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THESIS

WAVELET-GALERKIN METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

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A Thesis Submitted in Partial Fulfillment of
the Requirements for the Degree of
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In this thesis, the wavelet-Galerkin finite element method is presented for solving the one-dimensional heat equation, the singularly perturbed boundary value problem and the singularly perturbed parabolic problem. The multilevel augmentation method with wavelet bases is demonstrated to show as the fast technique for solving the problems. We consider two types of basis functions which are the Lagrange and wavelet bases for constructing the full form of matrix system. We consider both linear and quadratic bases in the Galerkin method. Our numerical results show that the rate of convergences for the linear Lagrange and wavelet bases are the same in order 2 while the rate of convergences for the quadratic Lagrange and wavelet bases are approximately in order 4. It also reveals that the wavelet basis provides an easy treatment to improve numerical resolutions that can be done by increasing just its desired levels in the multilevel process. We also applied the multilevel augmentation method with wavelet bases for the singularly perturbed problem. It is found that the multilevel augmentation method is faster than the standard multilevel method in the same accuracy for high multilevel basis applied.

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Student's signature	Thesis Advisor's signature	

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WAVELET-GALERKIN METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

INTRODUCTION

The Galerkin approach is one of the very successful methods for finding approximate solutions from the partial differential equation. The main concept is using an approximate basis function for the solution space of the governing equation, and then projecting the terms of approximate solution on the functional basis space. This process provides residual that should be minimized with respect to the functional basis. By this concept, the accuracy of numerical solution depends directly on the type of basis function.

In this thesis, we present the Galerkin method with wavelet bases called the wavelet-Galerkin method to solve numerically the linear one-dimensional heat equation, the singularly perturbed second-order boundary value problem and the singularly perturbed parabolic problem. The multilevel augmentation method using wavelet bases is also applied to solve the singularly perturbed second-order boundary value problem. Two types of the basis functions which are the Lagrange and wavelet bases, in the forms of linear and quadratic polynomial are employed in the Galerkin method. Full forms of stiffness matrices are also presented. The approximations in the time are performed by applying polynomial basis in time resulting to time discretized matrix operating to the approximation in space by tensor product operator. By this approximation, the order of accuracy in time discretization is easily increased that is unlike the standard time marching scheme such as the forward Euler or the Crank-Nicolson method.

The Gelerkin method of Lagrange and wavelet bases is applied to solve both steady and unsteady problems for smooth case which is demonstrated by the heat equation and for sharp gradient case which is presented by the singularly perturbed problem. Rates of convergence of numerical solutions for two types of linear and quadratic bases are also presented. The rates of convergence of both the Lagrange and wavelet bases are in the same order as in theoretical results. We have revealed in this thesis that the linear wavelet has more advantages than the linear Lagrange when high numerical resolutions are required. The accuracy by wavelet bases is easily improved by increasing just wavelet levels (multilevel concept). This concept is different from using the tradition as Lagrange bases that is required to calculate the whole system.

Other main objectives in this thesis is to steady the multilevel augmentation method by using wavelet bases to solve numerically the singularly perturbed boundary value problem. We consider both linear and quadratic wavelet bases in the method. The multilevel augmentation method is fast and accurate for solving differential equations. It has great advantage for solving large-scale problems. This method integrate the choices of bases and the design of numerical solvers for the discrete linear systems together.

The details of this thesis are organized as follows. Details of the Lagrange and wavelet basis functions including time discretization with polynomial basis are presented in the chapter of Basis functions. We introduce the concept of Galerkin finite element method to solve numerically the linear one-dimensional heat equation and numerical results are presented to demonstrate of this method in the chapter of Galerkin finite element method for heat equation. In the chapter of Galerkin finite element method for singularly perturbed problem, we study the concept of the Galerkin finite element method to solve numerically the singularly perturbed second-order boundary value problem and the unsteady singularly perturbed problem and presented numerical results of this method. We introduce the concept of the multilevel augmentation method using wavelet bases to solve numerically the singularly perturbed problem and numerical results are shown in the chapter of Multilevel augmentation method. Finally, we have made some conclusions in Conclusions and Recommendations.

OBJECTIVES

1. To study the Galerkin finite element method based on the Lagrange and wavelet bases to solve numerically the one-dimensional heat equation.

2. To study the Galerkin finite element method based on the Lagrange and wavelet bases to solve numerically the singularly perturbed problem.

3. To compare the order of accuracy in the Galerkin finite element method based on the Lagrange and wavelet bases by applying two classes of basis functions which are linear and quadratic bases.

4. To study the multilevel augmentation method using wavelet bases to solve numerically the singularly perturbed boundary value problem.

LITERATURE REVIEW

The Galerkin method with wavelet bases called the wavelet-Galerkin method is extensively applied to solve the partial differential equation. Wavelets in our consideration are compactly supported wavelets which are introduced previously by Zhongying Chen, Bin Wu and Yuesheng Xu (Chen *et al*., 2006). They introduced the multilevel augmentation method related to some wavelet bases for solving certain boundary value problems. This method has also been applied to solve the one-dimentional sine-Gordon equation by Jian Chen, Zhongying Chen and Sirui Cheng (Chen *et al*., 2011). Since the multilevel augmentation algorithm only needs to solve a fixed lower level nonlinear system and compensate the high level component by simple matrix-vector multiplications at each time step, it can reduce computational complexity largely. Jian Chen (Chen, 2011) developed a multilevel augmentation method for solving nonlinear boundary value problems. They described the multilevel augmentation method for solving the second kind of nonlinear operator equations and then applied the same method to solve the nonlinear two-point boundary value problems of second-order differential equations. In the case of solving the partial differential equations, the wavelet applications have been introduced by several authors, such as (Ho and Yang, 2001; El-Gamel, 2006; Choudhury and Deka, 2010; Chen and Xiang, 2011). In 2001, S.L. Ho and S.Y. Yang developed a new wavelet-Galerkin formulation for solving parabolic equations in finite domains, the formulation is based on the weak functional form that has the advantages of being able to deal with natural boundary conditions. Moreover, the lower order derivatives of wavelet bases are then involved in the connection coefficients. In 2006, Mohamed El-Gamel solving the singularly perturbed convection-dominated diffusion equation, there are few techniques available to numerically solve singularly perturbed parabolic problems. He has shown that the wavelet-Galerkin method is a very effective tool for solving the problems. The method was then tested on several examples and a comparison with the method of reduction order is made. It was shown that the wavelet-Galerkin method provided better results. In 2010, A.H. Choudhury and R.K. Deka applied the wavelet-Galerkin method based on the Daubechies basis functions to solve some types of the one-dimensional elliptic problems. They compared numerical solutions with the result by the finite difference method, and found that the wavelet-Galerkin method is a rigtht competator to the classical method. Recently, Xuefeng Chen and Jiawei Xiang have shown techniques to solve both steady and unsteady problems. They developed a numerical method based on wavelets of Hermite cubic splines for solving singularly perturbed convection-dominated diffusion equation. It was found that numerical accuracy can be improved in singular areas by using hierarchically configured wavelets for solving algebraic systems.

MATERIALS AND METHODS

Basis functions

In this chapter, we will show in details the derivation of matrix coefficients by two types of the basis function which are the Lagrange and the wavelet bases, that have two classes of the Lagrange basis function which are linear and quadratic bases and two classes of the wavelet basis function which are linear and quadratic bases. And study the basic concept of the time discretization with polynomial time basis.

1. Lagrange basis functions

1.1 Linear Lagrange basis functions

We begin defining nodal points in the domain $0 \le x \le l$ with K elements of uniform element size. Thus, there are $K+1$ nodes corresponding to the coordinates $x_1, x_2, ..., x_{K+1}$ as shown in Figure 1.

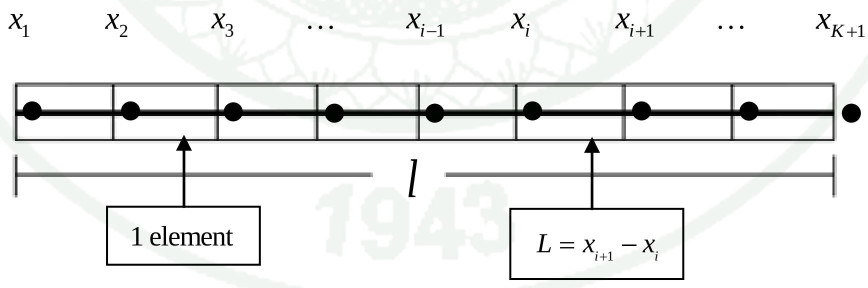

Figure 1 Linear elements

The linear Lagrange basis functions are

$$N_1(x) = 1 - \frac{x}{L} \quad , \tag{1}$$

$$N_2(x) = \frac{x}{L} \quad . \tag{2}$$

These are the well-known linear Lagrange basis functions. Their variations in an element are shown in Figure 2.

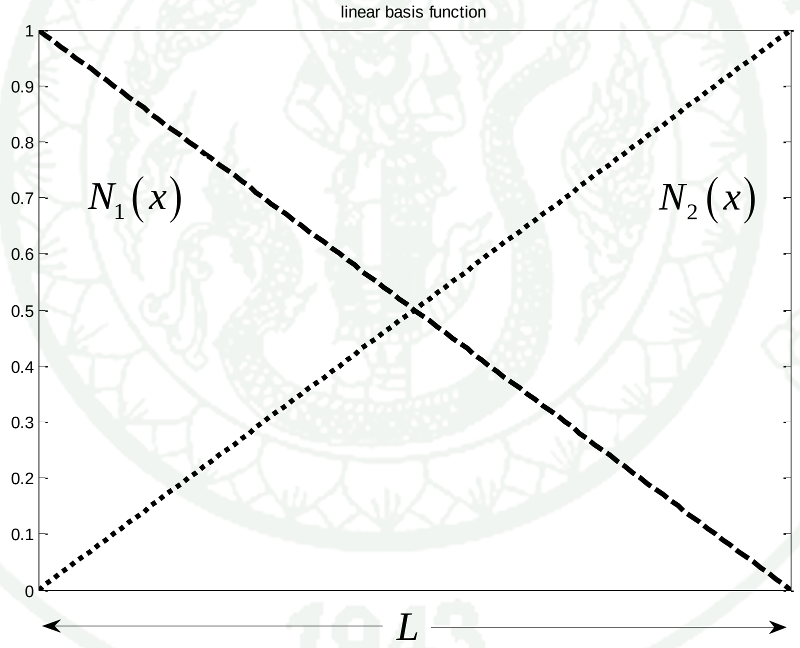

Figure 2 Linear Lagrange basis functions

1.2 Quadratic Lagrange basis functions

The nodal notation used in this case is shown in Figure 3. There are three nodes in one element. The i th element is defined on $x_{2i-1} \le x \le x_{2i+1}$, $i = 1, 2, ..., K$ and its element size is given by $L = x_{2i+1} - x_{2i-1}$.

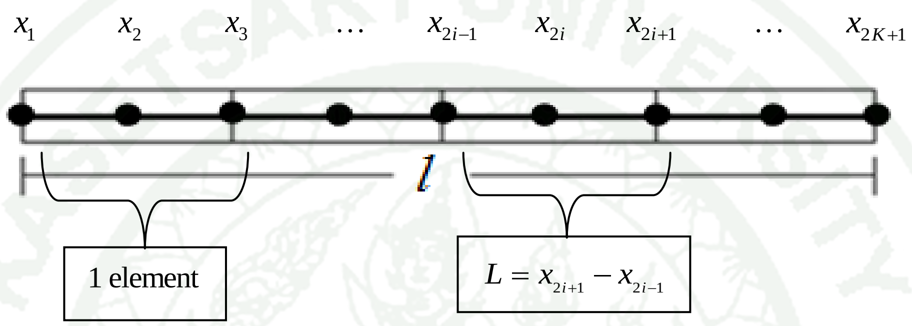

Figure 3 Quadratic elements

and the quadratic Lagrange basis functions are

$$N_1(x) = 1 - \frac{3x}{L} + 2\left(\frac{x}{L}\right)^2 , \tag{3}$$

$$N_2(x) = \frac{4x}{L}\left(1 - \frac{x}{L}\right) , \tag{4}$$

$$N_3(x) = \frac{x}{L}\left(\frac{2x}{L} - 1\right) . \tag{5}$$

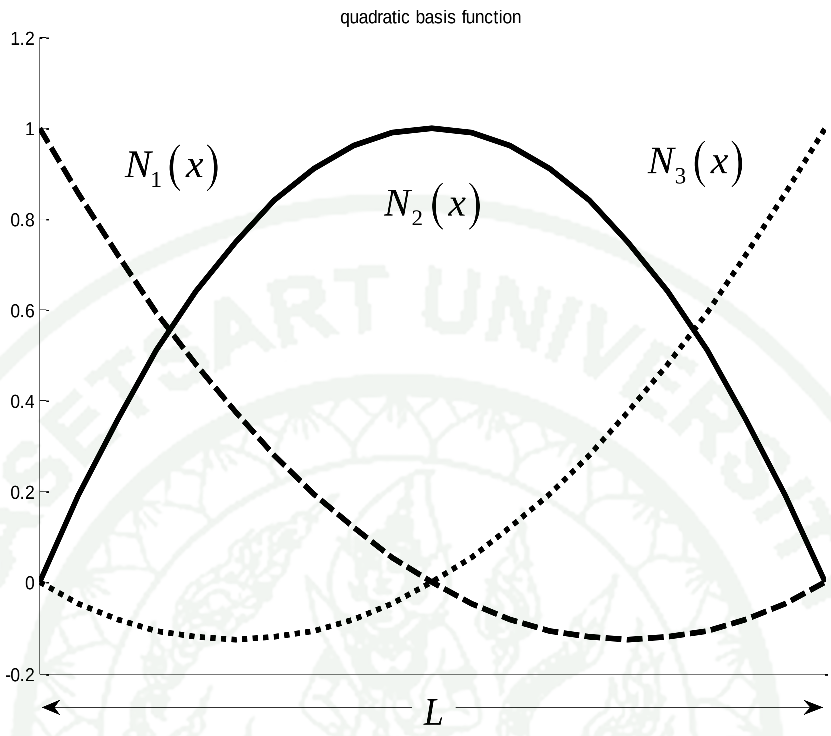

Figure 4 Quadratic Lagrange basis functions

2. Wavelet basis functions

The construction of the wavelet basis functions used in the Galerkin method follows the derivations proposed by Zhongying Chen , Bin Wu and Yuesheng Xu (Chen *et al*., 2006). We construct multi-scale orthonormal bases for the Sobolev space on the unit interval $I := [0,1]$. Specifically, we let m be a fixed positive integer and $H_0^m(I)$ denoted the Sobolev spaces. We let $j \in Z_m$, where $Z_m := \{0,1,2,...,m-1\}$. For any nonnegative integer n, we denote by X_n the subspace of $H_0^m(I)$ whose elements are the piecewise polynomials of order k with knots j/μ^n, $j-1 \in Z_{\mu^n-1}$, when $k > 2m$ and $\mu > 1$ be a fixed positive integer. We have that

$$\mathrm{X}_0 = \mathrm{span}\left\{x^{m+j}(1-x)^m : j \in Z_{k-2m}\right\},$$

so we let W_n be the orthonormal complement of X_{n+1} in X_n, i.e.,

$$\mathrm{X}_n = \mathrm{X}_{n-1} \oplus_m \mathrm{W}_n$$

and thus, repeatedly using this decomposition leads to

$$\mathrm{X}_n = \mathrm{X}_0 \oplus_m \mathrm{W}_1 \oplus_m \cdots \oplus_m \mathrm{W}_n .$$

Spaces W_n can be recursively constructed once W_1 has been given. To describe the construction, the family of affine mappings $\Phi_\mu := \{\phi_e : e \in Z_\mu\}$ is required where

$$\phi_e(x) = \frac{x+e}{\mu} \quad ; \; e \in Z_\mu .$$

Finally, the wavelet basis function $w_{ij} \in \mathrm{W}_i$, $i = 2,3,\ldots$, $j = 0,1,\ldots,\dim \mathrm{W}_i - 1$, $n = i-1$, $l \in Z_r$, $r = \dim \mathrm{W}_1$ can be constructed by the composition as follows.

$$w_{ij} = \mu^{n\left(\frac{1}{2}-m\right)} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x) \;\; ; \;\; \mathrm{e} \in Z_\mu^{i-1} . \qquad (6)$$

where $Z_\mu^i = \underbrace{Z_\mu \times Z_\mu \times \ldots \times Z_\mu}_{i \text{ times}}$, i times

This construction will be applied to obtain both linear and quadratic wavelet bases in the next sections.

2.1 Linear wavelet basis functions

Choose $k = 2, \mu = 2, r = 1$. In this case, we have that $\dim W_i = 2^{i-1}$ for $i > 0$. We also give l , Z_μ , and e by

$$l \in Z_r = \{0\} \quad ,$$

$$e \in Z_\mu = \{0,1\} \quad ,$$

and

$$\phi_0(x) = \frac{x}{2} \quad ,$$

$$\phi_1(x) = \frac{x+1}{2} \quad ,$$

The desired basis of W_1 (level 1) , see for the detail from Chen *et al.*, (2006), is obtained by

$$w_{10}(x) = \begin{cases} x & ; x \in \left[0, \frac{1}{2}\right) \\ 1-x & ; x \in \left[\frac{1}{2}, 1\right] \end{cases} . \qquad (7)$$

The wavelet basis function of W_2 (level 2) is given by $n = 1$, $e = \{0,1\} \in Z_2^1$ and $\Phi_2 := \{\phi_0, \phi_1\}$, where

$$\phi_0(x) = \frac{x}{2} \quad \text{and} \quad \phi_1(x) = \frac{x+1}{2} .$$

The inverse functions of ϕ_0 , ϕ_1 are

$$\phi_0^{-1}(x) = 2x \quad \text{and} \quad \phi_1^{-1}(x) = 2x - 1 .$$

Find w_{2j} from $\quad w_{ij} = \mu^{n\left(\frac{1}{2}-m\right)} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x)$

$$w_{2j} = \mu^{1\left(\frac{1}{2}-1\right)} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x) = \frac{1}{\sqrt{2}} w_{1l} \circ \phi_{\mathrm{e}}^{-1}(x)$$

Construction of W_2 from W_1

e	w_{1l}	$\Phi_{\mathrm{e}}^{-1}(x)$	w_{2j}
0	w_{10}	$2x$	$w_{20}(x) = \begin{cases} \frac{1}{\sqrt{2}}(2x) & ; x \in \left[0, \frac{1}{4}\right) \\ \frac{1}{\sqrt{2}}(1-2x) & ; x \in \left[\frac{1}{4}, \frac{1}{2}\right] \end{cases}$
1	w_{10}	$2x-1$	$w_{21}(x) = \begin{cases} \frac{1}{\sqrt{2}}(2x-1) & ; x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \frac{1}{\sqrt{2}}(2-2x) & ; x \in \left[\frac{3}{4}, 1\right] \end{cases}$

The wavelet basis function of W_3 (level 3) is that $n = 2$,

$\mathrm{e} = \{(0,0),(0,1),(1,0),(1,1)\} \in Z_2^2$ and $\Phi_3 := \{\phi_{(0,0)}, \phi_{(0,1)}, \phi_{(1,0)}, \phi_{(1,1)}\}$,

where

$$\phi_{(0,0)}(x) = \phi_0 \circ \phi_0(x) = \left(\frac{x}{2}\right) \circ \left(\frac{x}{2}\right) = \frac{x}{4} ,$$

$$\phi_{(0,1)}(x) = \phi_0 \circ \phi_1(x) = \left(\frac{x}{2}\right) \circ \left(\frac{x+1}{2}\right) = \frac{x+1}{4} ,$$

$$\phi_{(1,0)}(x) = \phi_1 \circ \phi_0(x) = \left(\frac{x+1}{2}\right) \circ \left(\frac{x}{2}\right) = \frac{x+2}{4} ,$$

$$\phi_{(1,1)}(x) = \phi_1 \circ \phi_1(x) = \left(\frac{x+1}{2}\right) \circ \left(\frac{x+1}{2}\right) = \frac{x+3}{4} .$$

The inverse functions of $\phi_{(0,0)},\phi_{(0,1)},\phi_{(1,0)},\phi_{(1,1)}$ are

$$\phi^{-1}_{(0,0)}(x)=4x \quad ,$$

$$\phi^{-1}_{(0,1)}(x)=4x-1 \ ,$$

$$\phi^{-1}_{(1,0)}(x)=4x-2 \ ,$$

$$\phi^{-1}_{(1,1)}(x)=4x-3 \ .$$

Find w_{3j} from $w_{ij} = \mu^{n\left(\frac{1}{2}-m\right)} w_{1l}\circ\phi_{e}^{-1}(x)$

$$w_{3j} = \mu^{2\left(\frac{1}{2}-1\right)} w_{1l}\circ\phi_{e}^{-1}(x) \quad = \frac{1}{2} w_{1l}\circ\phi_{e}^{-1}(x)$$

Construction of W_3 from W_1

e	w_{1l}	$\Phi_{e}^{-1}(x)$	w_{3j}
$(0,0)$	w_{10}	$4x$	$w_{30}(x)=\begin{cases}\frac{1}{2}(4x) & ;x\in\left[0,\frac{1}{8}\right) \\ \frac{1}{2}(1-4x) & ;x\in\left[\frac{1}{8},\frac{1}{4}\right]\end{cases}$
$(0,1)$	w_{10}	$4x-1$	$w_{31}(x)=\begin{cases}\frac{1}{2}(4x-1) & ;x\in\left[\frac{1}{4},\frac{3}{8}\right) \\ \frac{1}{2}(2-4x) & ;x\in\left[\frac{3}{8},\frac{1}{2}\right]\end{cases}$
$(1,0)$	w_{10}	$4x-2$	$w_{32}(x)=\begin{cases}\frac{1}{2}(4x-2) & ;x\in\left[\frac{1}{2},\frac{5}{8}\right) \\ \frac{1}{2}(3-4x) & ;x\in\left[\frac{5}{8},\frac{3}{4}\right]\end{cases}$

$(1,1)$	w_{10}	$4x-3$	$w_{33}(x)=\begin{cases}\frac{1}{2}(4x-3) & ;x\in\left[\frac{3}{4},\frac{7}{8}\right)\\ \frac{1}{2}(4-4x) & ;x\in\left[\frac{7}{8},1\right]\end{cases}$

The profiles of these three linear wavelet basis functions W_1, W_2 and W_3 are shown in Figure 5. In practice, any levels of the linear wavelet basis can be obtained recursively by the same process of this construction.

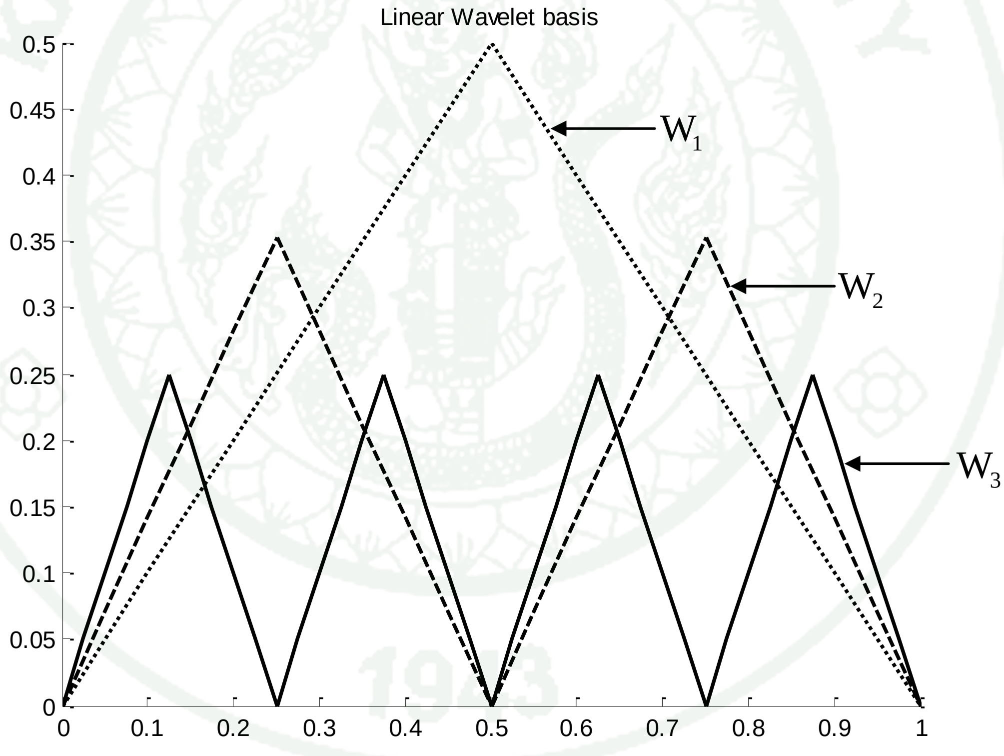

Figure 5 Linear basis functions of W_1 , W_2 and W_3

2.2 Quadratic wavelet basis functions

Choose $k = 3$, $\mu = 2$, $r = 2$. In this case, we have that $\dim \mathrm{W}_i = 2^i$ for $i > 0$. We also give l , Z_μ , and e by

$$l \in Z_r = \{0,1\} \quad ,$$
$$e \in Z_\mu = \{0,1\} \quad ,$$

and

$$\phi_0(x) = \frac{x}{2} \quad ,$$
$$\phi_1(x) = \frac{x+1}{2} \quad .$$

The desired bases of W_0 and W_1, see for the detail from Chen *et al*., (2006), are given by

$$w_{00}\left(x\right) = \sqrt{3}x(1-x) \qquad ; x \in \left[0,1\right] \ , \tag{8}$$

$$w_{10}\left(x\right) = \begin{cases} x\left(1-3x\right) & ; x \in \left[0, \frac{1}{2}\right) \\ \left(1-x\right)\left(3x-2\right) & ; x \in \left[\frac{1}{2}, 1\right] \end{cases} \ , \tag{9}$$

$$w_{11}\left(x\right) = \begin{cases} \sqrt{3}x\left(1-2x\right) & ; x \in \left[0, \frac{1}{2}\right) \\ \sqrt{3}\left(1-x\right)\left(1-2x\right) & ; x \in \left[\frac{1}{2}, 1\right] \end{cases} \ . \tag{10}$$

The quadratic wavelet basis of level two, W_2 is given by $n=1,\ \mathrm{e}=\{0,1\}\in Z_2^1$ and $\Phi_2 := \{\phi_0,\phi_1\}$, where

$$\phi_0(x)=\frac{x}{2} \quad \text{and} \quad \phi_1(x)=\frac{x+1}{2}\ .$$

The inverse functions of ϕ_0 , ϕ_1 are

$$\phi_0^{-1}(x)=2x \quad \text{and} \quad \phi_1^{-1}(x)=2x-1\ .$$

Find w_{2j} from $\quad w_{ij} = \mu^{n\left(\frac{1}{2}-m\right)} w_{1l}\circ\phi_{\mathrm{e}}^{-1}(x)$

$$w_{2j} = \mu^{1\left(\frac{1}{2}-1\right)} w_{1l}\circ\phi_{\mathrm{e}}^{-1}(x) = \frac{1}{\sqrt{2}} w_{1l}\circ\phi_{\mathrm{e}}^{-1}(x)$$

Construction of W_2 from W_1

e	w_{1l}	$\Phi_{\mathrm{e}}^{-1}(x)$	w_{2j}
0	w_{10}	$2x$	$w_{20}(x)=\begin{cases}\frac{1}{\sqrt{2}}(2x)(1-6x) & ;x\in\left[0,\frac{1}{4}\right) \\ \frac{1}{\sqrt{2}}(1-2x)(6x-2) & ;x\in\left[\frac{1}{4},\frac{1}{2}\right]\end{cases}$
0	w_{11}	$2x$	$w_{21}(x)=\begin{cases}\frac{\sqrt{3}}{\sqrt{2}}(2x)(1-4x) & ;x\in\left[0,\frac{1}{4}\right) \\ \frac{\sqrt{3}}{\sqrt{2}}(1-2x)(1-4x) & ;x\in\left[\frac{1}{4},\frac{1}{2}\right]\end{cases}$

1	w_{10}	$2x-1$	$w_{22}(x)=\begin{cases}\frac{1}{\sqrt{2}}(4-6x)(2x-1) & ;x\in\left[\frac{1}{2},\frac{3}{4}\right)\\ \frac{1}{\sqrt{2}}(2-2x)(6x-5) & ;x\in\left[\frac{3}{4},1\right]\end{cases}$
1	w_{11}	$2x-1$	$w_{23}(x)=\begin{cases}\frac{\sqrt{3}}{\sqrt{2}}(2x-1)(3-4x) & ;x\in\left[\frac{1}{2},\frac{3}{4}\right)\\ \frac{\sqrt{3}}{\sqrt{2}}(2-2x)(3-4x) & ;x\in\left[\frac{3}{4},1\right]\end{cases}$

The wavelet basis function of W_3 (level 3) is that $n=2$,

$e=\{(0,0),(0,1),(1,0),(1,1)\}\in Z_2^2$ and $\Phi_3:=\{\phi_{(0,0)},\phi_{(0,1)},\phi_{(1,0)},\phi_{(1,1)}\}$,

where

$$\phi_{(0,0)}(x)=\phi_0\circ\phi_0(x)=\left(\frac{x}{2}\right)\circ\left(\frac{x}{2}\right)=\frac{x}{4}\ ,$$

$$\phi_{(0,1)}(x)=\phi_0\circ\phi_1(x)=\left(\frac{x}{2}\right)\circ\left(\frac{x+1}{2}\right)=\frac{x+1}{4}\ ,$$

$$\phi_{(1,0)}(x)=\phi_1\circ\phi_0(x)=\left(\frac{x+1}{2}\right)\circ\left(\frac{x}{2}\right)=\frac{x+2}{4}\ ,$$

$$\phi_{(1,1)}(x)=\phi_1\circ\phi_1(x)=\left(\frac{x+1}{2}\right)\circ\left(\frac{x+1}{2}\right)=\frac{x+3}{4}\ .$$

The inverse functions of $\phi_{(0,0)},\phi_{(0,1)},\phi_{(1,0)},\phi_{(1,1)}$ are

$$\phi_{(0,0)}^{-1}(x)=4x\ ,$$

$$\phi_{(0,1)}^{-1}(x)=4x-1\ ,$$

$$\phi_{(1,0)}^{-1}(x)=4x-2\ ,$$

$$\phi_{(1,1)}^{-1}(x)=4x-3\ .$$

Find w_{3j} from $$w_{ij} = \mu^{n\left(\frac{1}{2}-m\right)} w_{1l} \circ \phi_{e}^{-1}(x)$$

$$w_{3j} = \mu^{2\left(\frac{1}{2}-1\right)} w_{1l} \circ \phi_{e}^{-1}(x) = \frac{1}{2} w_{1l} \circ \phi_{e}^{-1}(x)$$

Construction of W_3 from W_1

e	w_{1l}	$\Phi_{e}^{-1}(x)$	w_{3j}
$(0,0)$	w_{10}	$4x$	$w_{30}(x) = \begin{cases} \frac{1}{2}(4x)(1-12x) & ; x \in \left[0, \frac{1}{8}\right) \\ \frac{1}{2}(1-4x)(12x-2) & ; x \in \left[\frac{1}{8}, \frac{1}{4}\right] \end{cases}$
$(0,0)$	w_{11}	$4x$	$w_{31}(x) = \begin{cases} \frac{\sqrt{3}}{2}(4x)(1-8x) & ; x \in \left[0, \frac{1}{8}\right) \\ \frac{\sqrt{3}}{2}(1-4x)(1-8x) & ; x \in \left[\frac{1}{8}, \frac{1}{4}\right] \end{cases}$
$(0,1)$	w_{10}	$4x-1$	$w_{32}(x) = \begin{cases} \frac{1}{2}(4-12x)(4x-1) & ; x \in \left[\frac{1}{4}, \frac{3}{8}\right) \\ \frac{1}{2}(2-4x)(12x-5) & ; x \in \left[\frac{3}{8}, \frac{1}{2}\right] \end{cases}$
$(0,1)$	w_{11}	$4x-1$	$w_{33}(x) = \begin{cases} \frac{\sqrt{3}}{2}(4x-1)(3-8x) & ; x \in \left[\frac{1}{4}, \frac{3}{8}\right) \\ \frac{\sqrt{3}}{2}(2-4x)(3-8x) & ; x \in \left[\frac{3}{8}, \frac{1}{2}\right] \end{cases}$
$(1,0)$	w_{10}	$4x-2$	$w_{34}(x) = \begin{cases} \frac{1}{2}(7-12x)(4x-2) & ; x \in \left[\frac{1}{2}, \frac{5}{8}\right) \\ \frac{1}{2}(3-4x)(12x-8) & ; x \in \left[\frac{5}{8}, \frac{3}{4}\right] \end{cases}$

$(1,0)$	w_{11}	$4x-2$	$w_{35}(x)=\begin{cases}\frac{\sqrt{3}}{2}(4x-2)(5-8x) & ;x\in\left[\frac{1}{2},\frac{5}{8}\right)\\ \frac{\sqrt{3}}{2}(3-4x)(5-8x) & ;x\in\left[\frac{5}{8},\frac{3}{4}\right]\end{cases}$
$(1,1)$	w_{10}	$4x-3$	$w_{36}(x)=\begin{cases}\frac{1}{2}(10-12x)(4x-3) & ;x\in\left[\frac{3}{4},\frac{7}{8}\right)\\ \frac{1}{2}(4-4x)(12x-11) & ;x\in\left[\frac{7}{8},1\right]\end{cases}$
$(1,1)$	w_{11}	$4x-3$	$w_{36}(x)=\begin{cases}\frac{\sqrt{3}}{2}(4x-3)(7-8x) & ;x\in\left[\frac{3}{4},\frac{7}{8}\right)\\ \frac{\sqrt{3}}{2}(4-4x)(7-8x) & ;x\in\left[\frac{7}{8},1\right]\end{cases}$

The profiles of quadratic wavelet functions W_0, W_1 and W_2 are shown in Figure 6.

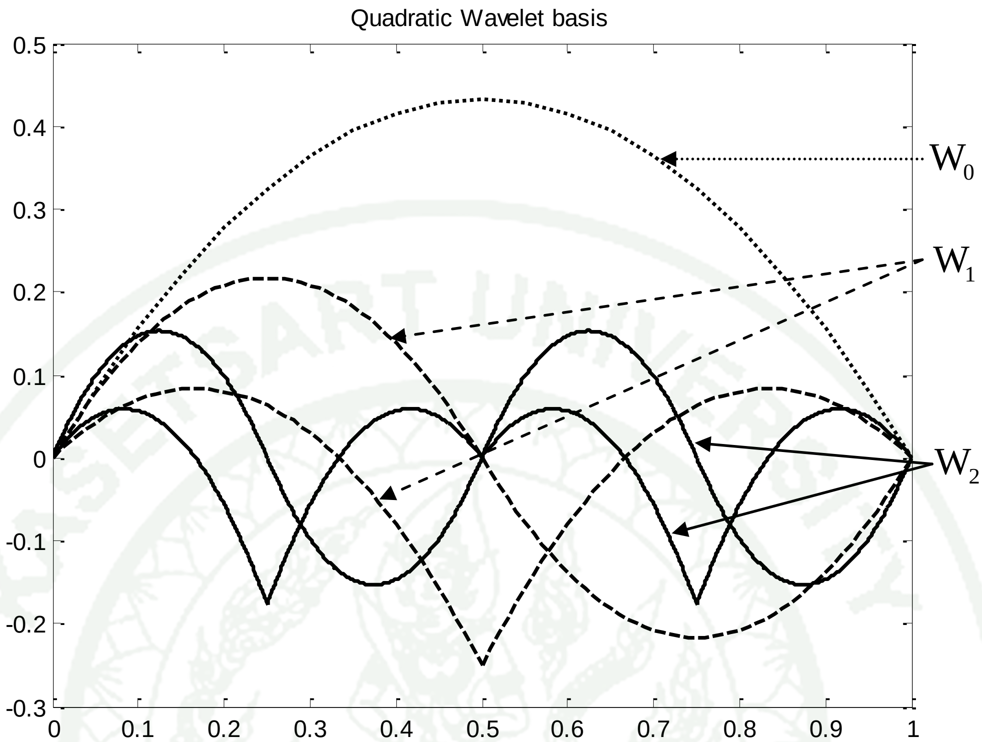

Figure 6 Quadratic basis functions of W_0 , W_1 and W_2

3. Time discretization

For the discretization in time, we give the basis function in time as $\theta_k\left(t\right)=\left(\left(t-t_{n-1}\right)/\Delta t\right)^k$ where

$$\begin{aligned}\{\theta\} &= \lfloor \theta_0 \quad \theta_1 \quad \theta_2 \quad \cdots \quad \theta_p \rfloor^T \\ &= \left\lfloor 1 \quad \frac{t-t_{n-1}}{\Delta t} \quad \left(\frac{t-t_{n-1}}{\Delta t}\right)^2 \quad \cdots \quad \left(\frac{t-t_{n-1}}{\Delta t}\right)^p \right\rfloor^T\end{aligned} \tag{11}$$

We give the notations $\left\{\theta^+\right\}=\left\{\theta\left(t_{n-1}\right)\right\}$ and $\left\{\theta^-\right\}=\left\{\theta\left(t_n\right)\right\}$ where $\Delta t=t_{n+1}-t_n$.

The coefficients of matrices $\left[Z_a\right]$, $\left[Z_b\right]$, $\left[Z_c\right]$, $\left[Z_+\right]$ and $\left[Z_-\right]$ can be calculated by

$$\begin{aligned}\left[Z_a\right] &= \int_{t_{n-1}}^{t_n}\{\theta\}\{\theta\}^T dt \\ &= \Delta t\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{p+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{p+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{p+1} & \frac{1}{p+2} & \frac{1}{p+3} & \cdots & \frac{1}{2p+1}\end{bmatrix},\end{aligned} \tag{12}$$

$$\left[Z_b\right] = \int_{t_{n-1}}^{t_n} \{\theta\}\left\{\frac{d\theta}{dt}\right\}^T dt$$

$$= \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & \frac{1}{2} & \frac{2}{3} & \cdots & \frac{p}{p+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{1}{p+1} & \frac{2}{p+2} & \cdots & \frac{p}{2p} \end{bmatrix}, \quad (13)$$

$$\left[Z_c\right] = \int_{t_{n-1}}^{t_n} \{\theta\}\, dt$$

$$= \Delta t \begin{bmatrix} 1 \\ \frac{1}{2} \\ \vdots \\ \frac{1}{p+1} \end{bmatrix}, \quad (14)$$

$$\left[Z_+\right] = \left\{\theta^+\right\}\left\{\theta^+\right\}^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (15)$$

$$[Z_-] = \{\theta^-\}\{\theta^+\}^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} . \qquad (16)$$

For example, we obtain discretization in time of level two by setting $\{\theta\} = \begin{bmatrix} 1 & \frac{t - t_{n-1}}{\Delta t} \end{bmatrix}^T$. So, we can find the matrices as follows

$$[Z_a] \;=\; \int_{t_{n-1}}^{t_n} \{\theta\}\{\theta\}^T dt \;=\; \Delta t \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} , \qquad (17)$$

$$[Z_b] \;=\; \int_{t_{n-1}}^{t_n} \{\theta\} \left\{\frac{d\theta}{dt}\right\}^T dt \;=\; \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} , \qquad (18)$$

$$[Z_c] \;=\; \int_{t_{n-1}}^{t_n} \{\theta\}\, dt \;=\; \Delta t \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} , \qquad (19)$$

$$[Z_+] \;=\; \{\theta^+\}\{\theta^+\}^T \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \;, \tag{20}$$

$$[Z_-] \;=\; \{\theta^-\}\{\theta^+\}^T \;=\; \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \;. \tag{21}$$

For example, we obtain discretization in time of level one by setting $\{\theta\} = [1]$. So, we can find the matrices as follows

$$[Z_a] \;=\; \int_{t_{n-1}}^{t_n} \{\theta\}\{\theta\}^T dt \;=\; \Delta t[1] \;, \tag{22}$$

$$[Z_b] \;=\; \int_{t_{n-1}}^{t_n} \{\theta\}\left\{\frac{d\theta}{dt}\right\}^T dt \;=\; [0] \;, \tag{23}$$

$$[Z_c] \;=\; \int_{t_{n-1}}^{t_n} \{\theta\}\, dt \;=\; \Delta t[1] \;, \tag{24}$$

$$[Z_+] \;=\; \{\theta^+\}\{\theta^+\}^T \;=\; [1] \;, \tag{25}$$

$$[Z_-] \;=\; \{\theta^-\}\{\theta^+\}^T \;=\; [1] \;. \tag{26}$$

In conclusion, we can derive the full form of all matrices resulting to the full system that can be solved iteratively to obtain approximate solutions when the initial and boundary conditions are specified. This time discretization method will be applied for solving the partial differential equations in the next chapter.

Galerkin finite element method for heat equation

In this chapter, we study the concept of the Galerkin finite element method to solve numerically the linear one-dimensional heat equation. Two types of basis functions which are the Lagrange and the wavelet bases are employed to derive the full form of matrix system. We consider both linear and quadratic bases in the Galerkin finite element method. Numerical results are presented to demonstrate of this method.

The time-dependent heat equation in terms of variable $T(x,t)$ is written in its one-dimensional form as $T(x,t)$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad , \qquad (0 < x < 1) \quad , \tag{27}$$

where T is temperature and α is the thermal diffusivity(constant). The domain is Ω $(0 < x < 1)$ with boundary Γ.

We give the boundary conditions as

$$T(0,t) = T(1,t) = 0 , \tag{28}$$

and initial condition as

$$T(x,0) = T_0(x) . \tag{29}$$

1. The Galerkin finite element method for Lagrange bases

By the weighted residual method, finite element formulation for an element L of equation (27) can be written as

$$\int_0^L W\left(\frac{\partial T}{\partial t}-\alpha\frac{\partial^2 T}{\partial x^2}\right)\mathrm{d}x=0\ , \qquad (30)$$

where W is a weighting function. Using integration by part, yields

$$\int_0^L W\left(\frac{\partial T}{\partial t}\right)dx \;+\; \int_0^L \alpha\left(\frac{\partial T}{\partial x}\frac{\partial W}{\partial x}\right)dx \;=0. \qquad (31)$$

Let us begin by approximating the unknown function in terms of the Lagrange basis as, for any each element,

$$T^n = T\left(x,t_n\right) \;=\; \sum_{i=1}^{m} N_i\left(x\right) T_i^n\ . \qquad (32)$$

where T_i^n denotes the variable's value at time $t=t_n$ at nodes, $N_i(x)$ is Lagrange basis function, m is the number of basis function, $m=2$ for linear Lagrange basis function and $m=3$ for quadratic Lagrange basis function.

1.1 Linear Lagrange basis function

For fixed $t = t_n$ and substituting equation (32) with linear Lagrange basis function into equation (31) yields the resulting matrix representation as

$$\left(\underset{(2\times2)}{\left[\mathrm{L}_{\mathrm{A}l}\right]} + \underset{(2\times2)}{\left[\mathrm{L}_{\mathrm{B}l}\right]} \right) \underset{(2\times1)}{\left\{T\right\}} = \underset{(2\times1)}{\left\{0\right\}} \tag{33}$$

The coefficients of matrices $\left[\mathrm{L}_{\mathrm{A}l}\right]$ and $\left[\mathrm{L}_{\mathrm{B}l}\right]$ can be evaluated as

$$\left[\mathrm{L}_{\mathrm{A}l}\right] = \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \lfloor N_1 \quad N_2 \rfloor dx$$

$$= \int_0^L \begin{Bmatrix} 1-\dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} \left\lfloor 1-\frac{x}{L} \quad \frac{x}{L} \right\rfloor dx$$

$$= \int_0^L \begin{bmatrix} \left(1-\dfrac{x}{L}\right)^2 & \dfrac{x}{L}-\dfrac{x^2}{L^2} \\ \dfrac{x}{L}-\dfrac{x^2}{L^2} & \dfrac{x^2}{L^2} \end{bmatrix} dx$$

$$= \frac{L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{34}$$

$$[\mathrm{L}_{\mathrm{B}l}] = \int_0^L \begin{Bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{Bmatrix} \left\lfloor \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right\rfloor dx$$

$$= \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor dx \tag{35}$$

$$= \int_0^L \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} dx$$

$$= \frac{1}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The region $0 < x < 1$ has been divided into K elements of equal length $L = \frac{1}{K}$. After assembling all elements at time $t = t_n$, we obtain the system of linear equation as

$$\left(\underset{(K+1)\times(K+1)}{[\mathrm{L}_{\mathrm{A}l}]_{\mathrm{sys}}} + \underset{(K+1)\times(K+1)}{[\mathrm{L}_{\mathrm{B}l}]_{\mathrm{sys}}} \right) \underset{(K+1)\times 1}{\{T\}} = \underset{(K+1)\times 1}{\{0\}} \tag{36}$$

where

$$[\mathrm{L}_{\mathrm{A}l}]_{\mathrm{sys}} = \frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2+2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2+2 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & 2+2 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 2+2 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{bmatrix}_{(K+1)\times(K+1)}$$

$$= \frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}_{(K+1)\times(K+1)} \quad (37)$$

$$[\mathrm{L}_{\mathrm{B}l}]_{\mathrm{sys}} = \frac{1}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1+1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1+1 & -1 & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & -1 & 1+1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1+1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{bmatrix}_{(K+1)\times(K+1)}$$

$$= \frac{1}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{(K+1)\times(K+1)} \quad (38)$$

1.2 Quadratic Lagrange basis function

For fixed $t = t_n$ and substituting equation (32) with quadratic Lagrange basis function into equation (31) yields the resulting matrix representation as

$$\left(\underset{(3\times3)}{\left[\mathrm{L}_{\mathrm{A}q}\right]}+\underset{(3\times3)}{\left[\mathrm{L}_{\mathrm{B}q}\right]}\right)\underset{(3\times1)}{\{T\}}=\underset{(3\times1)}{\{0\}} \tag{39}$$

The coefficients in matrices $\left[\mathrm{L}_{\mathrm{A}q}\right]$ and $\left[\mathrm{L}_{\mathrm{B}q}\right]$ can be evaluated as

$$\left[\mathrm{L}_{\mathrm{A}q}\right]= \int_0^L \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \lfloor N_1 \quad N_2 \quad N_3 \rfloor dx$$

$$= \int_0^L \begin{Bmatrix} 1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2 \\ \frac{4x}{L}\left(1-\frac{x}{L}\right) \\ \frac{x}{L}\left(\frac{2x}{L}-1\right) \end{Bmatrix} \left\lfloor 1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2 \quad \frac{4x}{L}\left(1-\frac{x}{L}\right) \quad \frac{x}{L}\left(\frac{2x}{L}-1\right) \right\rfloor dx$$

$$= \int_0^L \begin{bmatrix} \left(1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2\right)^2 & \left(\frac{4x}{L}\left(1-\frac{x}{L}\right)\right)\left(1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2\right) & \left(\frac{x}{L}\left(\frac{2x}{L}-1\right)\right)\left(1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2\right) \\ \left(1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2\right)\left(\frac{4x}{L}\left(1-\frac{x}{L}\right)\right) & \left(\frac{4x}{L}\left(1-\frac{x}{L}\right)\right)^2 & \left(\frac{x}{L}\left(\frac{2x}{L}-1\right)\right)\left(\frac{4x}{L}\left(1-\frac{x}{L}\right)\right) \\ \left(1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2\right)\left(\frac{x}{L}\left(\frac{2x}{L}-1\right)\right) & \left(\frac{4x}{L}\left(1-\frac{x}{L}\right)\right)\left(\frac{x}{L}\left(\frac{2x}{L}-1\right)\right) & \left(\frac{x}{L}\left(\frac{2x}{L}-1\right)\right)^2 \end{bmatrix} dx \tag{40}$$

$$= \frac{L}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$\left[\mathrm{L}_{\mathrm{B}q}\right]=\int_0^L\begin{Bmatrix}\dfrac{dN_1}{dx}\\\dfrac{dN_2}{dx}\\\dfrac{dN_3}{dx}\end{Bmatrix}\begin{bmatrix}\dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} & \dfrac{dN_3}{dx}\end{bmatrix}dx$$

$$=\int_0^L\begin{Bmatrix}\dfrac{1}{L}\left(\dfrac{4x}{L}-3\right)\\\dfrac{4}{L}\left(1-\dfrac{2x}{L}\right)\\\dfrac{1}{L}\left(\dfrac{4x}{L}-1\right)\end{Bmatrix}\begin{bmatrix}\dfrac{1}{L}\left(\dfrac{4x}{L}-3\right) & \dfrac{4}{L}\left(1-\dfrac{2x}{L}\right) & \dfrac{1}{L}\left(\dfrac{4x}{L}-1\right)\end{bmatrix}dx$$

$$=\int_0^L\begin{bmatrix}\left(\frac{1}{L}\left(\frac{4x}{L}-3\right)\right)^2 & \left(\frac{1}{L}\left(\frac{4x}{L}-3\right)\right)\left(\frac{4}{L}\left(1-\frac{2x}{L}\right)\right) & \left(\frac{1}{L}\left(\frac{4x}{L}-1\right)\right)\left(\frac{1}{L}\left(\frac{4x}{L}-3\right)\right)\\\left(\frac{1}{L}\left(\frac{4x}{L}-3\right)\right)\left(\frac{4}{L}\left(1-\frac{2x}{L}\right)\right) & \left(\frac{4}{L}\left(1-\frac{2x}{L}\right)\right)^2 & \left(\frac{1}{L}\left(\frac{4x}{L}-1\right)\right)\left(\frac{4}{L}\left(1-\frac{2x}{L}\right)\right)\\\left(\frac{1}{L}\left(\frac{4x}{L}-3\right)\right)\left(\frac{1}{L}\left(\frac{4x}{L}-1\right)\right) & \left(\frac{4}{L}\left(1-\frac{2x}{L}\right)\right)\left(\frac{1}{L}\left(\frac{4x}{L}-1\right)\right) & \left(\frac{1}{L}\left(\frac{4x}{L}-1\right)\right)^2\end{bmatrix}dx \quad (41)$$

$$=\frac{1}{3L}\begin{bmatrix}7 & -8 & 1\\-8 & 16 & -8\\1 & -8 & 7\end{bmatrix}$$

The region $0\le x\le 1$ has been divided into K elements of equal length $L=\dfrac{1}{K}$. After assembling all elements at time $t=t_n$, we obtain the system of linear equation as

$$\left(\underset{(2K+1)\times(2K+1)}{\left[\mathrm{L}_{\mathrm{A}q}\right]_{\mathrm{sys}}}+\underset{(2K+1)\times(2K+1)}{\left[\mathrm{L}_{\mathrm{B}q}\right]_{\mathrm{sys}}}\right)\underset{(2K+1)\times 1}{\{T\}}=\underset{(2K+1)\times 1}{\{0\}} \quad (42)$$

where

$$\left[\mathrm{L}_{\mathrm{A}q}\right]_{\mathrm{sys}} = \frac{L}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & 4+4 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 16 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & 4+4 & 2 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 2 & 4+4 & 2 & -1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 2 & 16 & 2 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 2 & 4+4 & 2 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 4 \end{bmatrix}_{(2K+1)\times(2K+1)}$$

$$= \frac{L}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & \cdots & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & \cdots & 0 & -1 & 2 & 4 \end{bmatrix}_{(2K+1)\times(2K+1)} \quad (43)$$

$$\left[\mathrm{L}_{\mathrm{B}q}\right]_{\mathrm{sys}} = \frac{1}{3L}\begin{bmatrix} 7 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -8 & 7+7 & -8 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -8 & 16 & -8 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -8 & 7+7 & -8 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -8 & 7+7 & -8 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -8 & 16 & -8 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & -8 & 7+7 & -8 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -8 & 7 \end{bmatrix}_{(2K+1)\times(2K+1)}$$

$$= \frac{1}{3L}\begin{bmatrix} 7 & -8 & 1 & 0 & \cdots & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & \cdots & 0 \\ 1 & -8 & 14 & -8 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -8 & 14 & -8 & 1 \\ 0 & \cdots & 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & \cdots & 0 & 1 & -8 & 7 \end{bmatrix}_{(2K+1)\times(2K+1)} \qquad (44)$$

Next, for marching in time, we define the coefficients in matrix element of time as

$$[A_L] = [\mathrm{L}_{\mathrm{A}r}]_{\mathrm{sys}} \otimes [Z_b] \quad (45)$$

$$[B_L] = \alpha [\mathrm{L}_{\mathrm{B}r}]_{\mathrm{sys}} \otimes [Z_a] \quad (46)$$

$$\left[M_L^{++}\right] = [\mathrm{L}_{\mathrm{A}r}]_{\mathrm{sys}} \otimes [Z_+] \quad (47)$$

$$\left[M_L^{-+}\right] = [\mathrm{L}_{\mathrm{A}r}]_{\mathrm{sys}} \otimes [Z_-] \quad (48)$$

where $[\mathrm{L}_{\mathrm{A}r}]_{\mathrm{sys}}$ and $[\mathrm{L}_{\mathrm{B}r}]_{\mathrm{sys}}$ are for the linear Lagrange bases (if r is l) or the quadratic Lagrange bases (if r is q). The matrices $[Z_a]$, $[Z_b]$, $[Z_+]$, $[Z_-]$ are defined in the chapter of basis function, and $\otimes$ is the outer tensor operation of two matrices defined below.

The operation of outer tensor $\otimes$ of two matrices is defined in a standard way, for example, when the 2×2 matrices A and B are given, the outer tensor operation is defined as follows.

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad , \qquad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad ,$$

$$A\otimes B=\begin{bmatrix} Ab_{11} & Ab_{12} \\ Ab_{21} & Ab_{22} \end{bmatrix}$$
$$=\left[\begin{array}{cc|cc} a_{11}b_{11} & a_{12}b_{11} & a_{112}b_{12} & a_{12}b_{12} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12} \\ \hline a_{11}b_{21} & a_{12}b_{21} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{21} & a_{22}b_{21} & a_{21}b_{22} & a_{22}b_{22} \end{array}\right] . \qquad (49)$$

For example, the full form matrices of linear Lagrange elements with time basis of level 2 are expressed as follows

$$[A_L] = \left[\frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}_{(K+1)\times(K+1)} \otimes \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \right]_{2(K+1)\times 2(K+1)} ,$$

$$[B_L] = \left[\alpha \frac{1}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{(K+1)\times(K+1)} \otimes \Delta t \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \right]_{2(K+1)\times 2(K+1)} ,$$

$$[M_L^{++}] = \left[\frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}_{(K+1)\times(K+1)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_{2(K+1)\times 2(K+1)} ,$$

$$[M_L^{-+}] = \left[\frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}_{(K+1)\times(K+1)} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right]_{2(K+1)\times 2(K+1)} .$$

When the quadratic Lagrange basis is applied with time basis of level 2, the matrices $[A_L]$, $[B_L]$, $\left[M_L^{++}\right]$ and $\left[M_L^{-+}\right]$ are obtained as follows

$$[A_L] = \left[\frac{L}{30} \begin{bmatrix} 4 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & \cdots & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & \cdots & 0 & -1 & 2 & 4 \end{bmatrix}_{(2K+1)\times(2K+1)} \otimes \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \right]_{2(2K+1)\times 2(2K+1)} ,$$

$$[B_L] = \alpha \left[\frac{1}{3L} \begin{bmatrix} 7 & -8 & 1 & 0 & \cdots & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & \cdots & 0 \\ 1 & -8 & 14 & -8 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -8 & 14 & -8 & 1 \\ 0 & \cdots & 0 & 0 & -8 & 16 & 2 \\ 0 & 0 & \cdots & 0 & 1 & -8 & 7 \end{bmatrix}_{(2K+1)\times(2K+1)} \otimes \Delta t \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \right]_{2(2K+1)\times 2(2K+1)} ,$$

$$\left[M_L^{++}\right] = \left[\frac{L}{30} \begin{bmatrix} 4 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & \cdots & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & \cdots & 0 & -1 & 2 & 4 \end{bmatrix}_{(2K+1)\times(2K+1)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_{2(2K+1)\times 2(2K+1)} ,$$

$$\left[M_L^{-+}\right]=\left[\frac{L}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & \cdots & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & \cdots & 0 & -1 & 2 & 4 \end{bmatrix}_{(2K+1)\times(2K+1)} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right]_{2(2K+1)\times 2(2K+1)} .$$

In general, the system of linear equation when applying the Lagrange bases can be expressed by

$$\left(\left[A_L\right]+\left[B_L\right]+\left[M_L^{++}\right]\right)\left\{T^n\right\}=\left[M_L^{-+}\right]\left\{T^{n-1}\right\}\ ,\ n=1,2,3,\dots\ . \qquad (50)$$

The system (50) can be solved iteratively when the initial condition $T\left(x,0\right)=T_0\left(x\right)$ is provided when the starting coefficients are obtained from

$$\left\{T^0\right\}=\left\{T_0(x_i)\right\}\ ,$$

where x_i are the element nodes.

In this work, we apply the Gauss-Seidel method to solve the system (50) when the tolerance (TOL) value is specified.

2. The Galerkin finite element method for wavelet bases

By the weighted residual method, equation (27) can be written as

$$\int_{\Omega} W\left(\frac{\partial T}{\partial t}-\alpha\frac{\partial^2 T}{\partial x^2}\right)\,\mathrm{d}\Omega = 0\ , \tag{51}$$

where W is a weighting function. Using integration by part, yields

$$\int_{\Omega} W\left(\frac{\partial T}{\partial t}\right)d\Omega \ + \int_{\Omega}\alpha\left(\frac{\partial T}{\partial x}\frac{\partial W}{\partial x}\right)d\Omega \ = 0. \tag{52}$$

Let us begin by approximating the unknown function in terms of the wavelet basis as

$$T^n = T(x,t_n) \ = \sum_{k=0}^{p}\sum_{i=1}^{M}\sum_{j=0}^{\dim(i)} w_{ij}(x)\ \theta_k(t_n)\ c_{ijk}^n\ , \tag{53}$$

where $T^n = T(x,t_n)$ denotes the variable's value , c_{ijk}^n are the coefficient to be approximated, $w_{ij}(x)$ is the wavelet basis function, $\theta_k(t_n)$ is the time basis function, M is the number of level in multi-level wavelet approach, p is the number of level for time discretization and $\dim(i) = \dim \mathrm{W}_i - 1$.

After setting $W = \{\mathrm{w}\}\otimes\{\theta\}$ where $\{\mathrm{w}\} = \{w_{ij}(x)\}$ and $\{c\} = \{c_{ijk}\}$, for $i = 1,2,3,\ldots,M$ that i is the ith level , $j = 0,1,\ldots,\dim \mathrm{W}_i - 1$, $k = 0,1,2,\ldots,p$ that k is the kth level in time and $e = \dim \mathrm{X}_M$, equation (52) can be written in the matrix form as

$$\left(\underset{(ek)\times(ek)}{[A]} + \underset{(ek)\times(ek)}{[B]} + \underset{(ek)\times(ek)}{\left[M^{++}\right]}\right)\underset{(ek)\times 1}{\left\{c^n\right\}} = \underset{(ek)\times(ek)}{\left[M^{-+}\right]}\underset{(ek)\times 1}{\left\{c^{n-1}\right\}}\ ,\ n = 1,2,3,\ldots\ . \tag{54}$$

The coefficients in each matrix element can be obtained. For brevity, the results are summarized as follows.

Let $\{\mathrm{w}\} = \{w_{ij}(x)\}$, for $i = 1,2,3,\ldots,M$ and $j = 0,1,\ldots,\dim \mathrm{W}_i - 1$ be the wavelet bases for each level.

$$
\begin{aligned}
[A] &= \int_0^1 \int_{t_{n-1}}^{t_n} (\{\mathrm{w}\} \otimes \{\theta\}) \left(\{\mathrm{w}\} \otimes \left\{ \frac{d\theta}{dt} \right\} \right)^T dtdx \\
&= \int_0^1 \{\mathrm{w}\}\{\mathrm{w}\}^T dx \otimes \int_{t_{n-1}}^{t_n} \{\theta\} \left\{ \frac{d\theta}{dt} \right\}^T dt \qquad (55) \\
&= [\mathrm{W}_{\mathrm{A}r}] \otimes [Z_b]
\end{aligned}
$$

$$
\begin{aligned}
[B] &= \alpha \int_0^1 \int_{t_{n-1}}^{t_n} \left(\left\{ \frac{d\mathrm{w}}{dx} \right\} \otimes \{\theta\} \right) \left(\left\{ \frac{d\mathrm{w}}{dx} \right\} \otimes \{\theta\} \right)^T dtdx \\
&= \alpha \int_0^1 \left\{ \frac{d\mathrm{w}}{dx} \right\} \left\{ \frac{d\mathrm{w}}{dx} \right\}^T dx \otimes \int_{t_{n-1}}^{t_n} \{\theta\}\{\theta\}^T dt \qquad (56) \\
&= \alpha [\mathrm{W}_{\mathrm{B}r}] \otimes [Z_a]
\end{aligned}
$$

$$
\begin{aligned}
\left[M^{++} \right] &= \int_0^1 \left(\{\mathrm{w}\} \otimes \{\theta^+\} \right) \left(\{\mathrm{w}\} \otimes \{\theta^+\} \right)^T dx \\
&= \int_0^1 \{\mathrm{w}\}\{\mathrm{w}\}^T dx \otimes \{\theta^+\}\{\theta^+\}^T \qquad (57) \\
&= [\mathrm{W}_{\mathrm{A}r}] \otimes [Z_+]
\end{aligned}
$$

$$\left[M^{-+}\right] = \int_0^1 \left(\{\mathrm{w}\}\otimes\{\theta^-\}\right)\left(\{\mathrm{w}\}\otimes\{\theta^+\}\right)^T dx$$

$$= \int_0^1 \{\mathrm{w}\}\{\mathrm{w}\}^T dx \otimes \{\theta^-\}\{\theta^+\}^T \qquad (58)$$

$$= [\mathrm{W}_{\mathrm{A}r}]\otimes[Z_-]$$

where $[\mathrm{W}_{\mathrm{A}r}]$ and $[\mathrm{W}_{\mathrm{B}r}]$ are for the linear wavelet basis (if r is l) or the quadratic wavelet basis (if r is q). The matrices $[Z_a]$, $[Z_b]$, $[Z_+]$ and $[Z_-]$ are defined in the chapter of basis function.

2.1 Linear wavelet basis function

For the linear wavelet basis functions, in case of using the first three levels of W_1 , W_2 and W_3 , the coefficients in matrices $[W_{Al}]$ and $[W_{Bl}]$ can be evaluated by

$$\left[W_{Al}\right]=\int_0^1 \{w\}\{w\}^T dx$$

$$=\begin{bmatrix} \frac{1}{12} & \frac{\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & \frac{1}{256} & \frac{3}{256} & \frac{3}{256} & \frac{1}{256} \\ \frac{\sqrt{2}}{64} & \frac{1}{48} & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} & 0 & 0 \\ \frac{\sqrt{2}}{64} & 0 & \frac{1}{48} & 0 & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} \\ \frac{1}{256} & \frac{\sqrt{2}}{256} & 0 & \frac{1}{192} & 0 & 0 & 0 \\ \frac{3}{256} & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 & 0 \\ \frac{3}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 \\ \frac{1}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & 0 & \frac{1}{192} \end{bmatrix}_{(7\times7)} \tag{59}$$

$$\left[W_{Bl}\right]=\int_0^1 \left\{\frac{dw}{dx}\right\}\left\{\frac{dw}{dx}\right\}^T dx$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(7\times7)} \tag{60}$$

In general, the full forms of matrix coefficients for linear wavelet bases of level M and operating with time basis level two can be expressed as follows.

$$[A] = \begin{bmatrix} \frac{1}{12} & \frac{\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & \frac{1}{256} & \frac{3}{256} & \frac{3}{256} & \frac{1}{256} & \cdots & \cdots & \cdots & \cdots \\ \frac{\sqrt{2}}{64} & \frac{1}{48} & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{\sqrt{2}}{64} & 0 & \frac{1}{48} & 0 & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{256} & \frac{\sqrt{2}}{256} & 0 & \frac{1}{192} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{3}{256} & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{3}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & 0 & \frac{1}{192} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \ddots \end{bmatrix}_{(2^M-1)\times(2^M-1)} \otimes \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots \end{bmatrix}_{(2^M-1)\times(2^M-1)} \otimes \Delta t \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

$$
\left[M^{++}\right] = \begin{bmatrix}
\frac{1}{12} & \frac{\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & \frac{1}{256} & \frac{3}{256} & \frac{3}{256} & \frac{1}{256} & \cdots & \cdots & \cdots & \cdots \\
\frac{\sqrt{2}}{64} & \frac{1}{48} & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{\sqrt{2}}{64} & 0 & \frac{1}{48} & 0 & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{256} & \frac{\sqrt{2}}{256} & 0 & \frac{1}{192} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{3}{256} & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{3}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & 0 & \frac{1}{192} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \ddots
\end{bmatrix}_{(2^M-1)\times(2^M-1)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
$$

$$
\left[M^{-+}\right] = \begin{bmatrix}
\frac{1}{12} & \frac{\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & \frac{1}{256} & \frac{3}{256} & \frac{3}{256} & \frac{1}{256} & \cdots & \cdots & \cdots & \cdots \\
\frac{\sqrt{2}}{64} & \frac{1}{48} & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{\sqrt{2}}{64} & 0 & \frac{1}{48} & 0 & 0 & \frac{\sqrt{2}}{256} & \frac{\sqrt{2}}{256} & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{256} & \frac{\sqrt{2}}{256} & 0 & \frac{1}{192} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{3}{256} & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{3}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & \frac{1}{192} & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{256} & 0 & \frac{\sqrt{2}}{256} & 0 & 0 & 0 & \frac{1}{192} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \ddots
\end{bmatrix}_{(2^M-1)\times(2^M-1)} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}
$$

2.2 Quadratic wavelet basis function

For the quadratic wavelet basis functions, in case of using the first three levels of W_0, W_1 and W_2, the coefficients in matrices $[W_{Aq}]$ and $[W_{Bq}]$ can be evaluated by

$$\left[W_{Aq}\right]=\int_0^1 \{w\}\{w\}^T dx$$

$$= \begin{bmatrix} \frac{1}{10} & \frac{-\sqrt{3}}{240} & 0 & \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{256} & \frac{-\sqrt{6}}{3840} & \frac{\sqrt{2}}{256} \\ \frac{-\sqrt{3}}{240} & \frac{1}{120} & 0 & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{768} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{768} \\ 0 & 0 & \frac{1}{40} & \frac{-\sqrt{6}}{1920} & 0 & \frac{\sqrt{6}}{1920} & 0 \\ \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{1920} & \frac{1}{480} & 0 & 0 & 0 \\ \frac{-\sqrt{2}}{256} & \frac{\sqrt{6}}{768} & 0 & 0 & \frac{1}{160} & 0 & 0 \\ \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{1920} & 0 & 0 & \frac{1}{480} & 0 \\ \frac{\sqrt{2}}{256} & \frac{-\sqrt{6}}{768} & 0 & 0 & 0 & 0 & \frac{1}{160} \end{bmatrix}_{(7\times 7)} \tag{61}$$

$$\left[W_{Bq}\right]=\int_0^1 \left\{\frac{dw}{dx}\right\}\left\{\frac{dw}{dx}\right\}^T dx$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(7\times 7)} \tag{62}$$

Similarly, the full forms of matrix coefficients for quadratic wavelet bases of level M and operating with time basis level two can be expressed as follows.

$$[A] = \begin{bmatrix} \frac{1}{10} & \frac{-\sqrt{3}}{240} & 0 & \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{256} & \frac{-\sqrt{6}}{3840} & \frac{\sqrt{2}}{256} & \cdots & \cdots & \cdots & \cdots \\ \frac{-\sqrt{3}}{240} & \frac{1}{120} & 0 & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{768} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{768} & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \frac{1}{40} & \frac{-\sqrt{6}}{1920} & 0 & \frac{\sqrt{6}}{1920} & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{1920} & \frac{1}{480} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{-\sqrt{2}}{256} & \frac{\sqrt{6}}{768} & 0 & 0 & \frac{1}{160} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{1920} & 0 & 0 & \frac{1}{480} & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{\sqrt{2}}{256} & \frac{-\sqrt{6}}{768} & 0 & 0 & 0 & 0 & \frac{1}{160} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \ddots \end{bmatrix}_{\left(2^{(M+1)}-1\right)\times\left(2^{(M+1)}-1\right)} \otimes \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots \end{bmatrix}_{\left(2^{(M+1)}-1\right)\times\left(2^{(M+1)}-1\right)} \otimes \Delta t \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

$$\left[M^{++}\right]=\begin{bmatrix}
\frac{1}{10} & \frac{-\sqrt{3}}{240} & 0 & \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{256} & \frac{-\sqrt{6}}{3840} & \frac{\sqrt{2}}{256} & \cdots & \cdots & \cdots & \cdots \\
\frac{-\sqrt{3}}{240} & \frac{1}{120} & 0 & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{768} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{768} & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \frac{1}{40} & \frac{-\sqrt{6}}{1920} & 0 & \frac{\sqrt{6}}{1920} & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{1920} & \frac{1}{480} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{-\sqrt{2}}{256} & \frac{\sqrt{6}}{768} & 0 & 0 & \frac{1}{160} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{1920} & 0 & 0 & \frac{1}{480} & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{\sqrt{2}}{256} & \frac{-\sqrt{6}}{768} & 0 & 0 & 0 & 0 & \frac{1}{160} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \ddots
\end{bmatrix}_{\left(2^{(M+1)}-1\right)\times\left(2^{(M+1)}-1\right)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[M^{-+}\right]=\begin{bmatrix}
\frac{1}{10} & \frac{-\sqrt{3}}{240} & 0 & \frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{256} & \frac{-\sqrt{6}}{3840} & \frac{\sqrt{2}}{256} & \cdots & \cdots & \cdots & \cdots \\
\frac{-\sqrt{3}}{240} & \frac{1}{120} & 0 & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{768} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{768} & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \frac{1}{40} & \frac{-\sqrt{6}}{1920} & 0 & \frac{\sqrt{6}}{1920} & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{-\sqrt{6}}{1920} & \frac{1}{480} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{-\sqrt{2}}{256} & \frac{\sqrt{6}}{768} & 0 & 0 & \frac{1}{160} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{-\sqrt{6}}{3840} & \frac{-\sqrt{2}}{1280} & \frac{\sqrt{6}}{1920} & 0 & 0 & \frac{1}{480} & 0 & \cdots & \cdots & \cdots & \cdots \\
\frac{\sqrt{2}}{256} & \frac{-\sqrt{6}}{768} & 0 & 0 & 0 & 0 & \frac{1}{160} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \ddots
\end{bmatrix}_{\left(2^{(M+1)}-1\right)\times\left(2^{(M+1)}-1\right)} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

For given an initial condition ,

$$T\left(x,0\right)=T_0\left(x\right) \quad ,$$

the system (54) can be solved iteratively by given $\left\{c^0\right\}$ which can be obtained from the following steps.

Recalling, we have assumed that

$$T^n\left(x,t_n\right) \;=\; \sum_{k=0}^{p}\sum_{i=1}^{M}\sum_{j=0}^{\dim(i)} w_{ij}\left(x\right)\theta_k\left(t_n\right)c_{ijk}^n \quad ,$$

so,

$$T\left(x,0\right) \;=\; \sum_{k=0}^{p}\sum_{i=1}^{M}\sum_{j=0}^{\dim(i)} w_{ij}\left(x\right)\theta_k\left(0\right)c_{ijk}^0 \quad . \tag{63}$$

After applying the initial values at the knots of wavelet basis, yields

$$\left\{T_0\left(x_s\right)\right\} \;=\; \left\{\left\{w_{ij}\left(x_s\right)\right\}\otimes\left\{\theta_k\left(0\right)\right\}\right\}^T\left\{c^0\right\} \quad ,$$

$$\left\{T_0\left(x_s\right)\right\} \;=\; \left\{\left\{w_{ij}\left(x_s\right)\right\}\otimes\left\{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right\}\right\}^T\left\{c^0\right\} \quad ,$$

$$\left\{T_0\left(x_s\right)\right\} \;=\; \left\{w_{ij}\left(x_s\right)\right\}^T\left\{c^0\right\} \quad .$$

Thus, in the case of linear wavelet basis function, we can find the coefficients $\{c^0\}$ from the system of $2^M + 1$ equations

$$\{w_{ij}(x_s)\}^T \{c^0\} = \{T_0(x_s)\} \ , \tag{64}$$

where $x_s = s/2^M$, $s = 0,1,...,2^M$, and $2^M + 1$ is the number of knots.

And, in the case of quadratic wavelet basis function, we can find the coefficients $\{c^0\}$ from the system of $2^{(M+1)} + 1$ equations

$$\{w_{ij}(x_s)\}^T \{c^0\} = \{T_0(x_s)\} \ , \tag{65}$$

where $x_s = s/2^{(M+1)}$, $s = 0,1,...,2^{(M+1)}$, and $2^{(M+1)} + 1$ is the number of knots.

Numerical results by the Galerkin finite element method for heat equation will be shown in the chapter of Results and Discussion.

Galerkin finite element method for singularly perturbed problem

In this chapter, we study the concept of the Galerkin finite element method to solve numerically the singularly perturbed second-order boundary value problem and the singularly perturbed one-dimensional (linear) parabolic problem. Two types of basis functions which are the Lagrange and the wavelet bases are employed to derive the full form of matrix system. We consider both linear and quadratic bases in the Galerkin finite element method. Numerical results are presented to demonstrate the efficiency of this method.

1. Singularly perturbed second-order boundary value problem

The partial differential equation for singularly perturbed second-order boundary value problem can be written as

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - u = 1 \quad , \quad 0 < x < 1 , \tag{66}$$

where ε is the diffusion coefficient or perturbation parameter $(0 < \varepsilon << 1)$.

The domain is Ω $(0 < x < 1)$ with boundary Γ.

The boundary conditions are

$$u(0) = 0 \text{ and } u(1) = 0 . \tag{67}$$

1.1 The Galerkin finite element method for Lagrange bases

By the weighted residual method, the finite element formulation for an element L of equation (66) can be written as

$$\int_0^L W\left(\varepsilon\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}-u-1\right)dx=0\ , \tag{68}$$

where W is a weighting function. Using integration by part, yields

$$\varepsilon\int_0^L \frac{\partial u}{\partial x}\frac{\partial W}{\partial x}dx\ -\int_0^L W\left(\frac{\partial u}{\partial x}\right)dx+\int_0^L W\left(u\right)dx+\int_0^L W dx=0\ . \tag{69}$$

Let us begin by approximating the unknown function in terms of the Lagrange basis as,

$$u=\ u\left(x\right)\ \ =\ \sum_{i=1}^{m} N_i\left(x\right)u_i\ . \tag{70}$$

where u_i are nodes values, $N_i(x)$ is the Lagrange basis function, m is the number of basis function, $m=2$ for linear Lagrange basis function and $m=3$ for quadratic Lagrange basis function.

1.1.1 Linear Lagrange basis function

Substituting equation (70) with linear Lagrange basis function into equation (69), yields the resulting matrix representation as

$$\left\{ \underset{(2\times2)}{\left[\mathrm{L}_{\mathrm{B}l}\right]} - \underset{(2\times2)}{\left[\mathrm{L}_{\mathrm{C}l}\right]} + \underset{(2\times2)}{\left[\mathrm{L}_{\mathrm{A}l}\right]} \right\} \underset{(2\times1)}{\{u\}} = -\underset{(2\times1)}{\left\{\mathrm{L}_{\mathrm{E}l}\right\}} \tag{71}$$

where $\left[\mathrm{L}_{\mathrm{A}l}\right]$ and $\left[\mathrm{L}_{\mathrm{B}l}\right]$ have been defined in the chapter of Galerkin finite element method for heat equation.

The coefficients of matrices $\left[\mathrm{L}_{\mathrm{C}l}\right]$ and $\left\{\mathrm{L}_{\mathrm{E}l}\right\}$ can be evaluated as

$$\begin{aligned}
\left[\mathrm{L}_{\mathrm{C}l}\right] &= \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \begin{bmatrix} \dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} \end{bmatrix} dx \\
&= \int_0^L \begin{Bmatrix} 1-\dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} dx \\
&= \int_0^L \begin{bmatrix} \dfrac{x}{L^2}-\dfrac{1}{L} & \dfrac{1}{L}-\dfrac{x}{L^2} \\ -\dfrac{x}{L^2} & \dfrac{x}{L^2} \end{bmatrix} dx \\
&= \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \quad ,
\end{aligned} \tag{72}$$

$$\{\mathrm{L}_{\mathrm{E}l}\} = \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} dx$$

$$= \int_0^L \begin{Bmatrix} 1-\dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} dx \qquad (73)$$

$$= \frac{L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad .$$

The domain $0 < x < 1$ has been divided into K elements of equal length $L = \dfrac{1}{K}$. After assembling all elements together, we obtain the system of linear equation as

$$\left\{ \underset{(K+1)\times(K+1)}{[\mathrm{L}_{\mathrm{B}l}]_{\mathrm{sys}}} - \underset{(K+1)\times(K+1)}{[\mathrm{L}_{\mathrm{C}l}]_{\mathrm{sys}}} + \underset{(K+1)\times(K+1)}{[\mathrm{L}_{\mathrm{A}l}]_{\mathrm{sys}}} \right\} \underset{(K+1)\times 1}{\{u\}} = -\underset{(K+1)\times 1}{\{\mathrm{L}_{\mathrm{E}l}\}_{\mathrm{sys}}} \qquad (74)$$

where

$$\left[\mathrm{L}_{Cl}\right]_{\mathrm{sys}} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1+1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1+1 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & -1 & -1+1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & -1+1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{bmatrix}_{(K+1)\times(K+1)} \tag{75}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}_{(K+1)\times(K+1)},$$

$$\left\{\mathrm{L}_{\mathrm{E}l}\right\}_{\mathrm{sys}} = \frac{L}{2}\begin{Bmatrix} 1 \\ 1+1 \\ 1+1 \\ \vdots \\ 1+1 \\ 1+1 \\ 1 \end{Bmatrix}_{(K+1)\times 1} \tag{76}$$

$$= \frac{L}{2}\begin{Bmatrix} 1 \\ 2 \\ \vdots \\ 2 \\ 1 \end{Bmatrix}_{(K+1)\times 1}.$$

1.1.2 Quadratic Lagrange basis function

Substituting equation (70) with quadratic Lagrange basis into equation (69), yields the resulting matrix representation as

$$\left\{ \underset{(3\times3)}{\left[\mathrm{L}_{\mathrm{B}q} \right]} - \underset{(3\times3)}{\left[\mathrm{L}_{\mathrm{C}q} \right]} + \underset{(3\times3)}{\left[\mathrm{L}_{\mathrm{A}q} \right]} \right\} \underset{(3\times1)}{\{u\}} = -\underset{(3\times1)}{\left\{ \mathrm{L}_{\mathrm{E}q} \right\}} \tag{77}$$

where $\left[\mathrm{L}_{\mathrm{A}q} \right]$ and $\left[\mathrm{L}_{\mathrm{B}q} \right]$ have been defined in the chapter of Galerkin finite element method for heat equation.

The coefficients of matrices $\left[\mathrm{L}_{\mathrm{C}q} \right]$ and $\left\{ \mathrm{L}_{\mathrm{E}q} \right\}$ can be evaluated as

$$\left[\mathrm{L}_{\mathrm{C}q} \right] = \int_0^L \begin{Bmatrix} N_1 \\ N_2 \\ N_2 \end{Bmatrix} \begin{bmatrix} \dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} & \dfrac{dN_2}{dx} \end{bmatrix} dx$$

$$= \int_0^L \begin{Bmatrix} 1 - \dfrac{3x}{L} + 2\left(\dfrac{x}{L}\right)^2 \\ \dfrac{4x}{L}\left(1 - \dfrac{x}{L}\right) \\ \dfrac{x}{L}\left(\dfrac{2x}{L} - 1\right) \end{Bmatrix} \begin{bmatrix} \dfrac{1}{L}\left(\dfrac{4x}{L} - 3\right) & \dfrac{4}{L}\left(1 - \dfrac{2x}{L}\right) & \dfrac{1}{L}\left(\dfrac{4x}{L} - 1\right) \end{bmatrix} dx$$

$$= \int_0^L \begin{bmatrix} \dfrac{8x^3}{L^4} - \dfrac{18x^2}{L^3} + \dfrac{13x}{L^2} - \dfrac{3}{L} & -\dfrac{16x^3}{L^4} + \dfrac{32x^2}{L^3} - \dfrac{20x}{L^2} + \dfrac{4}{L} & \dfrac{8x^3}{L^4} - \dfrac{14x^2}{L^3} + \dfrac{7x}{L^2} - \dfrac{1}{L} \\ -\dfrac{16x^3}{L^4} + \dfrac{28x^2}{L^3} - \dfrac{12x}{L^2} & \dfrac{32x^3}{L^4} - \dfrac{48x^2}{L^3} + \dfrac{16x}{L^2} & -\dfrac{16x^3}{L^4} + \dfrac{16x^2}{L^3} - \dfrac{4x}{L^2} \\ \dfrac{8x^3}{L^4} - \dfrac{10x^2}{L^3} + \dfrac{3x}{L^2} & -\dfrac{16x^3}{L^4} + \dfrac{20x^2}{L^3} - \dfrac{4x}{L^2} & \dfrac{8x^3}{L^4} - \dfrac{6x^2}{L^3} + \dfrac{x}{L^2} \end{bmatrix} dx \tag{78}$$

$$= \frac{1}{6} \begin{bmatrix} -3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} \quad ,$$

$$\left\{ \mathrm{L}_{\mathrm{E}q} \right\} = \int_0^L \begin{Bmatrix} N_1 \\ N_2 \\ N_2 \end{Bmatrix} dx$$

$$= \int_0^L \begin{Bmatrix} 1 - \frac{3x}{L} + 2\left(\frac{x}{L}\right)^2 \\ \frac{4x}{L}\left(1 - \frac{x}{L}\right) \\ \frac{x}{L}\left(\frac{2x}{L} - 1\right) \end{Bmatrix} dx$$

$$= \frac{L}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} \quad . \tag{79}$$

The domain $0 < x < 1$ has been divided by K elements of equal length $L = \frac{1}{K}$. After assembling all elements together, we obtain the system of linear equation as

$$\left\{ \left[\mathrm{L}_{\mathrm{B}q} \right]_{\substack{\mathrm{sys} \\ (2K+1)\times(2K+1)}} - \left[\mathrm{L}_{\mathrm{C}q} \right]_{\substack{\mathrm{sys} \\ (2K+1)\times(2K+1)}} + \left[\mathrm{L}_{\mathrm{A}q} \right]_{\substack{\mathrm{sys} \\ (2K+1)\times(2K+1)}} \right\} \left\{ u \right\}_{(2K+1)\times 1} = -\left\{ \mathrm{L}_{\mathrm{E}q} \right\}_{\substack{\mathrm{sys} \\ (2K+1)\times 1}} \tag{80}$$

where

$$
\left[\mathrm{L}_{Cq}\right]_{\mathrm{sys}} = \frac{1}{6}\begin{bmatrix}
-3 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & -4 & -3+3 & 4 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & -4 & 0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -4 & -3+3 & 4 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & -4 & -3+3 & 4 & -1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -4 & 0 & 4 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & -4 & -3+3 & 4 & -1 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 4 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 3
\end{bmatrix}_{(2K+1)\times(2K+1)}
$$

$$
= \frac{1}{6}\begin{bmatrix}
3 & 4 & -1 & 0 & \cdots & 0 & 0 \\
-4 & 0 & 4 & 0 & 0 & \cdots & 0 \\
1 & -4 & 0 & 4 & -1 & 0 & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & 0 & 1 & -4 & 0 & 4 & -1 \\
0 & \cdots & 0 & 0 & -4 & 0 & 4 \\
0 & 0 & \cdots & 0 & 1 & -4 & 3
\end{bmatrix}_{(2K+1)\times(2K+1)} , \tag{81}
$$

$$
\left\{\mathrm{L}_{\mathrm{E}q}\right\}_{\mathrm{sys}} = \frac{L}{6}\begin{Bmatrix} 1 \\ 4 \\ 1+1 \\ 4 \\ 1+1 \\ \vdots \\ 1+1 \\ 4 \\ 1+1 \\ 4 \\ 1 \end{Bmatrix}_{(2K+1)\times 1} = \frac{L}{6}\begin{Bmatrix} 1 \\ 4 \\ 2 \\ \vdots \\ 2 \\ 4 \\ 1 \end{Bmatrix}_{(2K+1)\times 1} . \tag{82}
$$

The linear system(80) can be solved by the Gauss-Seidel method. By setting $TOL = 10^{-12}$.

1.2 The Galerkin finite element method for wavelet bases

By the weighted residual method, equation (66) can be written as

$$\int_{\Omega} W\left(\varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - u - 1 \right) \mathrm{d}\Omega = 0 \ , \tag{83}$$

where W is a weighting function. Using integration by part, yields

$$\varepsilon \int_{\Omega} \frac{\partial u}{\partial x} \frac{\partial W}{\partial x} d\Omega \ - \int_{\Omega} W \left(\frac{\partial u}{\partial x} \right) \mathrm{d}\Omega + \int_{\Omega} W\left(u\right) \mathrm{d}\Omega + \int_{\Omega} W \mathrm{d}\Omega = 0 \ . \tag{84}$$

Let us begin by approximating the unknown function in terms of the wavelet basis as

$$u = u\left(x\right) \ = \sum_{i=1}^{M} \sum_{j=0}^{\dim(i)} w_{ij}\left(x\right) c_{ij} \ , \tag{85}$$

where $u\left(x\right)$ denotes the variable's value, c_{ij} are the coefficient to be approximated, $w_{ij}\left(x\right)$ is the wavelet basis function, M is the number of level in multi-level wavelet approach and $\dim\left(i\right) = \dim \mathrm{W}_i - 1$.

After setting $W = \mathrm{w}\left(x\right)$ where $\left\{\mathrm{w}\right\} = \left\{w_{ij}\left(x\right)\right\}$ and $\left\{c\right\} = \left\{c_{ij}\right\}$, for $i = 1, 2, 3, \ldots, M$ that i is the i th level , $j = 0, 1, \ldots, \dim \mathrm{W}_i - 1$ and $e = \dim \mathrm{X}_M$ equation (84) can be written in the matrix form as

$$\left\{ \underset{(e\times e)}{[\mathrm{W}_{\mathrm{B}r}]} - \underset{(e\times e)}{[\mathrm{W}_{\mathrm{C}r}]} + \underset{(e\times e)}{[\mathrm{W}_{\mathrm{A}r}]} \right\} \underset{(e\times 1)}{\left\{c\right\}} = - \underset{(e\times 1)}{\left\{\mathrm{W}_{\mathrm{E}r}\right\}} \ . \tag{86}$$

The linear system(86) can be solved by the Gauss-Seidel method. By setting $TOL = 10^{-12}$.

The coefficients in each matrix can be obtained. For brevity, the results are summarized as

Let $\{\mathrm{w}\} = \{w_{ij}(x)\}$, for $i = 1,2,3,\ldots,M$, $j = 0,1,\ldots,\dim \mathrm{W}_i - 1$ be the wavelet bases for each level.

$$[\mathrm{W}_{\mathrm{A}r}] = \int_0^1 \{\mathrm{w}\}\{\mathrm{w}\}^T \, dx \tag{87}$$

$$[\mathrm{W}_{\mathrm{B}r}] = \varepsilon \int_0^1 \left\{\frac{d\mathrm{w}}{dx}\right\}\left\{\frac{d\mathrm{w}}{dx}\right\}^T dx \tag{88}$$

$$[\mathrm{W}_{\mathrm{C}r}] = \int_0^1 \{\mathrm{w}\}\left\{\frac{d\mathrm{w}}{dx}\right\}^T dx \tag{89}$$

$$\{\mathrm{W}_{\mathrm{E}r}\} = \int_0^1 \{\mathrm{w}\} \, dx \tag{90}$$

where $[\mathrm{W}_{\mathrm{A}r}]$, $[\mathrm{W}_{\mathrm{B}r}]$, $[\mathrm{W}_{\mathrm{C}r}]$ and $\{\mathrm{W}_{\mathrm{E}r}\}$ are for the linear wavelet bases (if r is l) or the quadratic wavelet bases (if r is q). The coefficients in matrices $[\mathrm{W}_{\mathrm{A}r}]$ and $[\mathrm{W}_{\mathrm{B}r}]$ have been shown in the chapter of Galerkin finite element method for heat equation.

1.2.1 Linear wavelet basis function

For the linear wavelet basis functions, in the case of using the first three levels of W_1, W_2 and W_3, the coefficients in matrices $\left[W_{Cl}\right]$ and $\left\{W_{El}\right\}$ can be evaluated by

$$\left[W_{Cl}\right]=\int_0^1 \{w\}\left\{\frac{dw}{dx}\right\}^T dx$$

$$=\begin{bmatrix} 0 & -\frac{\sqrt{2}}{16} & \frac{\sqrt{2}}{16} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} \\ \frac{\sqrt{2}}{16} & 0 & 0 & -\frac{\sqrt{2}}{32} & \frac{\sqrt{2}}{32} & 0 & 0 \\ -\frac{\sqrt{2}}{16} & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{32} & \frac{\sqrt{2}}{32} \\ \frac{1}{32} & \frac{\sqrt{2}}{32} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{32} & -\frac{\sqrt{2}}{32} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{32} & 0 & \frac{\sqrt{2}}{32} & 0 & 0 & 0 & 0 \\ \frac{1}{32} & 0 & -\frac{\sqrt{2}}{32} & 0 & 0 & 0 & 0 \end{bmatrix}_{(7\times 7)}, \tag{91}$$

$$\left\{W_{El}\right\}=\int_0^1 \{w\}\, dx$$

$$=\begin{Bmatrix} \frac{1}{4} \\ \frac{\sqrt{2}}{16} \\ \frac{\sqrt{2}}{16} \\ \frac{1}{32} \\ \frac{1}{32} \\ \frac{1}{32} \\ \frac{1}{32} \end{Bmatrix}_{(7\times 1)}. \tag{92}$$

1.2.2 Quadratic Wavelet basis function

For the quadratic wavelet basis functions, in case of using the first three levels of W_0 , W_1 and W_2, the coefficients in matrices $\left[W_{Cq}\right]$ and $\left\{W_{Eq}\right\}$ can be evaluated by

$$\left[W_{Cq}\right]=\int_0^1 \{w\}\left\{\frac{dw}{dx}\right\}^T dx$$

$$=\begin{bmatrix} 0 & 0 & -\frac{1}{8} & 0 & -\frac{\sqrt{2}}{64} & 0 & -\frac{\sqrt{2}}{64} \\ 0 & 0 & \frac{\sqrt{3}}{24} & 0 & -\frac{\sqrt{6}}{64} & 0 & -\frac{\sqrt{6}}{64} \\ \frac{1}{8} & -\frac{\sqrt{3}}{24} & 0 & 0 & -\frac{\sqrt{2}}{32} & 0 & \frac{\sqrt{2}}{32} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{48} & 0 & 0 \\ \frac{\sqrt{2}}{64} & \frac{\sqrt{6}}{64} & \frac{\sqrt{2}}{32} & -\frac{\sqrt{3}}{48} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{48} \\ \frac{\sqrt{2}}{64} & \frac{\sqrt{6}}{64} & -\frac{\sqrt{2}}{32} & 0 & 0 & -\frac{\sqrt{3}}{48} & 0 \end{bmatrix}_{(7\times 7)}, \tag{93}$$

$$\left\{W_{Eq}\right\}=\int_0^1 \{w\}\, dx$$

$$=\begin{Bmatrix} \frac{\sqrt{3}}{6} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{(7\times 1)} . \tag{94}$$

Numerical results by the Galerkin finite element method for singularly perturbed second-order boundary value problem will be shown in the chapter of Results and Discussion.

2. Unsteady singularly perturbed problem

The unsteady singularly perturbed problem of the advection-diffusion-reaction type can be written as

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=\varepsilon\frac{\partial^2 u}{\partial x^2}+f\left(x,t\right)\ , \quad 0<x<1\ , \quad t>0\ , \tag{95}$$

subject to the boundary conditions as,

$$u\left(0,t\right)=u\left(1,t\right)=0\ , \quad t>0\ , \tag{96}$$

and the initial condition as

$$u\left(x,0\right)=u_0\left(x\right)\ , \quad 0<x<1\ , \tag{97}$$

where ε is the diffusion coefficient or perturbation parameter $\left(0<\varepsilon<<1\right)$, x and t denote the spatial coordinate and time, u is the dependent variable, a is a constant refers to speed and $f\left(x,t\right)$ is the reaction term.

2.1 The Galerkin finite element method for Lagrange bases

By the weighted residual method, the finite element formulation of an element L for equation (95) can be written as

$$\int_0^L W\left(\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}-\varepsilon\frac{\partial^2 u}{\partial x^2}-f\left(x,t\right)\right)\,\mathrm{d}x=0\ , \tag{98}$$

where W is a weighting function. Using integration by part, yields

$$\int_0^L W\left(\frac{\partial u}{\partial t}\right)\mathrm{d}x-\varepsilon\int_0^L W\left(\frac{\partial^2 u}{\partial x^2}\right)\mathrm{d}x+\int_0^L W\left(a\frac{\partial u}{\partial x}\right)\mathrm{d}x-\int_0^L W\left(f\left(x,t\right)\right)\mathrm{d}x=0\ . \tag{99}$$

Let us begin by approximating the unknown function in terms of the Lagrange basis as, for any each element,

$$u^n = u\left(x,t_n\right) = \sum_{i=1}^{m} N_i\left(x\right) u_i^n . \tag{100}$$

where u_i^n denotes the unknown value at time $t=t_n$ at nodes, $N_i(x)$ is the Lagrange basis function, m is the number of basis function, $m=2$ for linear Lagrange basis function and $m=3$ for quadratic Lagrange basis function.

2.1.1 Linear Lagrange basis function

For fixed $t = t_n$ and substituting equation (100) with linear Lagrange basis function into equation (99), yields the resulting matrix representation as

$$\left(\underset{(2\times2)}{\left[\mathrm{L}_{\mathrm{A}l}\right]} + \underset{(2\times2)}{\left[\mathrm{L}_{\mathrm{B}l}\right]} + \underset{(2\times2)}{\left[\mathrm{L}_{\mathrm{C}l}\right]} \right) \underset{(2\times1)}{\{u\}} = \underset{(2\times1)}{\left\{\mathrm{L}_{\mathrm{F}l}\right\}} \quad , \tag{101}$$

where $\left[\mathrm{L}_{\mathrm{A}l}\right]$ and $\left[\mathrm{L}_{\mathrm{B}l}\right]$ have been defined in the chapter of Galerkin finite element method for heat equation. $\left[\mathrm{L}_{\mathrm{C}l}\right]$ have been defined in the section of singularly perturbed second-order boundary value problem.

The coefficients of matrices $\left\{\mathrm{L}_{\mathrm{F}l}\right\}$ can be evaluated as

$$\left\{\mathrm{L}_{\mathrm{F}l}\right\} = \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \left(f\left(x,t\right) \right) dx \, . \tag{102}$$

The region $0 < x < 1$ has been divided into K elements of equal length $L = \dfrac{1}{K}$. After assembling all elements together at time $t = t_n$, we obtain the system of linear equation as

$$\left(\underset{(K+1)\times(K+1)}{\left[\mathrm{L}_{\mathrm{A}l}\right]_{\mathrm{sys}}} + \underset{(K+1)\times(K+1)}{\left[\mathrm{L}_{\mathrm{B}l}\right]_{\mathrm{sys}}} + \underset{(K+1)\times(K+1)}{\left[\mathrm{L}_{\mathrm{C}l}\right]_{\mathrm{sys}}} \right) \underset{(K+1)\times1}{\{u\}} = \underset{(K+1)\times1}{\left\{\mathrm{L}_{\mathrm{F}l}\right\}_{\mathrm{sys}}} \; . \tag{103}$$

2.1.2 Quadratic Lagrange basis function

Substituting equation (100) with quadratic Lagrange basis function into equation (99) yields the resulting matrix representation as

$$\left(\underset{(3\times3)}{\left[\mathrm{L}_{\mathrm{A}q}\right]}+\underset{(3\times3)}{\left[\mathrm{L}_{\mathrm{B}q}\right]}+\underset{(3\times3)}{\left[\mathrm{L}_{\mathrm{C}q}\right]}\right)\underset{(3\times1)}{\{u\}}=\underset{(3\times1)}{\left\{\mathrm{L}_{\mathrm{F}q}\right\}} \tag{104}$$

where $\left[\mathrm{L}_{\mathrm{A}q}\right]$ and $\left[\mathrm{L}_{\mathrm{B}q}\right]$ have been defined in the chapter of Galerkin finite element method for heat equation. $\left[\mathrm{L}_{\mathrm{C}q}\right]$ have been defined in the section of singularly perturbed second-order boundary value problem.

The coefficients of matrices $\left\{\mathrm{L}_{\mathrm{F}q}\right\}$ can be evaluated as

$$\left\{\mathrm{L}_{\mathrm{F}q}\right\}=\int_0^L\begin{Bmatrix}N_1\\N_2\\N_3\end{Bmatrix}\left(f\left(x,t\right)\right)\,dx\ . \tag{105}$$

The region $0\le x\le1$ has been divided into K elements of equal length $L=\dfrac{1}{K}$. After assembling all elements at time $t=t_n$, we obtain the system of linear equation as

$$\left(\underset{(2K+1)\times(2K+1)}{\left[\mathrm{L}_{\mathrm{A}q}\right]_{\mathrm{sys}}}+\underset{(2K+1)\times(2K+1)}{\left[\mathrm{L}_{\mathrm{B}q}\right]_{\mathrm{sys}}}+\underset{(2K+1)\times(2K+1)}{\left[\mathrm{L}_{\mathrm{C}q}\right]_{\mathrm{sys}}}\right)\underset{(2K+1)\times1}{\{u\}}=\underset{(2K+1)\times1}{\left\{\mathrm{L}_{\mathrm{F}q}\right\}_{\mathrm{sys}}}\ . \tag{106}$$

Next, for marching in time, we define the coefficients in matrix element of time as

$$\left[A_L\right] = [\mathrm{L}_{\mathrm{A}r}]_{\mathrm{sys}} \otimes [Z_b] \;, \tag{107}$$

$$\left[B_L\right] = \varepsilon[\mathrm{L}_{\mathrm{B}r}]_{\mathrm{sys}} \otimes [Z_a] \;, \tag{108}$$

$$\left[C_L\right] = a[\mathrm{L}_{\mathrm{C}r}]_{\mathrm{sys}} \otimes [Z_a] \;, \tag{109}$$

$$\left\{F_L\right\} = \int_{t_{n-1}}^{t_n} \left\{\mathrm{L}_{\mathrm{F}r}\right\}_{\mathrm{sys}} \otimes \left\{\theta\right\} dt \;, \tag{110}$$

$$\left[M_L^{++}\right] = [\mathrm{L}_{\mathrm{A}r}]_{\mathrm{sys}} \otimes [Z_+] \;, \tag{111}$$

$$\left[M_L^{-+}\right] = [\mathrm{L}_{\mathrm{A}r}]_{\mathrm{sys}} \otimes [Z_-] \;, \tag{112}$$

where $[\mathrm{L}_{\mathrm{A}r}]_{\mathrm{sys}}$, $[\mathrm{L}_{\mathrm{B}r}]_{\mathrm{sys}}$, $[\mathrm{L}_{\mathrm{C}r}]_{\mathrm{sys}}$ and $\left\{\mathrm{L}_{\mathrm{F}r}\right\}_{\mathrm{sys}}$ are for the linear Lagrange bases (if r is l) or the quadratic Lagrange bases (if r is q). The matrices $[Z_a]$, $[Z_b]$, $[Z_+]$, $[Z_-]$ are defined in the chapter of basis function.

In general, the system of linear equation when applying the Lagrange bases can be expressed by

$$\left\{[A_L]+[B_L]+[C_L]+\left[M_L^{++}\right]\right\}\left\{u^n\right\}=\left[M_L^{-+}\right]\left\{u^{n-1}\right\}+\left\{F_L\right\} \ , \ n=1,2,3,\dots \ . \quad (113)$$

The system (113) can be solved iteratively when the initial condition $u(x,0)=u_0(x)$ is provided when the starting coefficients are obtained from

$$\left\{u^0\right\}=\left\{u_0(x_i)\right\} \ , \quad (114)$$

where x_i are the element nodes.

2.2 The Galerkin finite element method for wavelet bases

By the weighted residual method, equation (95) can be written as

$$\int_{\Omega} W\left(\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}-\varepsilon\frac{\partial^2 u}{\partial x^2}-f\left(x,t\right)\right)\mathrm{d}\Omega=0\ , \tag{115}$$

where W is a weighting function. Using integration by part, yields

$$\int_{\Omega} W\left(\frac{\partial u}{\partial t}\right)\mathrm{d}\Omega-\varepsilon\int_{\Omega} W\left(\frac{\partial^2 u}{\partial x^2}\right)\mathrm{d}\Omega+\int_{\Omega} W\left(a\frac{\partial u}{\partial x}\right)\mathrm{d}\Omega-\int_{\Omega} W\left(f\left(x,t\right)\right)\mathrm{d}\Omega=0\,. \tag{116}$$

Let us begin by approximating the unknown function in terms of the wavelet basis as

$$u^n = u\left(x,t_n\right) = \sum_{k=0}^{p}\sum_{i=1}^{M}\sum_{j=0}^{\dim(i)} w_{ij}\left(x\right)\theta_k\left(t_n\right)c_{ijk}^n\,, \tag{117}$$

where $u^n = u\left(x,t_n\right)$ denotes the variable's value at time $t=t_n$, c_{ijk}^n are the coefficient to be approximated, $w_{ij}\left(x\right)$ is the wavelet basis function, $\theta_k(t_n)$ is the time basis function, M is the number of level in multi-level wavelet approach, p is the number of level for time discretization and $\dim\left(i\right)=\dim \mathrm{W}_i-1$.

After setting $W=\{\mathrm{w}\}\otimes\{\theta\}$ where $\{\mathrm{w}\}=\{w_{ij}\left(x\right)\}$ and $\{c\}=\{c_{ijk}\}$, for $i=1,2,3,\ldots,M$ that i is the i th level , $j=0,1,\ldots,\dim \mathrm{W}_i-1$, $k=0,1,2,\ldots,p$ that k is the k th level in time and $e=\dim \mathrm{X}_M$, equation (116) can be written in the matrix form as

$$\left\{ \underset{(ek)\times(ek)}{[A]} + \underset{(ek)\times(ek)}{[B]} + \underset{(ek)\times(ek)}{[C]} + \underset{(ek)\times(ek)}{\left[M^{++}\right]} \right\} \underset{(ek)\times 1}{\left\{c^{n}\right\}} = \underset{(ek)\times(ek)}{\left[M^{-+}\right]} \underset{(ek)\times 1}{\left\{c^{n-1}\right\}} + \underset{(ek)\times 1}{\{F\}} \ , \ n = 1,2,3,\ldots \ . \tag{118}$$

The matrices [A] , [B] , $\left[M^{++}\right]$ and $\left[M^{-+}\right]$ have been calculated in the chapter of Galerkin finite element method for heat equation. We next evaluate the coefficients in matrices [C] and $\{F\}$.

Let $\{\mathrm{w}\} = \{w_{ij}(x)\}$, for $i = 1,2,3,\ldots,M$ and $j = 0,1,\ldots,\dim \mathrm{W}_i - 1$ be the wavelet bases for each level.

$$\begin{aligned} [C] &= a\int_0^1 \int_{t_{n-1}}^{t_n} \left(\{\mathrm{w}\}\otimes\{\theta\}\right)\left(\left\{\frac{d\mathrm{w}}{dx}\right\}\otimes\{\theta\}\right)^T dtdx \\ &= a \int_0^1 \left(\{\mathrm{w}\}\left\{\frac{d\mathrm{w}}{dx}\right\}^T\right) dx \otimes \int_{t_{n-1}}^{t_n} \{\theta\}\{\theta\}^T dt \\ &= a\,[\mathrm{W}_{\mathrm{C}r}]\otimes[Z_a] \end{aligned} \tag{119}$$

$$\{F\} = \int_0^1 \int_{t_{n-1}}^{t_n} \left(\left\{\mathrm{w}\left(f(x,t)\right)\right\}\otimes\{\theta\}\right) dtdx \tag{120}$$

where $[\mathrm{W}_{\mathrm{C}r}]$ is for the linear wavelet basis (if r is l) or the quadratic wavelet basis (if r is q). The matrices $[Z_a]$ is defined in the chapter of basis function.

For given an initial condition ,

$$u(x,0) = u_0(x) \quad ,$$

the system (118) can be solved iteratively by given $\{c^0\}$ which can be obtained by the same method presented in the chapter of Galerkin finite element method for heat equation.

Thus, in the case of linear wavelet basis function, we can find the coefficients $\{c^0\}$ from the system of $2^M + 1$ equations

$$\{w_{ij}(x_s)\}^T \{c^0\} = \{u_0(x_s)\} \quad , \tag{121}$$

where $x_s = s/2^M$, $s = 0,1,...,2^M$, and $2^M + 1$ is the number of knots.

In the case of quadratic wavelet basis function, we can find the coefficients $\{c^0\}$ from the system of $2^{(M+1)} + 1$ equations

$$\{w_{ij}(x_s)\}^T \{c^0\} = \{u_0(x_s)\} \quad , \tag{122}$$

where $x_s = s/2^{(M+1)}$, $s = 0,1,...,2^{(M+1)}$, and $2^{(M+1)} + 1$ is the number of knots.

Numerical results by the Galerkin finite element method for unsteady singularly perturbed problem will be shown in the chapter of Results and Discussion.

Multilevel augmentation method

In this chapter, we study the concept of the multilevel augmentation method using wavelet bases to solve numerically the singularly perturbed second-order boundary value problem. This method integrate the choices of bases and the design of numerical solvers for the discrete linear systems together. The method is fast and accurate method for solving differential equations. It has great advantage for solving large-scale problems. We consider both the linear wavelet and the quadratic wavelet bases in this method. Numerical results are presented to demonstrate the efficiency of this method.

1. Numerical Methods

The partial differential equation for singularly perturbed second-order boundary value problem can be written as

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - u = 1 \quad , \qquad 0 < x < 1 , \tag{123}$$

where ε is the diffusion coefficient or perturbation parameter $(0 < \varepsilon << 1)$.

The domain is $\Omega \ (0 < x < 1)$ with boundary Γ, the boundary conditions are

$$u(0) = 0 \text{ and } u(1) = 0. \tag{124}$$

Recall from the chapter of Galerkin finite element method for singularly perturbed problem. We approximate the unknown function in terms of the wavelet basis as

$$u = u(x) = \sum_{i=1}^{M} \sum_{j=0}^{\dim(i)} w_{ij}(x)\, c_{ij} \ . \tag{125}$$

From equation (86),

$$\{[\mathrm{W}_{\mathrm{B}r}] - [\mathrm{W}_{\mathrm{C}r}] + [\mathrm{W}_{\mathrm{A}r}]\}\{c_{ij}\} = -\{\mathrm{W}_{\mathrm{E}r}\} \ .$$

The equation can be written in matrix form as

$$[\mathrm{P}]\{c_{ij}\} = \{\mathrm{S}\} \ . \tag{126}$$

where $[\mathrm{P}] = \{[\mathrm{W}_{\mathrm{B}r}] - [\mathrm{W}_{\mathrm{C}r}] + [\mathrm{W}_{\mathrm{A}r}]\}$, $\{\mathrm{S}\} = -\{\mathrm{W}_{\mathrm{E}r}\}$ and $\{c_{ij}\}$ denotes the variable's value for $i = 1, 2, 3, \ldots$ that i is the i th level and $j = 0, 1, \ldots, \dim \mathrm{W}_i - 1$.

The multilevel augmentation algorithm, see full details in Chen *et al.*, (2006).

Step 1:

Solve $\{c_{ij}\}$ with the multilevel method of level M ($i = 1, 2, \ldots, M$ for linear wavelet bases or $i = 0, 1, \ldots, M$ for quadratic wavelet bases and $j = 0, 1, \ldots, \dim \mathrm{W}_i - 1$) from equation (126).

Step 2:

Set $\left\{c_{ij}^{0}\right\}=\left\{c_{ij}\right\}$ for level M ($i=1,2,\ldots,M$ for linear wavelet bases or $i=0,1,\ldots,M$ for quadratic wavelet bases and $j=0,1,\ldots,\dim \mathrm{W}_i-1$).

Step 3:

Solve $\left\{c_{nj}^{1}\right\}$ for level $M+1$ ($n=1,2,\ldots,M+1$ for linear wavelet bases or $n=0,1,\ldots,M+1$ for quadratic wavelet bases and $j=0,1,\ldots,\dim \mathrm{W}_i-1$) in the following equation by the multilevel augmentation method.

$$[\mathrm{P}]_{(r\times r)}\left\{c_{nj}^{1}\right\}_{(r\times 1)}=\{\mathrm{S}\}_{(r\times 1)}\ , \tag{127}$$

- Splitting the matrices $[\mathrm{P}]_{r\times r}$, $\left\{c_{nj}^{1}\right\}_{r\times 1}$ and $\{\mathrm{S}\}_{r\times 1}$, equation (127) become

$$\begin{bmatrix}\mathrm{A} & \mathrm{B}\\ \mathrm{C} & \mathrm{D}\end{bmatrix}_{(r\times r)}\begin{Bmatrix}c_{ij}^{1}\\ c_{(M+1)j}^{1}\end{Bmatrix}_{(r\times 1)}=\begin{Bmatrix}\mathrm{S}_{ij}\\ \mathrm{S}_{(M+1)j}\end{Bmatrix}_{(r\times 1)}\ , \tag{128}$$

($i=1,2,\ldots,M$ for linear wavelet bases or $i=0,1,\ldots,M$ for quadratic wavelet bases and $j=0,1,\ldots,\dim \mathrm{W}_i-1$)

where

$\mathrm{A}=[\mathrm{A}]_{e\times e}$, $\mathrm{B}=[\mathrm{B}]_{e\times(r-e)}$,

$\mathrm{C}=[\mathrm{C}]_{(r-e)\times e}$, $\mathrm{D}=[\mathrm{D}]_{(r-e)\times(r-e)}$,

$e=\dim \mathrm{X}_M$,

$r=\dim \mathrm{X}_{M+1}$.

Equation(128) can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{r\times r} \begin{Bmatrix} c_{ij}^{1} \\ c_{(M+1)j}^{1} \end{Bmatrix}_{r\times 1} = \begin{Bmatrix} S_{ij} \\ S_{(M+1)j} \end{Bmatrix}_{r\times 1} ,$$

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}_{r\times r} \begin{Bmatrix} c_{ij}^{1} \\ c_{(M+1)j}^{1} \end{Bmatrix}_{r\times 1} + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}_{r\times r} \begin{Bmatrix} c_{ij}^{1} \\ 0 \end{Bmatrix}_{r\times 1} = \begin{Bmatrix} S_{ij} \\ S_{(M+1)j} \end{Bmatrix}_{r\times 1} ,$$

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}_{r\times r} \begin{Bmatrix} c_{ij}^{1} \\ c_{(M+1)j}^{1} \end{Bmatrix}_{r\times 1} + \begin{Bmatrix} 0 \\ C\left(c_{ij}^{1}\right) \end{Bmatrix}_{r\times 1} = \begin{Bmatrix} S_{ij} \\ S_{(M+1)j} \end{Bmatrix}_{r\times 1} ,$$

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}_{r\times r} \begin{Bmatrix} c_{ij}^{1} \\ c_{(M+1)j}^{1} \end{Bmatrix}_{r\times 1} + \begin{Bmatrix} 0 \\ C\left(c_{ij}^{1}\right) \end{Bmatrix}_{r\times 1} + \begin{Bmatrix} 0 \\ C\left(c_{ij}^{0}\right) \end{Bmatrix}_{r\times 1} - \begin{Bmatrix} 0 \\ C\left(c_{ij}^{0}\right) \end{Bmatrix}_{r\times 1} = \begin{Bmatrix} S_{ij} \\ S_{(M+1)j} \end{Bmatrix}_{r\times 1} ,$$

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}_{r\times r} \begin{Bmatrix} c_{ij}^{1} \\ c_{(M+1)j}^{1} \end{Bmatrix}_{r\times 1} + \begin{Bmatrix} 0 \\ C\left(c_{ij}^{0}\right) \end{Bmatrix}_{r\times 1} = \begin{Bmatrix} S_{ij} \\ S_{(M+1)j} \end{Bmatrix}_{r\times 1} - \begin{Bmatrix} 0 \\ C\left(c_{ij}^{1} - c_{ij}^{0}\right) \end{Bmatrix}_{r\times 1} ,$$

where $\begin{Bmatrix} 0 \\ C\left(c_{ij}^{1} - c_{ij}^{0}\right) \end{Bmatrix}_{r\times 1}$ is the error for approximating coefficients in the multilevel augmentation method, this error term converges to zero as the number of level of the multilevel augmentation method increased $(n \to \infty)$, see for the detail from Chen *et al*., (2005).

From multilevel augmentation method, the system(128) is approximated by

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{D} \end{bmatrix}_{r\times r} \begin{Bmatrix} c_{ij}^{1} \\ c_{(M+1)j}^{1} \end{Bmatrix}_{r\times 1} = \begin{Bmatrix} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{Bmatrix}_{r\times 1} - \begin{Bmatrix} 0 \\ \mathrm{C}\left(c_{ij}^{0}\right) \end{Bmatrix}_{r\times 1} . \tag{129}$$

- Next step is to find $\left\{c_{ij}^{1}\right\}_{e\times 1}$ and $\left\{c_{(M+1)j}^{1}\right\}_{(r-e)\times 1}$ by augmented equation (129)

$$\left[\mathrm{D}\right]_{(r-e)\times(r-e)} \left\{c_{(M+1)j}^{1}\right\}_{(r-e)\times 1} = \left\{\mathrm{S}_{(M+1)j}\right\}_{(r-e)\times 1} - \left[\mathrm{C}\right]_{(r-e)\times e} \left\{c_{ij}^{0}\right\}_{e\times 1} , \tag{130}$$

$$\left[\mathrm{A}\right]_{e\times e} \left\{c_{ij}^{1}\right\}_{e\times 1} = \left\{\mathrm{S}_{ij}\right\}_{e\times 1} - \left[\mathrm{B}\right]_{e\times(r-e)} \left\{c_{(M+1)j}^{1}\right\}_{(r-e)\times 1} . \tag{131}$$

We solve system (130) directly to find $\left\{c_{(M+1)j}^{1}\right\}_{(r-e)\times 1}$,

and then putting these values into the RHS of (131) to find $\left\{c_{ij}^{1}\right\}_{e\times 1}$.

This completes the overall steps in the multilevel augmentation method for level $M+1$.

For example, consider the multilevel augmentation method of level 3 with linear wavelet basis.

Step 1:

Solve $\{c_{ij}\}$ in the multilevel method of level 2 ($i=1,2$ for linear wavelet bases and $j=0,1,\ldots,\dim \mathrm{W}_i -1$) from equation

$$[\mathrm{P}]_{(3\times3)}\{c_{ij}\}_{(3\times1)}=\{\mathrm{S}\}_{(3\times1)}\ ,$$

$$\begin{bmatrix} \varepsilon+\frac{1}{12} & \frac{\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{64} \\ \frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \varepsilon+\frac{1}{48} & 0 \\ \frac{\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & 0 & \varepsilon+\frac{1}{48} \end{bmatrix} \begin{Bmatrix} c_{10} \\ c_{20} \\ c_{21} \end{Bmatrix} = -\begin{Bmatrix} \frac{1}{4} \\ \frac{1}{8\sqrt{2}} \\ \frac{1}{8\sqrt{2}} \end{Bmatrix}.$$

Step 2:

Set $\{c_{ij}^0\}=\{c_{ij}\}$ for level 2,

$$\begin{Bmatrix} c_{10}^0 \\ c_{20}^0 \\ c_{21}^0 \end{Bmatrix} = \begin{Bmatrix} c_{10} \\ c_{20} \\ c_{21} \end{Bmatrix}.$$

Step 3:

Solve $\left\{c_{nj}^{1}\right\}$ for level 3 ($n = 1,2,3$ for linear wavelet bases and $j = 0,1,\ldots,\dim \mathrm{W}_i - 1$) from equation (127) by the multilevel augmentation method.

$$[\mathrm{P}]_{(7\times7)}\left\{c_{nj}^{1}\right\}_{(7\times1)} = \{\mathrm{S}\}_{(7\times1)} \ ,$$

$$\left[\begin{array}{ccc|cccc}
\varepsilon+\frac{1}{12} & \frac{\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \frac{-1}{32}+\frac{1}{256} & \frac{-1}{32}+\frac{3}{256} & \frac{1}{32}+\frac{3}{256} & \frac{1}{32}+\frac{1}{256} \\
\frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \varepsilon+\frac{1}{48} & 0 & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 & 0 \\
\frac{\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & 0 & \varepsilon+\frac{1}{48} & 0 & 0 & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} \\
\hline
\frac{-1}{32}+\frac{1}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 & \varepsilon+\frac{1}{192} & 0 & 0 & 0 \\
\frac{-1}{32}+\frac{3}{256} & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 & 0 & \varepsilon+\frac{1}{192} & 0 & 0 \\
\frac{1}{32}+\frac{3}{256} & 0 & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 & 0 & \varepsilon+\frac{1}{192} & 0 \\
\frac{1}{32}+\frac{1}{256} & 0 & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 & 0 & 0 & \varepsilon+\frac{1}{192}
\end{array}\right]
\left\{\begin{array}{c} c_{10}^{1} \\ c_{20}^{1} \\ c_{21}^{1} \\ \hline c_{30}^{1} \\ c_{31}^{1} \\ c_{32}^{1} \\ c_{33}^{1} \end{array}\right\}
=
\left\{\begin{array}{c} -\frac{1}{4} \\ -\frac{1}{8\sqrt{2}} \\ -\frac{1}{8\sqrt{2}} \\ \hline -\frac{1}{32} \\ -\frac{1}{32} \\ -\frac{1}{32} \\ -\frac{1}{32} \end{array}\right\} .$$

- Splitting the matrices $[\mathrm{A}]$, $[\mathrm{B}]$, $[\mathrm{C}]$ and $[\mathrm{D}]$ from $[\mathrm{P}]$,

$$[\mathrm{P}]_{(7\times7)}\left\{c_{nj}^{1}\right\}_{(7\times1)} = \{\mathrm{S}\}_{(7\times1)} \ ,$$

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix}_{(7\times7)} \left\{c_{nj}^{1}\right\}_{(7\times1)} = \{\mathrm{S}\}_{(7\times1)} \ ,$$

where

$$A=\begin{bmatrix} \varepsilon+\frac{1}{12} & \frac{\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{64} \\ \frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \varepsilon+\frac{1}{48} & 0 \\ \frac{\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & 0 & \varepsilon+\frac{1}{48} \end{bmatrix}_{(3\times 3)},$$

$$B=\begin{bmatrix} \frac{1}{32}+\frac{1}{256} & \frac{1}{32}+\frac{3}{256} & \frac{-1}{32}+\frac{3}{256} & \frac{-1}{32}+\frac{1}{256} \\ \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} \end{bmatrix}_{(3\times 4)},$$

$$C=\begin{bmatrix} \frac{-1}{32}+\frac{1}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 \\ \frac{-1}{32}+\frac{1}{256} & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 \\ \frac{1}{32}+\frac{3}{256} & 0 & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} \\ \frac{1}{32}+\frac{1}{256} & 0 & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} \end{bmatrix}_{(4\times 3)},$$

$$D=\begin{bmatrix} \varepsilon+\frac{1}{192} & 0 & 0 & 0 \\ 0 & \varepsilon+\frac{1}{192} & 0 & 0 \\ 0 & 0 & \varepsilon+\frac{1}{192} & 0 \\ 0 & 0 & 0 & \varepsilon+\frac{1}{192} \end{bmatrix}_{(4\times 4)},$$

- Splitting matrices $\left\{c_{nj}^{1}\right\}_{(7\times1)}$ and $\left\{\mathrm{S}\right\}_{(7\times1)}$,

by setting

$$\left\{c_{nj}^{1}\right\}_{(7\times1)} = \begin{Bmatrix} c_{ij}^{1} \\ c_{(M+1)j}^{1} \end{Bmatrix}_{(7\times1)} = \begin{Bmatrix} c_{10}^{1} \\ c_{20}^{1} \\ c_{21}^{1} \\ \\ c_{30}^{1} \\ c_{31}^{1} \\ c_{32}^{1} \\ c_{33}^{1} \end{Bmatrix}_{(7\times1)},$$

and setting

$$\left\{\mathrm{S}\right\}_{(7\times1)} = \begin{Bmatrix} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{Bmatrix}_{(7\times1)} = \begin{Bmatrix} -\frac{1}{4} \\ -\frac{1}{8\sqrt{2}} \\ -\frac{1}{8\sqrt{2}} \\ \\ -\frac{1}{32} \\ -\frac{1}{32} \\ -\frac{1}{32} \\ -\frac{1}{32} \end{Bmatrix}_{(7\times1)}.$$

- Solve $\left\{c_{nj}^{1}\right\}_{7\times1}$ for level 3 from equation

$$\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{D} \end{bmatrix}_{(7\times7)} \begin{Bmatrix} c_{ij}^{1} \\ c_{(M+1)j}^{1} \end{Bmatrix}_{(7\times1)} = \begin{Bmatrix} \mathrm{S}_{ij} \\ \mathrm{S}_{(M+1)j} \end{Bmatrix}_{(7\times1)} - \begin{Bmatrix} 0 \\ \mathrm{C}\left(c_{ij}^{0}\right) \end{Bmatrix}_{(7\times1)} ,$$

Find $\left\{c_{(M+1)j}^{1}\right\}_{4\times1}$ of level 3 from equation

$$\left[\mathrm{D}\right]_{(4\times4)} \left\{c_{(M+1)j}^{1}\right\}_{(4\times1)} = \left\{\mathrm{S}_{(M+1)j}\right\}_{(4\times1)} - \left[\mathrm{C}\right]_{(4\times3)} \left\{c_{ij}^{0}\right\}_{(3\times1)} ,$$

$$\begin{bmatrix} \varepsilon+\frac{1}{192} & 0 & 0 & 0 \\ 0 & \varepsilon+\frac{1}{192} & 0 & 0 \\ 0 & 0 & \varepsilon+\frac{1}{192} & 0 \\ 0 & 0 & 0 & \varepsilon+\frac{1}{192} \end{bmatrix}_{(4\times4)} \begin{Bmatrix} c_{30}^{1} \\ c_{31}^{1} \\ c_{32}^{1} \\ c_{33}^{1} \end{Bmatrix}_{(4\times1)} = \begin{Bmatrix} -\frac{1}{32} \\ -\frac{1}{32} \\ -\frac{1}{32} \\ -\frac{1}{32} \end{Bmatrix}_{(4\times1)} - \begin{bmatrix} \frac{-1}{32}+\frac{1}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 \\ \frac{-1}{32}+\frac{1}{256} & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 \\ \frac{1}{32}+\frac{3}{256} & 0 & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} \\ \frac{1}{32}+\frac{1}{256} & 0 & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} \end{bmatrix}_{(4\times3)} \begin{Bmatrix} c_{10}^{0} \\ c_{20}^{0} \\ c_{21}^{0} \end{Bmatrix}_{(3\times1)}$$

This system can be solve easily because for linear wavelet bases the coefficients matrix is diagonal and for quadratic wavelet bases the coefficients matrix is tridiagonal.

Find $\left\{c_{ij}^{1}\right\}_{3\times1}$ of level 3 from equation

$$[\mathrm{A}]_{(3\times3)}\left\{c_{ij}^{1}\right\}_{(3\times1)}=\left\{\mathrm{S}_{ij}\right\}_{(3\times1)}-[\mathrm{B}]_{(3\times4)}\left\{c_{(M+1)j}^{1}\right\}_{(4\times1)}\ ,$$

$$\begin{bmatrix} \varepsilon+\frac{1}{12} & \frac{\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{64} \\ \frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & \varepsilon+\frac{1}{48} & 0 \\ \frac{\sqrt{2}}{16}+\frac{\sqrt{2}}{64} & 0 & \varepsilon+\frac{1}{48} \end{bmatrix}_{(3\times3)} \begin{Bmatrix} c_{10}^{1} \\ c_{20}^{1} \\ c_{21}^{1} \end{Bmatrix}_{(3\times1)} = \begin{Bmatrix} -\frac{1}{4} \\ -\frac{1}{8\sqrt{2}} \\ -\frac{1}{8\sqrt{2}} \end{Bmatrix}_{(3\times1)} - \begin{bmatrix} \frac{1}{32}+\frac{1}{256} & \frac{1}{32}+\frac{3}{256} & \frac{-1}{32}+\frac{3}{256} & \frac{-1}{32}+\frac{1}{256} \\ \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{256} & \frac{-\sqrt{2}}{32}+\frac{\sqrt{2}}{256} \end{bmatrix}_{(3\times4)} \begin{Bmatrix} c_{30}^{1} \\ c_{31}^{1} \\ c_{32}^{1} \\ c_{33}^{1} \end{Bmatrix}_{(4\times1)}$$

So, we can find all of coefficients in the solution expansion of level 3, and next we use these coefficients to find the coefficients in level 4, 5 and so on.

Numerical Example by the multilevel augmentation method for singularly perturbed second-order boundary value problem will be shown in the chapter of Results and Discussion.

RESULTS AND DISCUSSION

Numerical results by the Galerkin finite element method for heat equation

The time-dependent heat equation in terms of temperature $T(x,t)$ is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad , \qquad (0 < x < 1) \quad ,$$

where $\partial T / \partial t$ is the rate of change of temperature. We set the thermal diffusivity as $\alpha = 1$. The boundary and initial conditions are given by

$$T(0,t) = T(1,t) = 0 , \quad \text{for} \quad t \geq 0 , \quad \text{and}$$

$$T(x,0) = \sin(\pi x) , \quad \text{for} \quad 0 < x < 1 .$$

The exact solution for this problem is

$$T(x,t) = e^{-\pi^2 t} \sin(\pi x) .$$

The time basis is applied by setting $\theta = \begin{bmatrix} 1 & \dfrac{t - t_{n-1}}{\Delta t} \end{bmatrix}^T$ (level two),

or $\theta = [1]$ (level one).

We fix time step by $\Delta t = 0.005$ for all calculations.

To check the accuracy of the present numerical schemes, we use the RMS error defined by

$$RMS = \sqrt{\frac{\sum_{i=1}^{N} (T_i - T_{Exact})^2}{N}} \quad . \qquad (132)$$

The profiles of numerical solutions at various time steps (0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05) are shown in Figure 7. The temperature profile decreases dramatically as time increases. In this figure, the numerical solutions are obtained by the finite element method based on the linear wavelet basis with level 3 $\left(W_3\right)$. The numerical results are in good agreement with the exact solutions even if we have used small number of elements.

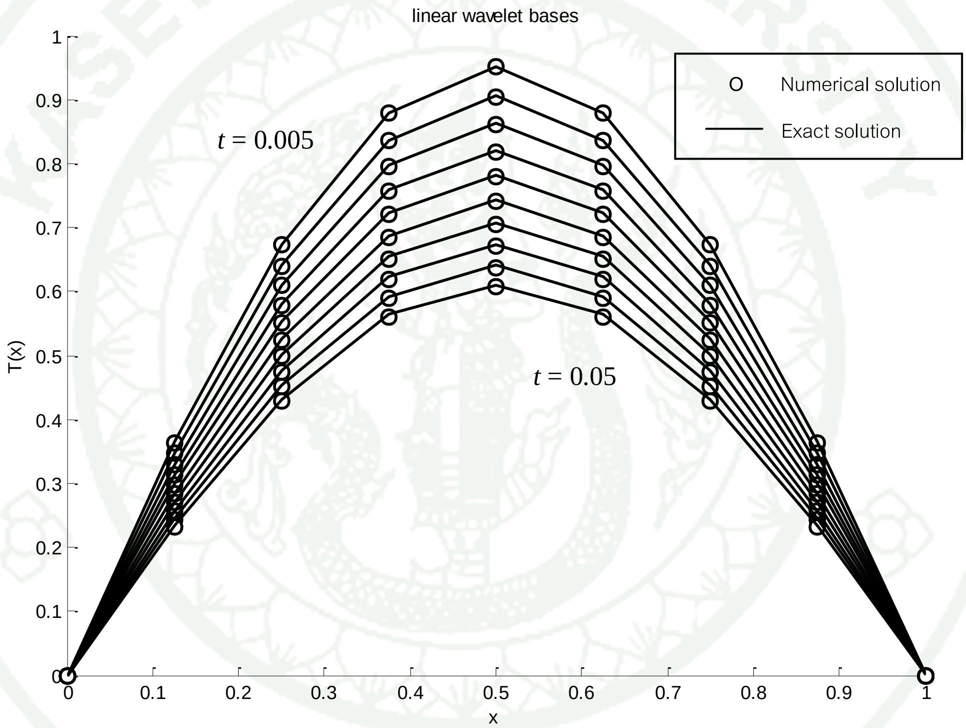

Figure 7 The finite element with linear wavelet basis results and exact solutions.

To investigate the convergent rate of numerical schemes for various types of basis functions, if we have observed in two cases of the element size, Δx and $\Delta x/2$, the rate of convergence (r) of the numerical method is defined by

$$r = \frac{\log\left(RMS_{\Delta x} / RMS_{\Delta x/2}\right)}{\log(2)} . \tag{133}$$

The numerical solutions at various time steps are shown in Tables 1- 4. Comparing Table 1 with Table 2, the RMS errors by the linear Lagrange basis are almost the same as the RMS errors by the linear wavelet basis. This implies that the accuracy is the same for these two types of basis function when using the same element size. The rate of convergence is approximately 2.1 as expected for the linear basis. The RMS errors and the rate of convergences for the quadratic Lagrange and wavelet bases are shown in Tables 3 and 4 respectively. The rates of convergence for the quadratic Lagrange and the quadratic wavelet are approximately 4.1 and 4.4 respectively. Plots of convergence rate are shown in Figure 8.

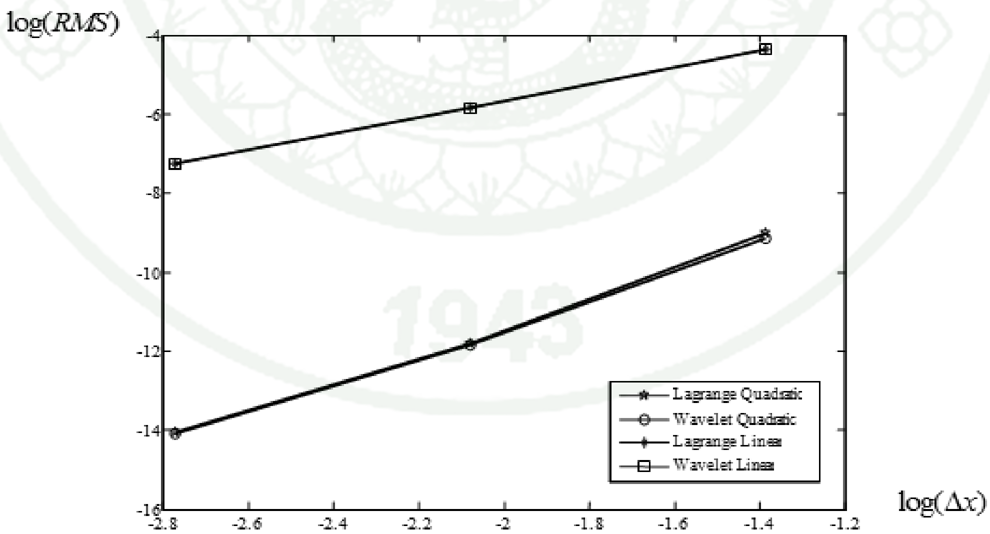

Figure 8 Rate of convergence for Lagrange and wavelet bases.

Table 1 RMS error linear Lagrange basis function for heat equation

Time \ Elements	0.01	0.02	0.03	0.04	0.05
8 elements	8.72626e-04	1.58022e-03	2.14619e-03	2.59100e-03	2.93250e-03
16 elements	2.10148e-04	3.80735e-04	5.17347e-04	6.24869e-04	7.07566e-04
32 elements	5.17174e-05	9.37099e-05	1.27348e-04	1.53833e-04	1.74212e-04
64 elements	1.29558e-05	2.34828e-05	3.19232e-05	3.85764e-05	4.37040e-05

($\Delta t = 0.005$, TOL = 10^(-8))

Approximate convergence rate = 2.1

Table 2 RMS error linear wavelet basis function for heat equation

Time \ Level	0.01	0.02	0.03	0.04	0.05
W_3	8.72628e-04	1.58022e-03	2.14620e-03	2.59101e-03	2.93251e-03
W_4	2.10147e-04	3.80734e-04	5.17346e-04	6.24867e-04	7.07563e-04
W_5	5.17171e-05	9.37094e-05	1.27348e-04	1.53833e-04	1.74212e-04
W_6	1.29009e-05	2.33765e-05	3.17686e-05	3.83766e-05	4.34614e-05

($\Delta t = 0.005$, TOL = 10^(-8))

Approximate convergence rate = 2.1

Table 3 RMS error quadratic Lagrange basis function for heat equation

Time \ Elements	0.01	0.02	0.03	0.04	0.05
2 elements	6.49752e-04	1.18419e-03	1.61835e-03	1.96275e-03	2.22884e-03
4 elements	6.16658e-05	7.39192e-05	9.10600e-05	1.07179e-04	1.20620e-04
8 elements	4.13478e-06	4.73808e-06	5.70161e-06	6.65048e-06	7.45794e-06
16 elements	2.98226e-07	4.29003e-07	5.66735e-07	6.84214e-07	7.78069e-07

($\Delta t = 0.005$, TOL = 10^(-12))

Approximate convergence rate = 4.1

Table 4 RMS error quadratic wavelet basis function for heat equation

Time \ Level	0.01	0.02	0.03	0.04	0.05
W_1	5.34864e-04	9.70882e-04	1.32503e-03	1.60590e-03	1.82285e-03
W_2	5.47707e-05	6.23686e-05	7.73105e-05	9.23327e-05	1.05184e-04
W_3	3.99457e-06	4.57742e-06	5.50828e-06	6.42497e-06	7.20506e-06
W_4	2.93376e-07	4.22027e-07	5.57519e-07	6.73088e-07	7.65417e-07

($\Delta t = 0.005$, TOL = 10^(-12))

Approximate convergence rate = 4.4

We have also investigated the rate of convergence of time basis. Two cases of time basis levels are considered which are θ_1 and θ_2. We set the final time as 0.8, and $\alpha = 0.5$. Numerical results are shown in Tables 5-7. The RMS errors using time basis θ_1 for the linear Lagrange and wavelet bases are almost the same as those errors obtained by the quadratic Lagrange and wavelet bases. The rates of convergence are approximately 1.1. Similarly, we have found that the rates of convergence are approximately in order 2 and 3 when using θ_2. This shows an advantage of this type of time basis such that the order of accuracy in time can be improved by just increasing the time basis level. That is unlike the other standard schemes such as the Euler method or the Runge-Kutta method that the order of accuracy is fixed. Applying the presented time basis is more flexible than the standard approach.

Table 5 RMS error by linear wavelet and Lagrange bases for heat equation

discretization in time in level (θ)	Time \ Level	$\Delta t = 0.4$	$\Delta t = 0.2$	$\Delta t = 0.1$	Approximate convergence rate (r)
		0.8	0.8	0.8	
level 1 (θ_1)	W_4	6.81299e-02	3.24632e-02	1.51638e-02	1.1
	16 element	6.81299e-02	3.24632e-02	1.51638e-02	1.1
level 2 (θ_2)	W_4	4.55056e-03	7.85250e-04	2.60605e-04	2.1
	16 element	4.55056e-03	7.85250e-04	2.60605e-04	2.1

($\alpha = 0.5$, TOL = 10^(-12))

Table 6 RMS error by linear wavelet and Lagrange bases for heat equation

discretization in time in level (θ)	Time \ Level	$\Delta t = 0.4$	$\Delta t = 0.2$	$\Delta t = 0.1$	Approximate convergence rate (r)
		0.8	0.8	0.8	
level 1 (θ_1)	W_5	6.72812e-02	3.21552e-02	1.51012e-02	1.1
	32 elements	6.72812e-02	3.21552e-02	1.51012e-02	1.1
level 2 (θ_2)	W_5	4.34828e-03	6.41622e-04	1.25339e-04	2.6
	32 elements	4.34828e-03	6.41622e-04	1.25339e-04	2.6

($\alpha = 0.5$, TOL = 10^(-12))

Table 7 RMS error by quadratic wavelet and Lagrange bases for heat equation

discretization in time in level (θ)	Time \ Level	$\Delta t = 0.4$	$\Delta t = 0.2$	$\Delta t = 0.1$	Approximate convergence rate (r)
		0.8	0.8	0.8	
level 1 (θ_1)	W_4	6.73676e-02	3.22285e-02	1.51627e-02	1.1
	16 elements	6.73676e-02	3.22285e-02	1.51627e-02	1.1
level 2 (θ_2)	W_4	4.30537e-03	5.97888e-04	8.15524e-05	2.9
	16 elements	4.37654e-03	6.07771e-04	8.29005e-05	2.9

($\alpha = 0.5$, TOL = 10^(-12))

Numerical results by the Galerkin finite element method for singularly perturbed problem

1. Singularly perturbed second-order boundary value problem

Recalling the singularly perturbed second-order boundary value problem is represented by

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - u = 1 \quad , \quad 0 < x < 1 .$$

We consider three cases of ε which are 0.1 , 0.01 and 0.001, subject to the boundary conditions,

$$u(0) = 0 \text{ and } u(1) = 0 .$$

The exact solution for this problem is

$$u(x) = \frac{\left(e^{m_2} - 1\right)e^{m_1 x} + \left(1 - e^{m_1}\right)e^{m_2 x}}{e^{m_2} - e^{m_1}} - 1 \ ,$$

where

$$m_1 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon} \ ,$$

$$m_2 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon} \ .$$

To check the accuracy of numerical schemes, we use the RMS error defined by

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{N} \left(u_i - u_{Exact}\right)^2}{N}} \ , \tag{134}$$

and the L^2 norms defined by

$$\|u_i - u_{Exact}\|_{L^2} = \left(\sum_{i=1}^{N}\left(u_i - u_{Exact}\right)^2\right)^{\frac{1}{2}} . \qquad (135)$$

where $u_i = u(x_i)$, x_i are the element nodes or knots.

The profiles of numerical solutions are shown in Figures 9, 10 and 11. In these Figures, the numerical solutions are obtained by the finite element method based on linear wavelet basis of level 9 (W_9). The numerical results are in good agreement with the exact solutions, except the case of very small in ε the very high gradient area.

The full system matrices can be solved iteratively by the Gauss-Seidel method and the build-in inverse function in Matlab. If we use the inverse function in Matlab, the solution does not converge when $\varepsilon < 10^{-4}$, The numerical solutions diverge rapidly if we set $\varepsilon \le 10^{-2}$ in the Gauss-Seidel method.

The accuracy of numerical solutions are shown in Tables 8-11. The RMS errors and L^2 norm error by the linear Lagrange basis are almost the same as those errors using the linear wavelet basis. This implies that the accuracy is approximately in the same order for these two types of bases when we use the same element size. The RMS errors and L^2 norms decreases when we increase the number of elements.

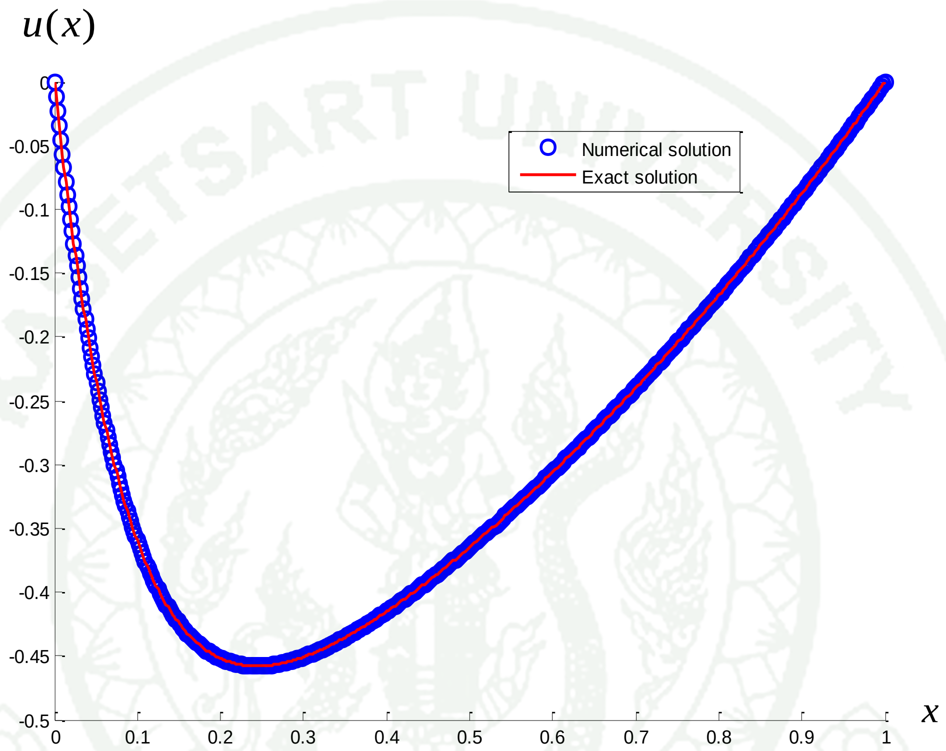

Figure 9 The Galerkin finite element results by linear wavelet basis of level 9 (W_9), and exact solution for $\varepsilon = 0.1$.

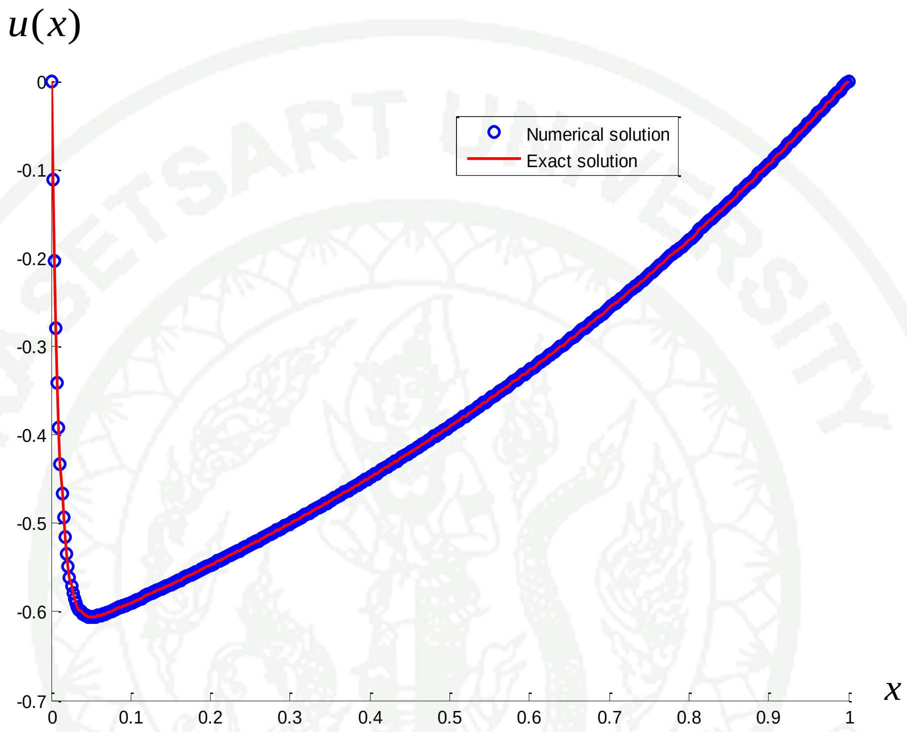

Figure 10 The Galerkin finite element results by linear wavelet basis of level 9 (W_9), and exact solution for $\varepsilon = 0.01$.

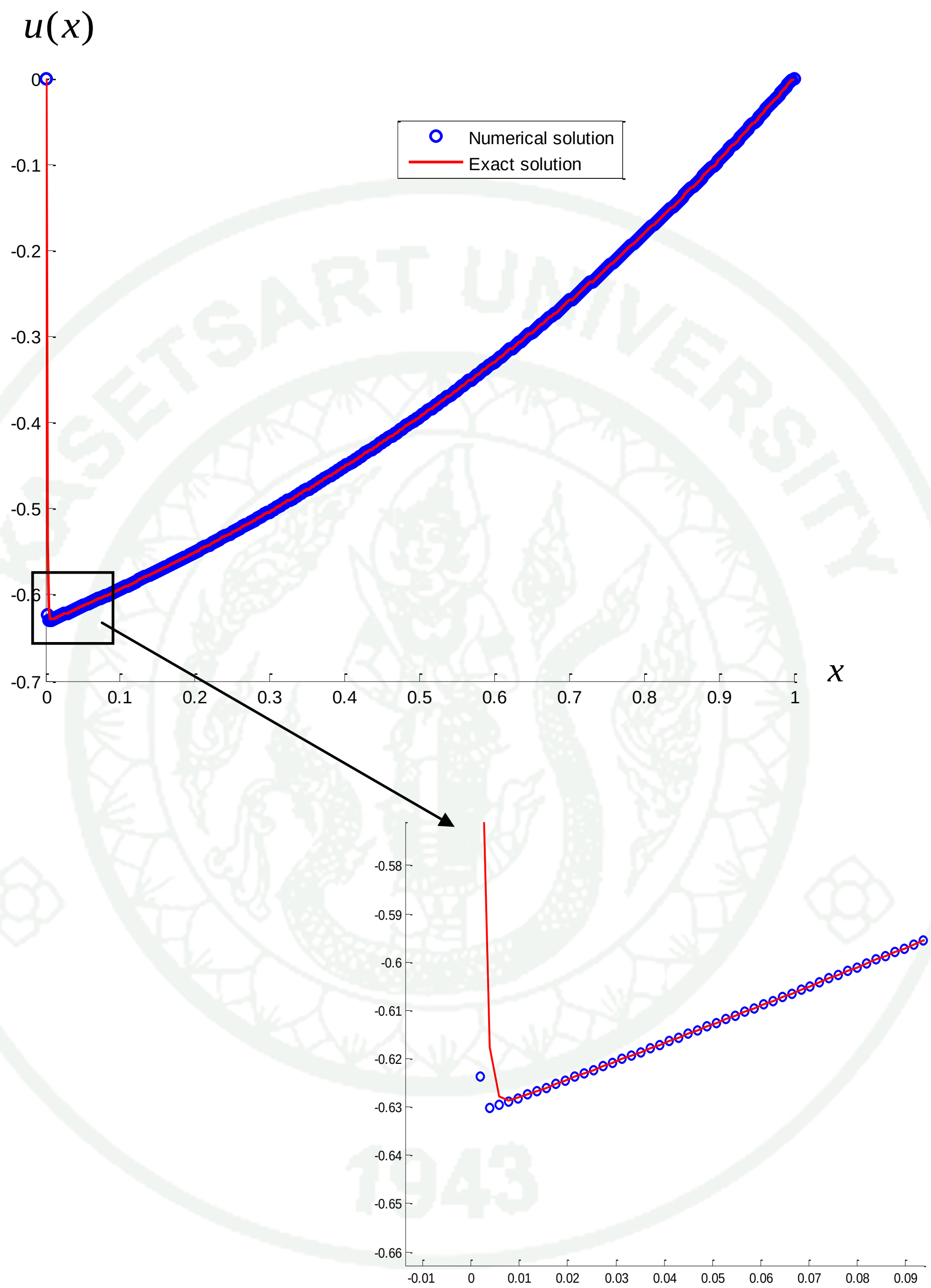

Figure 11 The Galerkin finite element results by linear wavelet basis of level 9 (W_9), and exact solution for $\varepsilon = 0.001$.

Table 8 RMS error and L^2 norms of Galerkin finite element method by linear wavelet basis function for singularly perturbed second-order boundary value problem.

Level	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	RMS error	L^2 norms	RMS error	L^2 norms	RMS error	L^2 norms
W_8	1.2727e-5	2.0323e-4	4.0764e-4	6.5095e-3	1.4233e-2	2.2728e-1
W_9	3.1783e-6	7.1846e-5	1.0083e-4	2.2794e-3	3.6773e-3	8.3127e-2
W_{10}	7.9416e-7	2.5400e-5	2.5133e-5	8.0387e-4	8.4807e-4	2.7125e-2
W_{11}	1.9849e-7	8.9805e-6	6.2778e-6	2.8403e-4	2.0252e-4	9.1632e-3
r	2.0	1.5	2.0	1.5	2.0	1.6

r is Approximate convergence rate

Table 9 RMS error and L^2 norms of Galerkin finite element method by linear Lagrange basis function singularly perturbed second-order boundary value problem.

Elements	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	RMS error	L^2 norms	RMS error	L^2 norms	RMS error	L^2 norms
256	1.2727e-5	2.0323e-4	4.0764e-4	6.5095e-3	1.4233e-2	2.2728e-1
512	3.1783e-6	7.1846e-5	1.0083e-4	2.2794e-3	3.6773e-3	8.3127e-2
1024	7.9416e-7	2.5400e-5	2.5133e-5	8.0387e-4	8.4807e-4	2.7125e-2
2048	1.9849e-7	8.9805e-6	6.2778e-6	2.8403e-4	2.0252e-4	9.1632e-3
r	2.0	1.5	2.0	1.5	2.0	1.6

r is Approximate convergence rate

Table 10 RMS error and L^2 norms of Galerkin finite element method by quadratic wavelet basis function for singularly perturbed second-order boundary value problem.

Level	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	RMS error	L^2 norms	RMS error	L^2 norms	RMS error	L^2 norms
W_4	4.0373e-5	2.2478e-4	2.2719e-2	1.2649e-1	1.8278e-1	1.0177e+0
W_5	2.5804e-6	2.0481e-5	3.6943e-3	2.9322e-2	8.5269e-2	6.7680e-1
W_6	1.6188e-7	1.8243e-6	3.5749e-4	4.0287e-3	3.5398e-2	3.9891e-1
W_7	1.0117e-8	1.6156e-7	2.5779e-5	4.1165e-4	1.1289e-2	1.8027e-1
r	4.0	3.5	3.3	2.8	1.3	1.0

r is Approximate convergence rate

Table 11 RMS error and L^2 norms of Galerkin finite element method by quadratic Lagrange basis function singularly perturbed second-order boundary value problem.

Elements	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	RMS error	L^2 norms	RMS error	L^2 norms	RMS error	L^2 norms
16	4.1040e-5	2.2478e-4	2.3095e-2	1.2649e-1	1.8580e-1	1.0177e+0
32	2.6011e-6	2.0481e-5	3.7239e-3	2.9322e-2	8.5954e-2	6.7680e-1
64	1.6252e-7	1.8243e-6	3.5891e-4	4.0287e-3	3.5398e-2	3.9891e-1
128	1.0137e-8	1.6156e-7	2.5829e-5	4.1165e-4	1.1311e-2	1.8027e-1
r	4.0	3.5	3.3	2.8	1.3	1.0

r is Approximate convergence rate

Figure 12 shows numerical results for 3 values of ε. Solution develops very high slope near $x = 0$ as ε decreases. The rate of convergence corresponds to the theoretical results, see for the detail from Chen *et al*., (2006), only for small ε (smooth case).

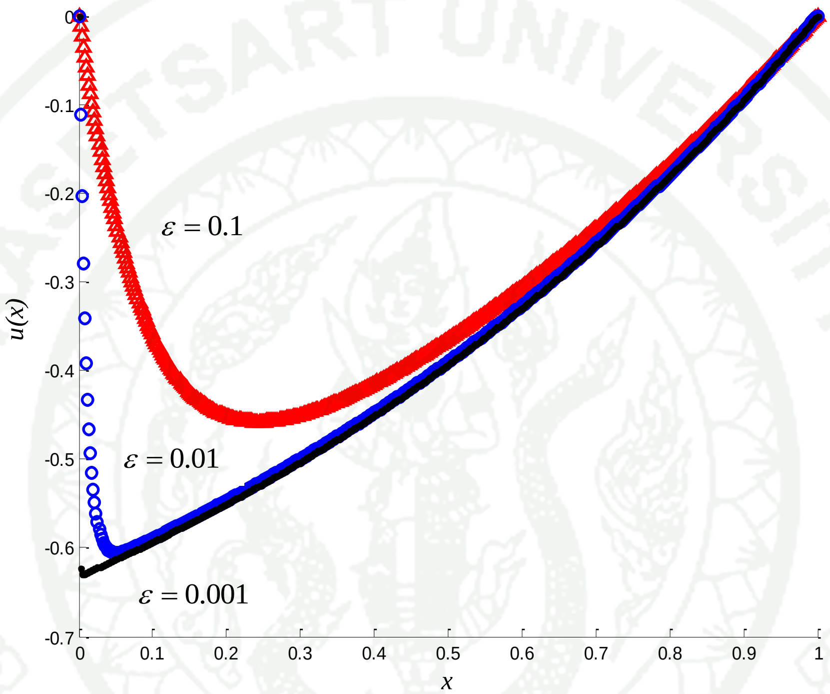

Figure 12 Graphs of the numerical solutions by linear wavelet basis of level 9 (W_9), for $\varepsilon = 0.1,\ 0.01$ and 0.001.

The rates of convergence for both the linear Lagrange and wavelet bases are in the same order as expected. The rate of convergence is 2.0 in RMS errors. The rates of convergence with L^2 norms for linear bases of both the Lagrange and wavelet are in the same order as expected of 1.5. The rates of convergence with RMS errors for quadratic bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 4.0 for $\varepsilon = 10^{-1}$, 3.3 for $\varepsilon = 10^{-2}$ and 1.3 for $\varepsilon = 10^{-3}$. The rates of convergence with L^2 norms for quadratic bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 3.5 for $\varepsilon = 10^{-1}$, 2.8 for $\varepsilon = 10^{-2}$ and 1.0 for $\varepsilon = 10^{-3}$. The plots of rate of convergence are shown in Figure 13.

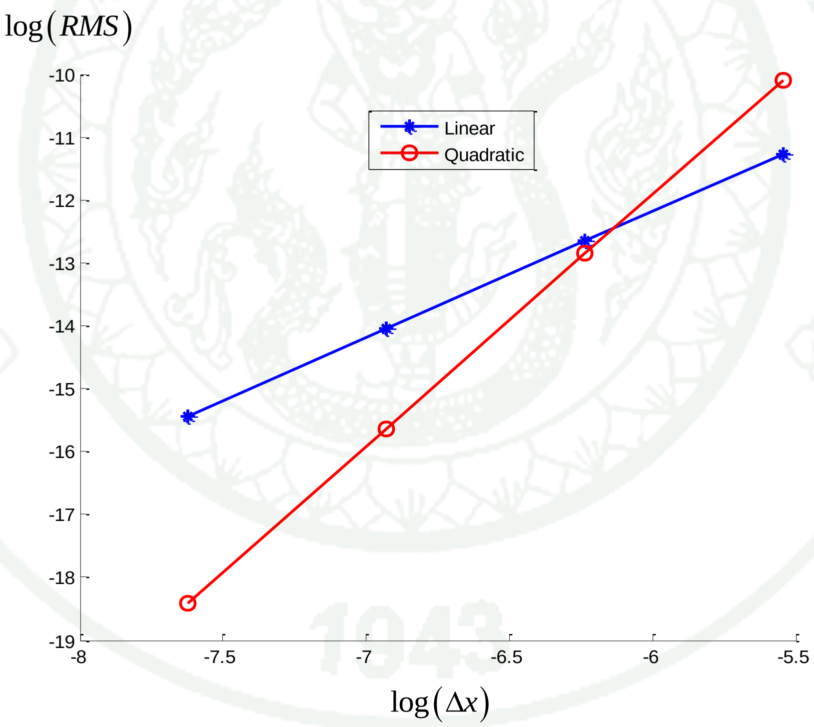

Figure 13 Rate of convergence with RMS errors of linear and quadratic bases for $\varepsilon = 0.1$.

2. Unsteady singularly perturbed problem

The singularly perturbed, one-dimensional (linear) parabolic problem of the advection-diffusion-reaction type can be written as

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2} + f\left(x,t\right) \ , \quad 0 < x < 1 \ , \quad t > 0 \ ,$$

We consider three case of ε which are 0.1 , 0.01 and 0.001 and $a = 1$,
and

$$f\left(x,t\right) = \exp\left(-t\right)\left[-2\exp\left(-\frac{1}{\varepsilon}\right) - \left(1-\exp\left(-\frac{1}{\varepsilon}\right)\right)x + \exp\left(-\frac{1-x}{\varepsilon}\right) + 1\right] \ ,$$

subject to the boundary conditions as

$$u\left(0,t\right) = u\left(1,t\right) = 0 \ , \quad t > 0 \ ,$$

and the initial condition as

$$u\left(x,0\right) = u_0\left(x\right)$$
$$= \exp\left(-\frac{1}{\varepsilon}\right) + \left(1-\exp\left(-\frac{1}{\varepsilon}\right)\right)x - \exp\left(-\frac{1-x}{\varepsilon}\right) \ .$$

The exact solution for this problem is

$$u\left(x,t\right) = \exp\left(-t\right)\left(C_1 + C_2 x - \exp\left(-\frac{1-x}{\varepsilon}\right)\right) \ ,$$

where $C_1 = \exp\left(-\frac{1}{\varepsilon}\right)$, $C_2 = 1 - C_1$.

The time basis is applied by setting $\theta = \begin{bmatrix} 1 & \frac{t - t_{n-1}}{\Delta t} \end{bmatrix}^T$ (level two),

or $\theta = [1]$ (level one).

We fix time step by $\Delta t = 0.005$ for all calculations.

The profiles of numerical solutions at various time steps (0.1, 0.2, 0.3, 0.4, 0.5) are shown in Figures 14-16. The temperature profile decreases dramatically as time increases. In these Figures, the numerical solutions are obtained by the finite element method based on quadratic wavelet basis of level 7 (W_7). The numerical results are in good agreement with the exact solutions even though we have used a small number of elements.

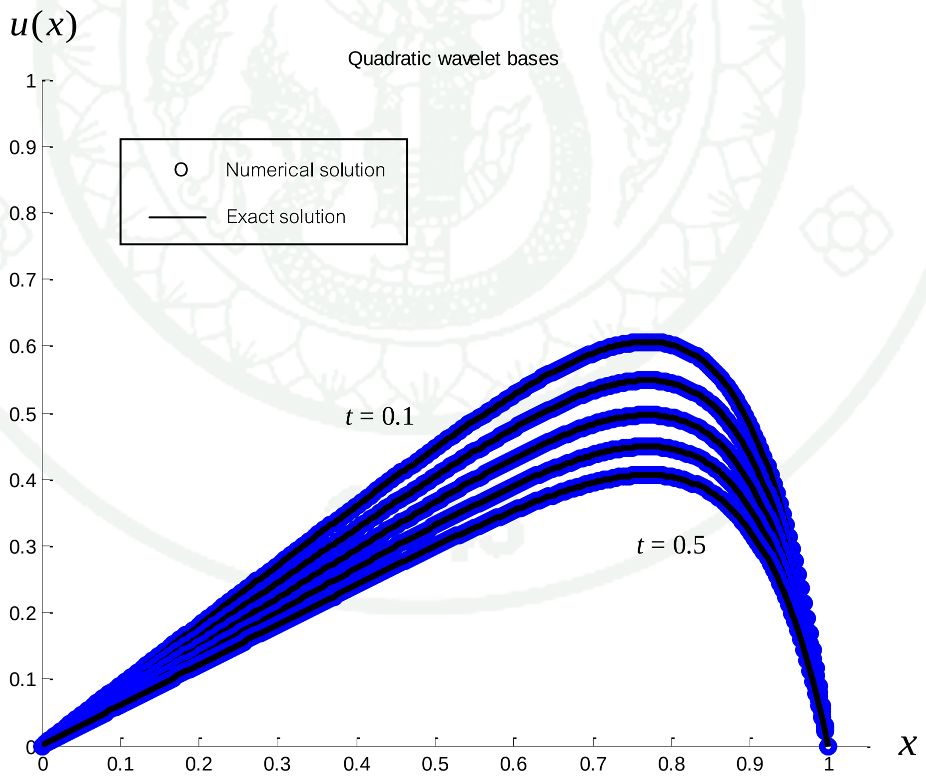

Figure 14 The Galerkin finite element results by quadratic wavelet basis of level 7 (W_7), and exact solution for $\varepsilon = 0.1$.

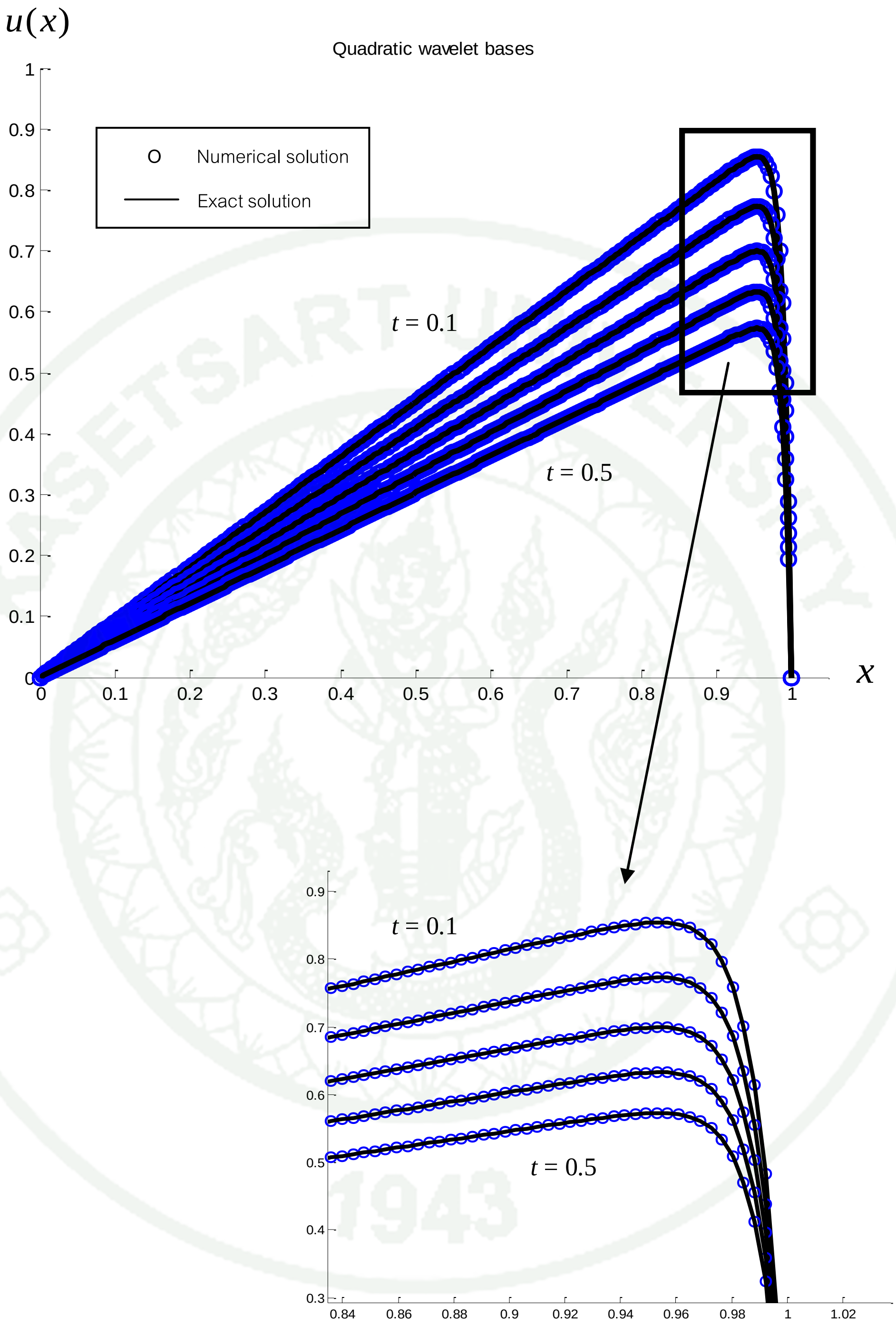

Figure 15 The Galerkin finite element results by quadratic wavelet basis of level 7 (W_7), and exact solution for $\varepsilon = 0.01$.

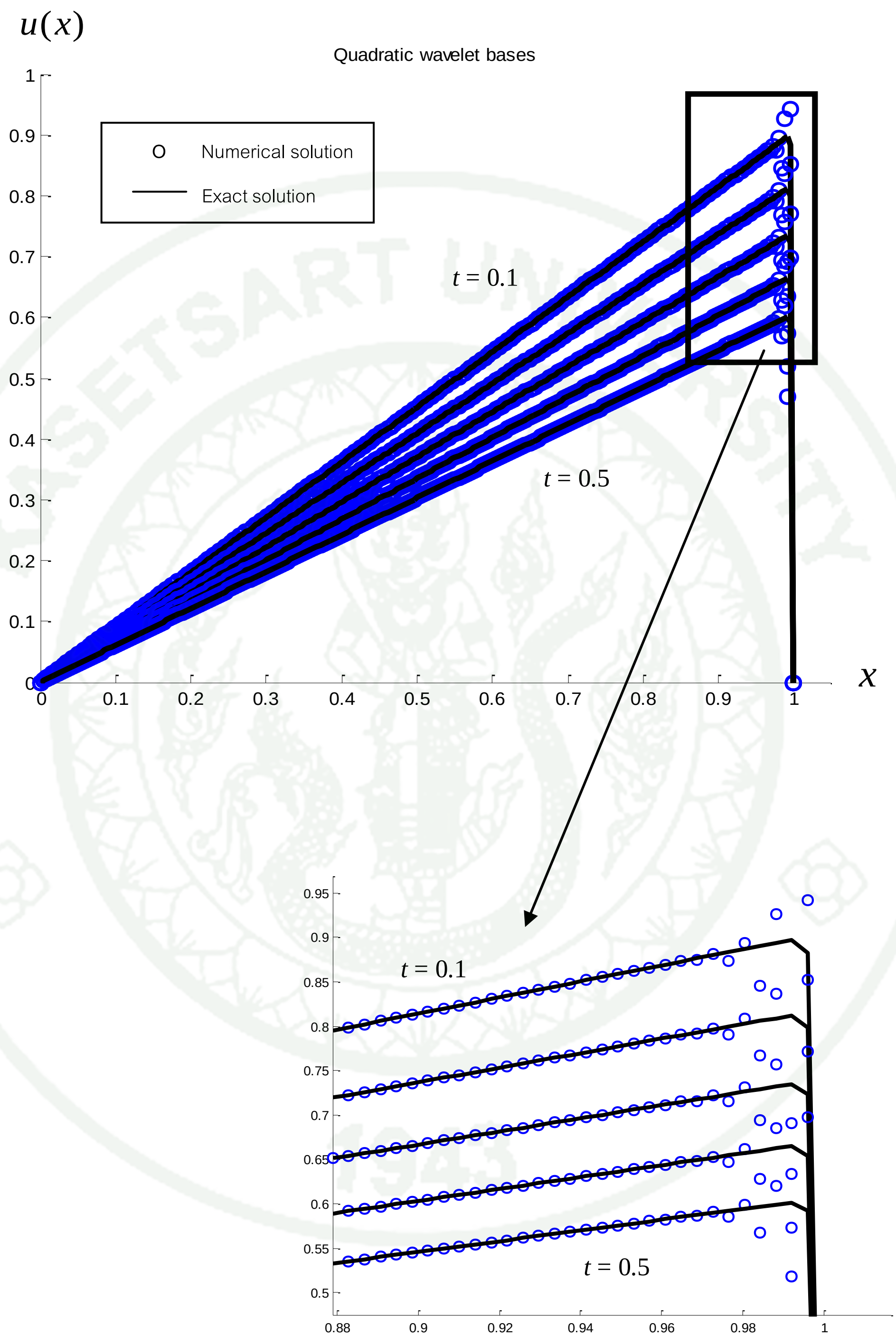

Figure 16 The Galerkin finite element results by quadratic wavelet basis of level 7 (W_7), and exact solution for $\varepsilon = 0.001$.

The full system matrices can be solved iteratively by the Gauss-Seidel method and the build-in inverse function in Matlab. If we use the inverse function in Matlab, the solution does not converge when $\varepsilon < 10^{-4}$, The numerical solutions diverge rapidly if we set $\varepsilon \leq 10^{-2}$ in the Gauss-Seidel method.

The accuracy of numerical solutions are shown in Tables 12-19. The RMS errors and L^2 norm error by the linear Lagrange basis are almost the same as using the linear wavelet basis. This implies that the accuracy is approximately in the same order for these two types of bases when we use the same element size. The RMS errors and L^2 norms decreases when we increase the number of elements.

The rates of convergence with RMS errors for linear bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 2 for $\varepsilon = 10^{-1}$, 2 for $\varepsilon = 10^{-2}$ and 1 for $\varepsilon = 10^{-3}$. The rates of convergence with L^2 norms for linear bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 1.5 for $\varepsilon = 10^{-1}$, 1.6 for $\varepsilon = 10^{-2}$ and 0.8 for $\varepsilon = 10^{-3}$. The rates of convergence with RMS errors for quadratic bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 3 for $\varepsilon = 10^{-1}$, 3 for $\varepsilon = 10^{-2}$ and 1 for $\varepsilon = 10^{-3}$. The rates of convergence with L^2 norms for quadratic bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 3 for $\varepsilon = 10^{-1}$, 3 for $\varepsilon = 10^{-2}$ and 1 for $\varepsilon = 10^{-3}$. The plots of rate of convergence are shown in Figure 17.

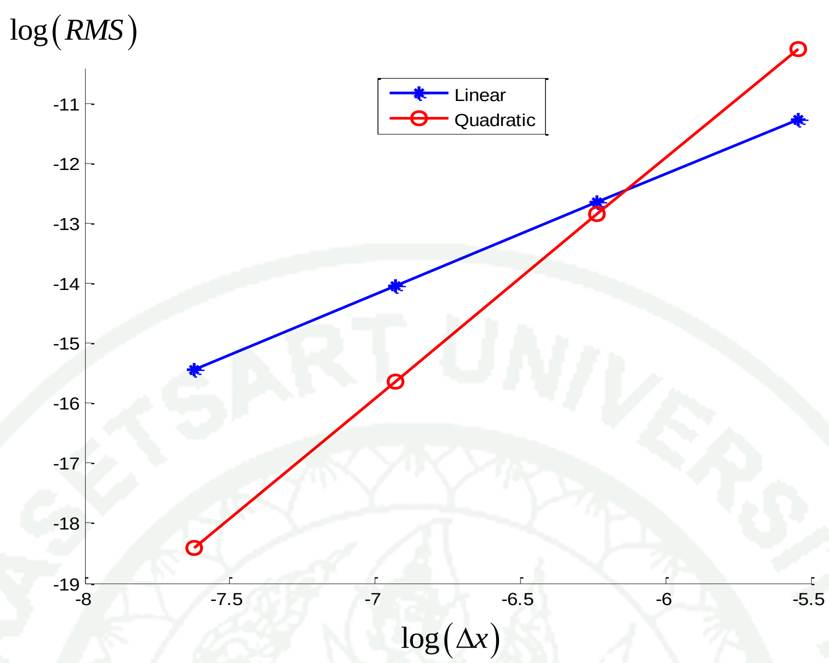

Figure 17 Rate of convergence with RMS errors of linear and quadratic bases for $\varepsilon = 0.1$.

Table 12 RMS error of Galerkin finite element method by linear wavelet bases for unsteady singularly perturbed problem.

Time \ Level	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	0.2	0.4	0.2	0.4	0.2	0.4
W_5	7.4787e-4	7.8955e-4	3.9758e-2	3.2582e-2	2.8268e-1	2.3349e-1
W_6	1.8442e-4	1.9472e-4	9.5326e-3	7.8095e-3	1.2731e-1	1.0424e-1
W_7	4.5857e-5	4.8418e-5	2.2021e-3	1.8040e-3	5.3642e-2	4.3918e-2
W_8	1.1437e-5	1.2076e-5	5.3197e-4	4.3581e-4	1.8481e-2	1.5131e-2
r	2.0		2.1		1.3	

($\Delta t = 0.005$, $\mathrm{TOL} = 10^{-12}$)

r is Approximate convergence rate

Table 13 L^2 norms of Galerkin finite element method by linear wavelet bases for unsteady singularly perturbed problem.

Time \ Level	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	0.2	0.4	0.2	0.4	0.2	0.4
W_5	4.1640e-3	4.3960e-3	2.2136e-1	1.8140e-1	1.5739e+0	1.3000e+0
W_6	1.4638e-3	1.5455e-3	7.5663e-2	6.1986e-2	1.0105e+0	8.2739e-1
W_7	5.1678e-4	5.4565e-4	2.4816e-2	2.0330e-2	6.0451e-1	4.9493e-1
W_8	1.8264e-4	1.9284e-4	8.4949e-3	6.9593e-3	2.9513e-1	2.4163e-1
r	1.5		1.6		0.8	

($\Delta t = 0.005$, $\text{TOL} = 10^{-12}$)

r is Approximate convergence rate

Table 14 RMS error of Galerkin finite element method by linear Lagrange bases for unsteady singularly perturbed problem.

Time \ Elements	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	0.2	0.4	0.2	0.4	0.2	0.4
32	8.1304e-4	8.6183e-4	3.9476e-2	3.2442e-02	2.8268e-01	2.3350e-1
64	2.1187e-4	2.2418e-4	9.4470e-3	7.7879e-03	1.2731e-01	1.0425e-1
128	5.4279e-5	5.7361e-5	2.1882e-3	1.8139e-03	5.3630e-02	4.3916e-2
256	1.3748e-5	1.4519e-5	5.3114e-4	4.4248e-04	1.8471e-02	1.5127e-2
r	2.0		2.1		1.3	

($\Delta t = 0.005$, $\text{TOL} = 10^{-12}$)

r is Approximate convergence rate

Table 15 L^2 norms of Galerkin finite element method by linear Lagrange bases for unsteady singularly perturbed problem.

Time \ Elements	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	0.2	0.4	0.2	0.4	0.2	0.4
32	4.5268e-3	4.7984e-3	2.1979e-1	1.8063e-1	1.5739e+0	1.3000e+0
64	1.6817e-3	1.7794e-3	7.4983e-2	6.1814e-2	1.0105e+0	8.2748e-1
128	6.1169e-4	6.4643e-4	2.4660e-2	2.0442e-2	6.0438e-1	4.9491e-1
256	2.1955e-4	2.3186e-4	8.4817e-3	7.0658e-3	2.9496e-1	2.4156e-1
r	1.5		1.6		0.8	

($\Delta t = 0.005$, $\text{TOL} = 10^{-12}$)

r is Approximate convergence rate

Table 16 RMS error of Galerkin finite element method by quadratic wavelet bases for unsteady singularly perturbed problem.

Time \ Level	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	0.2	0.4	0.2	0.4	0.2	0.4
W_4	4.3830e-5	3.6017e-5	2.9827e-2	2.4360e-2	2.5450e-1	2.0838e-1
W_5	2.7818e-6	2.2860e-6	4.7570e-3	3.8944e-3	1.1285e-1	9.2380e-2
W_6	2.8095e-6	2.3004e-6	2.8408e-3	2.3259e-3	5.3260e-2	4.3560e-2
W_7	1.0801e-8	8.9368e-9	3.2772e-5	2.6833e-5	1.4649e-2	1.1994e-2
r	3.6		3.0		1.3	

($\Delta t = 0.005$, $\text{TOL} = 10^{-12}$)

r is Approximate convergence rate

Table 17 L^2 norms of Galerkin finite element method by quadratic wavelet bases for unsteady singularly perturbed problem.

Time \ Level	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	0.2	0.4	0.2	0.4	0.2	0.4
W_4	2.4403e-4	2.0053e-4	1.6607e-1	1.3563e-1	1.4170e+0	1.1602e+0
W_5	2.2080e-5	1.8144e-5	3.7757e-2	3.0911e-2	8.9576e-1	7.3325e-1
W_6	3.1662e-5	2.5924e-5	3.2015e-2	2.6212e-2	6.0022e-1	4.9089e-1
W_7	1.7249e-7	1.4270e-7	5.2334e-4	4.2849e-4	2.3393e-1	1.9153e-1
r	3.1		2.5		1.0	

($\Delta t = 0.005$, $\text{TOL} = 10^{-12}$)

r is Approximate convergence rate

Table 18 RMS error of Galerkin finite element method by quadratic Lagrange bases for unsteady singularly perturbed problem.

Time \ Elements	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	0.2	0.4	0.2	0.4	0.2	0.4
16	1.0397e-4	1.0224e-4	3.0017e-2	2.4539e-2	2.5872e-1	2.1181e-1
32	1.9090e-5	1.7392e-5	4.7152e-3	3.8709e-3	1.1374e-1	9.3113e-2
64	3.3269e-6	2.9045e-6	4.4418e-4	3.6872e-4	4.6314e-2	3.7921e-2
128	5.2947e-7	4.8828e-7	3.1741e-5	2.8002e-5	1.4666e-2	1.2008e-2
r	3.2		3.3		1.4	

($\Delta t = 0.005$, $\text{TOL} = 10^{-12}$)

r is Approximate convergence rate

Table 19 L^2 norms of Galerkin finite element method by quadratic Lagrange bases for unsteady singularly perturbed problem.

Time \ Elements	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$	
	0.2	0.4	0.2	0.4	0.2	0.4
16	5.7630e-4	5.6002e-4	1.6441e-1	1.3440e-1	1.4170e+0	1.1601e+0
32	1.5031e-4	1.3694e-4	3.7127e-2	3.0480e-2	8.9561e-1	7.3317e-1
64	3.7345e-5	3.2603e-5	4.9859e-3	4.1389e-3	5.1987e-1	4.2566e-1
128	9.1605e-6	7.7819e-6	5.0586e-4	4.4628e-4	2.3374e-1	1.9139e-1
r	2.7		2.8		1.0	

($\Delta t = 0.005$, $\mathrm{TOL} = 10^{-12}$)

r is Approximate convergence rate

We have also investigated the rate of convergence of time basis. Two cases of time basis levels are considered which are θ_1 and θ_2. We set the final time as 0.8, and $\varepsilon = 0.1$. Numerical results are shown in Tables 20 and 21. The RMS errors using time basis θ_1 for the linear Lagrange and wavelet bases are almost the same as those errors obtained by the quadratic Lagrange and wavelet bases. The rates of convergence are approximately 1. Similarly, we have found that the rates of convergence are approximately in order 2 and 3 when using θ_2. This shows an advantage of this type of time basis such that the order of accuracy in time can be improved by just increasing the time basis level. That is unlike the other standard schemes such as the Euler method or the Runge-Kutta method that the order of accuracy is fixed. Applying the presented time basis is more flexible than the standard approach.

Table 20 RMS error by linear wavelet and Lagrange bases for unsteady singularly perturbed problem.

discretization in time in level (θ)	Time \ Level	$\Delta t = 0.4$	$\Delta t = 0.2$	$\Delta t = 0.1$	Approximate convergence rate
		0.8	0.8	0.8	
level 1 (θ_1)	W_8	4.11682e-02	2.24436e-02	1.17945e-02	1.0
	256 element	4.11590e-02	2.24337e-02	1.17841e-02	1.0
level 2 (θ_2)	W_8	4.05537e-04	4.91182e-05	7.52296e-06	2.8
	256 element	4.18002e-04	6.16836e-05	1.62143e-05	2.3

$\varepsilon = 0.1,\ TOL = 10^{-12}$

Table 21 RMS error by quadratic wavelet and Lagrange bases for unsteady singularly perturbed problem.

discretization in time in level (θ)	Time \ Level	$\Delta t = 0.4$	$\Delta t = 0.2$	$\Delta t = 0.1$	Approximate convergence rate
		0.8	0.8	0.8	
level 1 (θ_1)	W_7	4.11612e-02	2.24365e-02	1.17873e-02	1.0
	128 element	4.12421e-02	2.24805e-02	1.18103e-02	1.0
level 2 (θ_2)	W_7	4.11236e-04	5.43274e-05	7.02850e-06	2.9
	128 element	4.12220e-04	5.46118e-05	7.22594e-06	2.9

$\varepsilon = 0.1,\ TOL = 10^{-12}$

Numerical Example by the multilevel augmentation method

Recalling the singularly perturbed second-order boundary value problem,

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - u = 1 \quad , \quad 0 < x < 1 .$$

We consider the case of $\varepsilon = 0.1$, subject to the boundary conditions,

$$u(0) = 0 \text{ and } u(1) = 0 .$$

The exact solution for this problem is

$$u(x) = \frac{\left(e^{m_2} - 1\right)e^{m_1 x} + \left(1 - e^{m_1}\right)e^{m_2 x}}{e^{m_2} - e^{m_1}} - 1 \ ,$$

where

$$m_1 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon} \quad ,$$

$$m_2 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon} \quad .$$

We apply the linear and quadratic wavelet bases in our approximation. The numerical results are shown in Tables 22 and 23 respectively. The multilevel augmentation method is applied in the initial level 3 (W_3) for both linear and quadratic wavelet bases. The system of linear equations are solved iteratively by the Gauss-Seidel method with $TOL = 10^{-12}$ and by calling the build-in inverse function in Matlab.

The RMS errors by the multilevel augmentation method are almost the same as those errors obtained by using the multilevel method. The rate of convergence is approximately 2.0 as expected for the wavelet linear bases. The RMS errors and the rate of convergences for the quadratic wavelet bases are shown and is approximately in order 4.0.

Numerical results show that the proposed multilevel augmentation method is fast and accurate method for solving system equations. It has great advantage for solving large-scale problems, but the multilevel augmentation method must be applied for relatively high-initial level to reduce error collection in each level applied. The full system of multilevel augmentation method are smaller than the standard multilevel method for the same level, resulting to smaller memory to store coefficients in matrix. This explanation is shown in Table 22 for linear basis. We cannot solve the linear system by the Gauss-Seidel method in Matlab for the system of level 12 due to not enough memory in the computer of Intel Core i5-2410M CPU and 4.00GB memory, but the multilevel augmentation method can provide the results. Also, this method is suitable for multiscale orthonormal bases approximations.

Table 22 RMS error by linear wavelet bases for $\varepsilon = 0.1$

Level	Multilevel Augmentation Method	Multilevel Method
W_4	3.683109e-03	3.475348e-03
W_5	8.281858e-04	8.337140e-04
W_6	2.042934e-04	2.053123e-04
W_7	5.096677e-05	5.103239e-05
W_8	1.272315e-05	1.272720e-05
W_9	3.178067e-06	3.178319e-06
W_{10}	7.941528e-07	7.941685e-07
W_{11}	1.984912e-07	1.984922e-07
W_{12}	4.961684e-08	-
Approximate convergence rate	2.0	2.0

Table 23 RMS error by quadratic wavelet bases for $\varepsilon = 0.1$

Level	Multilevel Augmentation Method	Multilevel Method
W_4	6.335132e-05	4.037317e-05
W_5	3.645426e-06	2.580418e-06
W_6	1.785070e-07	1.618866e-07
W_7	1.029555e-08	1.011768e-08
Approximate convergence rate	4.0	4.0

CONCLUSION AND RECOMMENDATION

In this thesis, we have presented the Galerkin finite element method to solve numerically the one-dimensional heat equation. The purpose is to show and compare the order of accuracy in space and time for wavelet basis. Two types of basis functions which are the Lagrange (for comparing) and wavelet bases are employed to derive the full matrix system. We consider both linear and quadratic bases. Also, we have introduced a time basis for the time discretization process. When the initial and boundary conditions are specified, the full system matrices can be solved iteratively by the Gauss-Seidel method. Our numerical results show that the rate of convergence for the Linear Lagrange and the Linear Wavelet is the same in order of 2 while the rate of convergence for the quadratic Lagrange and the quadratic wavelet is approximately in order of 4. These two rates are in expected as theoretical results follow the Lagrange basis. The numerical resolutions can be increased by increasing the number of wavelet basis levels. This shows an advantage of the wavelet basis over using the Lagrange basis. By this point of investigation, we can apply the presented wavelet bases with multilevel approach to further solving other types of differential equation.

Then, we have presented the Galerkin finite element method to solve numerically the singularly perturbed boundary value problem. The purpose is to show and compare the order of accuracy in space for wavelet basis. Two types of basis functions which are the Lagrange (for comparing) and wavelet bases are employed to derive the full matrix system. When the initial and boundary conditions are specified, the full system matrices can be solved iteratively by the Gauss-Seidel method. The rates of convergence for both the linear Lagrange and wavelet bases are in the same order as expected. The rate of convergence is 2.0 in RMS errors. The rates of convergence with L^2 norms for linear bases of both the Lagrange and wavelet are in the same order as expected of 1.5. The rates of convergence with RMS errors for quadratic bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 4.0 for $\varepsilon = 10^{-1}$, 3.3 for $\varepsilon = 10^{-2}$ and 1.3 for

$\varepsilon = 10^{-3}$. The rates of convergence with L^2 norms for quadratic bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 3.5 for $\varepsilon = 10^{-1}$, 2.8 for $\varepsilon = 10^{-2}$ and 1.0 for $\varepsilon = 10^{-3}$.

Next, we have presented the Galerkin finite element method to solve the unsteady singularly perturbed problem. The purpose is to show and compare the order of accuracy in space and time for wavelet basis. Two types of basis functions which are the Lagrange (for comparing) and wavelet bases are employed to derive the full matrix system. When the initial and boundary conditions are specified, the full system matrices can be solved iteratively by the Gauss-Seidel method. The rates of convergence with RMS errors for linear bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 2 for $\varepsilon = 10^{-1}$, 2 for $\varepsilon = 10^{-2}$ and 1 for $\varepsilon = 10^{-3}$. The rates of convergence with L^2 norms for linear bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 1.5 for $\varepsilon = 10^{-1}$, 1.6 for $\varepsilon = 10^{-2}$ and 0.8 for $\varepsilon = 10^{-3}$. The rates of convergence with RMS errors for quadratic bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 3 for $\varepsilon = 10^{-1}$, 3 for $\varepsilon = 10^{-2}$ and 1 for $\varepsilon = 10^{-3}$. The rates of convergence with L^2 norms for quadratic bases of both the Lagrange and wavelet are in the same order as expected. The rates of convergence are 3 for $\varepsilon = 10^{-1}$, 3 for $\varepsilon = 10^{-2}$ and 1 for $\varepsilon = 10^{-3}$.

Finally, we have presented the concept of the multilevel augmentation method using wavelet bases to solve numerically the singularly perturbed second-order boundary value problem. Numerical methods show that the multilevel augmentation method is fast and accurate method for solving differential equations. It has great advantage for solving large-scale problems. This method integrate the choices of bases and the design of numerical solvers for the discrete linear systems together. We consider both the linear wavelet and the quadratic wavelet bases in the method. Numerical results are presented to demonstrate the efficiency of this method. The RMS errors by the multilevel augmentation method are almost the same as those errors obtained by using the multilevel method. The rate of convergence is approximately 2.0 as expected for the wavelet linear bases. The RMS errors and the rate of convergences for the quadratic wavelet bases are shown and is approximately in order 4.0. Numerical results show that the proposed multilevel augmentation method is fast and accurate method for solving system equations. It has great advantage for solving large-scale problems, but the multilevel augmentation method must be applied for relatively high-initial level to reduce error collection in each level applied. The full system of multilevel augmentation method are smaller than the standard multilevel method for the same level, resulting to smaller memory to store coefficients in matrix.

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APPENDIX

Galerkin with Lagrange and Wavelet Bases for the One-Dimensional Heat Equation

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Abstract

The Wavelet-Galerkin finite element method for solving the one-dimensional heat equation is presented in this work. Two types of basis functions which are the Lagrange and wavelet bases are employed to derive the full form of matrix system. Both linear and quadratic bases are considered in this work. The numerical results show that the rate of convergences for the linear Lagrange and the linear wavelet bases are the same and in order 2 while the rate of convergences for the quadratic Lagrange and the quadratic wavelet bases are approximately in order 4. It also reveals that the wavelet basis provides an easy treatment to improve numerical resolutions that can be done by increasing just its desired levels in the basis construction process.

Keywords: Wavelet basis functions; Lagrange basis functions; Heat equation; Galerkin finite element method;

1. Introduction

The Galerkin approach is one of the very successful methods for finding approximate solutions from the partial differential equation. The main concept is using an appropriate basis function for the solution space of the governing equation, and then projecting the terms of approximate solution on the functional basis space. This process provides residual that needed to be minimized with respect to the functional basis. By this concept, the accuracy of numerical solutions depends directly on the type of basis function.

In this work, we apply the Galerkin method with wavelet bases called the Wavelet-Galerkin method to solve numerically the linear one-dimensional heat equation. Wavelets in our consideration are compactly supported wavelets introduced by Chen et al. [1]. They introduced the multilevel augmentation method related with some wavelet bases for solving certain boundary value problems. This method has then been applied for solving the sine-Gordon equation in [2] and some types of nonlinear boundary value problems in [3]. For solving the partial differential equations, the wavelet applications have been introduced by several authors, such as a wavelet-Galerkin method for solving parabolic equations [4], the singularly perturbed convection-dominated diffusion equation [5], non-homogeneous heat and wave equations [6], some types of elliptic problems [7], and diffusion equation [8].

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We have presented in this work that the linear wavelet has more advantages than the linear Lagrange. The accuracy of numerical solution by the linear wavelet is easily improved by just increasing the wavelet levels (multilevel concept) and the computations for finding coefficients are performed by just calculating the extra coefficients included in the corresponding level. This concept is different from using the traditional Lagrange bases or the finite difference method that it is required to calculate the whole system when resolution increased.

The details of this presented work are organized as follows. In Section 2, we introduce the Galerkin finite element method and show how the Lagrange and wavelet basis functions are used to solve the heat equation in Sections 3 and 4 respectively. Time discretization is presented in Section 5. Some numerical examples and comparisons of numerical results are demonstrated in Section 6. We have made some conclusions in Section 7.

2. Galerkin finite element method

The time-dependent heat equation in terms of variable $T(x,t)$ is written in its one-dimensional form as

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad , \qquad \text{------ (1)}$$

where $\partial T / \partial t$ is the time rate of change of temperature, α is the thermal diffusivity, and time (t). The domain is $\Omega \; (0 \le x \le l)$ with boundary Γ.

We give the initial and boundary conditions as

$$T(0,t) = T(l,t) = 0 \;, \qquad T(x,0) = T_0(x) \;.$$

By the weighted residual method, Eq.(1) can be written as

$$\int_\Omega W \left(\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} \right) d\Omega = 0 \,,$$

where W is the weight function. Using integration by part, yields

$$\int_\Omega W \left(\frac{\partial T}{\partial t} \right) d\Omega \; + \int_\Omega \alpha \left(\frac{\partial T}{\partial x} \frac{\partial W}{\partial x} \right) d\Omega \; - \alpha W \frac{\partial T}{\partial x}\bigg|_\Gamma = 0 \;. \qquad \text{------ (2)}$$

Let us begin by approximating the unknown function in terms of the Lagrange basis as

$$T^n = T(x,t_n) \; = \sum_{k=0}^{p} \sum_{i=1}^{m} N_i(x) \theta_k(t_n) \, c_{ik}^n \;. \qquad \text{------ (3)}$$

Another choice is done by assuming the unknown variable in terms of the wavelet basis as

$$T^n = T(x,t_n) \; = \sum_{k=0}^{p} \sum_{i=1}^{M} \sum_{j=0}^{\dim(i)} w_{ij}(x) \theta_k(t_n) \, c_{ijk}^n \;, \qquad \text{------ (4)}$$

where $T^n = T(x,t_n)$ denotes the variable's value at time $t = t_n$, $N_i(x)$ is spatial basis function, $w_{ij}(x)$ is the wavelet basis function, $\theta_k(t_n)$ is the time basis function, m is the number of elements, M is the number of level in multi-level wavelet approach, and p is the number of level for time discretization.

After setting $W=N_i(x)$ where $N_i(x)$ is the Lagrange basis function, Eq.(2) can be written in the matrix form as

$$[C]\{c_{ik}^{n}\}+[K]\{c_{ik}^{n}\}+[M^{++}]\{c_{ik}^{n}\}-[M^{-+}]\{c_{ik}^{n-1}\}=0 \quad ,$$

$$\{[C]+[K]+[M^{++}]\}c_{ik}^{n}=[M^{-+}]c_{ik}^{n-1} \quad . \qquad \text{------ (5)}$$

Similarly, if we set $W=w_{ij}(x)$, Eq.(2) can be written in the matrix form as

$$[C]\{c_{ijk}^{n}\}+[K]\{c_{ijk}^{n}\}+[M^{++}]\{c_{ijk}^{n}\}-[M^{-+}]\{c_{ijk}^{n-1}\}=0 \quad ,$$

$$\{[C]+[K]+[M^{++}]\}c_{ijk}^{n}=[M^{-+}]c_{ijk}^{n-1} \quad . \qquad \text{------ (6)}$$

The coefficients in each matrix element can be obtained. For brevity, the results are summarized in the table shown below.
In the case of the Lagrange basis function, the initial condition provides us the starting unknown coefficients as

$$c_{ik}^{n-1}=[T_0(x)]\otimes[\theta(0)] \quad . \qquad \text{------ (7)}$$

For the wavelet basis function, the initial unknown coefficients are obtained by

$$c_{ijk}^{n-1}=[T_0(x)]\otimes[\theta(0)] \quad , \qquad \text{------ (8)}$$

where $\otimes$ is the outer tensor operation of two matrices.

Lagrange basis functions	Wavelet basis functions
$[C] = \int_\Omega W\frac{\partial T}{\partial t}d\Omega = \int_0^l\int_{t_{n-1}}^{t_n}(N_i\otimes\theta)\left(N_i\otimes\frac{\partial\theta}{\partial t}\right)^T dtdx$ $= \int_0^l(N_iN_i^T)dx\otimes\int_{t_{n-1}}^{t_n}\left(\theta_j\frac{\partial\theta^T}{\partial t}\right)dt$	$[C] = \int_\Omega W\frac{\partial T}{\partial t}d\Omega = \int_0^l\int_{t_{n-1}}^{t_n}(w_{ij}\otimes\theta)\left(w_{ij}\otimes\frac{\partial\theta}{\partial t}\right)^T dtdx$ $= \int_0^l(w_{ij}w_{ij}^T)dx\otimes\int_{t_{n-1}}^{t_n}\left(\theta_j\frac{\partial\theta^T}{\partial t}\right)dt$
$[K] = \int_\Omega\frac{\partial T}{\partial x}\frac{\partial W}{\partial x}d\Omega = \int_0^l\int_{t_{n-1}}^{t_n}\left(\frac{\partial N_i}{\partial x}\otimes\theta\right)\left(\frac{\partial N_i}{\partial x}\otimes\theta\right)^T dtdx$ $= \int_0^l\left(\frac{\partial N_i}{\partial x}\frac{\partial N_i}{\partial x}^T\right)dx\otimes\int_{t_{n-1}}^{t_n}(\theta\theta^T)dt$	$[K] = \int_\Omega\frac{\partial T}{\partial x}\frac{\partial W}{\partial x}d\Omega = \int_0^l\int_{t_{n-1}}^{t_n}\left(\frac{\partial w_{ij}}{\partial x}\otimes\theta\right)\left(\frac{\partial w_{ij}}{\partial x}\otimes\theta\right)^T dtdx$ $= \int_0^l\left(\frac{\partial w_{ij}}{\partial x}\frac{\partial w_{ij}}{\partial x}^T\right)dx\otimes\int_{t_{n-1}}^{t_n}(\theta\theta^T)dt$
$[M^{++}] = \int_0^l N_i(\theta^+)(N_i(\theta^+))^T dx$ $= \int_0^l N_iN_i^T dx\otimes\left[(\theta^+)(\theta^+)^T\right]$	$[M^{++}] = \int_0^l w_{ij}(\theta^+)(w_{ij}(\theta^+))^T dx$ $= \int_0^l w_{ij}w_{ij}^T dx\otimes\left[(\theta^+)(\theta^+)^T\right]$
$[M^{-+}] = \int_0^l N_i(\theta^-)(N_i(\theta^+))^T dx$ $= \int_0^l N_iN_i^T dx\otimes\left[(\theta^-)(\theta^+)^T\right]$	$[M^{-+}] = \int_0^l w_{ij}(\theta^-)(w_{ij}(\theta^+))^T dx$ $= \int_0^l w_{ij}w_{ij}^T dx\otimes\left[(\theta^-)(\theta^+)^T\right]$

For example, when matrices A and B are given in dimension 2×2, the operation is defined as follows.

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \quad , \quad B=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} \quad ,$$

$$A\otimes B=\begin{bmatrix}Ab_{11} & Ab_{12}\\ Ab_{21} & Ab_{22}\end{bmatrix}=\left[\begin{array}{cc|cc}a_{11}b_{11} & a_{12}b_{11} & a_{112}b_{12} & a_{12}b_{12}\\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12}\\ \hline a_{11}b_{21} & a_{12}b_{21} & a_{11}b_{22} & a_{12}b_{22}\\ a_{21}b_{21} & a_{22}b_{21} & a_{21}b_{22} & a_{22}b_{22}\end{array}\right] \quad . \qquad \text{------ (9)}$$

Finally, we have the systems of equation in Eq.(5), or in Eq.(6) that can be solved to find the coefficients $\{c_{ik}^n\}$, or $\{c_{ijk}^n\}$, and hence we know the approximate values, T^n. Note that the system of linear equation is solved iteratively by the Gauss-Seidel method in this work.

3. Lagrange basis function

In this section, we will show in details the derivation of matrix coefficients by two classes of the Lagrange basis function which are linear and quadratic bases.

3.1 Linear Lagrange basis function

We begin defining nodal points in the domain $0 \le x \le l$ with n elements of uniform element size. Thus, there are $n+1$ nodes corresponding to the coordinates $x_1, x_2, \ldots, x_{n+1}$ as shown in Fig.1.

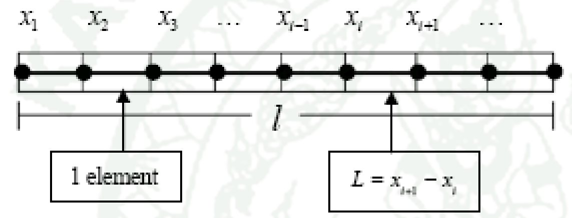

Fig.1

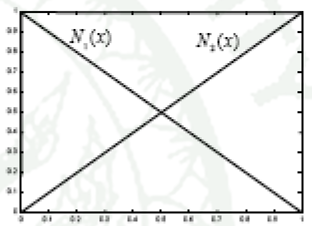

Fig.2 linear basis functions

In this case, we assume the approximate solution in Eq. (3) as

$$T^n(x,t_n) = \sum_{k=0}^{p}\left(N_1(x)\theta_k(t_n)c_{1k}^n + N_2(x)\theta_k(t_n)c_{2k}^n\right) ,$$

where

$$N_1(x) = 1-\frac{x}{L} \quad , \qquad N_2(x) = \frac{x}{L} \quad .$$

These are the well-known linear Lagrange basis functions. Their variations in an element are shown in Fig. 2.

Hence, some parts in the matrices $[C]$, $[K]$, $[M^{+}]$ and $[M^{-}]$ can be evaluated as

$$\int_0^L (N_i N_i^T)\,d\omega = \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix}\begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix}dx = \frac{L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} ,$$

$$\int_0^L \left(\frac{\partial N_i}{\partial x}\frac{\partial N_i}{\partial x}^T\right)d\omega = \int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}dx = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} .$$

The full definition of these matrices are obtained by using the outer tensor operator $\otimes$ in Eq.(9)

3.2 Quadratic Lagrange basis function

The nodal notation used in this case to relate the global system is shown in Fig.4. There are three nodes in one element. The i th element is defined on $x_{2i-1} \le x \le x_{2i+1}$, $i = 1, 2, \ldots, n$ and its element size is given by $L = x_{2i+1} - x_{2i-1}$.

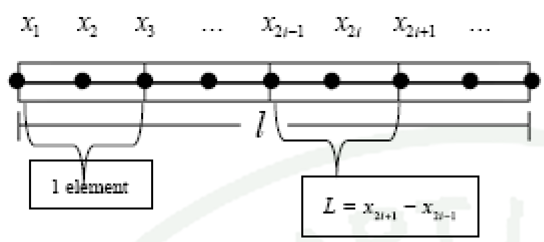

Fig.4

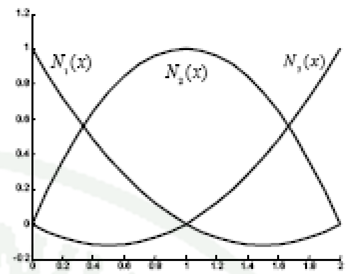

Fig.5 quadratic basis functions

By this basis function, we assume the approximate $T^n(x,t_n)$ as

$$T^n(x,t_n) = \sum_{k=0}^{p} \left(N_1(x)\theta_k(t_n)c_{1k}^n + N_2(x)\theta_k(t_n)c_{2k}^n + N_3(x)\theta_k(t_n)c_{3k}^n \right),$$

and the quadratic Lagrange basis functions are

$$N_1(x) = 1 - \frac{3x}{L} + 2\left(\frac{x}{L}\right)^2, \quad N_2(x) = \frac{4x}{L}\left(1 - \frac{x}{L}\right), \quad N_3(x) = \frac{x}{L}\left(\frac{2x}{L} - 1\right).$$

Some parts in the matrices $[C]$, $[K]$, $[M^{**}]$ and $[M^{*}]$ can be evaluated as

$$\int_0^L (N_i N_i^T) d\omega = \int_0^L \begin{Bmatrix} 1 - \frac{3x}{L} + 2\left(\frac{x}{L}\right)^2 \\ \frac{4x}{L}\left(1 - \frac{x}{L}\right) \\ \frac{x}{L}\left(\frac{2x}{L} - 1\right) \end{Bmatrix} \left[1 - \frac{3x}{L} + 2\left(\frac{x}{L}\right)^2 \quad \frac{4x}{L}\left(1 - \frac{x}{L}\right) \quad \frac{x}{L}\left(\frac{2x}{L} - 1\right) \right] dx = \frac{L}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$\int_0^L \left(\frac{\partial N_i}{\partial x} \frac{\partial N_i^T}{\partial x} \right) d\omega = \int_0^L \begin{Bmatrix} \frac{1}{L}\left(\frac{4x}{L} - 3\right) \\ \frac{4}{L}\left(1 - \frac{2x}{L}\right) \\ \frac{1}{L}\left(\frac{4x}{L} - 1\right) \end{Bmatrix} \left[\frac{1}{L}\left(\frac{4x}{L} - 3\right) \quad \frac{4}{L}\left(1 - \frac{2x}{L}\right) \quad \frac{1}{L}\left(\frac{4x}{L} - 1\right) \right] dx = \frac{1}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}.$$

Then we can apply the outer tensor operator $\otimes$ in Eq.(9) to find the full form of matrices.

4. Wavelet basis functions

The construction of the wavelet basis functions used in the Galerkin method follows the derivations proposed in [1]. We construct multi-scale orthonormal bases for the Sobolev space on the unit interval $I := [0,1]$. Specifically, we let m be a fixed positive integer and $H_0^m(I)$ denoted the Sobolev spaces of element that T satisfies the homogeneous boundary conditions of $T^{(j)}(0) = T^{(j)}(1) = 0$, $j \in Z_m$, where $Z_n := \{0,1,2,\ldots,n-1\}$. For any nonnegative integer n, we denote by X_n the subspace of $H_0^m(I)$ whose elements are the piecewise polynomials of order k with knots j/μ^n, $j-1 \in Z_{\mu^n-1}$, when $k > 2m$ and $\mu > 1$ be a fixed positive integer. We have that $X_0 = \text{span}\left\{x^{m+j}(1-x)^m : j \in Z_{k-2m}\right\}$, so we let W_n be the orthonormal complement of X_{n+1} in X_n, i.e., $X_n = X_{n-1} \oplus_m W_n$ and thus, repeatedly using this decomposition leads to $X_n = X_0 \oplus_m W_1 \oplus_m \cdots \oplus_m W_n$. Spaces W_n can be recursively constructed once W_1 has been given. To describe the construction, we need the family of affine mappings $\Phi_\mu := \{\phi_e : e \in Z_\mu\}$ where $\phi_e(x) = (x+e)/\mu$; $e \in Z_\mu$.

Finally, the wavelet basis function can be constructed by the composition as follows.

$$w_{ij} = \tau_e w_{1l} = \mu^{n\left(\frac{1}{2}-m\right)} w_{1l} \circ \phi_e^{-1}(x) \quad ; \quad e \in Z_\mu^{i-1}. \qquad \text{------} \quad (10)$$

This construction will be applied to obtain both linear and quadratic wavelet bases in the next sections.

4.1 Linear wavelet basis

From Eq. (10), we set $m=1$, $\mu=2$, $r=1$. We can give l , Z_μ , and e by

$$l \in Z_r = \{0\} \quad , \quad e \in Z_\mu = \{0,1\} \quad \text{and} \quad \phi_0(x) = \frac{x}{2} \quad , \quad \phi_1(x) = \frac{x+1}{2} .$$

The desired basis of W_1 (level 1) is obtained by

$$w_{10}(x) = \begin{cases} x & ; x \in [0,1/2] \\ 1-x & ; x \in [1/2,1] \end{cases} .$$

The wavelet basis function of W_2 (level 2) is given by $\left(n=1,\ e \in Z_2^1 = \{0,1\}\right)$

$$w_{2j} = \mu^{1\left(\frac{1}{2}-1\right)} w_{1l} \circ \phi_e^{-1}(x) = \frac{1}{\sqrt{2}} w_{1l} \circ \phi_e^{-1}(x) \ ,$$

$$w_{20}(x) = \begin{cases} 2x/\sqrt{2} & ; x \in [0,1/4] \\ (1-2x)/\sqrt{2} & ; x \in [1/4,1/2] \end{cases} , \quad w_{21}(x) = \begin{cases} (2x-1)/\sqrt{2} & ; x \in [1/2,3/4] \\ (2-2x)/\sqrt{2} & ; x \in [3/4,1] \end{cases} .$$

The wavelet basis function of W_3 (level 3) is that $\left(n=2,\ e \in Z_2^2 = \{(0,0),(0,1),(1,0),(1,1)\}\right)$

$$w_{3j} = \mu^{2\left(\frac{1}{2}-1\right)} w_{1l} \circ \phi_e^{-1}(x) = \frac{1}{2} w_{1l} \circ \phi_e^{-1}(x) \ ,$$

$$w_{30}(x) = \begin{cases} 4x/2 & ; x \in [0,1/8] \\ (1-4x)/2 & ; x \in [1/8,1/4] \end{cases} , \quad w_{31}(x) = \begin{cases} (4x-1)/2 & ; x \in [1/4,3/8] \\ (2-4x)/2 & ; x \in [3/8,1/2] \end{cases} ,$$

$$w_{32}(x) = \begin{cases} (4x-2)/2 & ; x \in [1/2,5/8] \\ (3-4x)/2 & ; x \in [5/8,3/4] \end{cases} , \quad w_{33}(x) = \begin{cases} (4x-3)/2 & ; x \in [3/4,7/8] \\ (4-4x)/2 & ; x \in [7/8,1] \end{cases} .$$

The profiles of these three linear wavelet basis functions W_1, W_2 and W_3 are shown in Fig.6. In practice, any levels of the linear wavelet basis can be obtained recursively by the same process of this construction.

Then we can apply the outer tensor operator $\otimes$ in Eq.(9) to find the full form of matrices.

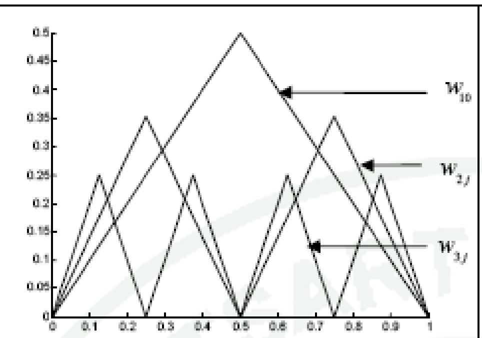

Fig.6 Linear wavelet basis functions of W_1, W_2, W_3

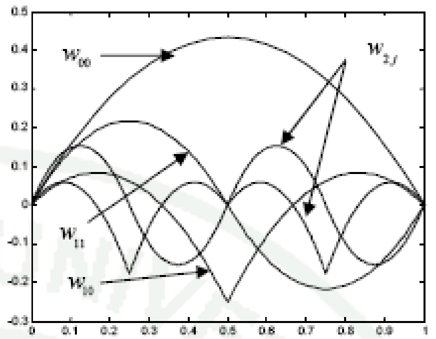

Fig.7 Quadratic wavelet basis functions of W_0, W_1, W_2

4.2 Quadratic wavelet basis

In this case, we set $m=1$, $\mu=2$, $r=2$, thus l, Z_μ, e are given by

$$l \in Z_r = \{0,1\} \quad , \quad e \in Z_\mu = \{0,1\} \quad \text{and} \quad \phi_0(x) = \frac{x}{2} \quad , \quad \phi_1(x) = \frac{x+1}{2} \ .$$

The desired bases of W_0 and W_1 are given by

$$w_{00}(x) = \sqrt{3}x(1-x) \quad ; x \in [0,1] \ ,$$

$$w_{10}(x) = \begin{cases} x(1-3x) & ; x \in [0,1/2] \\ (1-x)(3x-2) & ; x \in [1/2,1] \end{cases} \quad , \quad w_{11}(x) = \begin{cases} \sqrt{3}x(1-2x) & ; x \in [0,1/2] \\ \sqrt{3}(1-x)(1-2x) & ; x \in [1/2,1] \end{cases} \ .$$

The quadratic wavelet basis of level two, W_2 is given by $\left(n=1,\ e \in Z_2^1 = \{0,1\}\right)$,

$$w_{2j} = \mu^{1\left(\frac{1}{2}-1\right)} w_{1l} \circ \phi_e^{-1}(x) = \frac{1}{\sqrt{2}} w_{1l} \circ \phi_e^{-1}(x) \ ,$$

$$w_{20}(x) = \begin{cases} \frac{2x}{\sqrt{2}}(1-6x) & ; x \in [0,1/4] \\ \frac{2}{\sqrt{2}}(1-2x)(3x-1) & ; x \in [1/4,1/2] \end{cases} \quad , \quad w_{21}(x) = \begin{cases} \sqrt{6}x(1-4x) & ; x \in [0,1/4] \\ \frac{\sqrt{6}}{2}(1-2x)(3x-1) & ; x \in [1/4,1/2] \end{cases} \ ,$$

$$w_{22}(x) = \begin{cases} \frac{1}{\sqrt{2}}(4-6x)(2x-1) & ; x \in [1/2,3/4] \\ \frac{1}{\sqrt{2}}(2-2x)(6x-5) & ; x \in [3/4,1] \end{cases} \quad , \quad w_{23}(x) = \begin{cases} \frac{\sqrt{6}}{2}(2x-1)(3-4x) & ; x \in [1/2,3/4] \\ \frac{\sqrt{6}}{2}(2-2x)(3-4x) & ; x \in [3/4,1] \end{cases} \ .$$

The profiles of quadratic wavelet functions W_0, W_1 and W_2 are shown in Fig.7.
Then we can apply the outer tensor operator $\otimes$ in Eq.(9) to find the full form of matrices.

For a given initial condition , $T(x,0) = T_0(x)$,
and we have assumed that

$$T^n(x,t_n) = \sum_{k=0}^{p} \sum_{i=1}^{M} \sum_{j=0}^{\dim(i)} w_{ij}(x) \theta_k(t_n)\, c_{ijk}^n \ ,$$

so,

$$T(x,0) = \sum_{k=0}^{p} \sum_{i=1}^{M} \sum_{j=0}^{\dim(i)} w_{ij}(x) \theta_k(0)\, c_{ijk}^0 \ .$$

Thus, in the case of wavelet basis function, we can find the coefficients $\{c_{ijk}^0\}$ from the system

$$\left[w_{ij}(x_s)\right]\left[c_{ijk}^0\right] = \left[T_0(x_s)\right] \ ,$$

where $x_s = s\Delta x$, $\Delta x = 1/n$, $s = 1, 2, ..., n+1$, and $n+1$ is the number of knots.

5. Time Discretization

For the discretization in time, we give the basis function in time as $\theta_k(t) = ((t - t_{n-1})/\Delta t)^k$ where

$$\theta = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \cdots & \theta_p \end{bmatrix} = \begin{bmatrix} 1 & \frac{t - t_{n-1}}{\Delta t} & \left(\frac{t - t_{n-1}}{\Delta t}\right)^2 & \cdots & \left(\frac{t - t_{n-1}}{\Delta t}\right)^p \end{bmatrix}^T .$$

We give the notations $\theta^+ = \theta\left(t_{n-1}^+\right)$, and $\theta^- = \theta\left(t_{n-1}^-\right)$ referring to the right and left limits at time t_{n-1} respectively where $\Delta t = t_{n+1} - t_n$.

The coefficients of matrices $\left[\int_{t_{n-1}}^{t_n}\left(\theta\frac{\partial\theta^T}{\partial t}\right)dt\right]$, $\left[\int_{t_{n-1}}^{t_n}\left(\theta\theta^T\right)dt\right]$, $\left[(\theta^+)(\theta^+)^T\right]$ and $\left[(\theta^-)(\theta^-)^T\right]$ can be calculated. So, we can derive the full form of all matrices resulting to the full system that can be solved iteratively to obtain approximate solutions when the initial and boundary conditions have been specified.

6. Numerical Results

The time-dependent heat equation in terms of temperature $T(x,t)$ is

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2} \quad , \qquad \text{------ (11)}$$

where $\partial T/\partial t$ is the rate of change of temperature. We have set the thermal diffusivity as $k/\rho c = \alpha = 1$. The boundary and initial conditions are given by

$$T(0,t) = T(1,t) = 0 , \qquad T(x,0) = \sin(\pi x) .$$

The exact solution for this problem is $T(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

For discretization in time, we set $\theta = \begin{bmatrix} 1 & \frac{t - t_{n-1}}{\Delta t} \end{bmatrix}^T$ which corresponds to $k = 0$ and 1.

The time step is $\Delta t = 0.005$ for all calculations

To check the accuracy of the presented numerical schemes, we use the RMS error defined by $RMS\ error = \sqrt{\frac{\sum_{i=1}^{N}(T_i - T_{Exact})^2}{N}}$.

The profiles of numerical solutions at various time steps (0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05) are shown in Fig 8. The temperature profile decreases dramatically as time increases. In this figure, the numerical solutions are obtained by the finite element method based on linear Lagrange basis with 8 elements. The numerical results are in good agreement with the exact solutions even though we have used a small number of elements.

To investigate the convergent rate of our presented numerical schemes for various types of basis functions as described in the previous sections, if we have observed in two cases of the element size, Δx and $\Delta x/2$, the rate of convergence (r) of numerical method would be defined by

$$r = \frac{\log\left(RMS_{\Delta x}/RMS_{\Delta x/2}\right)}{\log(2)} .$$

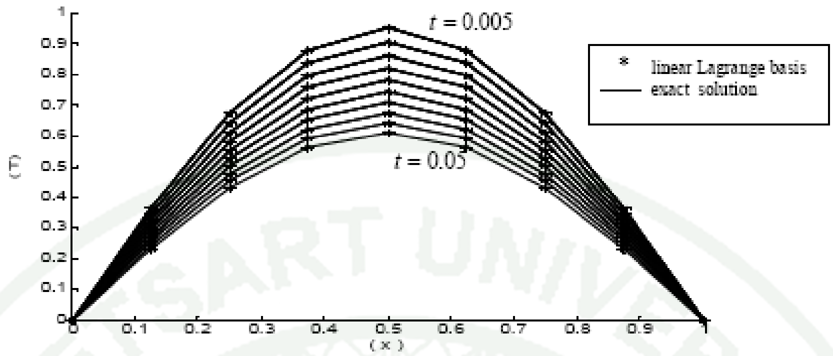

Fig.8 The finite element results with linear Lagrange basis and the exact solutions

The numerical solutions at various time steps are shown in Tables 1- 4. Comparing Table 1 with Table 2, the RMS errors by the linear Lagrange basis are almost the same as the RMS errors by the linear wavelet basis. This implies that the accuracy is the same for these two types of basis function when we use the same element size. The rate of convergence is approximately 2.1 as expected for the linear basis. The RMS errors and the rate of convergences for the quadratic Lagrange and wavelet bases are shown in Tables 3 and 4 respectively. The rate of convergences for the quadratic Lagrange and the wavelet bases are approximately 4.1 and 4.4 respectively. The plots of rate of convergence are shown in Fig. 9.

Table 1 : **RMS error** [Galerkin Finite Element Method (linear Lagrange basis function)]

elements \ time	0.01	0.02	0.03	0.04	0.05
8 element	8.726268905904809e-004	1.580224097384974e-003	2.146199506737274e-003	2.591009692454636e-003	2.932508760061338e-003
16 element	2.101481294762864e-004	3.807355277453610e-004	5.173477287369085e-004	6.248692519124532e-004	7.075661517082727e-004
32 element	5.171746870980369e-005	9.370994776400295e-005	1.273489192260620e-004	1.538339394825062e-004	1.742128449847294e-004
64 element	1.295589898436060e-005	2.348287635391392e-005	3.192320535664703e-005	3.857644890470036e-005	4.370406764447197e-005
Approximate convergence rate = 2.1					(Δt = 0.005 , TOL = 10^(-8))

Table 2 : **RMS error** [Galerkin Finite Element Method (linear wavelet basis function)]

Level \ time	0.01	0.02	0.03	0.04	0.05
W_3	8.726288567183947e-004	1.580227799601058e-003	2.146204899960899e-003	2.591016580311023e-003	2.932517129863043e-003
W_4	2.101477722397183e-004	3.807347698501974e-004	5.173463975690057e-004	6.248674850133021e-004	7.075637864254972e-004
W_5	5.171717657645989e-005	9.370948251299753e-005	1.273484125676908e-004	1.538335007287372e-004	1.742125961779128e-004
W_6	1.290094309298124e-005	2.337657080451988e-005	3.176862958491356e-005	3.837661665943470e-005	4.348140681479823e-005
Approximate convergence rate = 2.1					(Δt = 0.005 , TOL = 10^(-8))

Table 3 : **RMS error** [Galerkin Finite Element Method (quadratic Lagrange basis function)]

elements \ time	0.01	0.02	0.03	0.04	0.05
2 element	6.497522452264082e-004	1.184190592039042e-003	1.618358499025403e-003	1.962754030665724e-003	2.228842483549369e-003
4 element	6.166581330040138e-005	7.391927590789998e-005	9.106001552396346e-005	1.071794464504381e-004	1.206201025481296e-004
8 element	4.134782778254279e-006	4.738086410483837e-006	5.701617731023266e-006	6.650482265400279e-006	7.457947403317509e-006
16 element	2.982262211277284e-007	4.290036445807275e-007	5.667350507200760e-007	6.842148370459997e-007	7.780697815854959e-007
Approximate convergence rate = 4.1					(Δt = 0.005 , TOL = 10^(-12))

Table 4 : RMS error [Galerkin Finite Element Method (quadratic wavelet basis function)]

Level \ time	0.01	0.02	0.03	0.04	0.05
W_1	5.348649462580742e-004	9.708823628853893e-004	1.325039573628299e-003	1.605906839786903e-003	1.822858413772621e-003
W_2	5.477075921482611e-005	6.236986469117710e-005	7.731053327563377e-005	9.233279757502216e-005	1.051843778174137e-004
W_3	3.994579649279454e-006	4.577426359727216e-006	5.508296082458823e-006	6.424976337226647e-006	7.205061791732285e-006
W_4	2.933766984517604e-007	4.220275485367114e-007	5.575192941479182e-007	6.730887317638654e-007	7.654174944650751e-007
Approximate convergence rate = 4.4					(Δt = 0.005 , TOL = 10^(-12))

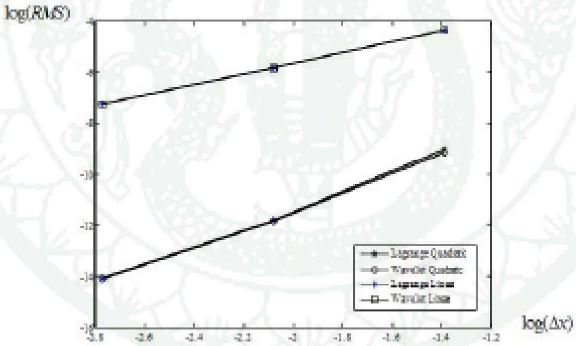

Fig.9 Rate of convergence

7. Conclusions

In this work, we have presented the Galerkin finite element method to solve numerically the one-dimensional heat equation. Two types of basis functions which are the Lagrange and wavelet bases are employed to derive the full system. We consider both linear and quadratic bases. Moreover, we have also introduced a time basis in the time discretization process. When the initial and boundary conditions are specified, the full system matrices can be solved iteratively by the Gauss-Seidel method. Our numerical results show that the rate of convergence for both linear Lagrange and linear wavelet bases is the same in order of 2 while the rate of convergence for both quadratic Lagrange and quadratic wavelet bases is approximately in order of 4. These two rates are in expected. The numerical resolutions can be increased by increasing the number of wavelet basis levels. This shows an advantage of the wavelet basis. By this point of investigation, we will apply the presented wavelet bases with multilevel approach to further investigate the other types of differential equation, especially for the singularly perturbed problem. Some results will be reported elsewhere further.

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