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Some Number-Theoretic Products

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ABSTRACT

For a positive integer n, let $P(n)$ and $P'(n)$ be the products of all elements in the finite sets $\{x : 1 \le x \le n, (x,n) = 1\}$ and $\{x : 1 \le x \le n/2, (x,n) = 1\}$, respectively. In this article, we verify the formula for $P(n)$ and use it to establish the formula for $P'(n)$. Explicit formulae for both $P(p^a)$ and $P'(p^a)$, where p^a is a prime power, are also derived.

Keywords: arithmetic function; Euler phi-function; Möbius function; product form of the Möbius inversion formula

1. Introduction

As usual (m,n) denotes the greatest common divisor of integers m and n and $|A|$ is the number of elements in a finite set A. By an *arithmetic function*, we mean a mapping f from the set of positive integers $\mathbb{N}$ into the field of complex numbers $\mathbb{C}$. There are many interesting examples of arithmetic functions. Two of them are the *Euler phi-function*,

$$\phi(n) = |\{x : 1 \le x \le n, (x,n) = 1\}|$$

and the *Möbius function*,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^2 | n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \text{ where} \\ & \text{all } p_i \text{ are distinct primes.} \end{cases}$$

For positive integers n and k, define finite sets of positive integers as follows:

$$R_k(n) = \{x^k : 1 \le x \le n, (x,n) = 1\},$$

$$R'_k(n) = \{x^k : 1 \le x \le \frac{n}{2}, (x,n) = 1\}.$$

Note that

$$|R_1(n)| = \phi(n) \quad (n \ge 1) \tag{1.1}$$

Let $S_k(n) = \sum R_k(n)$ and $S'_k(n) = \sum R'_k(n)$, where $\sum A$ denotes the sum of all elements in a finite set A of positive integers. It is well-known [3] that

$$S_1(n) = \frac{n\phi(n)}{2} \quad (n > 1)$$

and there is an exercise in [2] that

$$S_2(n) = \frac{2n^2\phi(n) + n\psi(n)}{6} \quad (n > 1),$$

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where ψ is an arithmetic function defined by $\psi(1) = 1$ and $\psi(n) = \prod_{p|n}(1-p)$ for $n > 1$, the product is over the prime divisors of n.

In another direction, Baum [2] provided the formula for $S_1'(n)$ and he advised the reader to prove $S_2'(n)$ as an exercise. The formulae for both $S_1'(n)$ and $S_2'(n)$ are as follows:

$$S_1'(n) = \frac{1}{8}\left(n\phi(n) - |r|\psi(n)\right) \quad (n > 2),$$

where $n \equiv r \pmod 4$ with $r \in \{-1, 0, 1, 2\}$ and

$$S_2'(n) = \begin{cases} \frac{n^2\phi(n)+2n\psi(n)}{24} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n^2\phi(n)-n\psi(n)}{24} & \text{if } n \equiv \pm 1 \pmod 4 \\ \frac{n^2\phi(n)-4n\psi(n)}{24} & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

for all $n > 2$.

In 2019, Kanasri, Pornsurat, and Tongron [4] established the general formulae for both $S_k(n)$ and $S_k'(n)$ for all positive integers n and k by the use of the Möbius inversion formula. They also confirmed that the known results for $k = 1, 2$, as mentioned above, follow from these general formulae.

Theorem 1.1. ***Möbius inversion formula*** *Let F and f be two arithmetic functions related by the formula*

$$F(n) = \sum_{d|n} f(d).$$

Then

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Note that the converse of the Möbius inversion formula is also true [5], [6]. The formulae for $S_k(n)$ and $S_k'(n)$ are as follows: For any positive integer k, we have

$$S_k(n) = \sum_{d|n} \mu(d) d^k g_k\left(\frac{n}{d}\right) \quad (n \geq 1)$$

and for $n > 2$, $S_k'(n) =$

$$\begin{cases} \sum_{d|(n/2)} \mu(d) d^k g_k\left(\frac{n}{2d}\right) \text{ if } n \equiv 0 \pmod 4 \\ \sum_{d|n} \mu(d) d^k g_k\left(\frac{n/d-1}{2}\right) \text{ if } n \equiv \pm 1 \pmod 4 \\ \sum_{d|(n/2)} \mu(d) d^k \left(g_k\left(\frac{n}{2d}\right) - 2^k g_k\left(\frac{n/2d-1}{2}\right)\right) \\ \qquad\qquad\qquad \text{if } n \equiv 2 \pmod 4, \end{cases}$$

where $g_k(n) = 1^k + 2^k + \cdots + n^k$.

Recently, the authors [8] established a generalization of $S_k(n)$ and $S_k'(n)$ by the use of Möbius inversion formula as follows: For positive integers k, m, and n with $n > m$, let $S_k^m(n)$ be the sum of all elements in the finite set $\{x^k : 1 \leq x \leq n/m, (x, n) = 1\}$. Then

$$S_k^m(n) = \sum_{d|n} \mu(d) d^k g_k\left(\lfloor n/dm \rfloor\right), \quad (1.2)$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to a real number x and let $g_k(0) = 0$ for $n < dm$. We also verified that the formulae for $S_k(n)$ and $S_k'(n)$ in [4] follow from Eq. (1.2) by letting $m = 1$ and $m = 2$, respectively.

We observe that all of the results mentioned above are verified by using the useful theorem, Möbius inversion formula. However, there is a product form of the Möbius inversion formula as an exercise in [1] and [6]. This form motivates us to study the products of all elements in $R_k(n)$ and $R_k'(n)$.

For positive integers n and k, we now let

$$P_k(n) = \prod R_k(n)$$

and

$$P_k'(n) = \prod R_k'(n),$$

where $\prod A$ denotes the product of all elements in a finite set A of positive integers. In this work, we are interested in establishing the formulae for both $P_k(n)$ and $P_k'(n)$. Since

$$P_k(n) = (P_1(n))^k \text{ and } P_k'(n) = (P_1'(n))^k,$$

it suffices to establish the formulae for $P(n) := P_1(n)$ and $P'(n) := P'_1(n)$. However, there is an exercise in [1] and [6] to verify the formula for $P(n)$ by using the product form of the Möbius inversion formula.

In this article, we first verify the product form of the Möbius inversion formula and use it to verify the formula for $P(n)$. We then establish a formula for $P'(n)$ by using both results mentioned above. Moreover, explicit formulae for $P(p^a)$ and $P'(p^a)$, where p^a is a prime power, are also derived.

2. Main results

Several well-known facts that we shall use in this article are collected in the following lemma [3], [7].

Lemma 2.1. *For each positive integer $n \geq 1$, we have*

(i) $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$

(ii) $\sum_{d|n} \phi(d) = n,$
the sums being extended over all positive divisors of n.

(iii) $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$
the product being taken over all primes which divide n.

By Lemma 2.1(iii), we have that

$$\phi(p^a) = p^a - p^{a-1} \tag{2.1}$$

for all primes p and $a \in \mathbb{N}$.

Before proceeding to our main results, we start with the following useful lemmas, the first one is the *product form of the Möbius inversion formula*.

Lemma 2.2. *Let f and g be two arithmetic functions such that $g(n) \neq 0$ for all $n \in \mathbb{N}$. Then*

$$f(n) = \prod_{d|n} g(d)$$

if and only if

$$g(n) = \prod_{d|n} f(d)^{\mu(n/d)},$$

where d runs through the positive divisors of n.

Proof. Assume that $f(n) = \prod_{d|n} g(d)$. Using the fact that

$$\{d \in \mathbb{N} : d \mid n\} = \{n/d : d \in \mathbb{N}, d \mid n\}, \tag{2.2}$$

we have

$$\begin{aligned} \prod_{d|n} f(d)^{\mu(n/d)} &= \prod_{d|n} f\left(\frac{n}{d}\right)^{\mu(d)} \\ &= \prod_{d|n} \prod_{e|(n/d)} g(e)^{\mu(d)}. \end{aligned}$$

Since $d|n$ and $e|(n/d)$ if and only if $e|n$ and $d|(n/e)$, the last equation becomes

$$\begin{aligned} \prod_{d|n} \prod_{e|(n/d)} g(e)^{\mu(d)} &= \prod_{e|n} \prod_{d|(n/e)} g(e)^{\mu(d)} \\ &= \prod_{e|n} g(e)^{\sum_{d|(n/e)} \mu(d)} \\ &= g(n)^{\mu(1)} \prod_{\substack{e|n \\ e<n}} g(e)^{\sum_{d|(n/e)} \mu(d)}. \end{aligned}$$

By Lemma 2.1(i), we have $\sum_{d|(n/e)} \mu(d) = 0$ for $e|n$ with $e < n$. It follows that $\prod_{d|n} f(d)^{\mu(n/d)} = g(n)$.

On the other hand, we suppose that $g(n) = \prod_{d|n} f(d)^{\mu(n/d)}$. Again by (2.2), we have

$$\prod_{d|n} g(d) = \prod_{d|n} g\left(\frac{n}{d}\right) = \prod_{d|n} \prod_{e|(n/d)} f(e)^{\mu\left(\frac{n/d}{e}\right)}.$$

Since $d|n$ and $e|\left(n/d\right)$ if and only if $e|n$ and $d|\left(n/e\right)$, the last equation becomes

$$\prod_{d|n}\prod_{e|(n/d)} f\left(e\right)^{\mu\left(\frac{n/d}{e}\right)} = \prod_{e|n}\prod_{d|(n/e)} f\left(e\right)^{\mu(n/de)}$$

$$= \prod_{e|n} f\left(e\right)^{\sum_{d|(n/e)}\mu\left(\frac{n/e}{d}\right)}$$

$$= \prod_{e|n} f\left(e\right)^{\sum_{d|(n/e)}\mu(d)}, \quad \text{by (2.2)}$$

$$= f\left(n\right)^{\mu(1)} \prod_{\substack{e|n\\ e<n}} f\left(e\right)^{\sum_{d|(n/e)}\mu(d)}$$

$$= f\left(n\right),$$

by Lemma 2.1(i). Hence, $\prod_{d|n} g\left(d\right) = f\left(n\right)$, as desired. □

Lemma 2.3. *For an odd integer $n > 1$, we have*

$$\left|\left\{x : 1 \le x \le \frac{n-1}{2}, (x,n) = 1\right\}\right| = \frac{\phi\left(n\right)}{2}.$$

Proof. Let $n > 1$ be an odd integer and let

$$A = \left\{x : 1 \le x \le \frac{n-1}{2}, (x,n) = 1\right\},$$

$$B = \left\{x : -\frac{n-1}{2} \le x \le \frac{n-1}{2}, (x,n) = 1\right\}.$$

For $i \in \{0, 1, 2, \ldots, (n-1)/2\}$, we have

$$i \equiv i \pmod{n} \quad \text{and} \quad -i \equiv n - i \pmod{n}.$$

By the fact that for $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{n}$, then $(a,n) = (b,n)$, we obtain

$$|B| = |\left\{x : 1 \le x \le n, (k,n) = 1\right\}| = \phi\left(n\right).$$

Since $(-x,n) = (x,n) = 1$ for all $x \in A$, we conclude that

$$|A| = \frac{|B|}{2} = \frac{\phi\left(n\right)}{2},$$

as desired. □

We are now ready to verify the formula for $P(n)$ as the following.

Theorem 2.4. *For each positive integer $n \ge 1$, we have*

$$P\left(n\right) = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)}.$$

Proof. For a positive divisor d of n, we define

$$A_d = \{x : 1 \le x \le n, (x,n) = d\}.$$

Note that $A_d \ne \emptyset$ since $d \in A_d$. Clearly, $\bigcup_{d|n} A_d = \{1, 2, \ldots, n\}$ and $A_{d_1} \cap A_{d_2} = \emptyset$ for $d_1 \ne d_2$. It follows that

$$\prod_{i=1}^{n} i = n! = \prod_{d|n}\prod A_d. \tag{2.3}$$

We next show that

$$A_d = dR_1\left(\frac{n}{d}\right). \tag{2.4}$$

If $x \in A_d$, then $1 \le x \le n$ and $(x,n) = d$. It follows that $x/d \in \mathbb{N}$, $1 \le x/d \le n/d$, and $(x/d, n/d) = 1$. We consequently have $x/d \in R_1\left(n/d\right)$ and so $x \in dR_1\left(n/d\right)$. If $y \in R_1\left(n/d\right)$, then $1 \le y \le n/d$ and $(y, n/d) = 1$. It follows that $1 \le d \le dy \le n$ and $(dy, n) = d$. This shows that $dy \in A_d$.

For $d|n$, we obtain by (1.1) and (2.4) that

$$\prod A_d = \prod dR_1\left(\frac{n}{d}\right)$$

$$= d^{\phi(n/d)} \prod R_1\left(\frac{n}{d}\right)$$

$$= d^{\phi(n/d)} P\left(\frac{n}{d}\right).$$

It follows from Lemma 2.1(ii), (2.2), and (2.3) that

$$n! = \prod_{d|n} d^{\phi(n/d)} P\left(\frac{n}{d}\right)$$

$$= \prod_{d|n} \left(\frac{n}{d}\right)^{\phi(d)} P(d)$$
$$= \prod_{d|n} n^{\phi(d)} \prod_{d|n} \frac{P(d)}{d^{\phi(d)}}$$
$$= n^{\sum_{d|n} \phi(d)} \prod_{d|n} \frac{P(d)}{d^{\phi(d)}}$$
$$= n^n \prod_{d|n} \frac{P(d)}{d^{\phi(d)}},$$

yielding

$$\frac{n!}{n^n} = \prod_{d|n} \frac{P(d)}{d^{\phi(d)}}.$$

Using Lemma 2.2 with $f(n) = \dfrac{n!}{n^n}$ and $g(n) = \dfrac{P(n)}{n^{\phi(n)}}$ for all $n \in \mathbb{N}$, we get

$$\frac{P(n)}{n^{\phi(n)}} = \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)},$$

yielding the desired result. □

Using Lemma 2.2, Lemma 2.3, and Theorem 2.4, we obtain the formula for $P'(n)$ as the following.

Theorem 2.5. *For each positive integer $n > 2$, we have $P'(n) =$*

$$\begin{cases} \left(\frac{n}{2}\right)^{\phi(n/2)} \prod_{d|(n/2)} \left(\frac{d!}{d^d}\right)^{\mu(n/2d)} & \text{if } n \equiv 0 \pmod 4 \\ n^{\phi(n)/2} \prod_{d|n} \left(\frac{\left(\frac{d-1}{2}\right)!}{d^{(d-1)/2}}\right)^{\mu(n/d)} & \text{if } n \equiv \pm 1 \pmod 4 \\ \left(\frac{n}{2}\right)^{\frac{\phi(n/2)}{2}} \left(\frac{1}{2}\right)^{\frac{\phi(n/2)}{2}} \cdot \prod_{d|(n/2)} \left(\frac{d(d-1)\cdots\left(\frac{d+1}{2}\right)}{d^{(d+1)/2}}\right)^{\mu(n/2d)} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. We prove this formula by considering three possible cases.

Case I: $n \equiv 0 \pmod 4$. Then n and $n/2$ are even. It follows that $(x, n) = 1$ if and only if $(x, n/2) = 1$ for any positive integer x. From Theorem 2.4, we have

$$\begin{aligned} P'(n) &= \prod \left\{x : 1 \le x \le \frac{n}{2}, (x, n) = 1\right\} \\ &= \prod \left\{x : 1 \le x \le \frac{n}{2}, \left(x, \frac{n}{2}\right) = 1\right\} \\ &= P\left(\frac{n}{2}\right) \\ &= \left(\frac{n}{2}\right)^{\phi(n/2)} \prod_{d|(n/2)} \left(\frac{d!}{d^d}\right)^{\mu(n/2d)}. \end{aligned}$$

Case II: $n \equiv \pm 1 \pmod 4$. Then n is odd. For $d|n$, we define

$$B_d = \{x : 1 \le x \le \frac{n}{2}, (x, n) = d\}.$$

Note that $B_d = \emptyset$ if and only if $d = n$, so we let $\prod B_n = 1$. Clearly,

$$\bigcup_{d|n} B_d = \left\{1, 2, \ldots, \frac{n-1}{2}\right\}$$

and

$$B_{d_1} \cap B_{d_2} = \emptyset$$

for $d_1 \ne d_2$. It follows that

$$\prod_{i=1}^{(n-1)/2} i = \left(\frac{n-1}{2}\right)! = \prod_{d|n} \prod B_d. \quad (2.5)$$

Next, we show that

$$B_d = dR'_1\left(\frac{n}{d}\right). \quad (2.6)$$

Observe that $R'_1(n/d) = \emptyset$ if and only if $d = n$, so we let $P'(1) = 1$. If $x \in B_d$, then $1 \le x \le n/2$ and $(x, n) = d$. It follows that $1 \le x/d \le n/2d$ and $(x/d, n/d) = 1$. Consequently, $x/d \in R'_1(n/d)$ and so $x \in dR'_1(n/d)$. If $y \in R'_1(n/d)$, then $1 \le y \le n/2d$ and $(y, n/d) = 1$. It follows that $1 \le d \le dy \le n/2$ and $(dy, n) = d$. This implies that $dy \in B_d$.

For $d|n$ with $d \ne n$, we get n/d is odd and $n/d > 1$. Using Lemma 2.3, we have

$$\begin{aligned}\left|R_1'\left(\frac{n}{d}\right)\right| &= \left|\left\{x : 1 \le x \le \frac{n}{2d}, \left(x, \frac{n}{d}\right) = 1\right\}\right| \\ &= \left|\left\{x : 1 \le x \le \frac{n/d - 1}{2}, \left(x, \frac{n}{d}\right) = 1\right\}\right| \\ &= \frac{\phi(n/d)}{2}.\end{aligned}$$

Form (2.6), we obtain

$$\begin{aligned}\prod B_d &= \prod dR_1'\left(\frac{n}{d}\right) \\ &= d^{\frac{\phi(n/d)}{2}} \prod R_1'\left(\frac{n}{d}\right) \\ &= d^{\frac{\phi(n/d)}{2}} P'\left(\frac{n}{d}\right).\end{aligned}$$

It follows from (2.5) that

$$\begin{aligned}\left(\frac{n-1}{2}\right)! &= \prod B_n \prod_{\substack{d|n \\ d<n}} \prod B_d \\ &= \prod_{\substack{d|n \\ d<n}} d^{\frac{\phi(n/d)}{2}} P'\left(\frac{n}{d}\right) \\ &= \frac{\prod_{d|n} d^{\frac{\phi(n/d)}{2}} P'\left(\frac{n}{d}\right)}{n^{1/2} P'(1)} \\ &= \frac{\prod_{d|n} \left(\frac{n}{d}\right)^{\phi(d)/2} P'(d)}{n^{1/2}},\end{aligned}$$

by Eq. (2.2) and so

$$\begin{aligned}n^{1/2}\left(\frac{n-1}{2}\right)! &= \prod_{d|n} \left(\frac{n}{d}\right)^{\phi(d)/2} P'(d) \\ &= \prod_{d|n} n^{\phi(d)/2} \prod_{d|n} \frac{P'(d)}{d^{\phi(d)/2}} \\ &= n^{\frac{1}{2}\sum_{d|n}\phi(d)} \prod_{d|n} \frac{P'(d)}{d^{\phi(d)/2}} \\ &= n^{n/2} \prod_{d|n} \frac{P'(d)}{d^{\phi(d)/2}},\end{aligned}$$

by Lemma 2.1(ii). This shows that

$$\frac{\left(\frac{n-1}{2}\right)!}{n^{(n-1)/2}} = \prod_{d|n} \frac{P'(d)}{d^{\phi(d)/2}}.$$

By Lemma 2.2 with $f(n) = \frac{\left(\frac{n-1}{2}\right)!}{n^{(n-1)/2}}$ and $g(n) = \frac{P'(n)}{n^{\phi(n)/2}}$ for all $n \in \mathbb{N}$, we get

$$\frac{P'(n)}{n^{\phi(n)/2}} = \prod_{d|n} \left(\frac{\left(\frac{d-1}{2}\right)!}{d^{(d-1)/2}}\right)^{\mu(n/d)}$$

as desired.

Case III: $n \equiv 2 \pmod 4$. Then we can write $n = 2m$ for some odd integer m. Thus, for any positive integer x, we have $(x, n) = 1$ if and only if $(x, m) = 1$ and x is odd. We also observe that $(2y, m) = 1$ if and only if $(y, m) = 1$ for any positive integer y. Using Lemma 2.2, Lemma 2.3, Theorem 2.4, and Case II, we obtain

$$\begin{aligned}P'(n) &= \prod\left\{x : 1 \le x \le \frac{n}{2}, (x, n) = 1\right\} \\ &= \prod\{x : 1 \le x \le m, (x, m) = 1, x \text{ is odd }\} \\ &= \prod\big(\{x : 1 \le x \le m, (x, m) = 1\}\setminus \\ &\quad \{x \mid 1 \le x \le m, (x, m) = 1, x \text{ is even}\}\big) \\ &= \prod\big(\{x : 1 \le x \le m, (x, m) = 1\}\setminus \\ &\quad \{2y \mid 1 \le 2y \le m, (2y, m) = 1\}\big) \\ &= \prod\big(\{x : 1 \le x \le m, (x, m) = 1\}\setminus \\ &\quad \left\{2y \mid 1 \le y \le \frac{m}{2}, (y, m) = 1\right\}\big)\end{aligned}$$

$$= \frac{\prod\{x : 1 \leq x \leq m, (x,m) = 1\}}{\prod\{2y : 1 \leq y \leq \frac{m}{2}, (y,m) = 1\}}$$

$$= \frac{P(m)}{2^{\phi(m)/2} P'(m)}$$

$$= \frac{m^{\phi(m)} \prod_{d|m} \left(\frac{d!}{d^d}\right)^{\mu(m/d)}}{2^{\phi(m)/2} m^{\phi(m)/2} \prod_{d|m} \left(\frac{\left(\frac{d-1}{2}\right)!}{d^{(d-1)/2}}\right)^{\mu(m/d)}}$$

$$= \left(\frac{m}{2}\right)^{\frac{\phi(m)}{2}} \prod_{d|m} \left(\frac{d(d-1)\cdots\left(\frac{d+1}{2}\right)}{d^{(d+1)/2}}\right)^{\mu(m/d)}$$

$$= \left(\frac{n}{2}\right)^{\frac{\phi(n/2)}{2}} \left(\frac{1}{2}\right)^{\frac{\phi(n/2)}{2}} \cdot \prod_{d|(n/2)} \left(\frac{d(d-1)\cdots\left(\frac{d+1}{2}\right)}{d^{(d+1)/2}}\right)^{\mu(n/2d)},$$

which completes the proof. □

3. Explicit formulae

We now proceed to the last section. In this section, we provide explicit formulae for $P(p^a)$ and $P'(p^a)$, where p^a is a prime power, as in the following propositions.

Proposition 3.1. *Let p be a prime and a be a positive integer. Then*

$$P(p^a) = \frac{p^a(p^a-1)\cdots(p^{a-1}+1)}{p^{p^{a-1}}}.$$

Proof. Let p be a prime and a be a positive integer. By Theorem 2.4 and (2.1), we obtain

$$P(p^a) = p^{a\phi(p^a)} \prod_{d|p^a} \left(\frac{d!}{d^d}\right)^{\mu(p^a/d)}$$

$$= p^{a(p^a-p^{a-1})} \left(\frac{p^{a-1}!}{p^{(a-1)p^{a-1}}}\right)^{\mu(p)} \left(\frac{p^a!}{p^{ap^a}}\right)^{\mu(1)}$$

$$= \frac{p^{a(p^a-p^{a-1})} \cdot p^{(a-1)p^{a-1}} \cdot p^a!}{p^{a-1}! \cdot p^{ap^a}}$$

$$= \frac{p^a(p^a-1)\cdots(p^{a-1}+1)}{p^{p^{a-1}}}.$$

□

Proposition 3.2. *Let p be an odd prime and a be a positive integer. Then*

(i) $P'(2^a) = \dfrac{2^{a-1}(2^{a-1}-1)\cdots(2^{a-2}+1)}{2^{2^{a-2}}}$ *for all $a \geq 2$.*

(ii) $P'(p^a) = \dfrac{\left(\frac{p^a-1}{2}\right)\left(\frac{p^a-3}{2}\right)\cdots\left(\frac{p^{a-1}+1}{2}\right)}{p^{(p^{a-1}-1)/2}}.$

Proof. Let p be an odd prime and a be a positive integer.

(i) If $a \geq 2$, then by Theorem 2.5, we obtain

$$P'(2^a) = \left(2^{a-1}\right)^{\phi(2^{a-1})} \prod_{d|2^{a-1}} \left(\frac{d!}{d^d}\right)^{\mu(2^{a-1}/d)}$$

$$= 2^{(a-1)\phi(2^{a-1})} \left(\frac{2^{a-2}!}{2^{(a-2)2^{a-2}}}\right)^{\mu(2)} \left(\frac{2^{a-1}!}{2^{(a-1)2^{a-1}}}\right)^{\mu(1)}$$

$$= \frac{2^{(a-1)(2^{a-1}-2^{a-2})} \cdot 2^{(a-2)2^{a-2}} \cdot 2^{a-1}!}{2^{a-2}! \cdot 2^{(a-1)2^{a-1}}}$$

$$= \frac{2^{a-1}(2^{a-1}-1)\cdots(2^{a-2}+1)}{2^{2^{a-2}}}.$$

(ii) By Theorem 2.5 and (2.1), we obtain

$$P'(p^a) = p^{a\phi(p^a)/2} \prod_{d|p^a} \left(\frac{\left(\frac{d-1}{2}\right)!}{d^{(d-1)/2}}\right)^{\mu(p^a/d)}$$

$$= p^{a(p^a-p^{a-1})/2} \left(\frac{\left(\frac{p^{a-1}-1}{2}\right)!}{p^{(a-1)(p^{a-1}-1)/2}}\right)^{\mu(p)} \cdot$$

$$\left(\frac{\left(\frac{p^a-1}{2}\right)!}{p^{a(p^a-1)/2}}\right)^{\mu(1)}$$

$$= \frac{p^{a(p^a-p^{a-1})/2}\cdot p^{(a-1)(p^{a-1}-1)/2}\cdot\left(\frac{p^a-1}{2}\right)!}{\left(\frac{p^{a-1}-1}{2}\right)!\cdot p^{a(p^a-1)/2}}$$

$$= \frac{\left(\frac{p^a-1}{2}\right)\left(\frac{p^a-3}{2}\right)\cdots\left(\frac{p^{a-1}+1}{2}\right)}{p^{(p^{a-1}-1)/2}},$$

which finishes the proof. □

Finally, we note by Proposition 3.1 and Proposition 3.2 (ii) that

$$P(p) = (p-1)!$$

for all primes p and

$$P^{'}(p) = \left(\frac{p-1}{2}\right)!$$

for all odd primes p, respectively.

4. Conclusion

For positive integers n and k, let $R_k(n) = \{x^k : 1 \le x \le n, (x,n) = 1\}$ and $R'_k(n) = \{x^k : 1 \le x \le n/2, (x,n) = 1\}$. Let $S_k(n)$ and $S'_k(n)$ be the sums of all elements in $R_k(n)$ and $R'_k(n)$, respectively. In the earlier work of Kanasri, Pornsurat, and Tongron, the formulae for both $S_k(n)$ and $S'_k(n)$ were established, which yield generalized versions of the formulae for $S_1(n), S_2(n)$ and $S'_1(n), S'_2(n)$, respectively. In this work, we are interested in studying the products of all elements in $R_k(n)$ and $R'_k(n)$, denoted by $P_k(n)$ and $P'_k(n)$, respectively. We obtain the formulae for both $P_k(n)$ and $P'_k(n)$. Moreover, explicit formulae for $P_k(p^a)$ and $P'_k(p^a)$, where p^a is a prime power, are also derived.

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