



# C-Class $F$ -Contraction in $C^*$ -Algebra Valued Metric Space

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## ABSTRACT

In the present manuscript, we enlarge the class of  $F$ -contraction in the framework of  $C^*$ -algebra valued metric space. We present some results on fixed points with the help of  $C$ -class function for different types of  $F$ -contractive condition. The result is an extension and generalization of several metric space results available. Moreover, some examples are presented here to illustrate the usability of obtained results.

**Keywords:**  $C$ -class function;  $C^*$ -algebra valued metric space;  $F$ -contraction

## 1. Introduction and Preliminaries

The study of fixed point theory for self mappings has been a fascinating area of research for the last few decades. The formal theoretical approach to the fixed point originated from Picard's work. However, it was Stefan Banach [1] who underlined the idea into an abstract framework and provided a constructive tool to establish the fixed points of mapping in the metric space. In 2014, Ma et al. [2] extended the Banach contraction principle to  $C^*$ -algebra valued metric spaces by replacing the set of real numbers with the set of all positive members of unital  $C^*$ -algebra. H. Massit and M. Rosaafi [3] inspired by the work of H. Piri et al. [4], introduced the concept of  $(\phi, F)$ -

contraction in  $C^*$ -algebra valued metric space and proved some fixed point results. Later on, M. Rossafi et al. [5] generalized  $(\phi, MF)$ -contraction and established some fixed point results on  $C^*$ -algebra valued metric space. Many researchers have obtained various results in this theory of fixed points and common fixed points (for references, see [6–17] and references therein).

Inspired by the work of Rossafi et al. [5], in this manuscript, we enlarge the class of  $F$ -contraction in the framework of  $C^*$ -algebra valued metric space. We present some results on fixed points with the help of  $C$ -class functions for different types of  $F$ -contractive conditions. The result is an extension and generalization of several met-

ric space results available. Moreover, some examples are presented here to illustrate the usability of obtained results.

We gave some notation and definition mentioned in [2], which will be required in sequel to prove the results.

**Definition 1.1** ([2]). Suppose  $X$  is a non-empty set. A function  $d : X \times X \rightarrow \mathbb{A}$  satisfies :

- (i)  $d(\sigma, \rho) \geq \theta_{\mathbb{A}}$  and  $d(\sigma, \rho) = \theta_{\mathbb{A}}$  if and only if  $\sigma = \rho$ ;
- (ii)  $d(\sigma, \rho) = d(\rho, \sigma)$ ;
- (iii)  $d(\sigma, \rho) \leq d(\sigma, \mu) + d(\mu, \rho)$ ;

for all  $\sigma, \rho, \mu \in X$ . Then,  $d$  is called a  $C^*$ -algebra valued metric and  $(X, \mathbb{A}, d)$  is called a  $C^*$ -algebra valued metric space.

**Definition 1.2** ([2]). A sequence  $\{\sigma_n\}$  in  $(X, \mathbb{A}, d)$  is said to be

1. (a) convergent with respect to  $\mathbb{A}$ , if for given  $\epsilon > 0$ , there exists a positive integer  $k$  such that  $\|d(\sigma_n, \sigma)\| < \epsilon$ , for all  $n > k$ ;
- (b) Cauchy sequence with respect to  $\mathbb{A}$  if for any  $\epsilon > 0$ , there exists  $k \in \mathbb{N}$  such that  $\|d(\sigma_n, \sigma_m)\| < \epsilon$ , for all  $n, m > k$ .
2.  $(X, \mathbb{A}, d)$  is called a complete  $C^*$ -algebra valued metric space if every Cauchy sequence with respect to  $\mathbb{A}$  is convergent.

**Definition 1.3** ([10]). Let  $\Psi_{\mathbb{A}}$  be the set of positive functions,  $\psi_{\mathbb{A}} : \mathbb{A}^+ \rightarrow \mathbb{A}^+$  satisfying the following conditions :

- (i)  $\psi_{\mathbb{A}}(a)$  is continuous and nondecreasing;

- (ii)  $\psi_{\mathbb{A}}(a) = \theta_{\mathbb{A}}$  if and only if  $a = \theta_{\mathbb{A}}$ .

**Definition 1.4** ([10]). ( $C$ - Class Function)

A continuous function  $F^* : \mathbb{A}^+ \times \mathbb{A}^+ \rightarrow \mathbb{A}^+$  is called a  $C$ -class function if for any  $P, Q \in \mathbb{A}^+$ , the following conditions hold :

- (i)  $F^*(P, Q) \leq P$ ;
- (ii)  $F^*(P, Q) = P$  implies that either  $P = \theta_{\mathbb{A}}$  or  $Q = \theta_{\mathbb{A}}$ .

An extra condition could be imposed on  $F^*$  if required such that  $F^*(\theta_{\mathbb{A}}, \theta_{\mathbb{A}}) = \theta_{\mathbb{A}}$ .

## 2. Main Result

In this section, we give some fixed point results using the different types of  $F$ -contractive conditions with  $C$ -class function. We also discuss some examples to illustrate the obtained results.

**Theorem 2.1.** Let  $(X, \mathbb{A}, d)$  be a complete  $C^*$ -algebra valued metric space. Define  $T$  be self mapping on  $X$  and there exists  $F : \mathbb{A}^+ \rightarrow \mathbb{A}$  satisfying the following conditions:

- (i)  $F$  is continuous and nondecreasing on  $\mathbb{A}^+$ ;
- (ii) For each sequence  $\{\alpha_n\} \subseteq \mathbb{A}^+$

$$\lim_{n \rightarrow \infty} \alpha_n = \theta_{\mathbb{A}} \text{ if and only if } \lim_{n \rightarrow \infty} F(\alpha_n) = \theta_{\mathbb{A}}; \quad (2.1)$$

- (iii) for every  $\zeta, \eta \in X$  with  $\tau > 0$ ,

$$\tau + F(\psi(d(T\zeta, T\eta))) \leq F^* \left( F(\psi(d(\zeta, \eta))), F(\phi(d(\zeta, \eta))) \right); \quad (2.2)$$

where  $F^*$  is  $C$ -class function and  $\psi, \phi \in \Psi$ . Then,  $T$  has unique fixed point.

*Proof.* First, let us observe that  $T$  has at most one fixed point. Indeed if

$$\zeta_1, \zeta_2 \in X; T\zeta_1 = \zeta_1 \neq \zeta_2 = T\zeta_2.$$

From Eq. (2.2), we have

$$\begin{aligned} \tau &+ F(\psi(d(T\zeta_1, T\zeta_2))) \\ &\leq F^*\left(F(\psi(d(\zeta_1, \zeta_2))), \right. \\ &\quad \left. F(\phi(d(\zeta_1, \zeta_2)))\right) \\ &\leq F(\psi(d(\zeta_1, \zeta_2))), \end{aligned}$$

implies that  $\tau \leq F(\psi(d(\zeta_1, \zeta_2))) - F(\psi(d(T\zeta_1, T\zeta_2))) = \theta_{\mathbb{A}}$ , which is a contradiction.

In order to show that a fixed point, let  $\zeta_0 \in X$ . We construct a sequence  $\{\zeta_n\}$  in  $X$  as follows  $\zeta_{n+1} = T\zeta_n$  for all  $n \in \mathbb{N}$ . If  $\zeta_n = \zeta_{n+1}$  for some  $n \in \mathbb{N}$ , then  $\zeta_n$  is a fixed point of  $T$ . Now, suppose that  $\zeta_n \neq \zeta_{n+1}$  for all  $n \in \mathbb{N}$ . Define  $d_n = d(\zeta_n, \zeta_{n+1})$ . Substitute  $\zeta = \zeta_{n+1}$  and  $\eta = \zeta_n$  in (2.2), we have

$$\begin{aligned} F(\psi(d(\zeta_{n+1}, \zeta_n))) &= F(\psi(d(T\zeta_n, T\zeta_{n-1}))) \\ &\leq \tau + F(\psi(d(T\zeta_n, T\zeta_{n-1}))) \\ &\leq F^*\left(F(\psi(d(\zeta_n, \zeta_{n-1}))), \right. \\ &\quad \left. F(\phi(d(\zeta_n, \zeta_{n-1})))\right) \leq F(\psi(d(\zeta_n, \zeta_{n-1}))). \end{aligned}$$

Therefore,

$$F(\psi(d_n)) \leq F(\psi(d_{n-1})). \quad (2.3)$$

Since,  $F$  is non-decreasing and so the sequence  $\{\psi(d_n)\}$  is monotonically decreasing in  $\mathbb{A}^+$ . Hence,  $\psi$  is non-decreasing in  $\mathbb{A}^+$  so the sequence  $\{d_n\}$  is monotonically decreasing in  $\mathbb{A}^+$ . Therefore, there exists  $\theta_{\mathbb{A}} \leq t \in \mathbb{A}^+$  such that

$$d(\zeta_n, \zeta_{n+1}) \rightarrow t \quad \text{as } n \rightarrow \infty.$$

From Eq. (2.3), we get  $\lim_{n \rightarrow \infty} F(\psi(d_n)) = \theta_{\mathbb{A}}$ . From Eq. (2.1), together we have

$$\lim_{n \rightarrow \infty} \psi(d_n) = \theta_{\mathbb{A}} \quad \text{implies} \quad \lim_{n \rightarrow \infty} d_n = \theta_{\mathbb{A}}. \quad (2.4)$$

Now, we shall show that  $\{\zeta_n\}$  is a Cauchy sequence in  $(X, \mathbb{A}, d)$ .

Assume that  $\{\zeta_n\}$  is not a Cauchy sequence in  $(X, \mathbb{A}, d)$ . Then, there exist  $\epsilon > 0$  and subsequences  $(\zeta_{m_k})$  and  $(\zeta_{n_k})$  with  $n_k > m_k > k$  such that

$$\|d(\zeta_{m_k}, \zeta_{n_k})\| \geq \epsilon. \quad (2.5)$$

Now, corresponding to  $m_k$  we can choose  $n_k$  such that it is the smallest integer with  $n_k > m_k$  and satisfying Eq. (2.5). Hence,

$$\|d(\zeta_{m_k}, \zeta_{n_k-1})\| < \epsilon. \quad (2.6)$$

Using Eqs. (2.5)-(2.6), we get

$$\begin{aligned} \epsilon &\leq \|d(\zeta_{m_k}, \zeta_{n_k})\| \\ &\leq \|d(\zeta_{m_k}, \zeta_{n_k-1})\| + \|d(\zeta_{n_k-1}, \zeta_{n_k})\| \\ &\leq \epsilon + \|d(\zeta_{n_k-1}, \zeta_{n_k})\|. \end{aligned} \quad (2.7)$$

From Eq. (2.4), we have

$$\lim_{k \rightarrow \infty} \|d(\zeta_{n_k-1}, \zeta_{n_k})\| = \theta_{\mathbb{A}}. \quad (2.8)$$

Taking limit as  $k \rightarrow \infty$  in Eq. (2.7) and using Eq. (2.8), we get

$$\lim_{k \rightarrow \infty} \|d(\zeta_{m_k}, \zeta_{n_k})\| = \epsilon. \quad (2.9)$$

Again,

$$\begin{aligned} &\|d(\zeta_{n_k}, \zeta_{m_k})\| \\ &\leq \|d(\zeta_{n_k}, \zeta_{n_k-1})\| + \|d(\zeta_{n_k-1}, \zeta_{m_k})\| \\ &\leq \|d(\zeta_{n_k}, \zeta_{n_k-1})\| + \|d(\zeta_{n_k-1}, \zeta_{m_k-1})\| \\ &\quad + \|d(\zeta_{m_k-1}, \zeta_{m_k})\|. \end{aligned} \quad (2.10)$$

Also,

$$\|d(\zeta_{n_k-1}, \zeta_{m_k-1})\| \leq$$

$$\begin{aligned} & \|d(\zeta_{n_k-1}, \zeta_{n_k})\| + \|d(\zeta_{n_k}, \zeta_{m_k-1})\| \\ \leq & \|d(\zeta_{n_k-1}, \zeta_{n_k})\| + \|d(\zeta_{n_k}, \zeta_{m_k})\| \\ & + \|d(\zeta_{m_k}, \zeta_{m_k-1})\|. \end{aligned} \quad (2.11)$$

Taking limit as  $k \rightarrow \infty$  in Eqs. (2.10)-(2.11) and using Eqs. (2.8)-(2.9), we get

$$\lim_{k \rightarrow \infty} \|d(\zeta_{n_k-1}, \zeta_{m_k-1})\| = \epsilon.$$

Since,  $d(\zeta_{n_k-1}, \zeta_{m_k-1}), d(\zeta_{n_k}, \zeta_{m_k}) \in \mathbb{A}^+$  and

$$\begin{aligned} \lim_{k \rightarrow \infty} \|d(\zeta_{n_k-1}, \zeta_{m_k-1})\| &= \\ \lim_{k \rightarrow \infty} \|d(\zeta_{n_k}, \zeta_{m_k})\| &= \epsilon, \end{aligned}$$

there exist  $s \in \mathbb{A}^+$  with  $\|s\| = \epsilon$  such that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|d(\zeta_{n_k-1}, \zeta_{m_k-1})\| &= \\ \lim_{k \rightarrow \infty} \|d(\zeta_{n_k}, \zeta_{m_k})\| &= s. \end{aligned}$$

By Eq. (2.2), we have

$$\begin{aligned} F(\psi(s)) &= \lim_{k \rightarrow \infty} F(\psi(d(\zeta_{n_k}, \zeta_{m_k}))) \\ &= \lim_{k \rightarrow \infty} F(\psi(d(T\zeta_{n_k-1}, T\zeta_{m_k-1}))) \\ &\leq \tau + \lim_{k \rightarrow \infty} F(\psi(d(T\zeta_{n_k-1}, T\zeta_{m_k-1}))) \\ &\leq \lim_{k \rightarrow \infty} F^* \left( F(\psi(d(\zeta_{n_k-1}, \zeta_{m_k-1})), \right. \\ & \quad \left. F(\phi(d(\zeta_{n_k-1}, \zeta_{m_k-1}))) \right) \\ &\leq \lim_{k \rightarrow \infty} F(\psi(d(\zeta_{n_k-1}, \zeta_{m_k-1}))) \\ &= F(\psi(s)). \end{aligned}$$

Therefore,  $F(\psi(s)) = \theta_{\mathbb{A}}$ . Hence,  $\psi(s) = \theta_{\mathbb{A}}$  implies  $s = \theta_{\mathbb{A}}$ , which is a contradiction. Hence, we get  $\{\zeta_n\}$  is a Cauchy sequence in a complete  $C^*$ -algebra valued metric space  $(X, \mathbb{A}, d)$ . Thus, there exists  $\zeta \in X$  such that  $\zeta_n \rightarrow \zeta$  as  $n \rightarrow \infty$ .

Consider,

$$\|F(\psi(d(\zeta, T\zeta)))\| \leq$$

$$\begin{aligned} & \tau + \lim_{n \rightarrow \infty} \|F(\psi(d(\zeta_n, T\zeta_n)))\| \\ & \leq \lim_{n \rightarrow \infty} \left\| F^* \left( F(\psi(d(\zeta_{n-1}, \zeta_n))), \right. \right. \\ & \quad \left. \left. F(\phi(d(\zeta_{n-1}, \zeta_n))) \right) \right\| \\ & \leq \lim_{n \rightarrow \infty} \|F(\psi(d(\zeta_{n-1}, \zeta_n)))\| \\ & = \|F(\psi(d(\zeta, \zeta)))\| \\ & = \|F(\psi(\theta_{\mathbb{A}}))\| \\ & = \|F(\theta_{\mathbb{A}})\| = \theta_{\mathbb{A}}. \end{aligned}$$

Therefore, we get  $\|F(\psi(d(\zeta, T\zeta)))\| = \theta_{\mathbb{A}}$  implies  $\psi(d(\zeta, T\zeta)) = \theta_{\mathbb{A}}$ . Hence,  $T\zeta = \zeta$ , i.e.  $\zeta$  is a fixed point of  $T$ . This completes the proof.  $\square$

Taking  $F^*(r, t) = r$  and  $\psi(t) = t = \phi(t)$ , we have the following result.

**Corollary 2.2.** *Let  $(X, \mathbb{A}, d)$  be a complete  $C^*$ -algebra valued metric space. Define  $T$  be self mapping on  $X$  and there exists  $F : \mathbb{A}^+ \rightarrow \mathbb{A}$  satisfying the following conditions:*

- (i)  *$F$  is continuous and nondecreasing on  $\mathbb{A}^+$ ;*
- (ii) *For each sequence  $\{\alpha_n\} \subseteq \mathbb{A}^+$*

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n &= \theta_{\mathbb{A}} \text{ if and only if} \\ \lim_{n \rightarrow \infty} F(\alpha_n) &= \theta_{\mathbb{A}}; \end{aligned}$$

- (iii) *for every  $\zeta, \eta \in X$  with  $\tau > 0$ ,*

$$\tau + F(d(T\zeta, T\eta)) \leq F(d(\zeta, \eta)).$$

*Then,  $T$  has a unique fixed point.*

### 3. Remarks

In Eq. (2.2), we can extend the following version of well known results of literature for self mapping in the framework of  $C^*$ -algebra valued metric space.

(1) (Kannan type, see [18]) for every  $\zeta, \eta \in X$  with  $\tau > 0$ ,

$$\begin{aligned} & \tau + F(\psi(d(T\zeta, T\eta))) \\ & \leq F^* \left( F \left( \psi \left( \frac{d(\zeta, T\zeta) + d(\eta, T\eta)}{2} \right) \right) \right), \\ & F \left( \phi \left( \frac{d(\zeta, T\zeta) + d(\eta, T\eta)}{2} \right) \right). \end{aligned} \quad (3.1)$$

(2) (Chatterjea type, see [19]) for every  $\zeta, \eta \in X$  with  $\tau > 0$ ,

$$\begin{aligned} & \tau + F(\psi(d(T\zeta, T\eta))) \\ & \leq F^* \left( F \left( \psi \left( \frac{d(\zeta, T\eta) + d(\eta, T\zeta)}{2} \right) \right) \right), \\ & F \left( \phi \left( \frac{d(\zeta, T\eta) + d(\eta, T\zeta)}{2} \right) \right). \end{aligned}$$

(3) (Reich type, see [20]) for every  $\zeta, \eta \in X$  with  $\tau > 0$ ,

$$\begin{aligned} & \tau + F(\psi(d(T\zeta, T\eta))) \\ & \leq F^* \left( F \left( \psi \left( \frac{d(\zeta, \eta) + d(\zeta, T\zeta) + d(\eta, T\eta)}{3} \right) \right) \right), \\ & F \left( \phi \left( \frac{d(\zeta, \eta) + d(\zeta, T\zeta) + d(\eta, T\eta)}{3} \right) \right). \end{aligned}$$

(4) (Hardy-Roger type, see [21]) for every  $\zeta, \eta \in X$  with  $\tau > 0$ ,

$$\begin{aligned} & \tau + F(\psi(d(T\zeta, T\eta))) \\ & \leq F^* (F(\psi(d(\zeta, \eta) + d(\zeta, T\zeta) + \\ & d(\eta, T\eta) + d(\eta, T\zeta) + d(\zeta, T\eta))/5), \\ & F(\phi(d(\zeta, \eta) + d(\zeta, T\zeta) + \\ & d(\eta, T\eta) + d(\eta, T\zeta) + d(\zeta, T\eta))/5)). \end{aligned}$$

(5) (Weak  $F$ -contraction type, see [22]) for every  $\zeta, \eta \in X$  with  $\tau > 0$ ,

$$\begin{aligned} & \tau + F(\psi(d(T\zeta, T\eta))) \\ & \leq F^* \left( F(\psi(M(\zeta, \eta))), \right. \\ & \left. F(\phi(M(\zeta, \eta))) \right), \end{aligned}$$

where

$$M(\zeta, \eta) = \max \left\{ d(\zeta, \eta), d(\zeta, T\zeta), d(\eta, T\eta), \frac{d(\zeta, T\eta) + d(\eta, T\zeta)}{2} \right\}.$$

**Example 3.1.** Let  $X = [0, 2]$  and  $\mathbb{A} = \mathbb{C}$ . Let  $F : \mathbb{A}^+ \rightarrow \mathbb{A}$  defined as  $F(a) = 25a$  and  $d : X \times X \rightarrow \mathbb{A}$  defined by

$$d(\rho, \sigma) = \begin{cases} |\rho| + |\sigma| & \text{if } \rho \neq \sigma \\ 0 & \text{if } \rho = \sigma. \end{cases}$$

Define,  $T\rho = \frac{\rho}{150}$ .

Then,

- (i)  $(X, \mathbb{A}, d)$  be a complete  $C^*$ -algebra valued metric space;
- (ii)  $F$  is non-decreasing;
- (iii)  $F$  is continuous;
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0$  and  $\lim_{n \rightarrow \infty} F(\sigma_n) = 0$ ;
- (v) for all  $\sigma, \rho \in X$  with  $\tau = 0.1$ ,  $F^*(r, t) = r$ ,  $\psi(t) = \phi(t) = 5t$ , we get  $d(T\rho, T\sigma) = \frac{\rho + \sigma}{150}$  and  $\psi(d(T\rho, T\sigma)) = \frac{\rho + \sigma}{30}$ . Hence,

$$\begin{aligned} & \tau + F(\psi(d(T\rho, T\sigma))) \\ & = 0.1 + F \left( \frac{\rho + \sigma}{30} \right) \\ & = 0.1 + \frac{5(\rho + \sigma)}{6} \\ & \leq 125(\rho + \sigma) \\ & = F(\psi(d(\rho, \sigma))) \\ & = F^* \left( F(\psi(d(\rho, \sigma))), \right. \\ & \left. F(\phi(d(\rho, \sigma))) \right). \end{aligned}$$

From Theorem (2.1), we get  $T$  has a unique fixed point. Indeed, 0 is the fixed point.

**Example 3.2.** Let  $X = [0, 2]$  and  $\mathbb{A} = \mathbb{C}$ . Let  $F : \mathbb{A}^+ \rightarrow \mathbb{A}$  defined as  $F(a) = 40a$  and  $d : X \times X \rightarrow \mathbb{A}$  defined by

$$d(\rho, \sigma) = |\rho - \sigma|.$$

Define,  $T\rho = \frac{\rho}{120}$ .

Then,

- (i)  $(X, \mathbb{A}, d)$  be a complete  $C^*$ -algebra valued metric space;
- (ii)  $F$  is non-decreasing;
- (iii)  $F$  is continuous;
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0$  and  $\lim_{n \rightarrow \infty} F(\sigma_n) = 0$ ;
- (v) for all  $\sigma, \rho \in X$  with  $\tau = 0.2$ ,  $F^*(r, t) = \frac{r}{2}$  and  $\psi(t) = \phi(t) = 6t$

$$\begin{aligned} d(T\rho, T\sigma) &= \frac{|\rho - \sigma|}{120} \\ d(\rho, T\rho) &= \frac{|119\rho|}{120} \\ d(\sigma, T\sigma) &= \frac{|119\sigma|}{120} \\ \psi(d(T\rho, T\sigma)) &= \frac{|\rho - \sigma|}{20} \\ \psi\left(\frac{d(\rho, T\rho) + d(\sigma, T\sigma)}{2}\right) &= \frac{119|\rho + \sigma|}{40}. \end{aligned}$$

Hence,

$$\begin{aligned} &\tau + F(\psi(d(T\rho, T\sigma))) \\ &= 0.2 + F\left(\frac{|\rho - \sigma|}{20}\right) \\ &= 0.2 + 2(|\rho - \sigma|) \\ &\leq \frac{119|\rho + \sigma|}{2} \\ &= \frac{1}{2}F\left(\psi\left(\frac{d(\rho, T\rho) + d(\sigma, T\sigma)}{2}\right)\right) \\ &= F^*\left(F\left(\psi\left(\frac{d(\rho, T\rho) + d(\sigma, T\sigma)}{2}\right)\right)\right), \end{aligned}$$

$$F\left(\phi\left(\frac{d(\rho, T\rho) + d(\sigma, T\sigma)}{2}\right)\right).$$

From Theorem (3.1), we get  $T$  has a unique fixed point. Indeed, 0 is the fixed point.

**Example 3.3.** Let  $X = [0, 2]$  and  $\mathbb{A} = \mathbb{C}$ . Let  $F : \mathbb{A}^+ \rightarrow \mathbb{A}$  defined as  $F(a) = 25a$  and  $d : X \times X \rightarrow \mathbb{A}$  defined by

$$d(\rho, \sigma) = \begin{cases} |\rho| + |\sigma| & \text{if } \rho \neq \sigma \\ 0 & \text{if } \rho = \sigma. \end{cases}$$

Define,  $T\rho = \begin{cases} 1/10, & \text{if } x \in [0, 1] \\ 1/20, & \text{otherwise.} \end{cases}$

Then,

- (i)  $(X, \mathbb{A}, d)$  be a complete  $C^*$ -algebra valued metric space;
- (ii)  $F$  is non-decreasing;
- (iii)  $F$  is continuous;
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0$  and  $\lim_{n \rightarrow \infty} F(\sigma_n) = 0$ ;
- (v) for all  $\sigma, \rho \in X$  with  $\tau = 0.1$ ,  $F^*(r, t) = r$ ,  $\psi(t) = \phi(t) = 5t$ , we get  $d(T\rho, T\sigma) = 0$  and  $\psi(d(T\rho, T\sigma)) = 0$ . Hence,

$$\begin{aligned} &\tau + F(\psi(d(T\rho, T\sigma))) \\ &= 0.1 + F(0) \\ &= 0.1 \\ &\leq 125(|\rho| + |\sigma|) \\ &= F(\psi(d(\rho, \sigma))) \\ &= F^*\left(F(\psi(d(\rho, \sigma))), F(\phi(d(\rho, \sigma)))\right). \end{aligned}$$

From Theorem (2.1), we get  $T$  has a unique fixed point. Indeed,  $\frac{1}{10}$  is the fixed point.

## Author Contributions

All the authors contributed equally for the preparation of the present manuscript.

## Competing interests

The author declare that they do not have any competing interests.

## Conflict of interest

All author declare that they have no conflict of interest.

## Ethical approval

This article does not contain any studies with human participants or animal performed by any of the authors.

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