambda} \cup \{x\}$,
$S_4 = S_{0\alpha} \cup S_{2\beta} \cup S_{0\rho} \cup S_{2\lambda} \cup \{x\}$
and $S_5 = S_{2\alpha} \cup S_{1\beta} \cup S_{1\rho} \cup S_{2\lambda}$.
We check at once that:

(1) if $\beta \equiv 2(\text{mod}3)$, then S_1 is a dominating set of PC and $|S_1| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$;

(2) if $\rho \equiv 2(\text{mod}3)$, then S_2 is a dominating set of PC and $|S_2| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$;

(3) if $\beta \equiv 0(\text{mod}3)$ and $\rho \not\equiv 2(\text{mod}3)$, then S_3 is a dominating set of PC and $|S_3| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$;

(4) if $\beta \equiv 1(\text{mod}3)$ and $\rho \equiv 0(\text{mod}3)$, then S_4 is a dominating set of PC and $|S_4| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$;

(5) if $\beta \equiv 1(\text{mod}3)$ and $\rho \equiv 1(\text{mod}3)$, then S_5 is a dominating set of PC and $|S_5| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$.

□

Lemma 3.3. *Let* $\alpha \equiv 0(\text{mod}3)$ *and* $\lambda \equiv 1(\text{mod}3)$.

(1) *If* $\beta \equiv 1(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil$.

(2) *If* $\rho \equiv 1(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-3}{3} \right\rceil$.

(3) *If* $\beta \equiv 2(\text{mod}3)$ *and* $\rho \not\equiv 1(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(4) *If* $\beta \equiv 0(\text{mod}3)$ *and* $\rho \equiv 2(\text{mod}3)$, *then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(5) If $\beta \equiv 0(\mathrm{mod}3)$ and $\rho \equiv 0(\mathrm{mod}3)$, then $\gamma(PC) \le \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

Proof. Set
$S_1 = S_{2\alpha} \cup S_{2\beta} \cup S_{0\rho} \cup S_{0\lambda} \cup \{y\}$,
$S_2 = S_{0\alpha} \cup S_{2\beta} \cup S_{0\rho} \cup S_{0\lambda} \cup \{x, y\}$,
$S_3 = S_{2\alpha} \cup S_{1\beta} \cup S_{0\rho} \cup S_{0\lambda} \cup \{y\}$,
$S_4 = S_{0\alpha} \cup S_{0\beta} \cup S_{0\rho} \cup S_{0\lambda} \cup \{x, y\}$,
$S_5 = S_{0\alpha} \cup S_{0\beta} \cup S_{0\rho} \cup S_{0\lambda} \cup \{x, y\}$
and $S_6 = S_{2\alpha} \cup S_{1\beta} \cup S_{2\rho} \cup S_{0\lambda} \cup \{y\}$.

(1) If $\beta \equiv 1(\bmod 3)$, then we get the following two cases.

a) If $\rho \equiv 0(\mathrm{mod}3)$, then S_1 is a dominating set of PC and $|S_1| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil$.

b) If $\rho \not\equiv 0(\mathrm{mod}3)$, then S_2 is a dominating set of PC and $|S_2| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil$.

(2) If $\rho \equiv 1(\mathrm{mod}3)$, then we also get the following two cases.

a) If $\beta \equiv 0(\mathrm{mod}3)$, then S_3 is a dominating set of PC and $|S_3| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-3}{3} \right\rceil$.

b) If $\beta \equiv 2(\mathrm{mod}3)$, then S_4 is a dominating set of PC and $|S_4| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-3}{3} \right\rceil$.

(3) If $\beta \equiv 2(\mathrm{mod}3)$ and $\rho \not\equiv 1(\mathrm{mod}3)$, then it follows easily that S_5 is a dominating set of PC and $|S_5| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(4) If $\beta \equiv 0(\mathrm{mod}3)$ and $\rho \equiv 2(\mathrm{mod}3)$, then it follows immediately that S_5 also is a dominating set of PC and $|S_5| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(5) If $\beta \equiv 0(\mathrm{mod}3)$ and $\rho \equiv 0(\mathrm{mod}3)$, then we see at once that S_6 is a dominating set of PC and $|S_6| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

□

Lemma 3.4. *Let $\alpha \equiv 0(\mathrm{mod}3)$ and $\lambda \equiv 2(\mathrm{mod}3)$.*

(1) If $\beta \equiv 0(\mathrm{mod}3)$, then $\gamma(PC) \le \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) If $\rho \equiv 0(\mathrm{mod}3)$, then $\gamma(PC) \le \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) If $\beta \equiv 1(\mathrm{mod}3)$ and $\rho \not\equiv 2(\mathrm{mod}3)$, then $\gamma(PC) \le \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(4) If $\beta \equiv 2(\mathrm{mod}3)$ and $\rho \equiv 1(\mathrm{mod}3)$, then $\gamma(PC) \le \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$.

(5) If $\beta \equiv 2(\mathrm{mod}3)$ and $\rho \equiv 2(\mathrm{mod}3)$, then $\gamma(PC) \le \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$.

Proof. Set
$S_1 = S_{2\alpha} \cup S_{2\beta} \cup S_{2\rho} \cup S_{1\lambda}$,
$S_2 = S_1 \cup \{x\}$,
$S_3 = S_{2\alpha} \cup S_{0\beta} \cup S_{2\rho} \cup S_{1\lambda} \cup \{x\}$,
$S_4 = S_{2\alpha} \cup S_{1\beta} \cup S_{2\rho} \cup S_{1\lambda}$,
$S_5 = S_{0\alpha} \cup S_{0\beta} \cup S_{2\rho} \cup S_{1\lambda} \cup \{x\}$,
$S_6 = S_{0\alpha} \cup S_{2\beta} \cup S_{2\rho} \cup S_{1\lambda}$
and $S_7 = S_{2\alpha} \cup S_{1\beta} \cup S_{2\rho} \cup S_{1\lambda}$.

(1) If $\beta \equiv 0(\bmod 3)$, then we get the following two cases.

a) If $\rho \equiv 2(\mathrm{mod}3)$, then S_1 is a dominating set of PC and $|S_1| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

b) If $\rho \not\equiv 2(\mathrm{mod}3)$, then S_2 is a dominating set of PC and $|S_2| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) If $\rho \equiv 0(\mod 3)$, then we get the following two cases.

a) If $\beta \equiv 1(\text{mod}3)$, then S_3 is a dominating set of PC and $|S_3| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

b) If $\beta \equiv 2(\text{mod}3)$, then S_4 is a dominating set of PC and $|S_4| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) If $\beta \equiv 1(\text{mod}3)$ and $\rho \not\equiv 2(\text{mod}3)$, then S_5 is a dominating set of PC and $|S_5| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(4) If $\beta \equiv 2(\text{mod}3)$ and $\rho \equiv 1(\text{mod}3)$, then S_6 is a dominating set of PC and $|S_6| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$.

(5) If $\beta \equiv 2(\text{mod}3)$ and $\rho \equiv 2(\text{mod}3)$, then S_7 is a dominating set of PC and $|S_7| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$.

□

From Lemma 3.3, we can now state the lemma below by substituting α and λ with λ and α, respectively.

Lemma 3.5. *Let $\alpha \equiv 1(\text{mod}3)$ and $\lambda \equiv 0(\text{mod}3)$.*

(1) If $\beta \equiv 1(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil$.

(2) If $\rho \equiv 1(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-3}{3} \right\rceil$.

(3) If $\beta \equiv 2(\text{mod}3)$ and $\rho \not\equiv 1(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(4) If $\beta \equiv 0(\text{mod}3)$ and $\rho \equiv 2(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(5) If $\beta \equiv 0(\text{mod}3)$ and $\rho \equiv 0(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

From now on, C stands for the cycle $PC \setminus \cup_{i\in\{1,2,3\}}(S_{i\alpha} \cup S_{i\lambda}) \cong PC(0, \beta, \rho, 0)$ (see Fig. 10).

Lemma 3.6. *If $\alpha \equiv 1(\mod 3)$ and $\lambda \equiv 1(\mod 3)$, then* $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

Proof. Since $C \setminus N[\{x, y\}] \cong P_{\beta-2} \cup P_{\rho-2}$, we can conclude that there is a dominating set D of $C \setminus N[\{x, y\}]$ such that $|D| = \left\lceil \frac{\beta-2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$. It follows immediately that $S = S_{0\alpha} \cup S_{0\lambda} \cup D \cup \{x, y\}$ is a dominating set of G and $|S| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$. □

Lemma 3.7. *Let $\alpha \equiv 1(\text{mod}3)$ and $\lambda \equiv 2(\text{mod}3)$.*

(1) If $\beta \equiv 2(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) If $\rho \equiv 2(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) If $\beta \not\equiv 2(\text{mod}3)$ and $\rho \not\equiv 2(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

Proof. Let $\alpha \equiv 1(\text{mod}3)$ and $\lambda \equiv 2(\text{mod}3)$. Set $S = S_{0\alpha} \cup S_{0\beta} \cup S_{0\rho} \cup S_{2\lambda} \cup \{x, y\}$.

(1) If $\beta \equiv 2(\text{mod}3)$, then it follows easily that S is a dominating set of PC and $|S| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) If $\rho \equiv 2(\text{mod}3)$, then it follows immediately that S is a dominating set of PC and $|S| = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) If $\beta \not\equiv 2(\text{mod}3)$ and $\rho \not\equiv 2(\text{mod}3)$, then, for the graph $H \setminus N[\{x, 1_4\}] \cong P_{\beta-1} \cup P_{\rho-1}$, there is a dominating

set D of $H \setminus N[\{x, 1_4\}]$ such that $|D| = \left\lceil \frac{\beta-1}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$. It follows easily that $S = S_{0\alpha} \cup S_{1\lambda} \cup D \cup \{x\}$ is a dominating set of PC and
$|S| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

□

By substituting α for λ and λ for α in Lemma 3.4 and Lemma 3.7, respectively, we have the following two lemmas.

Lemma 3.8. *Let $\alpha \equiv 2(\text{mod}3)$ and $\lambda \equiv 0(\text{mod}3)$.*

(1) If $\beta \equiv 0(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) If $\rho \equiv 0(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) If $\beta \equiv 1(\text{mod}3)$ and $\rho \not\equiv 0(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(4) If $\beta \equiv 2(\text{mod}3)$ and $\rho \equiv 1(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$.

(5) If $\beta \equiv 2(\text{mod}3)$ and $\rho \equiv 2(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$.

Lemma 3.9. *Let $\alpha \equiv 2(\text{mod}3)$ and $\lambda \equiv 1(\text{mod}3)$.*

(1) If $\beta \equiv 2(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) If $\rho \equiv 2(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) If $\beta \not\equiv 2(\text{mod}3)$ and $\rho \not\equiv 2(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

Lemma 3.10. *Let $\alpha \equiv 2(\text{mod}3)$ and $\lambda \equiv 2(\text{mod}3)$.*

(1) If $\beta \equiv 1(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(2) If $\rho \equiv 1(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$.

(3) If $\beta \not\equiv 1(\text{mod}3)$ and $\lambda \not\equiv 1(\text{mod}3)$, then $\gamma(PC) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$.

Proof. (1) If $\beta \equiv 1(\text{mod}3)$, then we consider the cycle $K = PC \setminus \cup_{i\in\{1,2,3\}}(S_{i\alpha} \cup S_{i\lambda})$. We see that $K \setminus N[\{x, 1_4\}] \cong P_{\beta-1} \cup P_{\rho-1}$. Thus, there is a dominating set D of this graph such that $|D| = \left\lceil \frac{\beta-1}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$. It is clear that $S = S_{2\alpha} \cup S_{1\lambda} \cup D \cup \{x\}$ is a dominating set of PC and $|S| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(2) The proof is based on the concept of (1) when replacing β and ρ by ρ and β, respectively.

(3) Set $M = PC \setminus \cup_{i\in\{1,2,3\}}(S_{i\alpha} \cup S_{i\lambda})$. We see that $M \setminus N[\{1_1, 1_4\}] \cong P_\beta \cup P_\rho$. Thus, there is a dominating set D of this graph such that $|D| = \left\lceil \frac{\beta}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$. It is easy to check that $S = S_{1\alpha} \cup S_{1\lambda} \cup D$ is a dominating set of PC and $|S| = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$.

□

We are now ready to give the domination numbers of PC. We will prove that the upper bounds which are given in Lemma 3.2-3.10 are the domination number of PC. Let us first give an observation of a γ-set H of PC. Consider two vertices x, y in PC (see Fig. 13). In order to dominate x and

y, $|H \cap N[\{x\}]| = 1$ and $|H \cap N[\{y\}]| = 1$. Since $N[\{x\}] = \{x, 1_1, 1_2, \rho_3\}$ and $N[\{y\}] = \{y, \beta_2, 1_3, 1_4\}$, there are sixteen cases: $x, y \in H; x, \beta_2 \in H; \ldots; \rho_3, 1_4 \in H$. If $x, y \in H$, then all vertices in $N[\{x, y\}] = \{x, y, 1_1, 1_2, \rho_3, \beta_2, 1_3, 1_4\}$ are dominated by H. Considering that $PC \setminus N[\{x, y\}] \cong P_{\alpha-1} \cup P_{\beta-2} \cup P_{\rho-2} \cup P_{\lambda-1}$ (see Fig. 14), we can deduce that $H \setminus \{x, y\}$ contains at least $\lceil \frac{\alpha-1}{3} \rceil + \lceil \frac{\beta-2}{3} \rceil + \lceil \frac{\rho-2}{3} \rceil + \lceil \frac{\lambda-1}{3} \rceil$ vetices. Consequently, we obtain the inequality $|H| \geq \lceil \frac{\alpha-1}{3} \rceil + \lceil \frac{\beta-2}{3} \rceil + \lceil \frac{\rho-2}{3} \rceil + \lceil \frac{\lambda-1}{3} \rceil + 2$. We can similarly derive observations for the remaining sixteen cases as follows:

Observation 3.11. *Let H be a γ-set of PC. In order to dominate x and y, there are sixteen cases.*

Case 1. $x, y \in H$. *Since* $PC \setminus N[\{x, y\}] \cong P_{\alpha-1} \cup P_{\beta-2} \cup P_{\rho-2} \cup P_{\lambda-1}$, *we get*

$$|H| \geq \left\lceil \frac{\alpha-1}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil + \left\lceil \frac{\lambda-1}{3} \right\rceil + 2. \quad (1)$$

Case 2. $x, \beta_2 \in H$. *Since* $PC \setminus N[\{x, \beta_2\}] \cong P_{\alpha-1} \cup P_{\beta-3} \cup P_{\rho-1} \cup P_{\lambda}$, *we get*

$$|H| \geq \left\lceil \frac{\alpha-1}{3} \right\rceil + \left\lceil \frac{\beta-3}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2. \quad (2)$$

Case 3. $x, 1_3 \in H$. *Since* $PC \setminus N[\{x, 1_3\}] \cong P_{\alpha-1} \cup P_{\beta-1} \cup P_{\rho-3} \cup P_{\lambda}$, *we get*

$$|H| \geq \left\lceil \frac{\alpha-1}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2. \quad (3)$$

Case 4. $x, 1_4 \in H$. *Since* $PC \setminus N[\{x, 1_4\}] \cong P_{\alpha-1} \cup P_{\beta-1} \cup P_{\rho-1} \cup P_{\lambda-2}$, *we get*

$$|H| \geq \left\lceil \frac{\alpha-1}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil + \left\lceil \frac{\lambda-2}{3} \right\rceil + 2. \quad (4)$$

Case 5. $1_1, y \in H$. *Since* $PC \setminus N[\{1_1, y\}] \cong P_{\alpha-2} \cup P_{\beta-1} \cup P_{\rho-1} \cup P_{\lambda-1}$, *we get*

$$|H| \geq \left\lceil \frac{\alpha-2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil + \left\lceil \frac{\lambda-1}{3} \right\rceil + 2. \quad (5)$$

Case 6. $1_1, \beta_2 \in H$. *Since* $PC \setminus N[\{1_1, \beta_2\}] \cong P_{\alpha-2} \cup P_{\beta-2} \cup P_{\rho} \cup P_{\lambda}$, *we get*

$$|H| \geq \left\lceil \frac{\alpha-2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2. \quad (6)$$

Case 7. $1_1, 1_3 \in H$. *Since* $PC \setminus N[\{1_1, 1_3\}] \cong P_{\alpha-2} \cup P_{\beta} \cup P_{\rho-2} \cup P_{\lambda}$, *we get*

$$|H| \geq \left\lceil \frac{\alpha-2}{3} \right\rceil + \left\lceil \frac{\beta}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2. \quad (7)$$

Case 8. $1_1, 1_4 \in H$. *Since* $PC \setminus N[\{1_1, 1_4\}] \cong P_{\alpha-2} \cup P_{\beta} \cup P_{\rho} \cup P_{\lambda-2}$, *we get*

$$|H| \geq \left\lceil \frac{\alpha-2}{3} \right\rceil + \left\lceil \frac{\beta}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil + \left\lceil \frac{\lambda-2}{3} \right\rceil + 2. \quad (8)$$

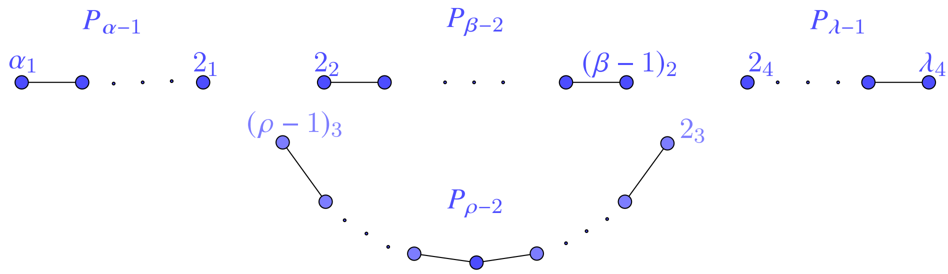

Fig. 14. $PC \setminus N[\{x, y\}] \cong P_{\alpha-1} \cup P_{\beta-2} \cup P_{\rho-2} \cup P_{\lambda-1}$.

Case 9. $1_2, y \in H$. Since $PC \setminus N[\{1_2, y\}] \cong P_\alpha \cup P_{\beta-3} \cup P_{\rho-1} \cup P_{\lambda-1}$, we get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta-3}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil + \left\lceil \frac{\lambda-1}{3} \right\rceil + 2. \tag{9}$$

Case 10. $1_2, \beta_2 \in H$. Since $PC \setminus N[\{1_2, \beta_2\}] \cong P_\alpha \cup P_{\beta-4} \cup P_\rho \cup P_\lambda$, we get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta-4}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2. \tag{10}$$

Case 11. $1_2, 1_3 \in H$. Since $PC \setminus N[\{1_2, 1_3\}] \cong P_\alpha \cup P_{\beta-2} \cup P_{\rho-2} \cup P_\lambda$, we get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2. \tag{11}$$

Case 12. $1_2, 1_4 \in H$. Since $PC \setminus N[\{1_2, 1_4\}] \cong P_\alpha \cup P_{\beta-2} \cup P_\rho \cup P_{\lambda-2}$, we get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil + \left\lceil \frac{\lambda-2}{3} \right\rceil + 2. \tag{12}$$

Case 13. $\rho_3, y \in H$. Since $PC \setminus N[\rho_3, y\}] \cong P_\alpha \cup P_{\beta-1} \cup P_{\rho-3} \cup P_{\lambda-1}$, we get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil + \left\lceil \frac{\lambda-1}{3} \right\rceil + 2. \tag{13}$$

Case 14. $\rho_3, \beta_2 \in H$. Since $PC \setminus N[\{\rho_3, \beta_2\}] \cong P_\alpha \cup P_{\beta-2} \cup P_{\rho-2} \cup P_\lambda$, we get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2. \tag{14}$$

Case 15. $\rho_3, 1_3 \in H$. Since $PC \setminus N[\{\rho_3, 1_3\}] \cong P_\alpha \cup P_\beta \cup P_{\rho-4} \cup P_\lambda$, we get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta}{3} \right\rceil + \left\lceil \frac{\rho-4}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2. \tag{15}$$

Case 16. $\rho_3, 1_4 \in H$. Since $PC \setminus N[\{\rho_3, 1_4\}] \cong P_\alpha \cup P_\beta \cup P_{\rho-2} \cup P_{\lambda-2}$, we get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil + \left\lceil \frac{\lambda-2}{3} \right\rceil + 2. \tag{16}$$

In Theorem 3.12-3.20, we will show that the order of a γ-set H of PC is greater than or equal to the upper bound given in Lemma 3.2-3.10, respectively. By Observation 3.11, in each theorem, we need to show that the lower bounds of $|H|$ in the the inequality (1)-(16) can be reduced to the upper bound given in the corresponding lemma.

Theorem 3.12. *Let* $\alpha \equiv 0(\mod 3)$ *and* $\lambda \equiv 0(\mod 3)$. *Then* $\gamma(G) =$

$$\begin{cases} \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil & \text{if } \beta \equiv 2(\text{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil & \text{if } \rho \equiv 2(\text{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil & \text{if } \beta \equiv 0(\text{mod}3) \\ & \text{and } \rho \not\equiv 2(\text{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil & \text{if } \beta \equiv 1(\text{mod}3) \\ & \text{and } \rho \equiv 0(\text{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil & \text{if } \beta \equiv 1(\text{mod}3) \\ & \text{and } \rho \equiv 1(\text{mod}3). \end{cases}$$

Proof. Let H be a γ-set of PC.

(1) It is clear that, for $i \in \{1, 2, ..., 16\} \setminus \{3, 13, 15\}$, the inequality (i) can be written in the form

$$|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil.$$

By Remark 3.1, we also get the inequalities (3), (13) and (15) can be written in the form

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta}{3} \right\rceil + \left\lceil \frac{\rho-4}{3} \right\rceil + \left\lceil \frac{\lambda-1}{3} \right\rceil + 2$$

$$\geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil.$$

Therefore, $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$. By (1) in Lemma 3.2 , $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) On substituting between β and ρ in (1), we obtain $|H| \geq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$. According to (2) in Lemma 3.2, we get $\gamma(G) = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) By Remark 3.1, for $i \in \{1, 2, ...16\} \setminus \{3, 13, 15\}$, we get the inequality (i) can be written in the form $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$. In other cases, by Remark 3.1, we also get

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta}{3} \right\rceil + \left\lceil \frac{\rho-4}{3} \right\rceil + \left\lceil \frac{\lambda-1}{3} \right\rceil + 2$$

$$\geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil.$$

Hence, $H \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$. From (3) in Lemma 3.2, we can conclude that $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(4) By replacing between β and ρ in (3), we get $|H| \geq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$. Therefore, $\gamma(G) = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$ by (4) in Lemma 3.2.

(5) It is evident that the inequalities (6), (8), (10) and (12) can be written in the form $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$. By Remark 3.1, we get the inequality (15) can be written in the form

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta+2}{3} \right\rceil + \left\lceil \frac{\rho-4}{3} \right\rceil$$

$$+\left\lceil\frac{\lambda}{3}\right\rceil+2$$
$$\geq\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho}{3}\right\rceil.$$

Also by Remark 3.1, for $i \in \{1,2,...16\}\backslash\{6,8,10,12,15\}$, we get the inequality (i) can be written in the form

$$|H|\geq\left\lceil\frac{\alpha}{3}\right\rceil+\left\lceil\frac{\beta-1}{3}\right\rceil+\left\lceil\frac{\rho-1}{3}\right\rceil$$
$$+\left\lceil\frac{\lambda}{3}\right\rceil+2$$
$$\geq\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho}{3}\right\rceil.$$

Hence, $|H|\geq\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho}{3}\right\rceil$. By (5) in Lemma 3.2, $\gamma(G)=\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho}{3}\right\rceil$.

□

Theorem 3.13. *Let* $\alpha\equiv 0(\mod 3)$ *and* $\lambda\equiv 1(\mod 3)$. *Then* $\gamma(G)=$

$$\begin{cases}\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-3}{3}\right\rceil & \text{if } \beta\equiv 1(\text{mod}3),\\ \left\lceil\frac{\alpha+\rho+\lambda+2}{3}\right\rceil+\left\lceil\frac{\beta-3}{3}\right\rceil & \text{if } \rho\equiv 1(\text{mod}3),\\ \left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-2}{3}\right\rceil & \text{if } \beta\equiv 2(\text{mod}3) \\ & \text{and } \rho\not\equiv 1(\text{mod}3),\\ \left\lceil\frac{\alpha+\rho+\lambda+2}{3}\right\rceil+\left\lceil\frac{\beta-2}{3}\right\rceil & \text{if } \beta\equiv 0(\text{mod}3) \\ & \text{and } \rho\equiv 2(\text{mod}3),\\ \left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-1}{3}\right\rceil & \text{if } \beta\equiv 0(\text{mod}3) \\ & \text{and } \rho\equiv 0(\text{mod}3).\end{cases}$$

Proof. (1) Obviously, we can rewrite the inequalities (1)-(16) except (15) as $H\geq\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil\left\lceil\frac{\rho-3}{3}\right\rceil$. By Remark 3.1, the inequality (15) is satisfied

$$|H|\geq\left\lceil\frac{\alpha}{3}\right\rceil+\left\lceil\frac{\beta}{3}\right\rceil+\left\lceil\frac{\rho-4}{3}\right\rceil+\left\lceil\frac{\lambda}{3}\right\rceil+2$$
$$\geq\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-3}{3}\right\rceil.$$

Now $\gamma(G)=\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-3}{3}\right\rceil$ which is due to (1) in Lemma 3.3.

(2) The proof of $|H|\geq\left\lceil\frac{\alpha+\rho+\lambda+2}{3}\right\rceil+\left\lceil\frac{\beta-3}{3}\right\rceil$ is similar in spirit to (1) when we substitute between β and ρ. By (2) in Lemma 3.3, we can assert that $\gamma(G)=\left\lceil\frac{\alpha+\rho+\lambda+2}{3}\right\rceil+\left\lceil\frac{\beta-3}{3}\right\rceil$.

(3) It is clear that, for $i\in\{1,2,...16\}\backslash\{3,13,15\}$, the inequality (i) can be written as

$$|H|\geq\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-2}{3}\right\rceil.$$

By Remark 3.1, we can write the inequalities (3), (13) and (15) in the form

$$|H|\geq\left\lceil\frac{\alpha}{3}\right\rceil+\left\lceil\frac{\beta+1}{3}\right\rceil+\left\lceil\frac{\rho-4}{3}\right\rceil$$
$$+\left\lceil\frac{\lambda-1}{3}\right\rceil+2$$
$$\geq\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-2}{3}\right\rceil.$$

By (3) in Lemma 3.3, we obtain that $\gamma(G)=\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-2}{3}\right\rceil$.

(4) By replacing between β and ρ in (3) and by (4) in Lemma 3.3, we obtain $\gamma(G)=\left\lceil\frac{\alpha+\rho+\lambda+2}{3}\right\rceil+\left\lceil\frac{\beta-2}{3}\right\rceil$.

(5) By Remark 3.1, for $i\in\{1,2,...16\}\backslash\{3,13,15\}$, the inequality (i) becomes $|H|\geq\left\lceil\frac{\alpha+\beta+\lambda+2}{3}\right\rceil+\left\lceil\frac{\rho-1}{3}\right\rceil$. By

Remark 3.1 again, we can conclude that (3), (13) and (15) satisfy

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta}{3} \right\rceil + \left\lceil \frac{\rho - 4}{3} \right\rceil + \left\lceil \frac{\lambda - 1}{3} \right\rceil + 2 \geq \left\lceil \frac{\alpha + \beta + \lambda + 2}{3} \right\rceil + \left\lceil \frac{\rho - 1}{3} \right\rceil.$$

Hence $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$ which is clear from (5) in Lemma 3.3. □

Theorem 3.14. *Let* $\alpha \equiv 0(\mod 3)$ *and* $\lambda \equiv 2(\mod 3)$. *Then* $\gamma(G) =$

$$\begin{cases} \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil & \text{if } \beta \equiv 0(\text{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-3}{3} \right\rceil & \text{if } \rho \equiv 0(\text{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil & \text{if } \beta \equiv 1(\text{mod}3) \text{ and } \rho \not\equiv 0(\text{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil & \text{if } \beta \equiv 2(\text{mod}3) \text{ and } \rho \equiv 1(\text{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil & \text{if } \beta \equiv 2(\text{mod}3) \text{ and } \rho \equiv 2(\text{mod}3). \end{cases}$$

Proof. (1) By Remark 3.1, we can write (3), (13) and (15) as

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta - 1}{3} \right\rceil + \left\lceil \frac{\rho - 4}{3} \right\rceil + \left\lceil \frac{\lambda + 1}{3} \right\rceil + 2 \geq \left\lceil \frac{\alpha + \beta + \lambda + 2}{3} \right\rceil + \left\lceil \frac{\rho - 2}{3} \right\rceil.$$

It is easy to check that, for $i \in \{1, 2, ...16\} \setminus \{3, 13, 15\}$, we can write the inequality (i) as $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil \left\lceil \frac{\rho-2}{3} \right\rceil$. We conclude from (1) in Lemma 3.4 that $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) By substituting between β and ρ in (1), we get $|H| \geq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$. By (2) in Lemma 3.4, we can conclude that $\gamma(G) = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) We see at once that (2), (4), (5), (6), (8), (9), (10) and (12) can be written as $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$. By Remark 3.1, we can write (1), (3), (7), (11), (13), (14) and (15) in the form

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta - 1}{3} \right\rceil + \left\lceil \frac{\rho - 4}{3} \right\rceil + \left\lceil \frac{\lambda}{3} \right\rceil + 2 \geq \left\lceil \frac{\alpha + \beta + \lambda + 2}{3} \right\rceil + \left\lceil \frac{\rho - 1}{3} \right\rceil.$$

By Remark 3.1 agian, we can write the inequality (15) as

$$|H| \geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta + 1}{3} \right\rceil + \left\lceil \frac{\rho - 2}{3} \right\rceil + \left\lceil \frac{\lambda - 2}{3} \right\rceil + 2 \geq \left\lceil \frac{\alpha + \beta + \lambda + 2}{3} \right\rceil + \left\lceil \frac{\rho - 1}{3} \right\rceil.$$

Therefore, $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$. We conclude from (3) in Lemma 3.4 that $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(4) By replacing between β and ρ in (3) and by Lemma 2.3 (4), we obtain $\gamma(G) = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$.

(5) We check at once that (2), (4), (5), (6), (8), (9), (10) and (12) can be written as $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$. By Remark 3.1, we can write (3), (7), (13) and (16) as

$$\begin{aligned} |H| &\geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta+1}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil \\ &\quad + \left\lceil \frac{\lambda-2}{3} \right\rceil + 2 \\ &\geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil . \end{aligned}$$

By Remark 3.1 again, we also can write (1), (11), (14) and (15) as

$$\begin{aligned} |H| &\geq \left\lceil \frac{\alpha}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil + \left\lceil \frac{\rho-3}{3} \right\rceil \\ &\quad + \left\lceil \frac{\lambda+1}{3} \right\rceil + 2 \\ &\geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil . \end{aligned}$$

Therefore, $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$. So $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$ which is clear from (5) in Lemma 3.4.

□

On substituting between α and λ in Theorem 3.13, we obtain the below theorem.

Theorem 3.15. *Let* $\alpha \equiv 1(\text{mod}3)$ *and* $\lambda \equiv 0(\text{mod}3)$. *Then* $\gamma(G) =$

$$\begin{cases} \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil & \text{if } \beta \equiv 1(\text{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil & \text{if } \rho \equiv 1(\text{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil & \text{if } \beta \equiv 2(\text{mod}3) \\ & \text{and } \rho \not\equiv 1(\text{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil & \text{if } \beta \equiv 0(\text{mod}3) \\ & \text{and } \rho \equiv 2(\text{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil & \text{if } \beta \equiv 0(\text{mod}3) \\ & \text{and } \rho \equiv 0(\text{mod}3). \end{cases}$$

Theorem 3.16. *If* $\alpha \equiv 1(\text{mod}3)$ *and* $\lambda \equiv 1(\text{mod}3)$, *then* $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

Proof. By Lemma 3.6, we get $\gamma(G) \leq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$. It remains to prove that there is a γ-set H of G such that

$$|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil . \quad (17)$$

We check at once that (1)-(16) except (3), (13) and (15) satisfy (17). By Remark 3.1, (3), (13) and (15) also can be written as in (17). Therefore, $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

□

Theorem 3.17. *Let* $\alpha \equiv 1(\mathrm{mod}3)$ *and* $\lambda \equiv 2(\mathrm{mod}3)$. *Then* $\gamma(G) =$

$$\begin{cases} \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil & \text{if } \beta \equiv 2(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil & \text{if } \rho \equiv 2(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil & \text{if } \beta \not\equiv 2(\mathrm{mod}3) \\ & \text{and } \rho \not\equiv 2(\mathrm{mod}3). \end{cases}$$

Proof. (1) It follows immediately that, for $i \in \{1, 2, ..., 16\} \setminus \{3, 13, 15\}$, we can write the inequality (i) as $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$. By Remark 3.1, the inequalities (3), (13) and (15) can be written as

$$\begin{aligned} |H| &\geq \left\lceil \frac{\alpha-1}{3} \right\rceil + \left\lceil \frac{\beta+1}{3} \right\rceil + \left\lceil \frac{\rho-4}{3} \right\rceil \\ &\quad + \left\lceil \frac{\lambda-1}{3} \right\rceil + 2 \\ &\geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil. \end{aligned}$$

Thus, $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$. We conclude from this inequality and (1) in Lemma 3.7 that $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil$.

(2) It is clear form substituting between β and ρ in (1) that $|H| \geq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$. By (2) in Lemma 3.7, we have $\gamma(G) = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil$.

(3) Since $\left\lceil \frac{\rho-2}{3} \right\rceil = \left\lceil \frac{\rho-1}{3} \right\rceil$, the inequality (i) can be written as $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$ for all $i \in \{1, 2, ...16\} \setminus \{3, 13, 15\}$. By Remark 3.1, we can write (3), (13) and (15) in the form $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$. From (3) in Lemma 3.7, we have $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

□

The two theorems we stated below come from substituting between α and λ in Theorem 3.14 and Theorem 3.17, respectively.

Theorem 3.18. *Let* $\alpha \equiv 2(\mathrm{mod}3)$ *and* $\lambda \equiv 0(\mathrm{mod}3)$. *Then* $\gamma(G) =$

$$\begin{cases} \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil & \text{if } \beta \equiv 0(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil & \text{if } \rho \equiv 0(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil & \text{if } \beta \equiv 1(\mathrm{mod}3) \\ & \text{and } \rho \not\equiv 0(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil & \text{if } \beta \equiv 2(\mathrm{mod}3) \\ & \text{and } \rho \equiv 1(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil & \text{if } \beta \equiv 2(\mathrm{mod}3) \\ & \text{and } \rho \equiv 2(\mathrm{mod}3). \end{cases}$$

Theorem 3.19. *Let* $\alpha \equiv 2(\mathrm{mod}3)$ *and* $\lambda \equiv 1(\mathrm{mod}3)$. *Then* $\gamma(G) =$

$$\begin{cases} \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-2}{3} \right\rceil & \text{if } \beta \equiv 2(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-2}{3} \right\rceil & \text{if } \rho \equiv 2(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil & \text{if } \beta \not\equiv 2(\mathrm{mod}3) \\ & \text{and } \rho \not\equiv 2(\mathrm{mod}3). \end{cases}$$

Theorem 3.20. *Let* $\alpha \equiv 2(\mathrm{mod}3)$ *and* $\lambda \equiv 2(\mathrm{mod}3)$. *Then* $\gamma(G) =$

$$\begin{cases} \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil & \text{if } \beta \equiv 1(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil & \text{if } \rho \equiv 1(\mathrm{mod}3), \\ \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil & \text{if } \beta \not\equiv 1(\mathrm{mod}3) \\ & \text{and } \rho \not\equiv 1(\mathrm{mod}3). \end{cases}$$

Proof. (1) It follows easily that (2), (4), (5), (6), (8), (9), (10) and (12) can be written as

$$|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil. \quad (18)$$

By Remark 3.1, the inequalities (1), (3), (7), (11), (13), (14), (15) and (16) are satisfied (18).

Thus, $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$. From (1) in Lemma 3.10, we have $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho-1}{3} \right\rceil$.

(2) It is clear from substituting between β and ρ in (1) that $|H| \geq \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$. By (2) in Lemma 3.10, we have $\gamma(G) = \left\lceil \frac{\alpha+\rho+\lambda+2}{3} \right\rceil + \left\lceil \frac{\beta-1}{3} \right\rceil$.

(3) We conclude from Remark 3.1 that the inequalities (1)-(16) can be written as $|H| \geq \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$. By (3) in Lemma 3.10, we obtain that $\gamma(G) = \left\lceil \frac{\alpha+\beta+\lambda+2}{3} \right\rceil + \left\lceil \frac{\rho}{3} \right\rceil$.

□

From Theorem 3.12-3.20, we obtain the exact value of $\gamma(PC(\alpha,\beta,\rho,\lambda))$ for all $\beta,\rho \in \mathbb{N}$ and for all $\alpha,\lambda \in \mathbb{N} \cup \{0\}$, not both zero. Consequently, we are able to derive the exact value of $\gamma(P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t)$ by substituting α,β,ρ and λ with $j, s-2, t-s$ and $r-(j+s)$, respectively.

Conclusion

In this research, our focus has been on investigating the domination number of the amalgamation $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$ with the condition that $H_1 \cong H_2 \cong P_s$. To tackle this problem, we employed congruence properties modulo 3. For cases where $s \in \{1, 2, \min\{r,t\}\}$, we utilized these congruence properties to identify a minimum dominating set and calculate the domination number of $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$. However, for the cases where $s \in \{3, 4, \ldots, \min\{r,t\}-1\}$, we adopted a different approach. We constructed a graph denoted as $PC(\alpha,\beta,\rho,\lambda)$ using four paths: $P_\alpha, P_\beta, P_\rho, P_\lambda$. Furthermore, we established that if $V(H_1) = \{v_{j+1}, v_{j+2}, \ldots, v_{j+s}\}$, then $P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t \cong PC(\alpha,\beta,\rho,\lambda)$, where $\alpha = j$, $\beta = s-2$, $\rho = t-s$, and $\lambda = r-(\alpha+s)$. Having established this correspondence, we then proceeded to determine the domination number $\gamma(P_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t)$ by using the corresponding value of $\gamma(PC(\alpha,\beta,\rho,\lambda))$. By employing this approach, we gained valuable insights into the domination number of the amalgamation and its relationship with the constructed graph $PC(\alpha,\beta,\rho,\lambda)$.

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