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Theorems on Common Fixed Point of ϕ-Weakly Expansive Mappings in Dislocated Metric Spaces

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Abstract

In this study, we will demonstrate a common fixed point theory for two weakly compatible self-maps G and H on a dislocated metric space (M, d^*) that satisfy the E.A. property and (CLR) property as well as the following ϕ-weakly expansive mappings.

Keywords: Common Fixed Point Theorem, Weakly Compatible, Weakly Expansive

1. Introduction

In 1922, Banach (1) established a common fixed point theorem, which guarantees the existence and uniqueness of a fixed point under certain conditions. Banach's result is known as the Banach contraction principle or Banach's fixed point theorem. In various ways, many authors have extended, generalized, and improved on Banach's fixed point theorem.

In order to generalize the Banach fixed point theorem, Jungck (2) established a common fixed point theorem for commuting maps. This theorem has a wide range of applications.

The idea of expansive mapping was first put forth by Wang et al. (3) as follows

$$d^*(Hx, Hy) \geq a d^*(x, y), \quad (1.1)$$

where H is a self-mapping of a metric space (M, d^*) for all $x, y \in M$ and $a > 1$.

Next, The ϕ-weakly expansive mapping were introduced by Kang et al. (4) as follows

$$d^*(Hx, Hy) \geq d^*(Gx, Gy) + \phi(d^*(Gx, Gy)), \quad (1.2)$$

where G and H are two self-mappings of a metric space (M, d^*) for all $x, y \in M$.
Let $\phi: [0, \infty) \to [0, \infty)$ be a continuous mapping with $\phi(0) = 0$ and $\phi(t) > t$ for all $t > 0$.

Kim et al. (5) extended (1.2) for find a common fixed point of two self maps satisfying ξ-weakly expansive mappings in dislocated metric space and researchers in (6) and (7) are study in dislocated quasi-metric spaces.

Motivated and inspired by above, we will show that two weakly compatible self-maps G and H on a dislocated metric space (M, d^*)

that satisfy the ϕ-weakly expansive condition have a common fixed point theorem.

2. Preliminaries

Definition 2.1 (5) Let $d^*: M \times M \to [0,\infty)$ be a function with $M \neq \emptyset$ and for all $x, y, z \in M$, the following conditions are satisfied

(a) $d^*(x,y) = d^*(y,x)$

(b) $d^*(x,y) = 0$, then $x = y$

(c) $d^*(x,y) \leq d^*(x,u) + d^*(u,y)$.

Then d^* is called dislocated metric (or simply d^*-metric) on M and the pair (M, d^*) is called dislocated metric space.

Definition 2.2 (6) Two self-mappings G and H defined on a metric space M are said to be weakly compatible if they commute at their coincidence points.

Definition 2.3 (7) Two self-mappings G and H of a metric space (M, d^*) are said to satisfy E.A. property if there exists a sequence $\{x_j\} \in M$ such that

$$\lim_{j\to\infty} H\,x_j = \lim_{j\to\infty} G\,x_j = s$$

for some $s \in M$.

Definition 2.4 (8) Two self-mappings G and H of a metric space (M, d^*) are said to satisfy (CLR_H) property if there exists a sequence $\{x_j\}$ in M such that

$$\lim_{j\to\infty} H\,x_j = \lim_{j\to\infty} G\,x_j = Hx$$

for some $x \in M$.

3. Main Results

Theorem 3.1 Let (M, d^*) be a dislocated metric space and let G and H be two self-maps on M satisfying

$$GM \subseteq HM. \quad (3.1)$$

There exists a continuous mapping $\phi: [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\phi(t) > t$ for all $t > 0$ such that

$$d^*(Hx, Hy) \geq d^*(Gx, Gy) + \phi(\Upsilon(Gx, Gy)), \quad (3.2)$$

where

$$\Upsilon(Gx, Gy) = max\{d^*(Gx, Gy), d^*(Gx, Hx), \ d^*(Gy, Hy), \frac{d^*(Gx,Hx)(1+d^*(Gy,Hy))}{1+d^*(Gx,Gy)}, \frac{d^*(Gy,Hy)(1+d^*(Gx,Hx))}{1+d^*(Hx,Hy)}, \frac{d^*(Gx,Hx)d^*(Gy,Hy)}{1+d^*(Gx,Gy)+d^*(Hx,Hy)}\}$$

for all $x, y \in M$.

If G and H are weakly compatible and GM or HM is complete, then G and H have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of M. We can define a sequence $\{x_j\}$ based on (2.1), such that

$$Gx_j = Hx_{j+1},$$

because $GM \subseteq HM$. Define a $\{y_j\}$ sequence in M by

$$y_j = Gx_j = Hx_{j+1}. \quad (3.3)$$

There is nothing to prove if $y_j = y_{j+1}$, $\exists j \in \mathbb{N}$. Now, assume that $y_j \neq y_{j+1}$, $\forall j \in \mathbb{N}$. We prove that

$$\lim_{j\to\infty} d^*(y_j\,, y_{j+1}) = 0. \quad (3.4)$$

Letting $x = x_j, y = x_{j+1}$ in (3.2) and using (3.3), we obtain

$$d^*(y_{j-1}, y_j) = d^*(Hx_j, Hx_{j+1}) \geq d^*(Gx_j, Gx_{j+1}) + \phi(\Upsilon(Gx_j, Gx_{j+1})) = d^*(y_j, y_{j+1}) + \phi(\Upsilon(y_j, y_{j+1})), \quad (3.5)$$

where

$$\Upsilon(y_j, y_{j+1}) = \Upsilon(Gx_j, Gx_{j+1}) = max\{d^*(Gx_j, Gx_{j+1}), d^*(Gx_j, Hx_j), d^*(Gx_{j+1}, Hx_{j+1}), \frac{d^*(Gx_j,Hx_j)(1+d^*(Gx_{j+1},Hx_{j+1}))}{1+d^*(Gx_j,Gx_{j+1})}, \frac{d^*(Gx_{j+1},Hx_{j+1})(1+d^*(Gx_j,Hx_j))}{1+d^*(Hx_j,Hx_{j+1})},$$

$$\frac{d^*(Gx_j,Hx_j)d^*(Gx_{j+1},Hx_{j+1})}{1+d^*(Gx_j,Gx_{j+1})+d^*(Hx_j,Hx_{j+1})}\}$$
$$= max\{d^*(y_j,y_{j+1}), d^*(y_j,y_{j-1}),$$
$$d^*(y_{j+1},y_j),$$
$$\frac{d^*(y_j,y_{j-1})(1+d^*(y_{j+1},y_j))}{1+d^*(y_j,y_{j+1})},$$
$$\frac{d^*(y_{j+1},y_j)(1+d^*(y_j,y_{j-1}))}{1+d^*(y_{j-1},y_j)},$$
$$\frac{d^*(y_j,y_{j-1})d^*(y_{j+1},y_j)}{1+d^*(y_j,y_{j+1})+d^*(y_{j-1},y_j)}\}$$
$$= max\{d^*(y_j,y_{j+1}), d^*(y_j,y_{j-1})\}.$$

If $d^*(y_j,y_{j+1}) < d^*(y_{j-1},y_j)$ and (3.5) then

$$d^*(y_{j-1},y_j) > d^*(y_j,y_{j+1}) + d^*(y_{j-1},y_j).$$

That is

$$d^*(y_j,y_{j+1}) < 0,$$

which is a contradiction.

If $d^*(y_{j-1},y_j) < d^*(y_j,y_{j+1})$ and (3.5) then

$$d^*(y_{j-1},y_j) > d^*(y_j,y_{j+1}) + \phi(d^*(y_j,y_{j+1})). \quad (3.6)$$

This mean that

$$d^*(y_{j-1},y_j) > d^*(y_j,y_{j+1})$$

Thus, the sequence $\{d^*(y_j,y_{j+1})\}$ is strictly decreasing and bounded below.

So, there exists $\gamma \geq 0$, such that

$$\lim_{j\to\infty} d^*(y_j,y_{j+1}) = \gamma,$$

taking $j \to \infty$ in (3.6), we get

$$\gamma \geq \gamma + \phi(\gamma),$$

which is a contradiction, hence we have $\gamma = 0$. Hence,

$$\lim_{j\to\infty} d^*(y_j,y_{j+1}) = 0. \quad (3.7)$$

Next, we prove that $\{y_j\}$ is a d^*-Cauchy sequence. Assume that $\{y_j\}$ is not a d^*-Cauchy sequence. Then there exists $\varepsilon > 0$, such that for $n \in \mathbb{N}$, there are $j(i), k(i) \in \mathbb{N}$ with $k(i) > j(i) > i$ satisfying
(a): $k(i)$ and $j(i)$ are positive integers.
(b): $d^*(y_{j(i)}, y_{k(i)}) > \varepsilon$.
(c): $k(i)$ is the smallest even number such that the condition (b) holds, that is,
$d^*(y_{j(i)}, y_{k(i)-1}) \leq \varepsilon$.
Thus,

$$\begin{aligned}\epsilon &< d^*(y_{j(i)}, y_{k(i)})\\ &\leq d^*(y_{j(i)}, y_{k(i)-1}) + d^*(y_{k(i)-1}, y_{k(i)})\\ &\leq \epsilon + d^*(y_{k(i)-1}, y_{k(i)}).\end{aligned} \quad (3.8)$$

Taking $i \to \infty$, we obtain

$$\lim_{i\to\infty} d^*(y_{j(i)}, y_{k(i)}) = \varepsilon. \quad (3.9)$$

We now have

$$\begin{aligned}&\epsilon < d^*(y_{j(i)-1}, y_{k(i)-1})\\ &\leq d^*(y_{j(i)-1}, y_{k(i)-2})\\ &\quad +d^*(y_{k(i)-2}, y_{k(i)-1})\\ &\leq \epsilon + d^*(y_{k(i)-2}, y_{k(i)-1}).\end{aligned} \quad (3.10)$$

Taking $i \to \infty$, we obtain

$$\lim_{i\to\infty} d^*(y_{j(i)-1}, y_{k(i)-1}) = \varepsilon. \quad (3.11)$$

Letting $x = x_{j(i)}$ and $y = x_{k(i)}$ in (3.2), we obtain

$$\begin{aligned}&d^*(y_{j(i)-1}, y_{k(i)-1}) = d^*(Hx_{j(i)}, Hx_{k(i)})\\ &\geq d^*(Gx_{j(i)}, Gx_{k(i)}) + \phi(\Upsilon(Gx_{j(i)}, Gx_{k(i)}))\\ &\geq d^*(y_{j(i)}, y_{k(i)}) + \phi(\Upsilon(y_{j(i)}, y_{k(i)})),\end{aligned} \quad (3.12)$$

where

$$\Upsilon(y_{j(i)}, y_{k(i)}) = \Upsilon(Gx_{j(i)}, Gx_{k(i)})$$
$$= max\{d^*(Gx_{j(i)}, Gx_{k(i)}),$$

$$d^*\left(Gx_{j(i)}, Hx_{j(i)}\right),\ d^*\left(Gx_{k(i)}, Hx_{k(i)}\right), \frac{d^*\left(Gx_{j(i)},Hx_{j(i)}\right)\left(1+d^*\left(Gx_{k(i)},Hx_{k(i)}\right)\right)}{1+d^*\left(Gx_{j(i)},Gx_{k(i)}\right)}, \frac{d^*\left(Gx_{k(i)},Hx_{k(i)}\right)\left(1+d^*\left(Gx_{j(i)},Hx_{j(i)}\right)\right)}{1+d^*\left(Hx_{j(i)},Hx_{k(i)}\right)}, \frac{d^*\left(Gx_{j(i)},Hx_{j(i)}\right)d^*\left(Gx_{k(i)},Hx_{k(i)}\right)}{1+d^*\left(Gx_{j(i)},Gx_{k(i)}\right)+d^*\left(Hx_{j(i)},Hx_{k(i)}\right)}\}$$
$$= max\{d^*\left(y_{j(i)}, y_{k(i)}\right), d^*\left(y_{j(i)}, y_{j(i)-1}\right),\ d^*\left(y_{k(i)}, y_{k(i)-1}\right), \frac{d^*\left(y_{j(i)},y_{j(i)-1}\right)\left(1+d^*\left(y_{k(i)},y_{k(i)-1}\right)\right)}{1+d^*\left(y_{j(i)},y_{k(i)}\right)}, \frac{d^*\left(y_{k(i)},y_{k(i)-1}\right)\left(1+d^*\left(y_{j(i)},y_{j(i)-1}\right)\right)}{1+d^*\left(y_{j(i)-1},y_{k(i)-1}\right)}, \frac{d^*\left(y_{j(i)},y_{j(i)-1}\right)d^*\left(y_{k(i)},y_{k(i)-1}\right)}{1+d^*\left(y_{j(i)},y_{k(i)}\right)+d^*\left(y_{j(i)-1},y_{k(i)-1}\right)}\}.$$

Taking limit as $i \to \infty$, we have

$$\lim_{i\to\infty} \Upsilon\left(y_{j(i)}, y_{k(i)}\right) = max\{\varepsilon, \varepsilon, \varepsilon, \frac{\varepsilon(1+\varepsilon)}{1+\varepsilon}, \frac{\varepsilon(1+\varepsilon)}{1+\varepsilon}, \frac{\varepsilon\cdot\varepsilon}{1+\varepsilon+\varepsilon}\} = \varepsilon.$$

From (3.12), we obtain

$$\varepsilon \geq \varepsilon + \phi(\varepsilon),$$

which is a contradiction, from $\phi(\varepsilon) \geq 0$. This implies that $\{y_j\}$ is a d^*-Cauchy sequence in M. Now, from HM is complete, there exists a point z in HM such that

$$\lim_{j\to\infty} y_j = \lim_{j\to\infty} Hx_{j+1} = g = \lim_{j\to\infty} Gx_j. \qquad (3.13)$$

Since $g \in HM$, we can find g^* in M such that $Hg^* = g$.

Now, we claim that $Hg^* = Gg^*$, let if possible $Hg^* \neq Gg^*$. Letting $x = x_{j+1}$ and $y = g^*$ in (3.2), we obtain

$$d^*(Gx_j, Hg^*) = d^*(Hx_{j+1}, Hg^*)$$

$$\geq d^*(Gx_{j+1}, Gg^*) + \phi(\Upsilon(Gx_{j+1}, Gg^*)), \quad (3.14)$$

where

$$\Upsilon(Gx_{j+1}, Gg^*) = max\{d^*\left(Gx_{j+1}, Gg^*\right), d^*\left(Gx_{j+1}, Hx_{j+1}\right),\ d^*(Gg^*, Hg^*), \frac{d^*\left(Gx_{j+1},Hx_{j+1}\right)\left(1+d^*(Gg^*,Hg^*)\right)}{1+d^*\left(Gx_{j+1},Gg^*\right)}, \frac{d^*(Gg^*,Hg^*)\left(1+d^*\left(Gx_{j+1},Hx_{j+1}\right)\right)}{1+d^*\left(Hx_{j+1},Hg^*\right)}, \frac{d^*\left(Gx_{j+1},Hx_{j+1}\right)d^*(Gg^*,Hg^*)}{1+d^*\left(Gx_{j+1},Gg^*\right)+d^*\left(Hx_{j+1},Hg^*\right)}\}.$$

Letting limit as $j \to \infty$, we have

$$d^*(Hg^*, Hg^*) \geq d^*(Hg^*, Gg^*) + \lim_{j\to\infty} \phi\left(\Upsilon\left(Gx_{j+1}, Gg^*\right)\right), \qquad (3.15)$$

where

$$\lim_{j\to\infty} \Upsilon(Gx_{j+1}, Gg^*) = max\{d^*(Hg^*, Gg^*), d^*(Hg^*, Hg^*),\ d^*(Gg^*, Hg^*), \frac{d^*(Hg^*,Hg^*)\left(1+d^*(Gg^*,Hg^*)\right)}{1+d^*(Hg^*,Gg^*)}, \frac{d^*(Gg^*,Hg^*)\left(1+d^*(Hg^*,Hg^*)\right)}{1+d^*(Hg^*,Hg^*)}, \frac{d^*(Hg^*,Hg^*)d^*(Gg^*,Hg^*)}{1+d^*(Hg^*,Gg^*)+d^*(Hg^*,Hg^*)}\} = max\{d^*(Hg^*, Gg^*), d^*(Hg^*, Hg^*)\}.$$

Next, we consider two cases.

Case (a): Let $\Upsilon(Gx_{j+1}, Gg^*) = d^*(Hg^*, Gg^*)$.

Using (3.14), we obtain

$$\begin{aligned} d^*(Hg^*, Hg^*) &\geq d^*(Hg^*, Gg^*) + \phi\left(d^*(Hg^*, Gg^*)\right) \\ &> d^*(Hg^*, Gg^*) + d^*(Hg^*, Gg^*) \\ &= 2d^*(Hg^*, Gg^*). \end{aligned}$$

Using triangular inequality, we obtain

$$\begin{aligned} d^*(Hg^*, Hg^*) &\leq d^*(Hg^*, Gg^*) + d^*(Hg^*, Gg^*) \\ &= 2d^*(Hg^*, Gg^*), \end{aligned}$$

which is a contradiction.

Case (b): Let $\Upsilon(Gx_{j+1}, Gg^*) = d^*(Hg^*, Hg^*)$.

Using (3.14), we obtain

$$d^*(H\mathcal{g}^*, H\mathcal{g}^*) \geq d^*(H\mathcal{g}^*, G\mathcal{g}^*) + \phi\big(d^*(H\mathcal{g}^*, H\mathcal{g}^*)\big) \geq d^*(H\mathcal{g}^*, G\mathcal{g}^*) + d^*(H\mathcal{g}^*, H\mathcal{g}^*).$$

This means that

$$d^*(H\mathcal{g}^*, G\mathcal{g}^*) < 0,$$

which is a contradiction. Therefore, $d^*(H\mathcal{g}^*, G\mathcal{g}^*) = 0$. This mean that

$$H\mathcal{g}^* = G\mathcal{g}^* = \mathcal{g}. \tag{3.16}$$

Thus, $\mathcal{g}$ is a coincidence point of G and H.

Now, we show that there exists a common fixed point of G and H. Because G and H are weakly compatible, using (3.16), we get

$GH\mathcal{g}^* = HG\mathcal{g}^*$ and $G\mathcal{g}^* = GH\mathcal{g}^* = HG\mathcal{g}^* = H\mathcal{g}$.

Consider

$$d^*(\mathcal{g}, G\mathcal{g}) = d^*(H\mathcal{g}^*, H\mathcal{g}) \geq d^*(G\mathcal{g}^*, G\mathcal{g}) + \phi(\Upsilon(G\mathcal{g}^*, G\mathcal{g})) = d^*(\mathcal{g}, G\mathcal{g}) + \phi(\Upsilon(G\mathcal{g}^*, G\mathcal{g})), \tag{3.17}$$

Where

$$\Upsilon(G\mathcal{g}^*, G\mathcal{g}) = max\{d^*(G\mathcal{g}^*, G\mathcal{g}), d^*(G\mathcal{g}^*, H\mathcal{g}^*), d^*(G\mathcal{g}, H\mathcal{g}), \frac{d^*(G\mathcal{g}^*,H\mathcal{g}^*)(1+d^*(G\mathcal{g},H\mathcal{g}))}{1+d^*(G\mathcal{g}^*,G\mathcal{g})}, \frac{d^*(G\mathcal{g},H\mathcal{g})(1+d^*(G\mathcal{g}^*,H\mathcal{g}^*))}{1+d^*(H\mathcal{g}^*,H\mathcal{g})}, \frac{d^*(G\mathcal{g}^*,H\mathcal{g}^*)d^*(G\mathcal{g},H\mathcal{g})}{1+d^*(G\mathcal{g}^*,G\mathcal{g})+d^*(H\mathcal{g}^*,H\mathcal{g})}\}$$

$$= max\{d^*(\mathcal{g}, G\mathcal{g}), d^*(\mathcal{g}, \mathcal{g}), d^*(G\mathcal{g}, G\mathcal{g}), \frac{d^*(\mathcal{g},\mathcal{g})(1+d^*(G\mathcal{g},H\mathcal{g}))}{1+d^*(\mathcal{g},G\mathcal{g})}, \frac{d^*(G\mathcal{g},G\mathcal{g})(1+d^*(\mathcal{g},\mathcal{g}))}{1+d^*(\mathcal{g},H\mathcal{g})}, \frac{d^*(\mathcal{g},\mathcal{g})d^*(G\mathcal{g},G\mathcal{g})}{1+d^*(\mathcal{g},G\mathcal{g})+d^*(\mathcal{g},G\mathcal{g})}\}$$

$$= max\{d^*(\mathcal{g}, G\mathcal{g}), 0, d^*(G\mathcal{g}, G\mathcal{g}), 0, \frac{d^*(G\mathcal{g},G\mathcal{g})}{1+d^*(\mathcal{g},H\mathcal{g})}, 0\}$$

$$= max\{d^*(\mathcal{g}, G\mathcal{g}), d^*(G\mathcal{g}, G\mathcal{g})\}.$$

We currently have two cases:

Case (i): Let $\Upsilon(G\mathcal{g}^*, G\mathcal{g}) = d^*(\mathcal{g}, G\mathcal{g})$.

From (3.16), we obtain

$$d^*(\mathcal{g}, G\mathcal{g}) \geq d^*(\mathcal{g}, G\mathcal{g}) + \phi(d^*(\mathcal{g}, G\mathcal{g})) > d^*(\mathcal{g}, G\mathcal{g}) + d^*(\mathcal{g}, G\mathcal{g}) = 2d^*(\mathcal{g}, G\mathcal{g}),$$

which is a contradiction.

Case (ii): Let $\Upsilon(G\mathcal{g}^*, G\mathcal{g}) = d^*(G\mathcal{g}, G\mathcal{g})$.

From (3.16), we obtain

$$d^*(\mathcal{g}, G\mathcal{g}) \geq d^*(\mathcal{g}, G\mathcal{g}) + \phi(d^*(G\mathcal{g}, G\mathcal{g})) > d^*(\mathcal{g}, G\mathcal{g}) + d^*(G\mathcal{g}, G\mathcal{g}) = d^*(\mathcal{g}, G\mathcal{g}),$$

which is again a contradiction. Thus,

$H\mathcal{g} = G\mathcal{g} = \mathcal{g}$. This mean that $\mathcal{g}$ is common fixed point of G and H.

For the uniqueness, let u be common fixed point of G and H and u^* be another common fixed point of G and H, such that $u \neq u^*$. Then

$$d^*(u, u^*) = d^*(Hu, Hu^*) \geq d^*(Gu, Gu^*) + \phi(\Upsilon(Gu, Gu^*)) = d^*(u, u^*) + \phi(d^*(u, u^*)) > d^*(u, u^*) + d^*(u, u^*),$$

which is a contradiction, so $u = u^*$. Thus, we proves the uniqueness of the common fixed point.

Theorem 3.2 Let (M, d^*) be a dislocate metric space and let G and H be two self-maps on M satisfying (3.2) and the followings:

G and H are weakly compatible, (3.18)
G and H satisfy the E.A. property. (3.19)

If either GM or HM is a complete subspace of M, then G and H have a unique common fixed point in M.

Proof. From G and H satisfy the E.A. property, there exists a sequence $\{x_j\}$ in M such that

$$\lim_{j\to\infty} Hx_j = \lim_{j\to\infty} Gx_j = x,\ \exists x \in M. \quad (3.20)$$

Now, assume HM is complete subspace of M. Then, there exists q in M such that $x = Hq$. After that, we obtain

$$\lim_{j\to\infty} Hx_j = \lim_{j\to\infty} Gx_j = x = Hq.$$

Now, we show that $Gq = Hq$.

From (3.2), we obtain

$$d^*(Hx_j, Hq) \geq d^*(Gx_j, Gq) + \phi(\Upsilon(Gx_j, Gq)).$$

Taking $j \to \infty$, we obtain

$$d^*(Hq, Hq) \geq d^*(Hq, Gq) + \lim_{j\to\infty} \phi(\Upsilon(Gx_j, Gq)), \quad (3.21)$$

where

$$\lim_{j\to\infty} \Upsilon(Gx_j, Gq) = \lim_{j\to\infty} max\{d^*(Gx_j, Gq), d^*(Gx_j, Hx_j), d^*(Gq, Hq), \frac{d^*(Gx_j,Hx_j)(1+d^*(Gq,Hq))}{1+d^*(Gx_j,Gq)}, \frac{d^*(Gq,Hq)(1+d^*(Gx_j,Hx_j))}{1+d^*(Hx_j,Hq)}, \frac{d^*(Gx_j,Hx_j)d^*(Gq,Hq)}{1+d^*(Gx_j,Gq)+d^*(Hx_j,Hq)}\}$$
$$= max\{d^*(Hq, Gq), d^*(Hq, Hq), d^*(Gq, Hq), \frac{d^*(Hq,Hq)(1+d^*(Gq,Hq))}{1+d^*(Hq,Gq)}, \frac{d^*(Gq,Hq)(1+d^*(Hq,Hq))}{1+d^*(Hq,Hq)}, \frac{d^*(Hq,Hq)d^*(Gq,Hq)}{1+d^*(Hq,Gq)+d^*(Hq,Hq)}\}$$
$$= max\{d^*(Hq, Gq), d^*(Hq, Hq)\}.$$

Now two cases arise:

Case (a): $\lim_{j\to\infty} \Upsilon(Gx_j, Gq) = d^*(Hq, Gq)$.

From (3.20), we obtain

$$\begin{aligned} d^*(Hq, Hq) &\geq d^*(Hq, Gq) + \phi(d^*(Hq, Gq)) \\ &> d^*(Hq, Gq) + d^*(Hq, Gq) \\ &= 2d^*(Hq, Gq). \end{aligned}$$

Using triangular inequality, we obtain

$$\begin{aligned} d^*(Hq, Hq) &\leq d^*(Hq, Gq) + d^*(Gq, Hq) \\ &= 2d^*(Hq, Gq), \end{aligned}$$

which is a contradiction.

Case (b): $\lim_{j\to\infty} \Upsilon(Gx_j, Gq) = d^*(Hq, Hq)$.

From (3.21), we obtain

$$\begin{aligned} d^*(Hq, Hq) &\geq d^*(Hq, Gq) + \phi(d^*(Hq, Hq)) \\ &> d^*(Hq, Gq) + d^*(Hq, Hq), \end{aligned}$$

which implies that

$$d^*(Hq, Gq) < 0,$$

which is a contradiction. Thus,

$d^*(Hq, Gq) = 0$ or $Hq = Gq$.

From G and H are weakly compatible,

$GHq = HGq$ implies that

$$HHq = HGq = GHq = GGq.$$

Now, we claim that Gq is the common fixed point of G and H. Using (3.2), we obtain

$$d^*(Gq, GGq) = d^*(Hq, HHq) \geq d^*(Gq, GGq) + \phi(\Upsilon(Gq, GGq)), \quad (3.22)$$

where

$$\Upsilon(Gq, GGq) = max\{d^*(Gq, GGq), d^*(Gq, Hq), d^*(GGq, HGq), \frac{d^*(Gq,Hq)(1+d^*(GGq,HGq))}{1+d^*(Gq,GGq)}, \frac{d^*(GGq,HGq)(1+d^*(Gq,Hq))}{1+d^*(Hq,HGq)}, \frac{d^*(Gq,Hq)d^*(GGq,HGq)}{1+d^*(Gq,GGq)+d^*(Hq,HGq)}\} = max\{d^*(Gq, GGq), 0, 0, 0, 0, 0\}.$$

This implies that

$$\Upsilon(Gq, GGq) = d^*(Gq, GGq).$$

Now, using (3.22), we obtain

$$d^*(Gq, GGq) \geq d^*(Gq, GGq) + \phi(d^*(Gq, GGq)) > d^*(Gq, GGq) + d^*(Gq, GGq) = 2d^*(Gq, GGq),$$

which implies that

$$Gq = GGq = HGq.$$

Thus, Gq is common fixed point of G and H.

For the uniqueness, let u be common fixed point of G and H and u^* be another common fixed point of G and H. Then, using (3.2), we obtain

$$d^*(u, u^*) = d^*(Hu, Hu^*) \geq d^*(Gu, Gu^*) + \phi(\Upsilon(Gu, Gu^*)), \quad (3.23)$$

Where

$$\Upsilon(Gu, Gu^*) = max\{d^*(Gu, Gu^*), d^*(Gu, Hu), d^*(Gu^*, Hu^*), \frac{d^*(Gu,Hu)(1+d^*(Gu^*,Hu^*))}{1+d^*(Gu,Gu^*)}, \frac{d^*(Gu^*,Hu^*)(1+d^*(Gu,Hu))}{1+d^*(Hu,Hu^*)}, \frac{d^*(Gu,Hu)d^*(Gu^*,Hu^*)}{1+d^*(Gu,Gu^*)+d^*(Hu,Hu^*)}\} = d^*(Gu, Gu^*).$$

From (3.23), we obtain

$$d^*(u, u^*) \geq d^*(Gu, Gu^*) + \phi\big(d^*(Gu, Gu^*)\big) > d^*(Gu, Gu^*) + d^*(Gu, Gu^*) = 2d^*(Gu, Gu^*) = 2d^*(u, u^*),$$

which is a contradiction. So $u = u^*$. Thus, we proves the uniqueness of common fixed point.

Theorem 3.3 Let (M, d^*) be a dislocated metric space, let G and H be self-maps on M satisfying (3.2) and (3.18). If G and H satisfy (CLR_H) property, then G and H have a unique common fixed point in M.

Proof. From G and H satisfy the (CLR_H) property, there exists a sequence $\{x_j\}$ in M such that

$$\lim_{j\to\infty} Hx_j = \lim_{j\to\infty} Gx_j = Hx, \ \exists x \in M.$$

First, we claim that $Hx = Gx$. Let $Hx \neq Gx$. Then from (3.2), we get

$$d^*\big(Hx_j, Hx\big) \geq d^*(Gx_j, Gx) + \phi\big(\Upsilon\big(Gx_j, Gx\big)\big), \quad (3.24)$$

where

$$\Upsilon\big(Gx_j, Gx\big) = max\{d^*\big(Gx_j, Gx\big), d^*\big(Gx_j, Hx_j\big), d^*(Gx, Hx), \frac{d^*\big(Gx_j,Hx_j\big)(1+d^*(Gx,Hx))}{1+d^*\big(Gx_j,Gx\big)},$$

$$\frac{d^*(Gx,Hx)(1+d^*(Gx_j,Hx_j))}{1+d^*(Hx_j,Hx)}, \frac{d^*(Gx_j,Hx_j)d^*(Gx,Hx)}{1+d^*(Gx_j,Gx)+d^*(Hx_j,Hx)}\}.$$

Letting limit as $j \to \infty$, we get

$$\lim_{j\to\infty} \Upsilon(Gx_j, Gx) = max\{d^*(Hx,Gx), d^*(Hx,Hx), d^*(Gx,Hx), \frac{d^*(Hx,Hx)(1+d^*(Gx,Hx))}{1+d^*(Hx,Gx)}, \frac{d^*(Gx,Hx)(1+d^*(Hx,Hx))}{1+d^*(Hx,Hx)}, \frac{d^*(Hx,Hx)d^*(Gx,Hx)}{1+d^*(Hx,Gx)+d^*(Hx,Hx)}\} = max\{d^*(Hx,Gx), d^*(Hx,Hx)\}.$$

Now, two cases arise:

Case (a): $\lim_{j\to\infty} \Upsilon(Gx_j, Gx) = d^*(Hx, Gx)$.

From (3.24), we obtain

$$d^*(Hx,Hx) \geq d^*(Hx,Gx) + \phi(d^*(Hx,Gx)) > d^*(Hx,Gx) + d^*(Hx,Gx) = 2d^*(Hx,Gx).$$

Using triangular inequality, we obtain

$$d^*(Hx,Hx) \leq d^*(Hx,Gx) + d^*(Gx,Hx) = 2d^*(Hx,Gx),$$

which is a contradiction.

Case (b): $\lim_{j\to\infty} \Upsilon(Gx_j, Gx) = d^*(Hx, Hx)$.

From (3.24), we obtain

$$d^*(Hx,Hx) \geq d^*(Hx,Gx) + \phi(d^*(Hx,Hx)) > d^*(Hx,Gx) + d^*(Hx,Hx),$$

which implies that

$$d^*(Hx,Gx) < 0,$$

which is a contradiction. This implies

$$d^*(Hx,Gx) = 0$$

which is a contradiction. Thus, $Hx = Gx$.
Let $y = Hx = Gx$. From $HGx = GHx$ implies that

$$Hy = HGx = GHx = Gy.$$

Now, we claim that $Gy = y$. From (3.2), we get

$$d^*(Gx,y) = d^*(Hy,Hx) \geq d^*(Gx,Gy) + \phi(\Upsilon(Gx,Gy)), \tag{3.25}$$

where

$$\Upsilon(Gx,Gy) = max\{d^*(Gx,Gy), d^*(Gx,Hx), d^*(Gy,Hy), \frac{d^*(Gx,Hx)(1+d^*(Gy,Hy))}{1+d^*(Gx,Gy)}, \frac{d^*(Gy,Hy)(1+d^*(Gx,Hx))}{1+d^*(Hx,Hy)}, \frac{d^*(Gx,Hx)d^*(Gy,Hy)}{1+d^*(Gx,Gy)+d^*(Hx,Hy)}\}$$

$$= max\{d^*(y,Gy), 0, 0, 0, 0, 0\}$$

$$= d^*(Gy,y).$$

Using (3.24), we get

$$d^*(Gy,y) \geq d^*(y,Gy) + \phi(d^*(y,Gy)) > d^*(y,Gy) + d^*(y,Gy) = 2d^*(y,Gy)$$

this is possible only when $Gy = y$. Thus, $Hy = Gy = y$. Hence, y is the common fixed point of G and H.

For the uniqueness, let u be common fixed point of G and H and u^* be another common fixed point of G and H. Using (3.2), we obtain

$$d^*(Hu,Hu^*) \geq d^*(Gu,Gu^*) + \phi(\Upsilon(Gu,Gu^*)), \tag{3.26}$$

where

$$\Upsilon(Gu, Gu^*) = max\{d^*(Gu, Gu^*), d^*(Gu, Hu), d^*(Gu^*, Hu^*), \frac{d^*(Gu,Hu)(1+d^*(Gu^*,Hu^*))}{1+d^*(Gu,Gu^*)}, \frac{d^*(Gu^*,Hu^*)(1+d^*(Gu,Hu))}{1+d^*(Hu,Hu^*)}, \frac{d^*(Gu,Hu)d^*(Gu^*,Hu^*)}{1+d^*(Gu,Gu^*)+d^*(Hu,Hu^*)}\} = d^*(Gu, Gu^*).$$

From (3.25), we have

$$d^*(Hu, Hu^*) \geq d^*(Gu, Gu^*) + \phi(d^*(Gu, Gu^*)) > d^*(Gu, Gu^*) + d^*(Gu, Gu^*) = 2d^*(Gu, Gu^*),$$

which is a contradiction, so $u = u^*$. Thus, we proves the uniqueness of common fixed point.

Example Let $M = [0,3]$ be equipped with the dislocated metric space and

$$d^*(x, y) = max\{|x|, |y|\} \quad \text{for all } x, y \in M.$$

Define, $G, H: M \to M$ by

$$Gx = \begin{cases} 0, & if\ x = 0 \\ \frac{x}{4}, & otherwise \end{cases}$$

and

$$Hx = \begin{cases} 0, & if\ x = 0 \\ \frac{3x}{4}, & otherwise. \end{cases}$$

Then we have $GM = [0,\frac{3}{4}] \subset [0,\frac{9}{4}] = HM$.

Let $\{x_n\}$ be a sequence in M such that $\{x_n\} = \frac{1}{n}$ for each n and $\phi : [0,\infty) \to [0,\infty)$ be define by

$$\phi(t) = \begin{cases} \frac{t}{9}, & if\ t > 0 \\ 0, & otherwise. \end{cases}$$

Clearly $H(0) = G(0) = 0$ and $HG(0) = GH(0) = 0$ for all $x, y \in M$, this show that G and H are weakly compatible.

Now, we have to check the inequality of Theorem 3.1 for the following cases

Case (I): Let $x = 0$ and $y = 0$, we get

$d^*(Hx, Hy) = 0$, $d^*(Gx, Gy) = 0$ and $\Upsilon(Gx, Gy) = 0$. Also, $\phi(\Upsilon(Gx, Gy)) = 0$,

hence

$$d^*(Hx, Hy) \geq d^*(Gx, Gy) + \phi(\Upsilon(Gx, Gy)).$$

Case (II): Let $x \neq 0$ and $y = 0$, we get

$$d^*(Hx, Hy) = d^*(\frac{3x}{4}, 0) = max\left\{\frac{3x}{4}, 0\right\} = \frac{3x}{4}$$

$$d^*(Gx, Gy) = d^*(\frac{x}{4}, 0) = max\left\{\frac{x}{4}, 0\right\} = \frac{x}{4},$$

where

$$\Upsilon(Gx, Gy) = max\{d^*(Gx, Gy), d^*(Gx, Hx), d^*(Gy, Hy), \frac{d^*(Gx,Hx)(1+d^*(Gy,Hy))}{1+d^*(Gx,Gy)}, \frac{d^*(Gy,Hy)(1+d^*(Gx,Hx))}{1+d^*(Hx,Hy)},$$

$$\frac{d^*(Gx,Hx)d^*(Gy,Hy)}{1+d^*(Gx,Gy)+d^*(Hx,Hy)}\}$$

$$= max\{d^*(\frac{x}{4}, 0), d^*\left(\frac{x}{4}, \frac{3x}{4}\right), d^*(0,0), \frac{d^*\left(\frac{x}{4}, \frac{3x}{4}\right)(1 + d^*(0,0))}{1 + d^*(\frac{x}{4}, 0)}, \frac{d^*(0,0)(1+d^*\left(\frac{x}{4}, \frac{3x}{4}\right))}{1+d^*\left(\frac{x}{4}, \frac{3x}{4}\right)},$$

$$\frac{d^*\left(\frac{x}{4},\frac{3x}{4}\right)d^*(0,0)}{1+d^*\left(\frac{x}{4},0\right)+d^*\left(\frac{3x}{4},0\right)}\}$$

$$= max\{\frac{x}{4},\frac{3x}{4},0,\frac{\frac{3x}{4}}{1+\frac{x}{4}},0,0\}$$
$$= max\{\frac{x}{4},\frac{3x}{4},0,\frac{3x}{4},0,0\}$$
$$=\frac{3x}{4}.$$

So, $\phi(\Upsilon(Gx,Gy)) = \frac{1}{9}\left(\frac{3x}{4}\right) = \frac{x}{12}$, clearly

$$d^*(Hx,Hy) > d^*(Gx,Gy) + \phi(\Upsilon(Gx,Gy)).$$

Case (III): Let $x = 0$ and $y \neq 0$, we get

$$d^*(Hx,Hy) = d^*(0,\frac{3y}{4})$$
$$= max\left\{0,\frac{3y}{4}\right\} = \frac{3y}{4}$$
$$d^*(Gx,Gy) = d^*(0,\frac{y}{4})$$
$$= max\left\{0,\frac{y}{4}\right\} = \frac{y}{4},$$

where

$$\Upsilon(Gx,Gy) = max\{d^*(Gx,Gy), d^*(Gx,Hx),\ d^*(Gy,Hy),\ \frac{d^*(Gx,Hx)(1+d^*(Gy,Hy))}{1+d^*(Gx,Gy)},\ \frac{d^*(Gy,Hy)(1+d^*(Gx,Hx))}{1+d^*(Hx,Hy)},$$

$$\frac{d^*(Gx,Hx)d^*(Gy,Hy)}{1+d^*(Gx,Gy)+d^*(Hx,Hy)}\}$$

$$= max\{d^*(0,\frac{y}{4}), d^*(0,0), d^*\left(\frac{y}{4},\frac{3y}{4}\right)\ \frac{d^*(0,0)(1+d^*\left(\frac{y}{4},\frac{3y}{4}\right))}{1+d^*(0,\frac{y}{4})},$$

$$\frac{d^*\left(\frac{y}{4},\frac{3y}{4}\right)(1+d^*(0,0))}{1+d^*(0,\frac{3y}{4})},$$

$$\frac{d^*\left(\frac{y}{4},\frac{3y}{4}\right)d^*(0,0)}{1+d^*\left(\frac{x}{4},0\right)+d^*\left(\frac{3x}{4},0\right)}\}$$

$$= max\{\frac{y}{4},0,\frac{3y}{4},0,\frac{\frac{3y}{4}}{1+\frac{3y}{4}},0\}$$
$$=\frac{3y}{4}.$$

So, $\phi(\Upsilon(Gx,Gy)) = \frac{1}{9}\left(\frac{3y}{4}\right) = \frac{y}{12}$, clearly

$$d^*(Hx,Hy) > d^*(Gx,Gy) + \phi(\Upsilon(Gx,Gy)).$$

Case (IV): Let $x \neq 0$ and $y \neq 0$, we discuss three subcases:

Case (i): If $x > y$, we get

$$d^*(Hx,Hy) = d^*(\frac{3x}{4},\frac{3y}{4})$$
$$= max\left\{\frac{3x}{4},\frac{3y}{4}\right\} = \frac{3x}{4},$$
$$d^*(Gx,Gy) = d^*(\frac{x}{4},\frac{y}{4})$$
$$= max\left\{\frac{x}{4},\frac{y}{4}\right\} = \frac{x}{4},$$

where

$$\Upsilon(Gx,Gy) = max\{d^*(Gx,Gy), d^*(Gx,Hx),\ d^*(Gy,Hy),\ \frac{d^*(Gx,Hx)(1+d^*(Gy,Hy))}{1+d^*(Gx,Gy)},\ \frac{d^*(Gy,Hy)(1+d^*(Gx,Hx))}{1+d^*(Hx,Hy)},$$

$$\frac{d^*(Gx,Hx)d^*(Gy,Hy)}{1+d^*(Gx,Gy)+d^*(Hx,Hy)}\}$$

$$= max\{d^*(\frac{x}{4},\frac{y}{4}), d^*\left(\frac{x}{4},\frac{3x}{4}\right), d^*\left(\frac{y}{4},\frac{3y}{4}\right)\ \frac{d^*\left(\frac{x}{4},\frac{3x}{4}\right)(1+d^*\left(\frac{y}{4},\frac{3y}{4}\right))}{1+d^*(\frac{x}{4},\frac{y}{4})},$$

$$\frac{d^*\left(\frac{y}{4},\frac{3y}{4}\right)(1+d^*\left(\frac{x}{4},\frac{3x}{4}\right))}{1+d^*(\frac{3x}{4},\frac{3y}{4})},$$

$$\frac{d^*\left(\frac{x}{4},\frac{3x}{4}\right)d^*\left(\frac{y}{4},\frac{3y}{4}\right)}{1+d^*\left(\frac{x}{4},\frac{y}{4}\right)+d^*\left(\frac{3x}{4},\frac{3y}{4}\right)}\}$$

$$= max\{\frac{x}{4},\frac{3x}{4},\frac{3y}{4},\frac{\frac{3x}{4}\left(1+\frac{3y}{4}\right)}{1+\frac{x}{4}},\frac{\frac{3y}{4}\left(1+\frac{3x}{4}\right)}{1+\frac{3x}{4}},$$
$$\frac{\frac{3x}{4}(\frac{3y}{4})}{1+\frac{x}{4}+\frac{3x}{4}}\}$$
$$=\frac{3x}{4}.$$

So, $\phi(\Upsilon(Gx,Gy))=\frac{1}{9}\left(\frac{3x}{4}\right)=\frac{x}{12}$, clearly

$$d^*(Hx,Hy)>d^*(Gx,Gy)+\phi(\Upsilon(Gx,Gy)).$$

Case (ii): If $x<y$, we get

$$d^*(Hx,Hy)=d^*\left(\frac{3x}{4},\frac{3y}{4}\right)$$
$$=max\left\{\frac{3x}{4},\frac{3y}{4}\right\}=\frac{3y}{4},$$
$$d^*(Gx,Gy)=d^*\left(\frac{x}{4},\frac{y}{4}\right)$$
$$=max\left\{\frac{x}{4},\frac{y}{4}\right\}=\frac{y}{4},$$

where

$$\Upsilon(Gx,Gy)$$
$$=max\{d^*(Gx,Gy),d^*(Gx,Hx),\ d^*(Gy,Hy),$$
$$\frac{d^*(Gx,Hx)(1+d^*(Gy,Hy))}{1+d^*(Gx,Gy)},$$
$$\frac{d^*(Gy,Hy)(1+d^*(Gx,Hx))}{1+d^*(Hx,Hy)},$$

$$\frac{d^*(Gx,Hx)d^*(Gy,Hy)}{1+d^*(Gx,Gy)+d^*(Hx,Hy)}\}$$

$$=max\{d^*(\frac{x}{4},\frac{y}{4}),d^*\left(\frac{x}{4},\frac{3x}{4}\right),d^*\left(\frac{y}{4},\frac{3y}{4}\right)$$
$$\frac{d^*\left(\frac{x}{4},\frac{3x}{4}\right)(1+d^*\left(\frac{y}{4},\frac{3y}{4}\right))}{1+d^*(\frac{x}{4},\frac{y}{4})},$$

$$\frac{d^*\left(\frac{y}{4},\frac{3y}{4}\right)(1+d^*\left(\frac{x}{4},\frac{3x}{4}\right))}{1+d^*(\frac{3x}{4},\frac{3y}{4})},$$

$$\frac{d^*\left(\frac{x}{4},\frac{3x}{4}\right)d^*\left(\frac{y}{4},\frac{3y}{4}\right)}{1+d^*\left(\frac{x}{4},\frac{y}{4}\right)+d^*\left(\frac{3x}{4},\frac{3y}{4}\right)}\}$$

$$= max\{\frac{y}{4},\frac{3x}{4},\frac{3y}{4},\frac{\frac{3x}{4}\left(1+\frac{3y}{4}\right)}{1+\frac{y}{4}},\frac{\frac{3y}{4}\left(1+\frac{3x}{4}\right)}{1+\frac{3y}{4}},$$
$$\frac{\frac{3x}{4}(\frac{3y}{4})}{1+\frac{y}{4}+\frac{3y}{4}}\}$$
$$=\frac{3y}{4}.$$

So, $\phi(\Upsilon(Gx,Gy))=\frac{1}{9}\left(\frac{3y}{4}\right)=\frac{y}{12}$, clearly

$$d^*(Hx,Hy)>d^*(Gx,Gy)+\phi(\Upsilon(Gx,Gy)).$$

Case (iii): If $x=y\neq 0$, we get

$$d^*(Hx,Hy)=d^*(\frac{3x}{4},\frac{3y}{4})$$
$$=max\left\{\frac{3x}{4},\frac{3x}{4}\right\}=\frac{3x}{4},$$
$$d^*(Gx,Gy)=d^*(\frac{x}{4},\frac{y}{4})$$
$$=max\left\{\frac{x}{4},\frac{y}{4}\right\}=\frac{x}{4},$$

where

$$\Upsilon(Gx,Gy)$$
$$=max\{d^*(Gx,Gy),d^*(Gx,Hx),\ d^*(Gy,Hy),$$
$$\frac{d^*(Gx,Hx)(1+d^*(Gy,Hy))}{1+d^*(Gx,Gy)},$$
$$\frac{d^*(Gy,Hy)(1+d^*(Gx,Hx))}{1+d^*(Hx,Hy)},$$

$$\frac{d^*(Gx,Hx)d^*(Gy,Hy)}{1+d^*(Gx,Gy)+d^*(Hx,Hy)}\}$$

$$=max\{d^*(\frac{x}{4},\frac{x}{4}),d^*\left(\frac{x}{4},\frac{3x}{4}\right),d^*\left(\frac{x}{4},\frac{3x}{4}\right)$$
$$\frac{d^*\left(\frac{x}{4},\frac{3x}{4}\right)(1+d^*\left(\frac{x}{4},\frac{3x}{4}\right))}{1+d^*(\frac{x}{4},\frac{x}{4})},$$

$$\frac{d^*\left(\frac{y}{4},\frac{3y}{4}\right)(1+d^*\left(\frac{x}{4},\frac{3x}{4}\right))}{1+d^*(\frac{3x}{4},\frac{3x}{4})},$$

$$\frac{d^*\left(\frac{x}{4},\frac{3x}{4}\right)d^*\left(\frac{x}{4},\frac{3x}{4}\right)}{1+d^*\left(\frac{x}{4},\frac{x}{4}\right)+d^*\left(\frac{3x}{4},\frac{3x}{4}\right)}\}$$

$$= max\{\frac{x}{4},\frac{3x}{4},\frac{3x}{4},\frac{\frac{3x}{4}\left(1+\frac{3x}{4}\right)}{1+\frac{x}{4}},\frac{\frac{3x}{4}\left(1+\frac{3x}{4}\right)}{1+\frac{3x}{4}},\frac{\frac{3x}{4}\left(\frac{3x}{4}\right)}{1+\frac{x}{4}+\frac{3x}{4}}\}$$

$$=\frac{3x}{4}.$$

So, $\phi(\Upsilon(Gx,Gy)) = \frac{1}{9}\left(\frac{3x}{4}\right) = \frac{x}{12}$, clearly

$$d^*(Hx,Hy) > d^*(Gx,Gy) + \phi(\Upsilon(Gx,Gy)).$$

Therefore, the inequality of Theorem 3.1 holds for all the cases.

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