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Unified Classes of Starlike and Convex Functions Associated with Touchard Polynomials

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ABSTRACT

The aim of the present paper is to further explore some description of Touchard polynomials associated with the unified subclasses of analytic functions and investigate inclusion relations for such subclasses in the open unit disk $\mathbb{D}$. Further, we discuss geometric properties of an integral operator related Touchard polynomials.

Keywords: Analytic; Convex; Hadamard (convolution) product; Starlike; Touchard polynomials; Univalent functions

1. Introduction

Let $\mathcal{A}$ represent the class of functions f of the form

$$f(\zeta) = z + \sum_{n=2}^{\infty} a_n \zeta^n, \qquad (1.1)$$

which are analytic in the open unit disk $\mathbb{D} = \{\zeta : \zeta \in \mathbb{C} \text{ and } |\zeta| < 1\}$ and gratify the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we represent by $\mathcal{S}$ the subclass of $\mathcal{A}$ comprising functions of the form Eq. (1.1) which are also univalent in $\mathbb{D}$.

Designate by $\mathcal{T}$ the subclass of $\mathcal{A}$ comprising functions whose non zero coefficients from second on, is given by

$$f(\zeta) = \zeta - \sum_{n=2}^{\infty} |a_n| \zeta^n. \qquad (1.2)$$

We recall the well-known subclasses of $\mathcal{A}$ are starlike and convex functions due to

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Robertson [21]. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\xi)$ if

$$\Re\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\} > \xi, \quad \zeta \in \mathbb{D}$$

holds.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}(\xi)$ if the condition

$$\Re\left\{\frac{(\zeta f'(\zeta))'}{f'(\zeta)}\right\} > \xi, \quad \zeta \in \mathbb{D},$$

is satisfied.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{P}_\varrho(\xi)$ if it satisfies the following condition

$$\Re\left\{\frac{\zeta f'(\zeta) + \varrho\zeta^2 f''(\zeta)}{(1-\varrho)f(\zeta) + \varrho\zeta f'(\zeta)}\right\} > \xi, \quad \zeta \in \mathbb{D}$$

and the function $f \in \mathcal{A}$ is said to be in the class $\mathfrak{D}_\varrho(\xi)$ if it satisfies the following condition

$$\Re\left(\frac{\varrho\zeta^3 f'''(\zeta) + (1+2\varrho)\zeta^2 f''(\zeta) + \zeta f'(\zeta)}{(1-\varrho)f(\zeta) + \varrho\zeta f'(\zeta)}\right) > \xi,$$

and $\zeta \in \mathbb{D}$. Further, we denote by $\mathcal{P}^*_\varrho(\xi) \equiv \mathcal{P}_\varrho(\xi) \cap \mathcal{T}$ and $\mathfrak{D}^*_\varrho(\xi) \equiv \mathfrak{D}_\varrho(\xi) \cap \mathcal{T}$.

The classes $\mathcal{P}_\varrho(\xi)$, $\mathfrak{D}_\varrho(\xi)$, $\mathcal{P}^*_\varrho(\xi)$ and $\mathfrak{D}^*_\varrho(\xi)$ were studied earlier by Altintas et al. [4]. It is worthy to note that for $\varrho = 0$ the classes $\mathcal{P}_\varrho(\xi)$ and $\mathfrak{D}_\varrho(\xi)$ reduce to the classes $\mathcal{S}^*(\xi)$ and $\mathcal{K}(\xi)$, respectively, and for $\varrho = 1$ the class $\mathcal{P}_\varrho(\xi)$ reduces to the class $\mathcal{K}(\xi)$, studied earlier by Robertson [21] and Silverman [23].

Definition 1.1. A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{R}^\tau(\vartheta, \delta)$, $(\tau \in \mathbb{C}\backslash\{0\},\ 0 < \vartheta \le 1; \delta < 1)$, if it satisfies the inequality

$$\left|\frac{(1-\vartheta)\frac{f(\zeta)}{\zeta} + \vartheta f'(\zeta) - 1}{2\tau(1-\delta) + (1-\vartheta)\frac{f(\zeta)}{\zeta} + \vartheta f'(\zeta) - 1}\right| < 1$$

and $\zeta \in \mathbb{D}$.

The class $\mathcal{R}^\tau(\vartheta, \delta)$ was introduced earlier by Swaminathan [26](for special cases see the references cited therein).

The Touchard polynomials, studied by Jacques Touchard [25] also named the exponential generating polynomials (see [8, 22,24]) and also called the Bell polynomials (see [3]) comprise a polynomial sequence of binomial type

for X is a random variable with a Poisson distribution with expected value ℓ, then its $n^t h$ moment is $E(X_\kappa) = \mathcal{T}(\kappa, \ell)$, leading to the form:

$$\mathcal{T}(\kappa, \ell) = e^\kappa \sum_{n=0}^{\infty} \frac{\kappa^n n^\ell}{n!} \zeta^n \qquad (1.3)$$

This is a new algorithm for answering linear and nonlinear integral equations. These polynomials were studied by Jacques Touchard and he generalized the Bell polynomials in order to inspect various problems of inventory of the permutations when the cycles possess certain properties. Besides, he introduced and studied a class of related polynomials. An exponential generating function(see [8,22,24]), recurrence relations and connections related to the other known polynomials were also examined. For some special cases, relations with the Stifling number of the first and second kind, as well as with other numbers recently examined, are derived. In general, the integral equations are problematic to be solved analytically, consequently in many equations we need to obtain the approximate solutions, and for this case, the "Touchard Polynomials method" for the solution of the linear "Volterra integro-differential equation" is implemented. The Touchard polynomials method has been applied to solve linear and nonlinear Volterra (Fredholm) integral equations. Further, Abdullah and Ali [2] provide some efficient numerical methods to solve linear Volterra integral equations and Volterra Integro differential equations of the first and second types, with exponential, singular, regular and convolution kernels. These methods are based on Touchard and Laguerre polynomials that convert these equations into a system of linear algebraic equations. On the other hand, there have been various papers on interesting applications of the Touchard polynomials in nonlinear Fredholm–Volterra integral equations comprising relations between bilinear and trilinear forms of nonlinear differential equations which hold soliton- wave solutions.

Lately,in [12] we consider Touchard polynomials coefficients after the second force as following

$$\Phi_m^\ell(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{(n-1)^\ell m^{n-1}}{(n-1)!} e^{-m} \zeta^n, \tag{1.4}$$

where $\zeta \in \mathbb{D}, \ell \geq 0; m > 0$ and we note that, by ratio test the radius of convergence of above series is infinity. We further define

$$\begin{aligned}\Lambda_m^\ell(\zeta) &= 2\zeta - \Phi_m^\ell(\zeta) \\ &= \zeta - \sum_{n=2}^{\infty} \frac{(n-1)^\ell m^{n-1}}{(n-1)!} e^{-m} \zeta^n,\end{aligned} \tag{1.5}$$

$\zeta \in \mathbb{D}$. It is worthy to note that for $\ell = 0$ the series given by Eq. (1.4) reduce to the Poisson distribution series introduced and studied by Porwal [15]. For functions $f \in \mathcal{A}$ given by Eq. (1.1) and $g \in \mathcal{A}$ given by $g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n, \quad \zeta \in \mathbb{D}. \tag{1.6}$$

Now, we define the linear operator

$$\mathcal{I}(l, m, \zeta) : \mathcal{A} \to \mathcal{A}$$

defined by the convolution or Hadamard product

$$\begin{aligned}\mathcal{I}(l, m, \zeta) f &= \Phi_m^\ell(\zeta) * f(\zeta) \\ &= \zeta + \sum_{n=2}^{\infty} \frac{(n-1)^l m^{n-1}}{(n-1)!} e^{-m} a_n \zeta^n.\end{aligned} \tag{1.7}$$

In our investigation, we shall require the following definition and lemmas.

Definition 1.2. The l^{th} moment of the Poisson distribution about origin is defined as

$$\mu_l' = \sum_{n=0}^{\infty} \frac{n^l m^n}{n!} e^{-m}.$$

Lemma 1.3. *[26] If $f \in \mathcal{R}^\tau(\vartheta, \delta)$ is of form Eq.* (1.1) *then*

$$|a_n| \leq \frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{1.8}$$

The bounds given in Eq. (1.8) *are sharp.*

Lemma 1.4. *([23]) Let $f \in \mathcal{A}$ be of the form Eq.* (1.1)*, then and satisfy the inequality*

$$\sum_{n=2}^{\infty} (n\varrho - \varrho + 1)(n - \xi)|a_n| \leq 1 - \xi, \tag{1.9}$$

then $f \in \mathcal{P}_\varrho(\xi)$, if

Lemma 1.5. *([23]) Let $f \in \mathcal{A}$ be of the form Eq.* (1.1)*, then and satisfy the inequality*

$$\sum_{n=2}^{\infty} n(n\varrho - \varrho + 1)(n - \xi)|a_n| \leq 1 - \xi, \tag{1.10}$$

then $f \in \mathfrak{D}_\varrho(\xi)$.

Remark 1.6. *Let $f \in \mathcal{A}$ be of the form Eq.* (1.2)*, then $f \in \mathcal{P}_\varrho^*(\xi)$, if and only if Eq.* (1.9) *is satisfied.*

Remark 1.7. *Let $f \in \mathcal{A}$ be of the form Eq.* (1.2)*, then $f \in \mathfrak{D}^*_\varrho(\xi)$, if and only if Eq.* (1.10) *is satisfied.*

By specializing the parameter $\varrho = 0$ and $\varrho = 1$ in the Lemmas 1.4 and 1.5, we obtain the results of Silverman [23].

The application of special functions in geometric function theory is a current and interesting topic of research. It is often used in areas such as mathematics, physics, and engineering. As a result of De Branges' study [6], the classic Bieberbach problem is successfully solved by applying a generalized hypergeometric function. Several types of special functions, including generalized hypergeometric Gaussian functions[11, 26] and references are cited therein. The study of distribution series is fascinating topic of research in geometric function theory and unlocks a new path of research. In fact, after the appearance of the paper of Porwal [15] several researchers familiarize Hypergeometric distribution series [1], confluent hypergeometric distribution series [17], hypergeometric distribution type series [19], Pascal distribution series [9, 10], binomial distribution series [14], generalized distribution series [16] and Mittag-Leffler type Poisson distribution series [18] and give nice solicitations on certain classes of univalent functions. Inspired by the work of Murugusundaramoorthy and Porwal [12], Murugusundaramoorthy et al. [13], Porwal and Kumar [20] , in this paper we investigated some sufficient conditions for the function $\Phi^\ell_m(\zeta)$ belonging to the classes $\mathcal{P}_\varrho(\xi)$ and $\mathfrak{D}_\varrho(\xi)$ and influences of these subclasses with $R^\tau(A, B)$.

2. Main Results

Theorem 2.1. *If $m > 0$, $\ell \in \mathbb{N}_0$ and the inequality*

$$\begin{cases} \varrho\mu'_{l+2} + (1+\varrho-\xi\varrho)\mu'_{l+1} + (1-\xi)\,\mu'_l, \\ l \geq 1 \\ \varrho m^2 + (1+\varrho-\xi\varrho)m + (1-\xi)(1-e^{-m}), \\ l = 0 \end{cases} \tag{2.1}$$

$$\leq 1-\xi,$$

then $\Phi^\ell_m(\zeta) \in \mathcal{P}_\varrho(\xi)$.

Proof. Since

$$\Phi^\ell_m(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{(n-1)^\ell m^{n-1}}{(n-1)!} e^{-m}\zeta^n.$$

By Lemma 1.4, it suffices to show that

$$P_1 = \textstyle\sum_{n=2}^{\infty}(n\varrho-\varrho+1)(n-\xi)\frac{(n-1)^\ell m^{n-1}}{(n-1)!}e^{-m} \leq 1-\xi.$$

Now

$$\begin{aligned} P_1 &= \sum_{n=2}^{\infty}(n\varrho-\varrho+1)(n-\xi)\frac{(n-1)^\ell m^{n-1}}{(n-1)!}e^{-m} \\ &= e^{-m}\left[\sum_{n=2}^{\infty}\left[\varrho(n-1)^2 + (1+\varrho-\xi\varrho)(n-1) \right.\right. \\ &\quad \left.\left. + (1-\xi)\right]\frac{(n-1)^\ell m^{n-1}}{(n-1)!}\right] \\ &= e^{-m}\left[\varrho\sum_{n=1}^{\infty}\frac{n^{\ell+2}m^n}{n!} \right. \\ &\quad + (1+\varrho-\xi\varrho)\sum_{n=1}^{\infty}\frac{n^{\ell+1}m^n}{n!} \\ &\quad \left. + (1-\xi)\sum_{n=1}^{\infty}\frac{n^\ell m^n}{n!}\right] \\ &= \begin{cases} \varrho\mu'_{l+2} + (1+\varrho-\xi\varrho)\mu'_{l+1} \\ +(1-\xi)\,\mu'_l, & l \geq 1 \\ \\ \varrho m^2 + (1+\varrho-\xi\varrho)m \\ +(1-\xi)(1-e^{-m}), & l = 0. \end{cases} \end{aligned}$$

But, this last expression is bounded above by $1-\xi$, if Eq. (2.1) holds. This completes the proof of Theorem 2.1. □

Theorem 2.2. *If $m > 0$, $\ell \in \mathbb{N}_0$ then $\Lambda_m^\ell(\zeta) \in \mathcal{P}_\varrho^*(\xi)$, if and only if Eq.* (2.1) *is satisfied.*

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1. Therefore, we omit the details involved. □

Theorem 2.3. *If $m > 0$, $\ell \in \mathbb{N}_0$ and the inequality*

$$\begin{cases} \varrho\mu'_{l+3} + (1+2\varrho-\xi\varrho)\mu'_{l+2} \\ +(2+\varrho-\xi-\xi\varrho)\mu'_{l+1} + (1-\xi)\,\mu'_l, l \geq 1 \\ \\ \varrho m^3 + (1+5\varrho-\xi\varrho)m^2 \\ +(3+4\varrho-\xi-2\xi\varrho)m \\ +(1-\xi)(1-e^{-m}), \quad l=0 \end{cases} \tag{2.2}$$

$$\leq 1-\xi$$

then $\Phi_m^\ell(\zeta) \in \mathfrak{D}_\varrho(\xi)$.

Proof. To prove that $\Phi_m^\ell(\zeta) \in \mathfrak{D}_\varrho(\xi)$. In virtue of Lemma 1.5, it is sufficient to

$$P_2 = \sum_{n=2}^\infty n(n\varrho-\varrho+1)(n-\xi)\frac{(n-1)^\ell m^{n-1}}{(n-1)!}e^{-m} \leq 1-\xi.$$

Now

$$\begin{aligned} P_2 &= \sum_{n=2}^\infty n(n\varrho-\varrho+1)(n-\xi)\frac{(n-1)^\ell m^{n-1}}{(n-1)!}e^{-m} \\ &= e^{-m}\left[\sum_{n=2}^\infty \left[\varrho(n-1)^3\right.\right. \\ &+ (1+2\varrho-\xi\varrho)(n-1)^2(2+\varrho-\xi-\xi\varrho) \\ &\times \left.\left.(n-1)(1-\xi)\right]\frac{(n-1)^\ell m^{n-1}}{(n-1)!}\right] \end{aligned}$$

$$\begin{aligned} &= e^{-m}\left[\sum_{n=1}^\infty \left[\varrho n^3 + (1+2\varrho-\xi\varrho)n^2\right.\right. \\ &+ \left.\left.(2+\varrho-\xi-\xi\varrho)n + (1-\xi)\right]\frac{n^\ell m^n}{n!}\right] \\ &= e^{-m}\left[\varrho\sum_{n=1}^\infty\frac{n^{\ell+3}m^n}{n!} + (1+2\varrho-\xi\varrho)\frac{n^{\ell+2}m^n}{n!}\right. \\ &+ \left.(2+\varrho-\xi-\xi\varrho)\sum_{n=1}^\infty\frac{n^{\ell+1}m^n}{n!} + (1-\xi)\sum_{n=1}^\infty\frac{n^\ell m^n}{n!}\right] \\ &= \begin{cases} \varrho\mu'_{l+3} + (1+2\varrho-\xi\varrho)\mu'_{l+2} \\ +(2+\varrho-\xi-\xi\varrho)\mu'_{l+1} + (1-\xi)\,\mu'_l, l \geq 1 \\ \\ \varrho m^3 + (1+5\varrho-\xi\varrho)m^2 \\ +(3+4\varrho-\xi-2\xi\varrho)m \\ +(1-\xi)(1-e^{-m}), \quad l=0 \end{cases} \end{aligned}$$

But, this last expression is bounded above by $1-\xi$, if Eq. (2.2) holds. This completes the proof of Theorem 2.3. □

Theorem 2.4. *If $m > 0$, $\ell \in \mathbb{N}_0$ then $\Lambda_m^\ell(\zeta) \in \mathfrak{D}_\varrho^*(\xi)$, if and only if Eq.* (2.2) *is satisfied.*

Proof. The proof of Theorem 2.4 is similar to that of Theorem 2.3 therefore we omit the details involved. □

Theorem 2.5. *If $m > 0$, $\ell \in \mathbb{N}_0$, $f \in \mathcal{R}^\tau(\vartheta,\delta)$ and the inequality*

$$\frac{2|\tau|(1-\delta)}{\vartheta} \times \begin{cases} \varrho\mu'_{l+2} + (1+\varrho-\xi\varrho)\mu'_{l+1} \\ +(1-\xi)\,\mu'_l, \quad l \geq 1 \\ \\ \varrho m^2 + (1+\varrho-\xi\varrho)m \\ +(1-\xi)(1-e^{-m}), \quad l=0 \end{cases} \leq 1-\xi, \tag{2.3}$$

is satisfied, then $\mathcal{I}(l,m,\zeta)f \in \mathfrak{D}_\varrho(\xi)$.

Proof. Let f be of the form Eq. (1.1) belong to the class $\mathcal{R}^\tau(\vartheta,\delta)$. To show that $\mathcal{I}(l,m,\zeta)f \in \mathfrak{D}_\varrho(\xi)$, enough to show that

$$P_3 = \sum_{n=2}^\infty n(n\varrho-\varrho+1)(n-\xi)$$

$$\times \quad \frac{(n-1)^l m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq \quad 1-\xi.$$

Since $f \in R^\tau(A,B)$, then by Lemma 1.3, we have

$$|a_n| \leq \frac{2\,|\tau|\,(1-\delta)}{1+\vartheta(n-1)}, \quad n \in \mathbb{N} \setminus \{1\}$$

and $1+\vartheta(n-1) \geq \vartheta n$. Hence

$$\begin{aligned} P_3 &= \sum_{n=2}^{\infty} n(n\varrho-\varrho+1)(n-\xi) \\ &\times \frac{(n-1)^l m^{n-1}}{(n-1)!} e^{-m} |a_n| \\ &\leq \sum_{n=2}^{\infty} n(n\varrho-\varrho+1)(n-\xi) \\ &\times \frac{(n-1)^l m^{n-1}}{(n-1)!} e^{-m} \frac{2\,|\tau|\,(1-\delta)}{1+\vartheta(n-1)} \\ &= \frac{2\,|\tau|\,(1-\delta)}{\vartheta} e^{-m} \left[\sum_{n=2}^{\infty} \left[\varrho(n-1)^2 \right.\right. \\ &+ \left.(1+\varrho-\xi\varrho)(n-1)+(1-\xi)\right] \\ &\times \left.\frac{(n-1)^\ell m^{n-1}}{(n-1)!}\right] \\ &= \frac{2\,|\tau|\,(1-\delta)}{\vartheta} e^{-m} \left[\sum_{n=1}^{\infty} \left[\varrho n^2 \right.\right. \\ &+ \left.\left.(1+\varrho-\xi\varrho)n+(1-\xi)\right] \frac{n^\ell m^n}{n!}\right] \\ &= \frac{2\,|\tau|\,(1-\delta)}{\vartheta} e^{-m} \left[\varrho \sum_{n=1}^{\infty} \frac{n^{\ell+2} m^n}{n!} \right. \\ &+ (1+\varrho-\xi\varrho) \sum_{n=1}^{\infty} \frac{n^{\ell+1} m^n}{n!} \\ &+ \left.(1-\xi) \sum_{n=1}^{\infty} \frac{n^\ell m^n}{n!}\right] \\ &= \frac{2\,|\tau|\,(1-\delta)}{\vartheta} \begin{cases} \varrho\mu'_{l+2} + (1+\varrho-\xi\varrho)\mu'_{l+1} \\ +(1-\xi)\,\mu'_l, \quad l \geq 1 \\ \\ \varrho m^2 + (1+\varrho-\xi\varrho)m \\ +(1-\xi)(1-e^{-m}), \quad l=0 \end{cases} \\ &\leq 1-\xi, \end{aligned}$$

by the given hypothesis. This completes the proof of Theorem 2.5. □

Theorem 2.6. *If* $m > 0$, $\ell \in \mathbb{N}_0$ *then* $L(m,\ell,\zeta) = \int_0^\zeta \left[2 - \frac{\Phi_m^\ell(t)}{t}\right] dt$ *is in* $\mathfrak{D}_\varrho^*(\xi)$, *if and only if Eq.* (2.1) *holds.*

Proof. It is easy to see that

$$L(m,\ell,\zeta) = \zeta - \sum_{n=2}^{\infty} \frac{(n-1)^l m^{n-1}}{n!} e^{-m} \zeta^n.$$

Using Remark 1.7, we only need to show that

$$\sum_{n=2}^{\infty} n(n\varrho-\varrho+1)(n-\xi) \frac{(n-1)^l m^{n-1}}{n!} e^{-m} \leq 1-\xi.$$

Now

$$\begin{aligned} P_4 &= \sum_{n=2}^{\infty} n(n\varrho-\varrho+1)(n-\xi) \frac{(n-1)^l m^{n-1}}{n!} e^{-m} \\ &= e^{-m} \left[\sum_{n=2}^{\infty} (\varrho(n-1)(n-2) \right. \\ &+ (1+2\varrho-\xi\varrho)(n-1) \\ &+ \left.(1-\xi)) \frac{(n-1)^l m^{n-1}}{(n-1)!}\right] \\ &= e^{-m} \left[\sum_{n=2}^{\infty} \left[\varrho(n-1)^2 \right.\right. \\ &+ (1+\varrho-\xi\varrho)(n-1) \\ &+ \left.\left.(1-\xi)\right] \frac{(n-1)^\ell m^{n-1}}{(n-1)!}\right] \\ &= e^{-m} \left[\sum_{n=1}^{\infty} \left[\varrho n^2 + (1+\varrho-\xi\varrho)n \right.\right. \\ &+ \left.\left.(1-\xi)\right] \frac{n^\ell m^n}{n!}\right] \\ &= e^{-m} \left[\varrho \sum_{n=1}^{\infty} \frac{n^{\ell+2} m^n}{n!} \right. \end{aligned}$$

$$+ \ (1+\varrho-\xi\varrho)\sum_{n=1}^{\infty}\frac{n^{\ell+1}m^n}{n!}$$

$$+ \ (1-\xi)\sum_{n=1}^{\infty}\frac{n^{\ell}m^n}{n!}\Bigg]$$

$$= \begin{cases} \varrho\mu_{l+2}^{'}+(1+\varrho-\xi\varrho)\mu_{l+1}^{'} \\ +(1-\xi)\,\mu_{l}^{'}, \qquad l\geq 1 \\ \\ \varrho m^2+(1+2\varrho-\xi\varrho)m \\ +(1-\xi)(1-e^{-m}), \qquad l=0. \end{cases}$$

But, this last expression is bounded above by $1-\xi$, if Eq. (2.1) holds. This completes the proof of Theorem 2.6.

□

Remark 2.7. *By fixing the parameters* $\varrho = 0$ *and* $\varrho = 1$ *one can easily deduce the above results for the function classes* $\mathcal{S}^*(\xi)$ *and* $\mathcal{K}(\xi)$*, respectively, and also for the class studied earlier by Robertson [21] and Silverman [23].*

3. Conclusion and Future Scope

The main conclusion of this paper is to obtain some necessary and sufficient conditions of a power series associated with Touchard polynomials for belonging to unified classes of starlike and convex functions. Here, we obtain the results for the case when $l = 0, 1, 2, \ldots$. Therefore, it is natural to ask what is the analogues results for $l > 0$ when l is not a positive integer. It is also interesting to investigate analogues Touchard polynomials associated with generalized distribution series, hypergeometric distribution series, confluent hypergeometric distribution[11, 17, 26] series and Pascal distribution series [9, 10].

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References

[1] Ahmad MS., Mehmood Q., Nazeer W. andḢaq AU. An application of a hypergeometric distribution series on certain analytic functions, Sci. Int., 2015;27: 2989-92.

[2] Abdullah, J.T., Ali, H.S. Laguerre and Touchard polynomials for linear Volterra integral and Integro differential equations. J. Phys. Conf. Ser. 2020, 1591, 012047.doi:10.1088/1742-6596/1591/1/012047.

[3] Al-Shaqsi K. On inclusion results of certain subclasses of analytic functions associated with generating function, AIP Conference Proceedings 2017; doi: 10.1063/1.4980979.

[4] Altintas O., Özkan Ö.and Srivastava HM. Neighborhoods of a class of analytic functions with negative coefficients, Appl. Math. Lett., 2000;13: 63-7.

[5] Altınkaya S. and Yalçın S. Poisson distribution series for analytic univalent functions, Complex Anal. Oper. Theory., 2018;12(5):1315-9.

[6] De Branges, L. A proof of the Bieberbach conjecture. Acta Math. 1985;154: 137-52.

[7] Bell ET. Exponential polynomials, Annal Math., 1934;35: 258-77.

[8] Boyadzhiev KN. Exponential polynomials, Stirling numbers, and evaluation of some Gamma integrals, Abstract Appl. Anal., 2009; Art. ID 168672: 1-18.

[9] Bulboaca T and Murugusundaramoorthy G.,Univalent functions with positive coefficients involving Pascal distribution series,Commun. Korean Math. Soc., 2020;35(3): 867-77.

[10] El-Deeb SM., Bulboacă T. and Dziok J. Pascal distribution series connected with certain subclasses of univalent functions, Kyungpook Math. J., 2019; 59(2): 301-14.

[11] Lin S.-D., Srivastava H. M. and Yao,J.-C. Some classes of generating relations associated with a family of the generalized Gauss type hypergeometric functions, Appl. Math. Inform. Sci.,2015,9:1731-8.

[12] Murugusundaramoorthy G. and porwal S. Univalent functions with positive coefficients involving Touchard polynomials, Al-Qadisiyah J. Pure Science, 2020;25(4): 1-8.

[13] Murugusundaramoorthy G., Vijaya K. and Kasthuri M. A note on subclasses of starlike and convex functions associated with Bessel functions, J. Nonlinear Funct. Anal.,2014; Art. ID 11: 1-11.

[14] Nazeer W., Mehmood Q. , Kang SM. and Haq AU. An application of Binomial distribution series on certain analytic functions, J.Comput. Anal. Appl.,2019;26: 11-7.

[15] Porwal S. An application of a Poisson distribution series on certain analytic functions, J. Complex Anal.,2014; Art. ID 984135: 1-3.

[16] Porwal S. Generalized distribution and its geometric properties associated with univalent functions, J. Complex Anal. 2018; Art. ID 8654506: 1-5.

[17] Porwal S. Confluent hypergeometric distribution and its applications on certain classes of univalent functions of conic regions, Kyungpook Math. J. 2018;58(3): 495-505.

[18] Porwal S. and Bajpai D. An application of Mittag-Leffler type Poisson distribution series on starlike and convex functions, Asian Pacific J. Math., 2020;7: 1-7.

[19] Porwal S. and Gupta A. Some properties of convolution for hypergeometric distribution type series on certain analytic univalent functions, Acta Univ. Apulensis Math. Inform., 2018;56: 69-80.

[20] Porwal S. and Manish Kumar, A unified study on starlike and convex functions associated with Poisson distribution series, Afr. Mat., 2016;27(5-6): 1021-7.

[21] Robertson MS. On the theory of univalent functions, Ann. Math.,1936;37: 374-408.

[22] Roman S. The Umbral calculus, Dover 1984.

[23] Silverman H. Univalent functions with negative coefficients, Proc. Amer. Math. Soc.,1975;51(1): 109-16.

[24] Simsek Y. Special numbers on analytic functions, Appl. Math.,2014; 5: 1091-8.

[25] Touchard J. Sur les cycles des substitutions, Acta Math.,1939; 70: 243-97.

[26] Swaminathan A. Certain suffienct conditions on Gaussian hypergeometric functions, Journal of Ineq. Pure & Appl. Math.,2004;5(4): 83,1-10.