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NAME: Mr. Thaweedet Yantassanavanich

THIS THESIS HAS BEEN ACCEPTED BY

________________ **THESIS ADVISOR**

(Associate Professor Vichian Laohakosol, Ph.D.)

________________ **COMMITTEE MEMBER**

(Miss Pattira Ruengsinsub, Ph.D.)

________________ **COMMITTEE MEMBER**

(Associate Professor Premjai Trisaranuwatana, M.Stat.)

________________ **DEPARTMENT HEAD**

(Assistant Professor Utsanee Leerawat, Ph.D.)

APPROVED BY THE GRADUATE SCHOOL ON ________________

________________ **DEAN**

(Associate Professor Vinai Artkongharn, M.A.)

THESIS

VANISHING SUM AND ROOTS OF UNITY

THAWEEDET YANTASSANAVANICH

A Thesis Submitted in Partial Fulfillment of
the Requirements for the Degree of
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A vanishing sum is an equation

$$u_1\alpha_1 + \cdots + u_k\alpha_k = 0,$$

where each α_i is an n^{th} root of unity and $u_i \in \mathbb{Q}$.

In 1958, Schoenberg constructed a $\mathbb{Z}$-basis for the module whose elements satisfy a vanishing sum and proved identities about cyclotomic polynomials. Moreover he proved that if $u \in \mathbb{Z}^n$ satisfies a vanishing sum, then it belongs to a $\mathbb{Z}$-module whose basis consists of all possible regular subpolygons inscribed in the regular n-gon.

In 2000, Lam and Leung proved that if $u \in \mathbb{Z}^n$ satisfies a vanishing sum, then u can be represented as a linear $\mathbb{Z}$-combination of elements belonging to subgroups of the cyclic group of order n.

In this thesis, we construct a $\mathbb{Z}$-basis for the module of elements satisfying more general vanishing sums. Next, we define general cyclotomic polynomials and give their basic properties. Furthermore, we extend Schoenberg's results about the $\mathbb{Z}$-basis of a vanishing sum module as well as its geometric meaning to this general setting. In the last part, we compare between the main theorem of Lam and Leung with that of Schoenberg.

Student's signature   Thesis Advisor's signature   —/—/—

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Thaweedet Yantassanavanich
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LIST OF SYMBOLS AND ABBREVIATIONS

VANISHING SUM AND ROOTS OF UNITY

INTRODUCTION

For a positive integer n, the n^{th} roots of unity are complex numbers α satisfying $\alpha^n - 1 = 0$. A vanishing sum is an equation of the shape

$$u_1\alpha_1 + \cdots + u_k\alpha_k = 0,$$

where each α_i is an n^{th} root of unity and $u_i \in \mathbb{Q}$.

In 1958, in his work about vanishing sums Schoenberg constructed a $\mathbb{Z}$-basis for the module whose elements satisfy a vanishing sum and proved certain identities about cyclotomic polynomials. Using a geometrical representation that all the n^{th} roots of unity give rise to a regular n-gon centered at the origin and radius 1, he proved that if $u \in \mathbb{Z}^n$ satisfies a vanishing sum, then it belongs to a $\mathbb{Z}$-module whose basis consists of all possible regular subpolygons inscribed in the regular n-gon.

In 2000, Lam and Leung studied the problem of vanishing sums using the notion of group-ring and, among other things, they proved that if $u \in \mathbb{Z}^n$ satisfies a vanishing sum, then u can be represented as a linear $\mathbb{Z}$-combination of elements belonging to subgroups of the cyclic group of order n.

In the first two sections of this thesis, basic results about cyclotomic polynomials and regular polygons are given. In Section 3, slightly extending Schoenberg's idea, we construct a $\mathbb{Z}$-basis for the module of elements satisfying more general vanishing sums. In Section 4, allowing the regular n-gon to have any fixed radius and moving it to a general position in the complex plane through the result of Skutnik, more general concepts of cyclotomic polynomials are introduced and their basic properties are verified. In Section 5, we extend Schoenberg's results about the $\mathbb{Z}$-basis of a vanishing sum module as well as its geometric meaning to this general setting. Section 6 gives a comparison between the main theorem of Lam and Leung with that of Schoenberg.

OBJECTIVES

1. Find a $\mathbb{Z}$-basis of a vanishing sum module.
2. Define general cyclotomic polynomials and study their basic properties.
3. Extend Schoenberg's result about $\mathbb{Z}$-basis of a vanishing sum module.
4. Compare the main result of Lam and Leung with that of Schoenberg.

LITERATURE REVIEW

Let $n \in \mathbb{N}$ be fixed throughout the entire work. The n^{th} roots of unity, $\omega^m := \exp\left(\frac{2\pi i m}{n}\right) \quad (m = 0, 1, \ldots, n-1)$, are complex numbers satisfying $x^n - 1 = 0$. If ω is a n^{th} root of unity which has the property that all the numbers $\omega^0 = 1, \omega^1, \omega^2, \ldots, \omega^{n-1}$ are distinct, then ω is said to be a **primitive** n^{th} root of unity. A necessary and sufficient condition for ω^m to be a primitive n^{th} root of unity is that $\gcd(m, n) = 1$.

The **cyclotomic polynomial** of degree $\phi(n)$(Euler phi function) is defined as

$$\Phi_n(x) = \prod_{\substack{m \\ \gcd(m,n)=1}} (x - \omega^m),$$

where the product runs through all the primitive n^{th} roots of unity.

Let $p_1, p_2, \ldots, p_r$ be all distinct prime fartors of n. Define

$$\prod_0 = x^n - 1, \quad \prod_v = \prod (x^{\frac{n}{p_{i_1} p_{i_2} \cdots p_{i_v}}} - 1) \quad (1 \leq v \leq r),$$

where the last product extends over all the v indices i_k which satisfy the conditions

$$1 \leq i_1 < i_2 < \ldots < i_v \leq r.$$

It is well-known, see e.g. p. 157-158 of Nagell, that

$$\Phi_n(x) \;=\; \frac{\prod_0 \prod_2 \cdots}{\prod_1 \prod_3 \cdots}.$$

The following proposition summarizes basic properties of cyclotomic polynomials.

Proposition 1. *Let p be a prime number.*

1. *If $p|n$, then $\Phi_{np}(x) \;=\; \frac{\Phi_n(x^p)}{\Phi_n(x)}$.*
2. *If $p \nmid n$, then $\Phi_{np}(x) \;=\; \Phi_n(x^p)$.*
3. *$\Phi_p(x) \;=\; \frac{x^p - 1}{x - 1}$.*

The first six cyclotomic polynomials are

$$\Phi_1(x) = x - 1, \Phi_2(x) = x + 1, \Phi_3(x) = x^2 + x + 1, \Phi_4(x) = x^2 + 1,$$
$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1, \Phi_6(x) = x^2 - x + 1.$$

1. The work of I.J. Schoenberg

Let $\omega = e^{\frac{2\pi i}{n}}$ be a primitive n^{th} root of unity. Since all the n^{th} roots of unity $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are clearly linearly dependent over $\mathbb{Q}$, then there exist $u_0, u_1, \ldots, u_{n-1} \in \mathbb{Q}$ are not all zero such that

$$u_0 + u_1\omega + \cdots + u_{n-1}\omega^{n-1} = 0. \tag{1}$$

In general, there are infinitely many such n-tuples $(u_0, u_1 \ldots, u_{n-1}) \in \mathbb{Q}^n$. A relation such as (1) is called a vanishing sum of the n^{th} roots of unity or **vanishing sum**, for short. Let

$$N_c := \left\{(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Q}^n;\ u_0 + u_1\omega + \cdots + u_{n-1}\omega^{n-1} = 0\right\}.$$

Clearly, $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$ with N_c being a subspace. Let

$$M_c := \{(u_0, u_1, \ldots, u_{n-1}) \in N_c;\ u_i \in \mathbb{Z}\}.$$

Clearly, M_c is a module over $\mathbb{Z}$, referred to as a **vanishing sum module**. Define the map $\varphi : \mathbb{Q}^n \longrightarrow \mathbb{Q}[x]$ by

$$\varphi((u_0, u_1, \ldots, u_{n-1})) = u_0 + u_1 x + \cdots + u_{n-1}x^{n-1},$$

and let

$$f_\nu = \varphi^{-1}(x^\nu \Phi_n(x)) \qquad (\nu = 0, 1, \ldots, n - \phi(n) - 1).$$

Schoenberg's first result is:

Theorem 1. *The set* $\left\{f_0, f_1, \ldots, f_{n-\phi(n)-1}\right\}$ *is a* $\mathbb{Z}$*-basis of the module* M_c*.*

Schoenberg's second, auxilaiary, result is:

Theorem 2. *The* n^{th} *cyclotomic polynomial can be written in the form*

$$\Phi_n(x) = \sum_{p|n} e(x;p)\Phi_p(x^{\frac{n}{p}}),$$

where $e(x;p) \in \mathbb{Z}[x]$.

Taking into account the n-fold symmetry of a *standard n-gon*, i.e., a regular n-gon whose vertices are all the n^{th} roots of unity situated on the unit circle in the complex plane, elements of M_c can be described via the following geometrical observation: $u \in M_c$ if all $u_i = 0$ except for certain k elements $u_i = 1$ corresponding to the points ω^k forming the vertices of a standard k-gon. This in turn necessitates that $k \mid n$. To analyze all the possibilities, let $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots$, where $p_1, p_2, p_3, \ldots$ are distinct primes and $a_i (i = 1, 2, 3, \ldots) \in \mathbb{N}$. In the large standard n-gon we can inscribe

- $\frac{n}{p_1}$ distinct standard p_1-gons,
- $\frac{n}{p_2}$ dintinct standard p_2-gons,
- $\frac{n}{p_3}$ dintinct standard p_3-gons, ...

Denote the set of all such standard

- p_1-gons by P_α $\left(\alpha = 0, \ldots, \frac{n}{p_1} - 1\right)$,
- p_2-gons by Q_β $\left(\beta = 0, \ldots, \frac{n}{p_2} - 1\right)$,
- p_3-gons by R_γ $\left(\gamma = 0, \ldots, \frac{n}{p_3} - 1\right)$, etc.,

where we adopt the convention that P_α starts with the vertex ω^α, etc.

Example 1. For $n = 6 = 2 \times 3$, then $p = 2, q = 3$. The three possible 2-gons, and two possible 3-gons are

$$P_0 = \langle 1,0,0,1,0,0 \rangle , \ P_1 = \langle 0,1,0,0,1,0 \rangle , \ P_2 = \langle 0,0,1,0,0,1 \rangle ,$$
$$Q_0 = \langle 1,0,1,0,1,0 \rangle , \ Q_1 = \langle 0,1,0,1,0,1 \rangle ,$$

where each 6-tuple correspond to all possible values of the 6-tuple $\langle u_0, \ldots, u_5 \rangle$ representing the 2-gons and 3-gons, respectively.

Schoenberg's third and main result reads:

Theorem 3. *If $u \in M_c$, then it is representable as a $\mathbb{Z}$-linear combination of all the possible subpolygons of the regular n-gon, i.e.,*

$$u = \sum a_\alpha P_\alpha + b_\beta Q_\beta + \cdots .$$

2. The work of Lam and Leung

Let R be a commutative ring with identity $1 \neq 0$, and let $G = \{g_1, \ldots, g_n\}$ be a finite multiplicative group. A group-ring RG is the set of all formal sums

$$a_1g_1 + a_2g_2 + \cdots + a_ng_n$$

where $a_i \in R$, $1 \leq i \leq n$. Addition is defined componentwise, i.e.,

$$(a_1g_1 + \cdots + a_ng_n) + (b_1g_1 + \cdots + b_ng_n) = (a_1 + b_1)g_1 + \cdots + (a_n + b_n)g_n.$$

Multiplication is performed by first defining

$$(ag_i)(bg_j) = (ab)g_k,$$

where the product $ab \in R$ and $g_ig_j = g_k \in G$. This product is then extended to all formal sums by the distributive laws so that the coefficient of g_k can be represented as the product

$$(a_1g_1 + \cdots + a_ng_n) \times (b_1g_1 + \cdots + b_ng_n) = \sum_{g_ig_j = g_k} a_ib_jg_k.$$

A group-ring RG is commutative if and only if G is a commutative group.

Example 2. Let $G = D_8$ be the dihedral group of order 8 with the usual generators r, s $(r^4 = s^2 = 1,\ rs = sr^{-1})$ and let $R = \mathbb{Z}$. Their corresponding group-ring is $\mathbb{Z}D_8$.

Let $p_1, p_2, ..., p_r$ be all distinct prime factors of $n \in \mathbb{N}$, and let G be the cyclic group of order $n = p_1^{a_1} \ldots p_r^{a_r}$ where $a_i \in \mathbb{N}$. We write G using its generator $z \in G$ as $G = < z >$. Let ω be a primitive n^{th} root of unity. The canonical homomorphism, referred to as the usual map, $\psi : \mathbb{Z}G \longrightarrow \mathbb{Z}[\omega]$ is defined by

$$\psi(a_0 + a_1z + \cdots + a_{n-1}z^{n-1}) = a_0 + a_1\omega + \cdots + a_{n-1}\omega^{n-1}.$$

For

$$x = \sum_{g \in G} x_gg \in \mathbb{Z}G,$$

clearly, $x \in ker(\psi)$ if and only if $\sum_{g \in G} x_g\psi(g) = 0 \in \mathbb{Z}[\omega]$. Let $S(P_i)$ $(1 \leq i \leq r)$ be the unique subgroup of G of order p_i, and define

$$\sigma(S(P_i)) = \sum_{g \in S(P_i)} g \in \mathbb{Z}G.$$

Clearly, for $g \in S(P_i)$, we have $\sigma(S(P_i))g = \sigma(S(P_i))$, and $\sigma(S(P_i)) \in ker(\psi)$.

A vanishing sum without any vanishing subsum is called a **minimal vanishing sum**.

Lam and Leung main results are:

- $ker(\psi) = \sum_{i=1}^{r} \mathbb{Z}G\sigma(S(P_i))$ and $ker(\psi) = \mathbb{Z}\sigma(S(P))$ if $n = p$.

- Let G be a cyclic group of order $n = p_1^{a_1} \ldots p_r^{a_r}$, where $p_1, \ldots, p_r$ are distinct primes, and let $\psi : \mathbb{Z}G \longrightarrow \mathbb{Z}[\omega]$ be the usual map, where ω is a primitive n^{th} root of unity. Let $G_0 \subseteq G$ be the (unique) subgroup of order $n_0 = p_1 \ldots p_r$ and let $\{g_j : 1 \leq j \leq [G : G_0]\}$ be a complete set of coset representatives of G with respect to G_0.

(i) Then

$$NG \cap ker(\psi) = \sum_j g_j(NG_0 \cap ker(\psi)).$$

(ii) If $\alpha_1 + \cdots + \alpha_k = 0$ is a minimal vanishing sum, then after a suitable rotation, all α_i's are n_0^{th} roots of unity where n_0 is square free.

(iii) If $r = 1$, then $NG \cap ker(\psi) = N\sigma(S(P_1))$.

(iv) If $r = 2$, then $NG \cap ker(\psi) = NS(P_1)\sigma(S(P_2)) + NS(P_2)\sigma(S(P_1))$.

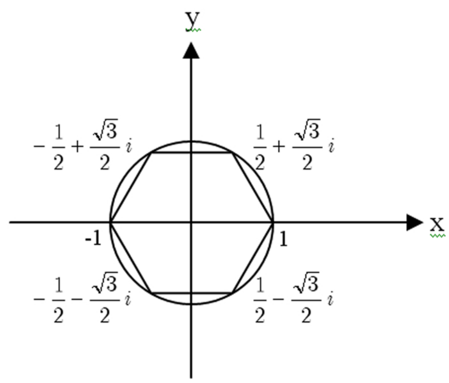
- Let G be a cyclic group of order $n = p_1^{a_1} p_2^{a_2}$, where p_1, p_2 are distinct primes. Then up to a rotation, the only minimal vanishing sums are $1 + \omega_{p_1} + \cdots + \omega_{p_1}^{p_1 - 1}$ and $1 + \omega_{p_2} + \cdots + \omega_{p_2}^{p_2 - 1}$ where ω_{p_1} is a primitive p_1^{th} root of unity and ω_{p_2} is a primitive p_2^{th} root of unity.

MATERIALS AND METHODS

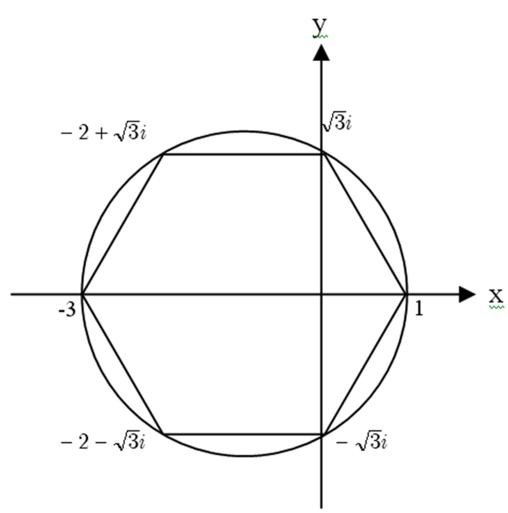
First, recall the following well-known result about integer polynomials.

Lemma 1. ***(Gauss)*** (Pollard and Diamond, 1975) *If $p(x) \in \mathbb{Z}[x]$ can be factored over $\mathbb{Q}$, then it can be factored into polynomials in $\mathbb{Z}[x]$.*

It is well-known that all the n^{th} roots of unity lie on the unit circle at the n vertices of a regular n-gon whose center is the origin. The following diagram illustrates the case $n = 6$.



It is natural to ask whether there are any polynomials in $\mathbb{Z}[x]$ all of whose roots represent an arbitrary regular n-gon. This is witnessed by the following example: $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x - 63$



For convenience, we refer to a polynomial $f(x) \in \mathbb{C}[x],\ \deg f = n$ as a (general) n**-gon polynomial** if all its roots of $f(x)$ form a regular n-gon in the complex plane. B.J. Skutnik, in 1978 gave a complete characterization of n-gon polynomials, which we now elaborate.

Theorem 4. (Skutnik, 1978) *Let* $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \in \mathbb{C}[x]$*. Then* $f(x)$ *is an* n*-gon polynomial if and only if*

$$a_k = \frac{\binom{n}{k}a_1^k}{n^k}\ (k = 2, 3, \ldots, n-1),\ a_n \neq \left(\frac{a_1}{n}\right)^n.$$

Proof. The hypothesis about the coefficients holds if and only if

$$\begin{aligned}
f(x) &= x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \\
&= x^n + a_1x^{n-1} + \binom{n}{2}\frac{a_1^2x^{n-2}}{n^2} + \cdots + \binom{n}{n-1}\frac{a_1^{n-1}x}{n^{n-1}} + a_n \\
&= \frac{1}{n^n}\left\{(xn)^n + \binom{n}{1}a_1(xn)^{n-1} + \binom{n}{2}a_1^2(xn)^{n-2} + \cdots + \binom{n}{n-1}a_1^{n-1}(xn)\right\} + a_n \\
&= \frac{1}{n^n}(xn + a_1)^n + a_n - \left(\frac{a_1}{n}\right)^n := \left(x + \frac{a_1}{n}\right)^n - c^n,
\end{aligned}$$

where $c \in \mathbb{C}$ satisfies $c^n = \left(\frac{a_1}{n}\right)^n - a_n \neq 0$. This last polynomial is an n-gon polynomial with center $-\frac{a_1}{n}$ and radius $|c|$. □

Note that the above proof of Skutnik's theorem also provides us with a map

$$\begin{aligned}
T : \mathbb{C} &\longrightarrow \mathbb{C} \\
y &\longmapsto Y = T(y) = \frac{y}{c} + \frac{a_1}{nc},
\end{aligned} \tag{2}$$

and the inverse map is

$$T^{-1}:\ Y \longmapsto y = T^{-1}(Y) = cY - \frac{a_1}{n}.$$

Observe that

1. the map T sends the n-gon polynomial
$$\left\{z \in \mathbb{C}\ :\ \left(\frac{z}{c} + \frac{a_1}{nc}\right)^n - 1 = 0\right\} = \left\{z \in \mathbb{C}\ :\ \left(z + \frac{a_1}{n}\right)^n - c^n = 0\right\}$$
with center at $-\frac{a_1}{n}$ and radius $|c|$ onto the standard n-gon polynomial $\{Z \in \mathbb{C} : Z^n - 1 = 0\}$;

2. the inverse map T^{-1} sends each root ω^i of the standard n-gon polynomial $Z^n - 1$ onto the corresponding root

$$\zeta_i = T^{-1}(\omega^i) = c\omega^i - \frac{a_1}{n} \quad (i = 0, 1, \ldots, n-1)$$

of the n-gon polynomial $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$ with $a_k = \frac{\binom{n}{k} a_1^k}{n^k}$ $(k = 2, 3, \ldots, n-1)$, $c^n = \left(\frac{a_1}{n}\right)^n - a_n \neq 0$.

Location and Duration of Research

Location Department of Mathematics, Kasetsart University.

Duration of Research May 2006-May 2007.

RESULTS AND DISCUSSION

$\mathbb{Z}$-basis of the vanishing sum module

Let $P(x) \in \mathbb{Z}[x]$, be monic with $\deg P = n$ and let ξ be a root of $P(x)$. Since ξ is an algebraic integer, say, of degree m, denote its minimal polynomial by

$$M(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0 \in \mathbb{Z}[x].$$

It is well-known that $M(x)|P(x)$ and $m \leq n$. If $m = n$, then it is easy to see that $P(x)$ is a constant multiple of $M(x)$. In practice we prefer to have $P(x)$ of a very simple form in contrast to $M(x)$. This generally requires n to be much larger than m. Henceforth, we assume that $n > m$.

Let

$$N_P = \{(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Q}^n : u_0 + u_1\xi + \cdots + u_{n-1}\xi^{n-1} = 0\},$$

and

$$M_P = \{(u_0, u_1, \ldots, u_{n-1}) \in N_P : u_i \in \mathbb{Z}\}.$$

Clearly, N_P is a vector space over $\mathbb{Q}$, and M_P is a module over $\mathbb{Z}$. Using Schoenberg's approach in the proof of his first main result, a $\mathbb{Z}$-basis for M_P is now determined. To do so, we define the map

$$\begin{aligned} \varphi : \mathbb{Q}^n &\longrightarrow \mathbb{Q}[x] \bmod P(x) \\ (u_0, u_1, \ldots, u_{n-1}) &\longmapsto u_0 + u_1x + \cdots + u_{n-1}x^{n-1} := \varphi_u(x). \end{aligned}$$

It is easy to show that φ is an isomorphism. Let

$$l_\nu = \varphi^{-1}\left(x^\nu M(x)\right) \qquad (\nu = 0, 1, \ldots, n - m - 1).$$

Clearly, $l_\nu \in M_P$ for all $\nu \in \{0, 1, \ldots, n - m - 1\}$.

Theorem 5. *The set $\{l_0, l_1, \ldots, l_{n-m-1}\}$ is a $\mathbb{Z}$-basis of the vanishing sum module M_P .*

Proof. Let $u = (u_0, u_1, \ldots, u_{n-1}) \in M_P$. Then $\varphi_u(x) = u_0 + u_1x + \cdots + u_{n-1}x^{n-1}$ satisfies $\varphi_u(\xi) = 0$, and so $M(x) \mid \varphi_u(x)$ in $\mathbb{Q}[x]$. By Gauss lemma, $M(x) \mid \varphi_u(x)$ in $\mathbb{Z}[x]$, i.e.,

$$\varphi_u(x) = u_0 + u_1x + \cdots + u_{n-1}x^{n-1} = M(x)(c_0 + c_1x + \cdots + c_{n-m-1}x^{n-m-1}), \quad (3)$$

where $c_0 + c_1x + \cdots + c_{n-m-1}x^{n-m-1} \in \mathbb{Z}[x]$.

We next show that the set $\{l_0, l_1, \ldots, l_{n-m-1}\}$ spans the vanishing sum $\mathbb{Z}$-module M_P. If $u = (u_0, \ldots, u_{n-1}) \in M_P$, from (3) we have

$$\varphi^{-1}(u_0 + u_1x + \cdots + u_{n-1}x^{n-1}) = \varphi^{-1}\left(M(x)(c_0 + c_1x + \cdots + c_{n-m-1}x^{n-m-1})\right).$$

Since φ is an isomorphism, we have

$$\begin{aligned} u &= c_0\varphi^{-1}(M(x)) + c_1\varphi^{-1}(x^1M(x)) + \cdots + c_{n-m-1}\varphi^{-1}(x^{n-m-1}M(x)) \\ &= c_0l_0 + c_1l_1 + \cdots + c_{n-m-1}l_{n-m-1}. \end{aligned}$$

There remains to check that $l_0, l_1, l_2, \ldots, l_{n-m-1}$ are linearly independent over $\mathbb{Z}$. Assume that there are integers $a_0, a_1, \ldots, a_{n-m-1}$ not all zero such that

$$a_0l_0 + a_1l_1 + \cdots + a_{n-m-1}l_{n-m-1} = (0, \ldots, 0).$$

Then

$$\begin{aligned} 0 &= a_0\varphi(l_0) + a_1\varphi(l_1) + \cdots + a_{n-m-1}\varphi(l_{n-m-1}) \\ &= a_0M(x) + a_1xM(x) + \cdots + a_{n-m-1}x^{n-m-1}M(x). \end{aligned} \tag{4}$$

Since $\deg M(x) = m$ and $M(x)$ is monic, comparing the leading coefficients in (4), we get $a_{n-m-1} = 0$, and (4) becomes

$$a_0M(x) + a_1xM(x) + \cdots + a_{n-m-2}x^{n-m-2}M(x) = 0.$$

Repeating the procedure, we see that all $a_i = 0$. Consequently, $l_0, l_1, \ldots, l_{n-m-1}$ are linearly independent over $\mathbb{Z}$. □

Example 3. Let $P(x) = x^4 - 1$. The four roots of $P(x)$ are $1, -1, i, -i$. For i, the minimal polynomial of i is $M(x) = x^2 + 1$. Let

$$M_P = \{(u_0, u_1, u_2, u_3) \in \mathbb{Z}^4 \mid u_0 + u_1i - u_2 - u_3i = 0\}.$$

Here,

$$l_0 = \varphi^{-1}(M(x)) = (1, 0, 1, 0), \quad l_1 = \varphi^{-1}(xM(x)) = (0, 1, 0, 1).$$

In this example, elements in M_P are of the form $al_0 + bl_1 = (a, b, a, b) \ (a, b \in \mathbb{Z})$.

Example 4. Let $P(x) = x^4 - 16$. The four roots of $P(x)$ are $2, -2, 2i, -2i$. For $\xi = -2i$, the minimal polynomial of ξ is $M(x) = x^2 + 4$. Since $\xi^2 = -4,\ \xi^3 = 8i$, we have

$$M_P \ = \ \{\ (u_0, u_1, u_2, u_3) \in \mathbb{Z}^4 \mid u_0 - 2iu_1 - 4u_2 + 8iu_3 \ = \ 0\ \}.$$

Here,

$$l_0 = \varphi^{-1}(M(x)) = (4, 0, 1, 0), \quad l_1 = \varphi^{-1}(xM(x)) = (0, 4, 0, 1).$$

Each element of M_P is of the form $al_0 + bl_1 = (4a, 4b, a, b) \ \ (a, b \in \mathbb{Z})$.

General cyclotomic polynomials

Recall that the classical cyclotomic polynomial of order $n \in \mathbb{N}$ is defined as

$$\Phi_n(x) = \prod_{\substack{i \\ \gcd(i,n)=1}} (x - \omega^i),$$

which reflects the position of its roots on the unit circle about the origin. Using Skutnik's transformation (2), we define the **general cyclotomic polynomial**, abbreviated as GCP, of order n by

$$F_n(x) := \Phi_n\left(\frac{x}{c} + \frac{a_1}{nc}\right) = \prod_{\substack{i \\ \gcd(i,n)=1}} \left(\frac{x}{c} + \frac{a_1}{nc} - \omega^i\right),$$

where $a_1, a_n, c \in \mathbb{C}$ with $c^n = \left(\frac{a_1}{n}\right)^n - a_n$. A simple change of variable yields

$$\Phi_n(x) = F_n\left(cx - \frac{a_1}{n}\right).$$

The rest of this section is devoted to the derivation of basic properties of GCP to be used later. The main result in this section says that a GCP of order n can be written as a linear combination of GCP's of prime orders. More precisely, we have

Theorem 6. *For $n \in \mathbb{N}$, then*

$$F_n(x) = \sum_{p|n} E(X;p) F_p\left(cX^{\frac{n}{p}} - \frac{a_1}{n}\right),$$

where $X := \frac{x}{c} + \frac{a_1}{nc}$, $E(X;p) \in \mathbb{Z}[X]$.

Proof. From Schoenberg's second result, we have

$$\Phi_n(x) = \sum_{p|n} e(x;p)\Phi_p(x^{\frac{n}{p}}),$$

where $e(x;p) \in \mathbb{Z}[x]$. Thus,

$$F_n(x) = \Phi_n\left(\frac{x}{c} + \frac{a_1}{nc}\right) = \sum_{p|n} e\left(\frac{x}{c} + \frac{a_1}{nc};p\right)\Phi_p\left(\left(\frac{x}{c} + \frac{a_1}{nc}\right)^{\frac{n}{p}}\right)$$

$$= \sum_{p|n} E(X;p)F_p\left(c\left(\frac{x}{c} + \frac{a_1}{nc}\right)^{\frac{n}{p}} - \frac{a_1}{n}\right) = \sum_{p|n} E(X;p)F_p\left(cX^{\frac{n}{p}} - \frac{a_1}{n}\right).$$

□

Corollary 1. *If p is a prime dividing n, then*

$$F_{np}(x) = \frac{F_n\left(c(\frac{x}{c}+\frac{a_1}{nc})^p - \frac{a_1}{n}\right)}{F_n(x)}.$$

Proof. From Proposition 1,

$$\Phi_{np}(x) = \frac{\Phi_n(x^p)}{\Phi_n(x)},$$

and so

$$F_{np}(x) = \Phi_{np}\left(\frac{x}{c}+\frac{a_1}{nc}\right) = \frac{\Phi_n\left((\frac{x}{c}+\frac{a_1}{nc})^p\right)}{\Phi_n(\frac{x}{c}+\frac{a_1}{nc})}.$$

Therefore

$$F_{np}(x) = \frac{F_n\left(c(\frac{x}{c}+\frac{a_1}{nc})^p - \frac{a_1}{n}\right)}{F_n(x)}.$$

□

Corollary 2. *Let p be prime number. If $p \nmid n$, then*

$$F_{np}(x) = F_n\left(c\left(\frac{x}{c}+\frac{a_1}{nc}\right)^p - \frac{a_1}{n}\right).$$

Proof. Let $p \nmid n$. By Proposition 1,

$$\Phi_{np}(x) = \Phi_n(x^p),$$

and so

$$\Phi_{np}\left(\frac{x}{c}+\frac{a_1}{nc}\right) = \Phi_n\left(\left(\frac{x}{c}+\frac{a_1}{nc}\right)^p\right).$$

Therefore

$$F_{np}(x) = F_n\left(c\left(\frac{x}{c}+\frac{a_1}{nc}\right)^p - \frac{a_1}{n}\right).$$

□

Corollary 3. *We have*

$$F_n(x) \in \mathbb{Z}\left[\frac{a_1}{nc}\right]\left(\frac{x}{c}\right).$$

Proof. Note that

$$
\begin{aligned}
&F_n(x) = \Phi_n\left(\frac{x}{c} + \frac{a_1}{nc}\right)\\
&= \left(\frac{x}{c} + \frac{a_1}{nc}\right)^{\phi(n)} + b_1\left(\frac{x}{c} + \frac{a_1}{nc}\right)^{\phi(n)-1} + \cdots + b_{\phi(n)-1}\left(\frac{x}{c} + \frac{a_1}{nc}\right) + b_{\phi(n)} \quad (b_i \in \mathbb{Z})\\
&= \left(\frac{x}{c}\right)^{\phi(n)} + \left\{\binom{\phi(n)}{1}\left(\frac{a_1}{nc}\right) + b_1\right\}\left(\frac{x}{c}\right)^{\phi(n)-1} + \cdots\\
&+ \left\{\binom{\phi(n)}{\phi(n)-1}\left(\frac{a_1}{nc}\right)^{\phi(n)-1} + b_1\binom{\phi(n)-1}{\phi(n)-2}\left(\frac{a_1}{nc}\right)^{\phi(n)-2} + \cdots + b_{\phi(n)-1}\right\}\left(\frac{x}{c}\right)\\
&+ \left\{\binom{\phi(n)}{\phi(n)}\left(\frac{a_1}{nc}\right)^{\phi(n)-1} + b_1\binom{\phi(n)-1}{\phi(n)-1}\left(\frac{a_1}{nc}\right)^{\phi(n)-1} + \cdots + b_{\phi(n)}\right\}\\
&\in \mathbb{Z}\left[\frac{a_1}{nc}\right]\left(\frac{x}{c}\right).
\end{aligned}
$$

□

The following example illustrates the result of Theorem 6.

Example 5. Since

$$
\begin{aligned}
F_6(x) &= \Phi_6\left(\frac{x}{c} + \frac{a_1}{nc}\right) = \left(\frac{x}{c} + \frac{a_1}{nc}\right)^2 - \left(\frac{x}{c} + \frac{a_1}{nc}\right) + 1,\\
F_2(x) &= \Phi_2\left(\frac{x}{c} + \frac{a_1}{nc}\right) = \left(\frac{x}{c} + \frac{a_1}{nc}\right) + 1,\\
F_3(x) &= \Phi_3\left(\frac{x}{c} + \frac{a_1}{nc}\right) = \left(\frac{x}{c} + \frac{a_1}{nc}\right)^2 + \left(\frac{x}{c} + \frac{a_1}{nc}\right) + 1,
\end{aligned}
$$

by Theorem 6 we have

$$
\begin{aligned}
F_6(x) &= \left(\frac{x}{c} + \frac{a_1}{nc}\right)^2 - \left(\frac{x}{c} + \frac{a_1}{nc}\right) + 1\\
&= -\left(\frac{x}{c} + \frac{a_1}{nc}\right)\left\{\left(\frac{x}{c} + \frac{a_1}{nc}\right)^3 + 1\right\} + \left\{\left(\frac{x}{c} + \frac{a_1}{nc}\right)^4 + \left(\frac{x}{c} + \frac{a_1}{nc}\right)^2 + 1\right\}\\
&= E(X;2)\Phi_2\left(\left(\frac{x}{c} + \frac{a_1}{nc}\right)^3\right) + E(X;3)\Phi_3\left(\left(\frac{x}{c} + \frac{a_1}{nc}\right)^2\right)\\
&= E(X;2)F_2\left(cX^3 - \frac{a_1}{n}\right) + E(X;3)F_3\left(cX^2 - \frac{a_1}{n}\right).
\end{aligned}
$$

The next example illustrates the result of Corollary 3 for a general cyclotomic polynomial of order 6.

Example 6. Consider the polynomial

$$x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x - 63 \in \mathbb{Z}[x].$$

By Skutnik's theorem, this polynomial is a 6-gon with $\frac{a_1}{n} = \frac{6}{6} = 1$ and $c = 2$. Its corresponding general cyclotomic polynomial of order 6 is

$$F_6(x) = \Phi_6\left(\frac{x}{c} + \frac{a_1}{nc}\right) = \left(\frac{x}{2} + \frac{1}{2}\right)^2 - \left(\frac{x}{2} + \frac{1}{2}\right) + 1 = \left(\frac{x}{c}\right)^2 - \left(\frac{1}{2}\right)^2 + 1.$$

Example 7. Consider the polynomial $x^4 + 2$. By Skutnik's theorem, this is a 4-gon polynomial with $\frac{a_1}{n} = 0$ and $c = \sqrt[4]{-2}$. The corresponding general cyclotomic polynomial of order 4 is

$$F_4(x) \quad = \quad \Phi_4\left(\frac{x}{\sqrt[4]{-2}}\right) \quad = \quad \left(\frac{x}{\sqrt[4]{-2}}\right)^2 + 1.$$

We also have

$$F_4(x) = F_2\left(\sqrt[4]{-2}X^2\right),$$

confirming the result of Theorem 6.

Vanishing sum relative to general n-gon polynomials

In this section, Schoenberg's third result is generalized through the use of n-gon polynomials and general cyclotomic polynomials. We begin by extending a result about the vector space

$$N_c = \left\{(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Q}^n \mid u_0 + u_1\omega + \cdots + u_{n-1}\omega^{n-1} \;=\; 0\right\},$$

where ω is a primitive n^{th} root of unity. Define

$$N := \left\{(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{C}^n \mid u_0 + u_1\omega + \cdots + u_{n-1}\omega^{n-1} \;=\; 0\right\}.$$

Theorem 7. *For each $u \in N$, if there exists $l \in \mathbb{C} - \{0\}$ such that $lu \in \mathbb{Z}^n$, then*

$$u \in \mathbb{Z}\left[\frac{1}{l}\right] P_\alpha + \mathbb{Z}\left[\frac{1}{l}\right] Q_\beta + \cdots,$$

where P_α, $Q_\beta, \ldots$ are standard subpolygons inscribed in the standard n-gon as defined in Section 1.

Proof. For $u = (u_0, \ldots, u_{n-1}) \in N$, from $u_0 + u_1\omega + \cdots + u_{n-1}\omega^{n-1} = 0$, if there exists $l \in \mathbb{C} - \{0\}$ such that $lu \in \mathbb{Z}^n$, then the relation

$$lu_0 + lu_1\omega + \cdots + lu_{n-1}\omega^{n-1} = 0$$

shows that lu belongs to Schoenberg's vanishing sum module. By Schoenberg's third result, we deduce

$$lu = \sum a_\alpha P_\alpha + b_\beta Q_\beta + \cdots.$$

Hence

$$u \in \mathbb{Z}\left[\frac{1}{l}\right] P_\alpha + \mathbb{Z}\left[\frac{1}{l}\right] Q_\beta + \cdots.$$

□

Example 8. The six roots of the polynomial $x^6 - 1$ are

$$1, \frac{1}{2} + \frac{\sqrt{3}i}{2}, -\frac{1}{2} + \frac{\sqrt{3}i}{2}, -1, -\frac{1}{2} - \frac{\sqrt{3}i}{2}, \frac{1}{2} - \frac{\sqrt{3}i}{2}.$$

Let ω be a primitive 6^{th} root of unity. Here,

$$N = \left\{\ (u_0, u_1, u_2, u_3, u_4, u_5) \in \mathbb{C}^6 \mid u_0 + u_1\omega + u_2\omega^2 + u_3\omega^3 + u_4\omega^4 + u_5\omega^5 = 0\right\}.$$

Observe that

$$u = (1 - i, 2 - 2i, 1 - i, 2 - 2i, 1 - i, 2 - 2i) \in N.$$

For $l = 2 + 2i \in \mathbb{C} - \{0\}$, note that

$$lu = (2 + 2i)(1 - i, 2 - 2i, 1 - i, 2 - 2i, 1 - i, 2 - 2i) \in \mathbb{Z}^6,$$

and by Theorem 7, we have

$$\begin{aligned}&(1 - i, 2 - 2i, 1 - i, 2 - 2i, 1 - i, 2 - 2i)\\ &= \frac{4}{2+2i}\langle 1, 0, 1, 0, 1, 0\rangle + \frac{8}{2+2i}\langle 0, 1, 0, 1, 0, 1\rangle.\end{aligned}$$

Let the n-gon polynomial be

$$G(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$

with

$$a_k = \frac{\binom{n}{k} a_1^k}{n^k}\ (k = 2, 3, \ldots, n-1),\ c^n = \left(\frac{a_1}{n}\right)^n - a_n \neq 0.$$

Define the **general vanishing sum module** corresponding to $G(z)$ by

$$M_G := \{u = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Z}^n \mid u_0\zeta_0 + u_1\zeta_1 + \cdots + u_{n-1}\zeta_{n-1} = -a_1\},$$

where ζ_i are all distinct roots of $G(z)$.

Theorem 8. *Let $n \in \mathbb{N}$; $a_1, a_n, c \in \mathbb{C}$ with $c^n = (\frac{a_1}{n})^n - a_n \neq 0$. Assume that $\frac{a_1}{nc} \in \mathbb{Q}$. If*

$$u = (u_0, u_1, \ldots, u_{n-1}) \in M_G$$

and

$$t := \frac{a_1}{nc}(u_0 + u_1 + \cdots + u_{n-1} - n) \in \mathbb{Z},$$

then

$$u \in \frac{a_1}{nc}\mathbb{Z}\langle 1, 0, \ldots, 0\rangle + \mathbb{Z}P_\alpha + \mathbb{Z}Q_\beta + \cdots.$$

Proof. Since $G(z)$ is an n-gon polynomial whose roots are ζ_i, Skutnik's transformation yields

$$\zeta_i = c\omega^i - \frac{a_1}{n} \quad (i = 0, 1, \ldots, n-1)$$

where ω is its corresponding primitive n^{th} root of unity. Then

$$\begin{aligned} M_G &= \left\{(u_0, \dots, u_{n-1}) \in \mathbb{Z}^n : u_0 \left(c - \frac{a_1}{n}\right) + \cdots + u_{n-1}\left(c\omega^{n-1} - \frac{a_1}{n}\right) = -a_1\right\} \\ &= \left\{(u_0, \dots, u_{n-1}) \in \mathbb{Z}^n : u_0 + \cdots + u_{n-1}\omega^{n-1} = \frac{a_1}{nc}\left(u_0 + \cdots + u_{n-1} - n\right) = t\right\} \\ &= \left\{(u_0, \dots, u_{n-1}) \in \mathbb{Z}^n : (u_0 - t) + u_1\omega + \cdots + u_{n-1}\omega^{n-1} = 0\right\} \end{aligned}$$

It follows that $(u_0 - t, u_1, u_2, \dots, u_{n-1}) \in M_c$, the classical vanishing sum module. By Schoenberg's third result, we have

$$(u_0 - t, u_1, u_2, \dots, u_{n-1}) = \sum a_\alpha P_\alpha + b_\beta Q_\beta + \cdots ,$$

where $a_\alpha, b_\beta, .. \in \mathbb{Z}$ and so

$$\begin{aligned} u = (u_0, u_1, u_2, \dots, u_{n-1}) &= t\left\langle 1, 0, \dots, 0\right\rangle + \sum a_\alpha P_\alpha + b_\beta Q_\beta + \cdots \\ &\in \frac{a_1}{nc}\mathbb{Z}\left\langle 1, 0, \dots, 0\right\rangle + \mathbb{Z}P_\alpha + \mathbb{Z}Q_\beta + \cdots . \end{aligned}$$

□

Example 9. Consider a 6-gon polynomial

$$G(x) = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x - 63$$

all of whose roots are $1, \sqrt{3}i, -2 + \sqrt{3}i, -3, -2 - \sqrt{3}i, -\sqrt{3}i$. Here, $n = 6, a_1 = 6, a_6 = -63, c = 2,$

$$M_G = \left\{(u_0, u_1, u_2, u_3, u_4, u_5) \in \mathbb{Z}^6 \mid u_0 \cdot 1 + \cdots + u_5(-\sqrt{3}i) = -6\right\}.$$

For $(1, 1, 1, 1, 1, 1) \in M_G$, we see that

$$(1, 1, 1, 1, 1, 1) = \left(\frac{1}{2}\right)(0)\left\langle 1, 0, 0, 0, 0, 0\right\rangle + \left\langle 1, 0, 1, 0, 1, 0\right\rangle + \left\langle 0, 1, 0, 1, 0, 1\right\rangle .$$

Since $(-3, 0, 0, 1, 0, 0) \in M_G$, we have

$$(-3, 0, 0, 1, 0, 0) = \left(\frac{1}{2}\right)(-8)\left\langle 1, 0, 0, 0, 0, 0\right\rangle + \left\langle 1, 0, 0, 1, 0, 0\right\rangle ,$$

and since $(0, -3, 3, 0, 0, 0) \in M_G$, we have

$$(0, -3, 3, 0, 0, 0) = \left(\frac{1}{2}\right)(-6)\left\langle 1, 0, 0, 0, 0, 0\right\rangle + 3\left\langle 1, 0, 1, 0, 1, 0\right\rangle + (-3)\left\langle 0, 1, 0, 0, 1, 0\right\rangle ,$$

both relations confirm the results of Theorem 8.

Schoenberg's third result is a special case of the following corollary which follows immediately from Theorem 8 by taking a simple general n-gon polynomial.

Corollary 4. *Let $G(x) = x^n - a^n \in \mathbb{Q}[x]$, $a \neq 0$ be an n-gon polynomial. Let its corresponding vanishing sum module be*

$$M_G = \{(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Z}^n \ : \ u_0\zeta_0 + u_1\zeta_1 + \cdots + u_{n-1}\zeta_{n-1} = -a_1\} \,,$$

where $\zeta_i = a\omega^i$ and ω is a primitive n^{th} root of unity. If $u \in M_G$, then

$$u = \sum a_\alpha P_\alpha + b_\beta Q_\beta + \cdots \,.$$

Corollary 5. *Let*

$$G(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$

with $a_k = \frac{\binom{n}{k} a_1^k}{n^k}$ $(k = 2, 3, \ldots, n-1)$, $c^n = \left(\frac{a_1}{n}\right)^n - a_n \neq 0$ be an n-gon polynomial. For $u \in M_G$, then $u \in M_c$ if and only if $a_1 = 0$.

Proof. As seen in the proof of Theorem 8, for $u = (u_0, \ldots, u_{n-1}) \in M_G$, we have

$$u_0 + u_1\omega + \cdots + u_{n-1}\omega^{n-1} = \frac{a_1}{nc}\left(u_0 + u_1 + \cdots + u_{n-1} - n\right),$$

from which the result follows. □

Schoenberg's thrid result vs Lam-Leung's main result

Let $n = p_1^{a_1} p_2^{a_2} \cdots (p_1, p_2, \ldots$ distinct primes; $a_1, a_2, \ldots \in \mathbb{N})$ and let ω be a primitive n^{th} root of unity. Define

$$N_c = \left\{ (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Q}^n : u_0 + u_1 \omega + \cdots + u_{n-1} \omega^{n-1} = 0 \right\}$$

and the vanishing sum module

$$M_c = \{ (u_0, u_1, \ldots, u_{n-1}) \in N_c \mid u_i \in \mathbb{Z} \} .$$

Schoenberg's third and main result reads:

If $u \in M_c$ then it is representable as a $\mathbb{Z}$-linear combination of subpolygons of the standard n-gon

$$u = \sum a_\alpha P_\alpha + b_\beta Q_\beta + \cdots .$$

Since $P_\alpha, Q_\beta, \ldots$ are the standard subpolygons corresponding to the prime factors $p_1, p_2, \ldots$ of n. There are $\frac{n}{p_1}$ distinct standard p_1-gons, the first of which is $P_0 = (1, 0, ...0, 1, 0, ...0, 1, 0, ...0, 1, 0, \ldots, 0)$, where the digit 1 appears at the

$$0^{th}, \left(\frac{n}{p_1} \right)^{th}, \left(\frac{2n}{p_1} \right)^{th}, \ldots, \left(\frac{(p_1 - 1)n}{p_1} \right)^{th}$$

positions, i.e.,

$$P_0 = 1 + \omega^{\frac{n}{p_1}} + \omega^{\frac{2n}{p_1}} + \cdots + \omega^{\frac{(p_1-1)n}{p_1}} .$$

The remaining standard p_1-gons are

$$P_j = \omega^j + \omega^{\frac{n}{p_1}+j} + \omega^{\frac{2n}{p_1}+j} + \cdots + \omega^{\frac{(p_1-1)n}{p_1}+j} \quad \left(j = 1, \ldots, \frac{n}{p_1} - 1 \right)$$

Similarly, there are $\frac{n}{p_2}$ distinct standard p_2-gons, the first of which is $Q_0 = (1, 0, ...0, 1, 0, ...0, 1, 0, ...0, 1, 0, \ldots, 0)$, where the digit 1 appears at the

$$0^{th}, \left(\frac{n}{p_2} \right)^{th}, \left(\frac{2n}{p_2} \right)^{th}, \ldots, \left(\frac{(p_2 - 1)n}{p_2} \right)^{th}$$

positions, i.e.,

$$Q_0 = 1 + \omega^{\frac{n}{p_2}} + \omega^{\frac{2n}{p_2}} + \cdots + \omega^{\frac{(p_2-1)n}{p_2}} ,$$

and the remaining standard p_2-gons are

$$Q_j = \omega^j + \omega^{\frac{n}{p_2}+j} + \omega^{\frac{2n}{p_2}+j} + \cdots + \omega^{\frac{(p_2-1)n}{p_2}+j} \quad \left(j = 1, \ldots, \frac{n}{p_2} - 1 \right),$$

and so are all the other standard subpolygons.

Let $G =< z >$ be a cyclic group of order n with generator $z \in G$. Let ω be a primitive n^{th} root of unity. Define the usual map $\psi : \mathbb{Z}G \longrightarrow \mathbb{Z}[\omega]$ by

$$\psi(a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}) = a_0 + a_1 \omega + \cdots + a_{n-1} \omega^{n-1}.$$

Each element in $\mathbb{Z}G$ is of the form $x = \sum_{g \in G} x_g g$. Let $S(P_i)$ be the unique subgroup of G of order p_i, and define

$$\sigma(S(P_i)) = \sum_{g \in S(P_i)} g.$$

Lam and Leung's Theorem reads:

$$ker(\psi) = \sum_{p_i | n} \mathbb{Z}G\, \sigma(S(P_i)), \quad ker(\psi) = \mathbb{Z}\sigma(S(P)) \text{ if } n = p.$$

Since $ker(\psi) = M_c,\ \sigma(S(P_i)) \in ker(\psi)$ and

$$\sigma(S(P_i)) = \sum_{g \in S(P_i)} g = 1 + z^{\frac{n}{p_i}} + z^{\frac{2n}{p_i}} + \cdots + z^{\frac{(p_i - 1)n}{p_i}}.$$

Then

$$z^j \sigma(S(P_i)) = \sum_{g \in S(P_i)} z^j g = z^j + z^{\frac{n}{p_i}+j} + z^{\frac{2n}{p_i}+j} + \cdots + z^{\frac{(p_i-1)n}{p_i}+j} \quad \left(j = 1, \ldots, \frac{n}{p_i} - 1\right).$$

It is thus clear that Schoenberg's third result and Lam-Leung theorem are equivalent via the correspondence:

$$P_j \longleftrightarrow \psi(\sigma(z^j S(P_1)) \quad \left(j = 0, 1, \ldots, \frac{n}{p_1} - 1\right)$$
$$Q_j \longleftrightarrow \psi(\sigma(z^j S(P_2)) \quad \left(j = 0, 1, \ldots, \frac{n}{p_2} - 1\right)$$
$$\vdots$$

CONCLUSION

Let $P(x)$ be a monic integer polynomial of degree n, ξ a root of $P(x)$ and $M(x)$ be a minimal polynomial of ζ of degree m. Put

$$N_P = \{(u_0, u_1, \dots, u_{n-1}) \in \mathbb{Q}^n : u_0 + u_1\xi + \cdots + u_{n-1}\xi^{n-1} = 0\},$$

and

$$M_P = \{(u_0, u_1, \dots, u_{n-1}) \in N_P : u_i \in \mathbb{Z}\}.$$

Define

$$\begin{aligned} \varphi : \mathbb{Q}^n &\longrightarrow \mathbb{Q}[x] \bmod P(x) \\ (u_0, u_1, \dots, u_{n-1}) &\longmapsto u_0 + u_1 x + \cdots + u_{n-1}x^{n-1} = \varphi_u(x). \end{aligned}$$

This is an isomorphism and define

$$l_\nu = \varphi^{-1}\left(x^\nu M(x)\right) \qquad (\nu = 0, 1, \dots, n-m-1).$$

We have proved:

Theorem. The set $\{l_0, l_1, \dots, l_{n-m-1}\}$ is a $\mathbb{Z}$-basis of the vanishing sum module M_P .

For an n-gon polynomial, which is a polynomial all of whose roots represent all vertices of a regular n-gon in the complex plane,

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \in \mathbb{C}[x],$$

where $c^n = \left(\frac{a_1}{n}\right)^n - a_n$ and the cyclotomic polynomial $\Phi_n(x)$ of order n. The general cyclotomic polynomial $F_n(x)$ is defined by

$$F_n(x) := \Phi_n\left(\frac{x}{c} + \frac{a_1}{nc}\right).$$

We have proved:

Theorem. For $n \in \mathbb{N}$, then

$$F_n(x) = \sum_{p|n} E(X;p) F_p\left(cX^{\frac{n}{p}} - \frac{a_1}{n}\right),$$

where $X = \frac{x}{c} + \frac{a_1}{nc}$, $E(X;p) \in \mathbb{Z}[X]$.

Let

$$N_c = \left\{(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Q}^n \mid u_0 + u_1\omega + \cdots + u_{n-1}\omega^{n-1} \; = \; 0\right\},$$

where ω is a primitive n^{th} root of unity. Define

$$N = \left\{(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{C}^n \mid u_0 + u_1\omega + \cdots + u_{n-1}\omega^{n-1} \; = \; 0\right\}.$$

We have proved:

Theorem. For each $u \in N$, if there exists $l \in \mathbb{C} - \{0\}$ such that $lu \in \mathbb{Z}^n$, then

$$u \in \mathbb{Z}\left[\frac{1}{l}\right] P_\alpha + \mathbb{Z}\left[\frac{1}{l}\right] Q_\beta + \cdots,$$

where P_α, $Q_\beta, \ldots$ are standard subpolygons inscribed in the standard n-gon.

Let

$$G(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1}z + a_n$$

be an n-gon polynomial and $c^n = \left(\frac{a_1}{n}\right)^n - a_n \neq 0$.
Let

$$M_G := \{u = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Z}^n \mid u_0\zeta_0 + u_1\zeta_1 + \cdots + u_{n-1}\zeta_{n-1} = -a_1\},$$

where ζ_i are distinct roots of $G(z)$.

We have proved:

Theorem. Let $n \in \mathbb{N}$; $a_1, a_n, c \in \mathbb{C}$ with $c^n = (\frac{a_1}{n})^n - a_n \neq 0$. Assume that $\frac{a_1}{nc} \in \mathbb{Q}$. If $u = (u_0, u_1, \ldots, u_{n-1}) \in M_G$ and $t := \frac{a_1}{nc}(u_0 + u_1 + \cdots + u_{n-1} - n) \in \mathbb{Z}$, then

$$u \in \frac{a_1}{nc}\mathbb{Z}\langle 1, 0, \ldots, 0\rangle + \mathbb{Z}P_\alpha + \mathbb{Z}Q_\beta + \cdots.$$

LITERATURE CITED

Lam, T. Y. and K. H. Leung. 1995. On vanishing sums of roots of unity. **J. Algebra** 224: 91-109.

Nagell, T. 1981. **Introduction to Number Theory.** Chelsea, New York

Pollard, H. and H.G. Diamond. 1975. **The Theory of Algebraic Numbers**. The Mathematical Association of America.

Schoenberg, I. J. 1964. A note on the cyclotomic polynomial. **Mathematika** 11: 131-136.

Skutnik, B.J. 1978. Algebraic equation whose roots form regular n-gon in the complex plane. **Amer. Math. Monthly** 85: 770-771.

CURRICULM VITAE

NAME	: Mr. Thaweedet Yantassanavanich		
BIRTH DATE	: August 4, 1981		
BIRTH PLACE	: Bangkok, Thailand		
ECUCATION	: YEAR	INSTITUTION	DEGREE/DIPLOMA
	2003	Kasetsart University	B.S.(Mathematics)