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ปริมา 68 เล่มที่ 709 มกราคม – เมษายน 2566
http://www.mathassociation.net Email: MathThaiOrg@gmail.com

ไฮเปอร์ไอเดนติตี้ในกราฟวาไรตี้ที่ก่อกำเนิดโดยกราฟซีโรโพเทนต์และยูนิโพเทนต์

Hyperidentities in Graph Variety Generated by Zeropotent and Unipotent Graphs

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เชาวน์วัฒน์ มั่นยืน
สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยราชภัฏมหาสารคาม
มหาสารคาม 44000

Chaowat Manyuen
Department of Mathematics, Faculty of Science, Rajabhat Maha Sarakham University,
Mahasarakham 44000

Email: chaowat.ma@rmu.ac.th

วันที่รับบทความ : 18 กรกฎาคม 2564 วันที่แก้ไขบทความ : 19 สิงหาคม 2565 วันที่ตอบรับบทความ : 25 กุมภาพันธ์ 2566

บทคัดย่อ

พีชคณิตกราฟสร้างจากกราฟระบุทิศทางที่ไม่มีเส้นเชื่อมขนาน และพีชคณิตแบบพิเศษชนิด $(2,0)$ และจะกล่าวว่ากราฟ G สอดคล้องกับเอกลักษณ์ $s \approx t$ ถ้าพีชคณิตกราฟ $A(G)$ ที่สมนัยกับ G สอดคล้องกับ $s \approx t$ เซตของเอกลักษณ์ $s \approx t$ ทั้งหมดซึ่งกราฟ G สอดคล้องเขียนแทนด้วย $\mathrm{Id}(\{G\})$ เซตของกราฟทั้งหมดซึ่งสอดคล้องกับเอกลักษณ์ $s \approx t$ ใน $\mathrm{Id}(\{G\})$ เรียกว่ากราฟวาไรตี้ที่ก่อกำเนิดโดย $\{G\}$ เขียนแทนด้วย $\mathcal{V}_g(\{G\})$ เอกลักษณ์ $s \approx t$ จะเป็นเอกลักษณ์ใน

$\mathcal{V}_g(\{G\})$ ถ้า $A(G')$ สอดคล้อง $s \approx t$ ทุก $G \in V_g(\{G\})$ เอกลักษณ์ $s \approx t$ ของพจน์ s และ t ชนิด τ จะเรียกว่าไฮเปอร์ไอเด็นติตี้ของพีชคณิต $\underline{A}$ ถ้าไม่ว่าเมื่อใดก็ตามที่เอกลักษณ์การดำเนินการใด ๆ ที่ปรากฏในพจน์ s และ t ถูกแทนที่ด้วยการดำเนินการพจน์ใด ๆ ที่เหมาะสมของ $\underline{A}$ และผลลัพธ์ที่ได้เป็นจริงใน $\underline{A}$ ในบทความนี้จะพิจารณาลักษณะทั้งหมดของไฮเปอร์ไอเด็นติตี้ใน $\mathcal{V}_g(\{G\})$ เมื่อ G เป็นกราฟซีโรโพเทนต์และยูนิโพเทนต์

คำสำคัญ: กราฟวาไรตี้ กราฟก่อกำเนิด พจน์ เอกลักษณ์ พีชคณิตทวิภาค พีชคณิตกราฟ ไฮเปอร์ไอเด็นติตี้

ABSTRACT

Directed graphs without multiple edges can be represented as algebras of type $(2,0)$, so-called graph algebras. We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $A(G)$ satisfies $s \approx t$. The set of all term equations $s \approx t$ which the graph G satisfies denoted by $\mathrm{Id}(\{G\})$. The class of all graph algebras satisfying all term equations in $\mathrm{Id}(\{G\})$ is called the graph variety generated by G, denoted by $\mathcal{V}_g(\{G\})$. A term equation $s \approx t$ is called an identity in $\mathcal{V}_g(\{G\})$ if $A(G')$ satisfies $s \approx t$ for all $G \in \mathcal{V}_g(\{G\})$. An identity $s \approx t$ of terms s and t of any type τ is called a hyperidentity of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identities hold in $\underline{A}$. In this paper, we characterize all identities, all graphs and all hyperidentities in $\mathcal{V}_g(\{G\})$ where G is the zeropotent and unipotent.

Keywords: Graph varieties, Generated graphs, Terms, Identities, Binary algebras, Graph algebras, Hyperidentities

1. Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identity holds in $\underline{A}$.

Hyperidentities can be defined more precisely by using the concept of a hypersubstitution, which was introduced by Denecke, Lau, Pöschel and Schweigert in [2].

We fix a type $\tau = (n_i)_{i \in I}, n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is $n_i - \mathrm{ary}$. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $\mathrm{Alg}(\tau)$ be the class of all algebras of type τ. Then, a mapping

$$\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every $n_i - \mathrm{ary}$ operation symbol f_i an $n_i - \mathrm{ary}$ term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). We denote the extension of the hypersubstitution σ by a mapping

$$\hat{\sigma}: W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X, and

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras were introduced by Shallon [13] in 1979 with the purpose of providing examples of nonfinitely based finite algebras. Let us briefly recall this concept. Given a directed graph $G = (V, E)$ without multiple edges, the *graph algebra* associated with G is the algebra $A(G) = (V \cup \{\infty\},\ \circ,\ \infty)$ of type $(2, 0)$, where ∞ is an element not belonging to V and the binary operation $\circ$ is defined by the rule

$$u \circ v := \begin{cases} u & \text{if } (u, v) \in E, \\ \infty & \text{otherwise,} \end{cases}$$

for all $u, v \in V \cup \{\infty\}$. We will denote the product $u \circ v$ simply by juxtaposition uv.

In [12], Pöschel and Wessel, graph varieties were investigated for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [11], these investigations were extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by identities for their corresponding graph algebras. The answer is a theorem of **Birkhoff-type**, which uses graph theoretic closure operations. A class of finite directed graphs is equational (i.e., a graph variety) if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.

Let $s \approx t$ be a term equation. Poomsa-ard and et. al. characterized the graph variety $\mathcal{V} = \mathrm{Mod}_g(\{s \approx t\})$ in various kind of terms s and t. Further they characterized identities and hyperidentities in these graph varieties, too. But these results are not convenient for apply to the real-world situation. Because at first we will check that what kind of terms s and t which the graph variety $\mathcal{V} = \mathrm{Mod}_g(\{s \approx t\})$ contains the graph algebra of the diagram of that real-world situation. It is not easy to do this. So we will characterize the graph variety generate by the graph G of the diagram directly. Then characterize identities of this graph variety. In [5], Jampachon and Poomsa-ard characterized all identities, all graphs and all hyperidentities in graph variety generated by $((xx)(y((zx)z)))z$ graph. In [7], Lehtonen and Manyuen characterized all the graph varieties axiomatized by certain noteworthy groupoid identities that are of general interest in algebra, such as the zeropotent, unipotent, commutative, alternative, semimedial, and medial identities.

In this paper, we characterized all identities, all graphs and all hyperidentities in $\mathcal{V} = \mathrm{Mod}_g(\{s \approx t\})$ where G is the zeropotent and unipotent graphs.

2. Terms, identities and graph varieties

Dealing with terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞ too).

Definition 2.1 A term over the alphabet

$$X = \{x_1, x_2, x_3, \dots\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1,2,3, \dots$ and ∞ are terms;

(ii) if t_1 and t_2 are terms, then $t_1 t_2$ is a term.

Let $W_\tau(X)$ be the set of all terms which can be obtained from (i) and (ii) in finitely many steps. Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $W_\tau(X_2)$. The leftmost variable of a term t is denoted by $L(t)$, the rightmost variable of a term t is denoted by $R(t)$. A term in which the symbol ∞ occurs is called a *trivial term*.

Definition 2.2 For each non-trivial term t of type $\tau = (2,0)$, one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1 t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1 t_2$ is a compound term. $L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is called the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3 Let $G = (V, E)$ be a graph. Let $h : X \to V \cup \{\infty\}$ be a map, called an *assignment*. Extend h to a map $\bar{h} : W_\tau(X) \to V \cup \{\infty\}$ by the rule $\bar{h}(t) = h(t)$ if $t = x \in X$, and $\bar{h}(t) = \bar{h}(t_1)\bar{h}(t_2)$ if $t = t_1 t_2$, where the product is taken in $A(G)$. Then $\bar{h}$ is called the *valuation* of the term t in the graph G with respect to assignment h. Although the graph G does not appear in the notation $\bar{h}$, it will always be clear from the context.

Definition 2.4 An *identity* (in the language of graph algebras) is an ordered pair (s,t) of terms $s,t \in W_\tau(X)$, usually written as $s \approx t$. Let $A(G)$ be a graph algebra corresponding to $G=(V,E)$. We say that $A(G)$ *satisfies* $s \approx t$, and we write $A(G) \models s \approx t$ if $\bar{h}(s)=\bar{h}(t)$ for every assignment $h : X \to V \cup \{\infty\}$. In this case, we also say that G satisfies $s \approx t$ and we write $G \models s \approx t$.

The above notation extends to an arbitrary class $\mathcal{G}$ of graphs and to any set Σ of identities as follows:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$,

$\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,

$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$.

The relation of satisfaction of an identity by a graph induces a Galois connection between graphs and identities via the polarities

$\mathrm{Id}\,\mathcal{G} = \{s \approx t \mid s,t \in W_\tau(X), \mathcal{G} \models s \approx t\}$,

$\mathrm{Mod}_g \Sigma = \{G \mid G$ is a graph and $G \models \Sigma\}$.

It follows from the general theory of Galois connections (see [1]) that $\mathrm{Mod}_g\,\mathrm{Id}$ is a closure operator on graphs, which we denote simply by $\mathcal{V}_g$. The closed sets of graphs, i.e., sets $\mathcal{G}$ satisfying $\mathcal{V}_g(\mathcal{G})=\mathcal{G}$, are called *graph varieties*.

3. Identities in graph variety generated by zeropotent and unipotent graphs

In [7], we recall that a graph G is *zeropotent* and *unipotent*, if it satisfies the identities $(xx)y \approx xx \approx y(xx)$ and $xx \approx yy$ when $x,y \in X$, respectively.

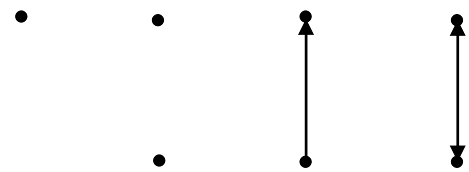
Proposition 3.1 (Lehtonen and Manyuen [7, Theorem 3.1]) Let G be a graph. The following conditions are equivalent:

(i) G is zeropotent.

(ii) G is unipotent.

(iii) G has no loops.

From Proposition 3.1 we see that G_1, G_2, G_3, G_4 graphs are zeropotent and unipotent.



G_1 G_2 G_3 G_4

Let $\mathcal{K}' = \mathrm{Mod}_g\ \{(xx)y \approx xx \approx y(xx), xx \approx yy\}$. This means $\mathcal{K}'$ is the set of all loopless graphs. We want to characterized all identities in $\mathcal{K}'$. Before to do this we need some results for reference as the following:

The equational theory of the class of all graphs (all graph algebras) was described by Kiss, Pöschel and Pröhle in [6] as follows.

Proposition 3.2 (Kiss, Pöschel and Pröhle [6, Lemma 2.2(3)]). Let $s \approx t$ be an identity and let $\mathcal{G}$ be the class of all graphs. Then $\mathcal{G} \models s \approx t$ if and only if s and t are trivial terms or $G(s) = G(t)$ and $L(s) = L(t)$.

The following results provide useful tools for checking whether a graph satisfies an identity.

Proposition 3.3 (Kiss, Pöschel and Pröhle [6, Lemma 2.2(2)]). Let $G = (V, E)$ be a graph and let $h : X \to V \cup \{\infty\}$ be an evaluation of the variables. Consider the canonical extension $\bar{h}$ of h to the set of all terms. Then the following holds. If t is a trivial term, then $\bar{h}(t) = \infty$. Otherwise, if $h : G(t) \to G$ is a homomorphism of graphs, then $\bar{h}(t) = \bar{h}\big(L(t)\big) = h(L(t))$, and if h is not a homomorphism of graphs, then $\bar{h}(t) = \infty$.

Proposition 3.4 (Pöschel and Wessel [12, Proposition 1.5(2)]). Let s and t be non-trivial terms such that $V(s) = V(t)$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if G has the following property: a mapping $h : V(s) \to$

$V \cup \{\infty\}$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G.

Now we characterize all identities in $\mathcal{K}'$. Clearly, if $s \approx t$ is a trivial equation (i.e. $G(s) = G(t),\ L(s) = L(t)$) or both of them are trivial terms, then $s \approx t$ is an identity in $\mathcal{K}'$. Now we consider the case $s \approx t$ is a non-trivial equation. Then all identities in $\mathcal{K}'$ are characterized by the following theorem:

Theorem 3.1 Let $s \approx t$ be non-trivial equation. Then, $\mathcal{K}' \models s \approx t$ if and only if the following conditions are satisfied:

(i) $G(s)$ has a loop if and only if $G(t)$ has a loop,

(ii) if $G(s)$ and $G(t)$ have no loop then $V(s) = V(t)$,

(iii) if $G(s)$ and $G(t)$ have no loop then for any $x, y \in V(s)$, $(x, y) \in E(s)$ if and only if $(x, y) \in E(t)$,

(iv) if $G(s)$ and $G(t)$ have no loop then $L(s) = L(t)$.

Proof. (i) Suppose that $G(s)$ has a loop but $G(t)$ has no loop. Let $G = G(t)$. By Proposition 3.1, we have $G \in \mathcal{K}'$. Let $h : V(s) \cup V(t) \to V(G) \cup \{\infty\}$ such that $h(x) = x$ for all $x \in V(t)$ and $h(y) = \infty$ for all $y \in V(s) \setminus V(t)$. It is clear that h is a homomorphism from $G(t)$ into G but h is not a homomorphism from $G(s)$ into G. By Proposition 3.3, we have $\bar{h}(t) = h(L(t)) = L(t) \neq \infty = \bar{h}(s)$. Therefore $G \not\models s \approx t$. Hence $\mathcal{K}' \not\models s \approx t$.

(ii) Suppose that $G(s)$ and $G(t)$ have no loop but $V(s) \neq V(t)$. Then there exists $y \in V(s)$ but $y \notin V(t)$. Let $G = G(t)$. By Proposition 3.2, we have $G \in \mathcal{K}'$. Let $h : V(s) \cup V(t) \to V(G) \cup \{\infty\}$ such that $h(x) = x$ for all $x \in V(t)$ and $h(y) = \infty$ for all $y \in V(s) \setminus V(t)$. We have h is a homomorphism from $G(t)$ into G but h is not a homomorphism from $G(s)$ into G. By Proposition 3.3, we have $\bar{h}(t) = h(L(t)) = L(t) \neq \infty = \bar{h}(s)$. Therefore $G \not\models s \approx t$. Hence $\mathcal{K}' \not\models s \approx t$.

(iii) Suppose that $G(s)$ and $G(t)$ have no loop, $(x,y) \in E(s)$ but $(x,y) \notin E(t)$. Let $G = G(t)$. By Proposition 3.1, we have $G \in \mathcal{K}'$. By (ii), we have $V(s) = V(t)$. Let $h : V(t) \to V(G) \cup \{\infty\}$ such that $h(x) = x$ for all $x \in V(t)$. Since $G = G(t)$ and $(x,y) \notin E(t)$, we have h is a homomorphism from $G(t)$ into G and $(h(x), h(y)) \notin E(G)$. Because $(x,y) \in E(s)$, we get h is not a homomorphism from $G(s)$ into G. By Proposition 3.3, we have $\bar{h}(t) = h(L(t)) = L(t) \neq \infty = \bar{h}(s)$. Therefore $G \nvDash s \approx t$. Hence $\mathcal{K}' \nvDash s \approx t$.

(iv). Suppose that $G(s)$ and $G(t)$ have no loop and $L(s) \neq L(t)$. Let $G = G(t)$. By Proposition 3.1, we have $G \in \mathcal{K}'$. By (ii), we have $V(s) = V(t)$. Let $h : V(t) \to V(G) \cup \{\infty\}$ such that $h(x) = x$ for all $x \in V(t)$. Since $G = G(t)$, we get h is a homomorphism from $G(t)$ into G. By (iii), we have $(x,y) \in E(s)$ if and only if $(x,y) \in E(t)$. Thus h is a homomorphism from $G(s)$ into G. By Proposition 3.3, $\bar{h}(s) = h(L(s)) = L(s) \neq L(t) = h(L(t)) = \bar{h}(t)$. Therefore $G \nvDash s \approx t$. Hence $\mathcal{K}' \nvDash s \approx t$.

Conversely, suppose that $s \approx t$ is a non-trivial equation satisfying (i), (ii), (iii) and (iv). Let $G \in \mathcal{K}'$. Then G has no loop. By (i), $G(s)$ has a loop if and only if $G(t)$ has a loop.

Case I If $G(s)$ and $G(t)$ have a loop then any h is not a homomorphism from $G(s)$ into G and it is not a homomorphism from $G(t)$ into G. Thus $\bar{h}(s) = \infty = \bar{h}(t)$. Hence $G \vDash s \approx t$.

Case II If $G(s)$ and $G(t)$ have no loop then by (ii) and (iv), $V(s) = V(t)$ and $L(s) = L(t)$. Let $h : V(s) \to V(G) \cup \{\infty\}$ be a function. Suppose h is a homomorphism from $G(s)$ into G. Let $(x,y) \in E(s)$. Then $\big(h(x), h(y)\big) \in E(G)$. By (iii), we get $(x,y) \in E(t)$. Thus h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G then it is a homomorphism from $G(s)$ into G. Hence by Proposition 3.4, we get $G \vDash s \approx t$.

Therefore $\mathcal{K}' \vDash s \approx t$. ☐

Corollary 3.1 Let $s \approx t$ be a non-trivial equation such that $G(s)$ and $G(t)$ have no loop. Then $\mathcal{K}' \models s \approx t$ if and only if $L(s) = L(t)$ and $G(s) = G(t)$.

Proof. Let $s \approx t$ be a non-trivial equation such that $G(s)$ and $G(t)$ have no loop. Assume that $\mathcal{K}' \models s \approx t$. Then by condition (ii), (iii), (iv) of Theorem 3.1, we have $L(s) = L(t)$, $V(s) = V(t)$ and $E(s) = E(t)$. Hence $G(s) = G(t)$.

Conversely, assume that $L(s) = L(t)$ and $G(s) = G(t)$. So $V(s) = V(t)$ and $E(s) = E(t)$. Thus s and t satisfy condition (i), (ii), (iii) and (iv) of Theorem 3.1. Hence $\mathcal{K}' \models s \approx t$. □

4. Hyperidentities in graph variety generated by zeropotent and unipotent graphs

Let $\mathcal{K}$ be any graph variety. Now, we want to formulate precisely the concept of a graph hypersubstitution for graph algebras.

Definition 4.1 A mapping $\sigma : \{f, \infty\} \to W_\tau(X_2)$, where $X_2 = \{x_1, x_2\}$ and f is the operation symbol corresponding to the binary operation of a graph algebra is called the *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in W_\tau(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2 An identity $s \approx t$ is a $\mathcal{K}$ *graph hyperidentity* if and only if for all graph hypersubstitutions $\sigma, \mathcal{K} \models \hat{\sigma}[s] \approx \hat{\sigma}[t]$.

If we want to check that an identity $s \approx t$ is a hyperidentity in $\mathcal{K}$ we can restrict our consideration to a (small) subset of $Hyp\mathcal{G}$ - the set of all graph hypersubstitutions. In [8], the following relation between hypersubstitutions was defined:

Definition 4.3 Two graph hypersubstitutions σ_1, σ_2 are called $\mathcal{K}$- *equivalent* if and only if $\sigma_1(f) \approx \sigma_2(f)$ is an identity in $\mathcal{K}$. In this case we write $\sigma_1 \sim_{\mathcal{K}} \sigma_2$.

The following lemma was proven in [9].

Lemma 4.1 If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in \mathrm{Id}\,\mathcal{K}$ and $\sigma_1 \sim_{\mathcal{K}} \sigma_2$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in \mathrm{Id}\,\mathcal{K}$. Therefore, it is enough to consider the quotient set $Hyp\mathcal{G} /\sim_{\mathcal{K}}$.

In [10], it showed that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Without difficulties one shows $G\big(NF(t)\big) = G(t),\ L(NF(t)) = L(t)$.

The following definition was given in [3].

Definition 4.4 The graph hypersubstitution $\sigma_{NF(t)}$, is called *normal form graph hypersubstitution*. Here $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in \mathrm{Id}\,\mathcal{K}$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in W_\tau(X_2)$, one obtains $\sigma_t \sim_{\mathcal{K}} \sigma_{NF(t)}$.

Table 1. In [3], all rooted graphs with at most two vertices were considered. Then, we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the following table:

Normal form term	graph hypers	Normal form term	graph hypers
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$(x_2(x_1x_2))x_2$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$x_2((x_1x_1)x_2)$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

Since $G(\sigma_3), G(\sigma_4), G(\sigma_6), G(\sigma_7), G(\sigma_8), G(\sigma_9), G(\sigma_{10}), G(\sigma_{11}), G(\sigma_{14}),$ $G(\sigma_{15}), G(\sigma_{16}), G(\sigma_{17}), G(\sigma_{18})$ and $G(\sigma_{19})$ have loops, hence they satisfy conditions (i), (ii), (iii) and (iv) of Theorem 3.1. Thus we have the following relations:

$$\sigma_3 \sim_{\mathcal{K}'} \sigma_4 \sim_{\mathcal{K}'} \sigma_6 \sim_{\mathcal{K}'} \sigma_7 \sim_{\mathcal{K}'} \sigma_8 \sim_{\mathcal{K}'} \sigma_9 \sim_{\mathcal{K}'} \sigma_{10} \sim_{\mathcal{K}'} \sigma_{11}$$
$$\sim_{\mathcal{K}'} \sigma_{14} \sim_{\mathcal{K}'} \sigma_{15} \sim_{\mathcal{K}'} \sigma_{16} \sim_{\mathcal{K}'} \sigma_{17} \sim_{\mathcal{K}'} \sigma_{18} \sim_{\mathcal{K}'} \sigma_{19}$$

Let $M_{\mathcal{K}^{'}}$ be the set of all normal form graph hypersubstitutions in $\mathcal{K}^{'}$. Then we get,

$$M_{\mathcal{K}^{'}} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_{12}, \sigma_{13}\}$$

We defined the product of two normal form graph hypersubstitutions in $M_{\mathcal{K}^{'}}$ as follows.

Definition 4.5 The *product* $\sigma_{1N} \circ_N \sigma_{2N}$ *of two normal form graph hypersubstitutions* is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\hat{\sigma}_{2N}(f)])$.

Table 2. The following table gives the multiplication of elements in $M_{\mathcal{K}^{'}}$.

$\circ_N$	σ_0	σ_1	σ_2	σ_3	σ_5	σ_{12}	σ_{13}
σ_0	σ_0	σ_1	σ_2	σ_3	σ_5	σ_{12}	σ_{13}
σ_1	σ_1	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2
σ_2	σ_2	σ_1	σ_2	σ_1	σ_1	σ_1	σ_2
σ_3	σ_3	σ_1	σ_2	σ_3	σ_3	σ_3	σ_3
σ_5	σ_5	σ_1	σ_2	σ_3	σ_0	σ_3	σ_3
σ_{12}	σ_{12}	σ_1	σ_2	σ_3	σ_{13}	σ_{12}	σ_{13}
σ_{13}	σ_{13}	σ_1	σ_2	σ_3	σ_{12}	σ_3	σ_3

The concept of a proper hypersubstitution of a class of algebras was introduced in [9].

Definition 4.6 A hypersubstitution σ is called *proper with respect to a class* $\mathcal{K}$ *of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \mathrm{Id}\,\mathcal{K}$ for all $s \approx t \in \mathrm{Id}\,\mathcal{K}$.

A graph hypersubstitution with the property that $\sigma(f)$ contains both variables x_1 and x_2 is called *regular*. It is easy to check that the set of all regular graph hypersubstitutions forms a groupoid M_{reg}.

The following lemma was proved in [3].

Lemma 4.2 For each non-trivial term s, $(s \neq x \in X)$ and for all $u, v \in X$, we have

$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u) | (u,v) \in E(s)\}$,

$E(\hat{\sigma}_8[s]) = E(s) \cup \{(v,v) | (u,v) \in E(s)\}$, and $\quad E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v,u) | (u,v) \in E(s)\}$.

In the similar way we can prove that, $E(\hat{\sigma}_3[s]) = \{(L(s), L(s))\}$.

We want to find all proper graph hypersubstitutions with respect to $\mathcal{K}'$. Then we obtain:

Theorem 4.1 $\{\sigma_0, \sigma_3, \sigma_{12}\}$ is the set of all proper graph hypersubstitutions with respect to $\mathcal{K}'$.

Proof. If $s \approx t \in \mathrm{Id}\,\mathcal{K}'$ and s, t are trivial terms, then for every graph hypersubstitution $\sigma \in \{\sigma_0, \sigma_3, \sigma_{12}\}$ the term $\hat{\sigma}[s]$ and $\hat{\sigma}[t]$ are also trivial and thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \mathrm{Id}\,\mathcal{K}'$. In the same manner, we see that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \mathrm{Id}\,\mathcal{K}'$ for every $\sigma \in \{\sigma_0, \sigma_3, \sigma_{12}\}$ if $s = t = x$.

Now assume that s and t are non-trivial terms, different from variables, and $s \approx t \in \mathrm{Id}\,\mathcal{K}'$.

Consider σ_0. Since $\sigma_0 = x_1 x_2$, it is clear that $E(\hat{\sigma}_0[s]) = E(s)$ and $L(\hat{\sigma}_0[s]) = L(s)$ for all non-trivial term s. Hence $\hat{\sigma}_0[s] \approx \hat{\sigma}_0[t] \in \mathrm{Id}\,\mathcal{K}'$.

Consider σ_3. By Lemma 4.2, $E(\hat{\sigma}_3[s]) = \{(L(s), L(s))\}$ and $E(\hat{\sigma}_3[t]) = \{(L(t), L(t))\}$. That is $G(s)$ and $G(t)$ have a loop. By Theorem 3.1 (i), we get $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \in \mathrm{Id}\,\mathcal{K}'$. Therefore σ_3 is a proper hypersubstitution.

Consider σ_{12}. If s and t have a loop then $\hat{\sigma}_{12}[s]$ and $\hat{\sigma}_{12}[t]$ have a loop. By Theorem 3.1 (i), $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in \mathrm{Id}\,\mathcal{K}'$. If s and t have no loop then $G(s) = G(t)$ and $L(s) = L(t)$. By Lemma 4.2, we get $G(\hat{\sigma}_{12}[s]) = G(\hat{\sigma}_{12}[t])$, $L(\hat{\sigma}_{12}[s]) = L(\hat{\sigma}_{12}[t])$ So $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in \mathrm{Id}\,\mathcal{K}'$. Therefore σ_{12} is a proper hypersubstitution.

For any $\sigma \notin \{\sigma_0, \sigma_3, \sigma_{12}\}$. Let $s_1 = x_1 x_1$ and $t_1 = x_2 x_2$. Then $s_1 \approx t_1 \in \mathrm{Id}\,\mathcal{K}'$. We see that, $\hat{\sigma}_1[s_1] = x_1$, $\hat{\sigma}_1[t_1] = x_2$, $\hat{\sigma}_2[s_1] = x_1$ and $\hat{\sigma}_2[t_1] = x_2$. Thus $\hat{\sigma}_1[s_1] \approx \hat{\sigma}_1[t_1]$, $\hat{\sigma}_2[s_1] \approx \hat{\sigma}_2[t_1] \notin \mathrm{Id}\,\mathcal{K}'$.

Let $s_2 = x_1 x_2$ and $t_2 = (x_1 x_2) x_2$. Then $s_2 \approx t_2 \in \mathrm{Id}\,\mathcal{K}'$. We see that, $\hat{\sigma}_5[s_2] = x_2 x_1$, $\hat{\sigma}_5[t_2] = x_2(x_2 x_1)$, $\hat{\sigma}_{13}[s_2] = x_2(x_1 x_2)$ and

$\hat{\sigma}_{13}[t_2] = x_2((x_2(x_1x_2))x_2)$. We see that $\hat{\sigma}_5[s_2]$, $\hat{\sigma}_5[s_2]$ have no loop but $\hat{\sigma}_5[t_2]$, $\hat{\sigma}_{13}[t_2]$ have a loop. Hence $\hat{\sigma}_5[s_2] \approx \hat{\sigma}_5[t_2],\ \hat{\sigma}_{13}[s_2] \approx \hat{\sigma}_{13}[t_2] \notin \mathrm{Id}\,\mathcal{K}'$.

Therefore $\{\sigma_0, \sigma_3, \sigma_{12}\}$ is the set of all proper graph hypersubstitutions with respect to $\mathcal{K}'$. □

Now, we apply our results to characterize all hyperidentities in $\mathcal{K}'$. Clearly, if s and t are trivial terms, then $s \approx t$ is a hyperidentity in $\mathcal{K}'$ if and only if $L(s) = L(t)$, $R(s) = R(t)$ and $x \approx x,\ x \in X$ is a hyperidentity in $\mathcal{K}'$, too. So, we consider the case that s and t are non-trivial terms and different from variables.

Theorem 4.2 An identity $s \approx t$ in $\mathcal{K}'$, where s and t are non-trivial terms and different from variables is a hyperidentity in $\mathcal{K}'$ if and only if $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t],\ \hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ and $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ are also identities in $\mathcal{K}'$.

Proof. Let $s \approx t \in \mathrm{Id}\,\mathcal{K}'$, where s, t are non-trivial and $s \neq x, t \neq x$. If $s \approx t$ is a hyperidentity in $\mathcal{K}'$, then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in \mathrm{Id}\,\mathcal{K}',\ \hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in \mathrm{Id}\,\mathcal{K}'$ and $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in \mathrm{Id}\,\mathcal{K}'$.

Assume that $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t],\ \hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ and $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ are identity in $\mathcal{K}'$. Since $\sigma_0, \sigma_3, \sigma_{12}$ are the proper graph hypersubstitutions, we have $\hat{\sigma}_0[s] \approx \hat{\sigma}_0[t],\ \hat{\sigma}_3[s] \approx \hat{\sigma}_3[t]$ and $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t]$ are identity in $\mathcal{K}'$.

Because of $\sigma_{12} \circ_N \sigma_5 = \sigma_{13}$ and σ_{12} is a proper graph hypersubstitution with respect to the class $\mathcal{K}'$, we have $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ is an identities in $\mathcal{K}'$. □

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