P-ISSN 2586-9000
E-ISSN 2586-9027
Homepage : *https://tci-thaijo.org/index.php/SciTechAsia* Science & Technology Asia
Vol.27 No.4 October - December 2022 Page: [175-194]
Original research article

Adaptive Tseng's Type Methods for Inclusion and Signal Processing

Duangkamon Kitkuan, Anantachai Padcharoen*, Naknimit Akkasriworn

Department of Mathematics, Faculty of Science and Technology, Rambhai Barni Rajabhat University, Chanthaburi 22000, Thailand

Received 12 October 2021; Received in revised form 11 May 2022
Accepted 20 May 2022; Available online 31 December 2022

ABSTRACT

This research presents a new non-monotone adaptive step size for Tseng's type method to solve the monotone inclusion problem. The results were applied to the problem of convex minimization and signal processing, using a basic assumption, numerical testing, and comparisons with other operations to show that the recommended operation was weakly and strongly convergent.

Keywords: Inclusion problem; Tseng's type method; Signal processing

1. Introduction

The topic of inclusion problems and fixed point issues has piqued the interest of many mathematicians. This is because comparable challenges can be applied to a range of other problems. Convex programming, the minimization problem, variational inequalities, and the split feasibility problem, for example, can all be solved using these concerns. As a result, machine learning, signal processing, image restoration, computerized tomography sensor networks, data compression, and intensity modulated radiation therapy treatment planning can all be considered now (see [1–15]).

Allow $\mathcal{H}$ to be a real Hilbert with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let $\mathcal{K}:\mathcal{H}\to\mathcal{H}$ and $\mathcal{B}:\mathcal{H}\to 2^{\mathcal{H}}$ be, respectively, single-valued and multi-valued operators. In this paper, we study the so-called monotone inclusion problem:

$$\text{find} \quad z^*\in\mathcal{H} \quad \text{such that} \quad 0\in(\mathcal{K}+\mathcal{B})z^*. \tag{1.1}$$

The set of solutions of the problem (1.1) is denoted by $\Omega := (\mathcal{K}+\mathcal{B})^{-1}(0)$. This problem (1.1) has a lot of interest because it is at the heart of many mathematical problems, such as split feasibility problem, e.g., let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the real Hilbert spaces, $\mathcal{K}$ is bounded linear operator and $\mathcal{S}:\mathcal{H}_1\to\mathcal{H}_2$

***Corresponding author:** anantachai.p@rbru.ac.th **doi:** xx.xxxxx/scitechasia.xxxx.xx

which $\mathcal{S}^*$ is the inverse of $\mathcal{S}$. $\mathcal{C} \subset \mathcal{H}_1$ and $\mathcal{Q} \subset \mathcal{H}_2$ are not empty, closed and convex. The convex split feasibility problem [2], has a pattern as follows:

$$\text{find} \quad z^* \in \mathcal{C} \quad \text{such that} \quad \mathcal{S}z^* \in \mathcal{Q}. \tag{1.2}$$

Let $\mathcal{K}z^* = \nabla(\frac{1}{2}\|\mathcal{S}z^* - \mathcal{P}_{\mathcal{Q}}\mathcal{S}z^*\|^2) = \mathcal{S}^*(\mathcal{I} - \mathcal{P}_{\mathcal{Q}})\mathcal{S}z^*$, where $\mathcal{P}_{\mathcal{Q}}$ is a metric projection onto $\mathcal{Q}$ and ∇ is a gradient and $\mathcal{B} = \partial i_{\mathcal{C}}$ is the sub-differential $\partial i_{\mathcal{C}}$ of $i_{\mathcal{C}}$ and a maximal monotone operator defined by

$$\partial i_{\mathcal{C}}(x) = \{z^* \in \mathcal{H} : i_{\mathcal{C}}(x) \leq \langle z^*, x-y\rangle + i_{\mathcal{C}}(y)\},$$

for all $y \in \mathcal{H}$. Therefore, the convex split feasibility problem has the same structure as (1.1).

The forward-backward splitting algorithm (FBSA) [4,16] is a classical method for addressing the problem (1.1) in Hilbert space $\mathcal{H}$. This uses the following procedure to build an iterative sequence $\{x_n\}$:

$$\begin{cases} x_1 \in \mathcal{H}, \\ x_{n+1} = J_{\eta}^{\mathcal{B}}(\mathcal{I} - \eta\mathcal{K})x_n, \quad \forall\, n \geq 1, \end{cases} \tag{1.3}$$

where $J_{\eta}^{\mathcal{B}} := (\mathcal{I} + \eta\mathcal{B})^{-1}$ is the resolvent operator of an operator $\mathcal{B}$ and $\mathcal{I}$ denotes the identity operator on $\mathcal{H}$. It was proved that the sequence generated by (1.3) converges weakly to an element in $\mathbf{\Omega} := (\mathcal{K}+\mathcal{B})^{-1}(0)$ under the assumption of the α-cocoercivity of the operator $\mathcal{K}$, that is,

$$\langle \mathcal{K}x - \mathcal{K}y, x - y\rangle \geq \alpha\|\mathcal{K}x - \mathcal{K}y\|^2,$$

for all $x, y \in \mathcal{H}$ and η is chosen in $(0, 2\alpha)$.

Moudafi and Oliny [17] investigated the monotone inclusion problem (1.1). They developed the inertial proximal point algorithm, which combines the heavy ball method and the proximal point algorithm. The inertial proximal point algorithm is defined as follows:

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ z_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\eta_n}^{\mathcal{B}}(\mathcal{I} - \eta_n\mathcal{K})x_n, \quad \forall\, n \geq 1, \end{cases} \tag{1.4}$$

where $\{\theta_n\} \subset [0, 1)$ such that

$$\sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty \tag{1.5}$$

which $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ and $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ are single and multi-valued mappings, respectively. It was proven that if $\eta_n < 2/\mathcal{L}$ with the Lipschitz constant $\mathcal{L}$ of the monotone operator $\mathcal{K}$ and the condition (1.5) holds, then the sequence $\{x_n\}$ generated by the algorithm (1.4) converges weakly to a solution of the inclusion problem (1.1).

In recent years, many authors in a variety of settings have studied and modified the FBSA for solving the monotone inclusion problem (1.1) when $\mathcal{K}$ is α-cocoercive (see [18–23]). It is worth noting that the α-cocoercivity of the operator $\mathcal{K}$ is a strong assumption. Tseng [24] proposed the Tseng's splitting method to alleviate this assumption:

$$\begin{cases} x_1 \in \mathcal{H}, \\ z_n = J_{\eta_n}^{\mathcal{B}}(\mathcal{I} - \eta_n\mathcal{K})x_n, \\ x_{n+1} = z_n - \eta_n(\mathcal{K}z_n - \mathcal{K}x_n), \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\mathcal{K}$ is monotone and $\mathcal{L}$-Lipschitz continuous with $\mathcal{L} > 0$. It was proved that the sequence $\{x_n\}$ generated by (1.5) converges weakly to an element in $\mathbf{\Omega} := (\mathcal{K} + \mathcal{B})^{-1}(0)$ provided the step size η_n is chosen in $(0, 1/\mathcal{L})$. It's worth noting that Tseng's splitting method necessitates knowledge of the Lipschitz constant for the mapping. Unfortunately, Lipschitz constants are generally unknown or difficult to approximate.

Then problem (1.1) becomes the following minimization problem:

$$\min_{x\in\mathcal{H}} \mathcal{S}(x) + \mathcal{T}(x). \tag{1.7}$$

Methods for solving the problem (1.7), in case $\mathcal{F} = \nabla\mathcal{S}$ without the Lipschitz continuity of $\mathcal{F}$ often use a linesearch procedure, which runs in each iteration of the algorithm until a stopping criterion is satisfied. Because linesearch methods require several computations of $\mathcal{F}$ values as well as projections onto a feasible set, they are time-consuming. In addition, complexity calculations in linesearch algorithms become less useful. This is clear because they only show the number of outside iterations required to achieve the required accuracy, but not the number of inner linesearch iterations.

In this paper, we propose two modifications of Tseng's splitting method with monotone adaptive step sizes for solving the problems (1.1) and (1.7) in the framework of Hilbert spaces, inspired by Shehu [25]. Step size in our methods does not require prior knowledge of the operator's Lipschitz constant, nor does it require any linesearch procedure.

2. Preliminaries

Lemma 2.1 ([26]). *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping.*

(a) *$\mathcal{K}$ is called nonexpansive mapping if*

$$\|\mathcal{K}x - \mathcal{K}y\| \le \|x - y\|,$$

for all $x, y \in \mathcal{H}$.

(b) *$\mathcal{K}$ is called firmly nonexpansive mapping if*

$$\|\mathcal{K}x - \mathcal{K}y\|^2 + \|(\mathcal{I} - \mathcal{K})x - (\mathcal{I} - \mathcal{K})y\|^2 \le \|x - y\|^2,$$

for all $x, y \in \mathcal{H}$.

(c) *$\mathcal{K}$ is called monotone mapping if*

$$\langle \mathcal{K}x - \mathcal{K}y, y - x\rangle \ge 0.$$

(d) *$\mathcal{K}$ is called $\mathcal{L}$-Lipschitz continuous mapping if*

$$\|\mathcal{K}x - \mathcal{K}y\| \le \mathcal{L}\|x - y\|,$$

for all $\mathcal{L} > 0$.

Example 2.2. Define a mapping $\mathcal{K}$ on $[0,\infty)$ by $\mathcal{K}x = -x$. Then

(a) $\mathcal{K}$ is a nonexpansive mapping.

(b) $\mathcal{K}$ is not a firmly nonexpansive mapping.

(c) $\mathcal{K}$ is a monotone mapping.

Consider, for all $x, y \in [0,\infty)$

(a)

$$\begin{aligned}\|\mathcal{K}x - \mathcal{K}y\| &= \|-x + y\| \\ &= \|x - y\|.\end{aligned}$$

Thus, $\mathcal{K}$ is a nonexpansive mapping.

(b)

$$\begin{aligned}&\|\mathcal{K}x - \mathcal{K}y\|^2 + \|(\mathcal{I} - \mathcal{K})x - (\mathcal{I} - \mathcal{K})y\|^2 \\ &= \|-x + y\|^2 + \|x + x - (y + y)\|^2 \\ &= \|x - y\|^2 + \|2x - 2y\|^2 \\ &= \|x - y\|^2 + \|2(x - y)\|^2 \\ &= \|x - y\|^2 + 4\|x - y\|^2 \\ &= 5\|x - y\|^2 > \|x - y\|^2.\end{aligned}$$

Thus, $\mathcal{K}$ is not a firmly nonexpansive mapping.

(c)

$$\begin{aligned}\langle \mathcal{K}x - \mathcal{K}y, y - x\rangle &= \langle -x + y, y - x\rangle \\ &= \langle y - x, y - x\rangle \\ &= \|y - x\|^2 \\ &\ge 0.\end{aligned}$$

Thus, $\mathcal{K}$ is a monotone mapping.

In fact, every firmly nonexpansive mapping is nonexpansive, but the inverse is not always true. Moreover, if $\mathcal{L} = 1$, $\mathcal{L}$-Lipschitz continuous mapping is nonexpansive.

Lemma 2.3 ([26]). *Let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a mapping. Then, the following items are equivalent:*

(a) $\mathcal{K}$ *is firmly nonexpansive*;

(b) $(\mathcal{I} - \mathcal{K})$ *is firmly nonexpansive*;

(c) $\|\mathcal{K}x - \mathcal{K}y\|^2 \leq \langle x - y, \mathcal{K}x - \mathcal{K}y\rangle$, *for all* $x, y \in \mathcal{K}$.

Let $\mathcal{C}$ be a nonempty convex subset of $\mathcal{H}$. A subset $\mathcal{C} \subset \mathcal{H}$ is said to be proximinal if, for each $x \in \mathcal{H}$, there exists $y \in \mathcal{C}$ such that

$$\|x - y\| = d(x, \mathcal{C}) = \inf\{\|x - w\| : w \in \mathcal{C}\}.$$

We denote by $C\mathcal{B}(\mathcal{C})$ the collection of all nonempty closed bounded subsets of $\mathcal{C}$. The Hausdorff metric on $C\mathcal{B}(\mathcal{C})$ is defined by

$$\mathcal{H}(\mathcal{P}, \mathcal{Q}) = \max\left\{\sup_{x\in\mathcal{P}} d(x, \mathcal{Q}), \sup_{y\in\mathcal{Q}} d(y, \mathcal{P})\right\}$$

for all $\mathcal{P}, \mathcal{Q} \in C\mathcal{B}(\mathcal{C})$, where $d(x, \mathcal{Q}) = \inf_{q\in\mathcal{Q}} \|x - q\|$. A multivalued mapping $\mathcal{B} : \mathcal{C} \to C\mathcal{B}(\mathcal{C})$ is said to be nonexpansive if

$$\mathcal{H}(\mathcal{B}x, \mathcal{B}y) \leq \|x - y\|$$

for all $x, y \in \mathcal{C}$.

Definition 2.4 ([26]). A set-valued operator $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ is called maximal monotone if $\mathcal{B}$ is monotone, i.e.,

$$\langle x - y, u - v\rangle \geq 0,$$

for all $x, y \in \mathcal{H}$, $u \in \mathcal{B}x$ and $v \in \mathcal{B}y$. The graph $\mathcal{G}(\mathcal{B})$ defined by

$$\mathcal{G}(\mathcal{B}) = \{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in \mathcal{B}(x)\}$$

is not properly contained in the graph of any other monotone operator.

Example 2.5. Define a mapping $\mathcal{B}$ on $[0, 4]$ by

$$\mathcal{B}x = \begin{cases} [0, \frac{x}{4}], & \text{if } x \in [0, 3], \\ \{0\}, & \text{if } x \in (3, 4]. \end{cases}$$

Then $\mathcal{B}$ is monotone mapping.

Indeed, if $x \geq y$ for $x \in (3, 4]$ and $y \in [0, 3]$, then for $u_x \in \mathcal{B}x = \{0\}$, there exists $u_y = 0 \in \mathcal{B}y$ such that $u_x \geq u_y$. Thus, $\mathcal{K}$ is a monotone mapping.

Example 2.6 ([27]). Let $\mathcal{A}$ be an $n \times n$ matrix with real entries. Consider the operator $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathcal{T}x = \mathcal{A}x$. Then $\mathcal{T}$ is maximal monotone if $\mathcal{T}$ is a positive linear operator.

Lemma 2.7 ([28]). *Let $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ be any set-valued operator and $J_\eta^{\mathcal{B}}$ be the resolvent of $\mathcal{B}$ with parameter $\eta > 0$. Then we have the following*:

(a) *if $\mathcal{B}$ is a maximal monotone operator, then a point $x^* \in \mathcal{H}$ is a fixed point of $J_\eta^{\mathcal{B}}$ if and only if $x^* \in \mathcal{B}^{-1}(0) = \{x \in \mathcal{H} : 0 \in \mathcal{B}x\}$*;

(b) *$\mathcal{B}$ is monotone if and only if the resolvent $J_\eta^{\mathcal{B}}$ is single-valued and firmly nonexpansive*;

(c) *$\mathcal{B}$ is maximal monotone if and only if $J_\eta^{\mathcal{B}}$ is single-valued, firmly nonexpansive and $\mathcal{D}(J_\eta^{\mathcal{B}}) = \mathcal{H}$, where $\mathcal{D}(J_\eta^{\mathcal{B}})$ is a domain of the operator $J_\eta^{\mathcal{B}}$.*

Lemma 2.8 ([29]). *Let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous and monotone mapping and $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone mapping, then the mapping $\mathcal{K} + \mathcal{B}$ is a maximal monotone mapping.*

Lemma 2.9 ([31]). *Let $\alpha \in (0, 1)$ for $x, y \in \mathcal{H}$, we have the following statements*:

(a) $|\langle x, y\rangle| \leq \|x\|\|y\|$;

(b) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(c) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;

(d) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Lemma 2.10 ([32]). *Assume that $\{\eta_n\}$ and $\{\varphi_n\}$ are two nonnegative real sequences such that*

$$\eta_{n+1} \leq \eta_n + \varphi_n, \quad \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \varphi_n < \infty$, then $\lim_{n\to\infty} \eta_n$ exists.

Lemma 2.11 ([33]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \ \forall n \geq 1,$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$ then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Assumption 1.

(A1) The feasible set of (1.1) is a nonempty closed and convex subset of $\mathcal{H}$.

(A2) The solution set Ω of (1.1) is nonempty.

(A3) $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ is monotone, $\mathcal{L}$-Lipschitz continuous on $\mathcal{H}$, and $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone.

Algorithm 1. Adaptive Tseng's type method for inclusion problem

initialization: Given $x_0, x_1 \in \mathcal{H}$, $\mu \in (0, 1)$, $\alpha > 0$, $\eta_1 > 0$ and select a nonnegative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{\infty} \varphi_n < \infty$. Moreover, select a sequence $\{\psi_n\} \subset (0, 1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \psi_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \psi_n = \infty.$$

Step 1: Compute

$$z_n = (1 - \psi_n)(x_n + \theta_n(x_n - x_{n-1})),$$

where $\{\theta_n\}$ is a sequence such that $0 \leq \theta_n \leq \hat{\theta}_n$ and

$$\hat{\theta}_n = \begin{cases} \min\left\{\dfrac{n-1}{n+\alpha-1}, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \dfrac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$s_n = J_{\eta_n}^{\mathcal{B}}(\mathcal{I} - \eta_n\mathcal{K})z_n.$$

If $s_n = z_n$, then stop and s_n is a solution of (1.1). Else, go to **Step 3.**

Step 3: Compute

$$x_{n+1} = s_n - \eta_n(\mathcal{K}s_n - \mathcal{K}z_n)$$

where the sizes are adaptively updated as follows:

$$\eta_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|s_n - z_n\|}{\|\mathcal{K}s_n - \mathcal{K}z_n\|}, \eta_n + \varphi_n\right\}, & \text{if } \mathcal{K}s_n \neq \mathcal{K}z_n, \\ \eta_n + \varphi_n, & \text{otherwise.} \end{cases} \tag{3.1}$$

Set $n := n + 1$ and go back to Step 1.

Lemma 3.1. *The sequence $\{\eta_n\}$ generated by Eq.* (3.1) *is monotonically decreasing and bounded from below by* $\min\left\{\frac{\mu}{\mathcal{L}}, \eta_1\right\}$.

Proof. It is clear that the sequence $\{\eta_n\}$ is monotonically decreasing. Since $\mathcal{K}$ is a Lipschitz function with Lipschitz's constant $\mathcal{L}$, for $\mathcal{K}s_n \neq \mathcal{K}z_n$, we have

$$\frac{\mu\|s_n - z_n\|}{\|\mathcal{K}s_n - \mathcal{K}z_n\|} \geq \frac{\mu\|s_n - z_n\|}{\mathcal{L}\|s_n - z_n\|} = \frac{\mu}{\mathcal{L}}. \quad (3.2)$$

Therefore, it follows that $\eta_n \geq \min\left\{\frac{\mu}{\mathcal{L}}, \eta_1\right\}$. □

Lemma 3.2. *The sequence* $\{\eta_n\}$ *generated by Eq.* (3.1) *and* $\lim_{n\to\infty}\eta_n = \eta \in [\min\left\{\frac{\mu}{\mathcal{L}}, \eta_1\right\}, \eta_1+\varphi]$, *where* $\varphi = \sum_{n=1}^{\infty}\varphi_n$. *Then*

$$\|\mathcal{K}s_n - \mathcal{K}z_n\| \leq \frac{\mu}{\eta_{n+1}}\|s_n - z_n\|. \quad (3.3)$$

Proof. Using Eq. (3.1) and mathematical induction, we have the sequence $\{\eta_n\}$ has upper bound $\eta_1 + \varphi$ and lower bound $\min\{\frac{\mu}{\mathcal{L}}, \eta_1\}$. Using Lemma 2.10, we have $\lim_{n\to\infty}\eta_n$ exists and we denote $\eta = \lim_{n\to\infty}\eta_n$. It is obvious something which $\eta \in [\min\left\{\frac{\mu}{\mathcal{L}}, \eta_1\right\}, \eta_1 + \varphi]$. By the definition of $\{\eta_n\}$, we have

$$\begin{aligned}\eta_{n+1} &= \min\left\{\frac{\mu\|s_n - z_n\|}{\|\mathcal{K}s_n - \mathcal{K}z_n\|}, \eta_n + \varphi_n\right\}\\ &\leq \frac{\mu\|s_n - z_n\|}{\|\mathcal{K}s_n - \mathcal{K}z_n\|}.\end{aligned}$$

This means that

$$\|\mathcal{K}s_n - \mathcal{K}z_n\| \leq \frac{\mu}{\eta_{n+1}}\|s_n - z_n\|, \quad \forall\, n \geq 1. \quad (3.4)$$

□

Lemma 3.3. *Let* $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ *be a mapping that satisfies Assumption 1 and a sequence* $\{x_n\}$ *generated by Algorithm 1. Then, we have*

$$\begin{aligned}&\|x_{n+1} - z\|^2\\ &\leq \|z_n - z\|^2 - \left(1 - \mu^2\frac{\eta_n^2}{\eta_{n+1}^2}\right)\|z_n - s_n\|^2,\end{aligned}$$

for each $z \in \Omega$.

Proof. Let $z \in \Omega$. Then, by the definition of $\{x_{n+1}\}$ and using Lemma 2.9, we obtain

$$\begin{aligned}&\|x_{n+1} - z\|^2\\
&= \|s_n + \eta_n(\mathcal{K}z_n - \mathcal{K}s_n) - z\|^2\\
&= \|s_n - z\|^2 + \eta_n^2\|\mathcal{K}z_n - \mathcal{K}s_n\|^2\\
&\quad + 2\eta_n\langle s_n - z, \mathcal{K}z_n - \mathcal{K}s_n\rangle\\
&= \|s_n + z_n - z_n - z\|^2\\
&\quad + \eta_n^2\|\mathcal{K}z_n - \mathcal{K}s_n\|^2\\
&\quad + 2\eta_n\langle s_n - z, \mathcal{K}z_n - \mathcal{K}s_n\rangle\\
&= \|s_n - z_n\|^2 + \|z_n - z\|^2\\
&\quad + 2\langle s_n - z_n, z_n - z\rangle\\
&\quad + \eta_n^2\|\mathcal{K}z_n - \mathcal{K}s_n\|^2\\
&\quad + 2\eta_n\langle s_n - z, \mathcal{K}z_n - \mathcal{K}s_n\rangle\\
&= \|z_n - z\|^2 + \|s_n - z_n\|^2\\
&\quad + \eta_n^2\|\mathcal{K}z_n - \mathcal{K}s_n\|^2\\
&\quad + 2\langle s_n - z_n, s_n - z\rangle\\
&\quad + 2\langle s_n - z_n, z_n - s_n\rangle\\
&\quad + 2\eta_n\langle s_n - z, \mathcal{K}z_n - \mathcal{K}s_n\rangle\\
&= \|z_n - z\|^2 + \|s_n - z_n\|^2\\
&\quad + \eta_n^2\|\mathcal{K}z_n - \mathcal{K}s_n\|^2\\
&\quad + 2\langle s_n - z_n, s_n - z\rangle\\
&\quad - 2\langle s_n - z_n, s_n - z_n\rangle\\
&\quad - 2\eta_n\langle \mathcal{K}z_n - \mathcal{K}s_n, z - s_n\rangle\\
&= \|z_n - z\|^2 + \|s_n - z_n\|^2\\
&\quad + \eta_n^2\|\mathcal{K}z_n - \mathcal{K}s_n\|^2\\
&\quad - 2\langle s_n - z_n, z - s_n\rangle\\
&\quad - 2\|s_n - z_n\|\\
&\quad - 2\eta_n\langle \mathcal{K}z_n - \mathcal{K}s_n, z - s_n\rangle\\
&= \|z_n - z\|^2 - \|s_n - z_n\|^2\\
&\quad + \eta_n^2\|\mathcal{K}z_n - \mathcal{K}s_n\|^2\\
&\quad - 2\langle z_n - s_n - \eta_n(\mathcal{K}z_n - \mathcal{K}s_n), s_n - z\rangle.
\end{aligned} \quad (3.5)$$

By combining (3.4) and (3.5), we obtain

$$\|x_{n+1} - z\|^2$$

$$\leq \|z_n - z\|^2 - \|s_n - z_n\|^2 + \mu^2 \frac{\eta_n^2}{\eta_{n+1}^2} \|z_n - s_n\|^2 - 2\langle z_n - s_n - \eta_n(\mathcal{K}z_n - \mathcal{K}s_n), s_n - z\rangle. \tag{3.6}$$

In fact, the resolvent $J_{\eta_n}^{\mathcal{B}}$ is firmly nonexpansive and $s_n = J_{\eta_n}^{B}(\mathcal{I} - \eta_n\mathcal{K})z_n = (\mathcal{I} + \eta_n\mathcal{B})^{-1}(\mathcal{I} - \eta_n\mathcal{K})z_n$, since $\mathcal{B}$ is maximal monotone, there exists $\rho_n \in \mathcal{B}z_n$ such that

$$(\mathcal{I} + \eta_n\mathcal{K})z_n = s_n + \eta_n\rho_n.$$

This means that

$$\rho_n = \frac{1}{\eta_n}(z_n - s_n - \eta_n\mathcal{K}z_n). \tag{3.7}$$

However, we have $0 \in (\mathcal{K} + \mathcal{B})z$ and $\mathcal{K}s_n + \rho_n \in (\mathcal{K} + \mathcal{B})z_n$. From $\mathcal{K} + \mathcal{B}$ is maximal monotone, we have

$$\langle \mathcal{K}s_n + \rho_n, s_n - z\rangle \geq 0. \tag{3.8}$$

Using (3.7) and (3.8), we have

$$\frac{1}{\eta_n}\langle z_n - s_n - \eta_n(\mathcal{K}z_n - \mathcal{K}s_n), s_n - z\rangle \geq 0.$$

This means that

$$\langle z_n - s_n - \eta_n(\mathcal{K}z_n - \mathcal{K}s_n), s_n - z\rangle \geq 0. \tag{3.9}$$

From (3.6) and (3.9), we have

$$\|x_{n+1} - z\|^2 \leq \|z_n - z\|^2 - \left(1 - \mu^2 \frac{\eta_n^2}{\eta_{n+1}^2}\right) \|z_n - s_n\|^2. \tag{3.10}$$

□

Theorem 3.4. *Let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a mapping that satisfies Assumption 1 and a sequence $\{z_n\}$ generated by Algorithm 1. If there exists a subsequence $\{z_{n_k}\}$ weakly convergent to $z \in \mathcal{H}$ with $\lim_{n\to\infty} \|z_n - s_n\| = 0$, then $z \in \mathbf{\Omega}$.*

Proof. From $\lim_{n\to\infty} \eta_n$ exists and $\mu \in (0, 1)$, it follows that

$$\lim_{n\to\infty} \left(1 - \mu^2 \frac{\eta_n^2}{\eta_{n+1}^2}\right) = 1 - \mu^2 > 0.$$

From the above explanation there exists a fixed number $n_0 \in \mathbb{N}$ such that

$$1 - \mu^2 \frac{\eta_n^2}{\eta_{n+1}^2} > 0, \quad \forall n \geq n_0. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\|x_{n+1} - z\|^2 \leq \|z_n - z\|^2, \quad \forall n \geq n_0. \tag{3.12}$$

This means that $\lim_{n\to\infty} \|z_n - z\|$ exists and so $\{\|z_n - z\|\}$ is bounded. From (3.10), we have

$$\left(1 - \mu^2 \frac{\eta_n^2}{\eta_{n+1}^2}\right) \|z_n - s_n\|^2 \leq \|z_n - z\|^2 - \|x_{n+1} - z\|^2. \tag{3.13}$$

Therefore,

$$\lim_{n\to\infty} \|z_n - s_n\|^2 = 0. \tag{3.14}$$

Also,

$$\lim_{n\to\infty} \|z_n - s_n\| = 0. \tag{3.15}$$

Using the fact that $\mathcal{K}$ is Lipschitz continuous, we have

$$\lim_{n\to\infty} \|\mathcal{K}z_n - \mathcal{K}s_n\| = 0. \tag{3.16}$$

From the boundedness of $\{z_n\}$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup z \in \mathcal{H}$. From Eq. (3.15), we have $s_{n_k} \rightharpoonup z$. Let $(w, v) \in \mathcal{G}(\mathcal{K} + \mathcal{B})$, we have $v - \mathcal{K}w \in \mathcal{B}w$. Since $s_{n_k} = J_{\eta_{n_k}}^{\mathcal{B}}(\mathcal{I} - \eta_{n_k}\mathcal{K})z_{n_k} = (\mathcal{I} + \eta_{n_k}\mathcal{B})^{-1}(\mathcal{I} - \eta_{n_k}\mathcal{K})z_{n_k}$, we obtain

$$(\mathcal{I} - \eta_{n_k}\mathcal{K})z_{n_k} \in (\mathcal{I} + \eta_{n_k}\mathcal{B})s_{n_k}.$$

This means that

$$\frac{1}{\eta_{n_k}}(z_{n_k} - s_{n_k} - \eta_{n_k}\mathcal{K}z_{n_k}) \in \mathcal{B}s_{n_k}.$$

Using the maximal monotonicity of $\mathcal{B}$, we have

$$\langle w - z_{n_k}, v - \mathcal{K}w - \frac{1}{\eta_{n_k}}(z_{n_k} - s_{n_k} - \eta_{n_k}\mathcal{K}z_{n_k})\rangle \geq 0.$$

and using the monotonicity of $\mathcal{K}$, we have

$$\begin{aligned}
&\langle w - s_{n_k}, v\rangle \\
&\geq \langle w - s_{n_k}, \mathcal{K}w + \frac{1}{\eta_{n_k}}(z_{n_k} - s_{n_k} - \eta_{n_k}\mathcal{K}z_{n_k})\rangle \\
&= \langle w - s_{n_k}, \mathcal{K}w - \mathcal{K}z_{n_k}\rangle \\
&\quad + \frac{1}{\eta_{n_k}}\langle w - s_{n_k}, z_{n_k} - s_{n_k}\rangle \\
&= \langle w - s_{n_k}, \mathcal{K}w - \mathcal{K}s_{n_k}\rangle \\
&\quad + \langle w - s_{n_k}, \mathcal{K}s_{n_k} - \mathcal{K}z_{n_k}\rangle \\
&\quad + \frac{1}{\eta_{n_k}}\langle w - s_{n_k}, z_{n_k} - s_{n_k}\rangle \\
&\geq \langle w - s_{n_k}, \mathcal{K}s_{n_k} - \mathcal{K}z_{n_k}\rangle \\
&\quad + \frac{1}{\eta_{n_k}}\langle w - s_{n_k}, z_{n_k} - s_{n_k}\rangle.
\end{aligned}$$

In fact, the $\mathcal{K}$ is Lipschitz continuous and $\lim_{n\to\infty}\|z_n - s_n\| = 0$, it follows that $\lim_{n\to\infty}\|\mathcal{K}z_n - \mathcal{K}s_n\| = 0$. From $\lim_{n\to\infty}\eta_n$ exists, we obtain

$$\langle w - z, v\rangle = \lim_{k\to\infty}\langle w - s_{n_k}, v\rangle \geq 0.$$

The preceding inequality, together with the maximal monotonicity of $\mathcal{K}+\mathcal{B}$ implies that $0 \in (\mathcal{K}+\mathcal{B})z$ that is $z \in \Omega$. □

Algorithm 2. Adaptive Tseng's type method for inclusion problem

initialization: Given $x_0, x_1 \in \mathcal{H}$, $\varepsilon_n \in (0,1)$, $\mu \in (0,1)$, $\alpha > 0$, $\eta_1 > 0$ and select a nonnegative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{\infty}\varphi_n < \infty$. Moreover, select a sequence $\{\psi_n\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{n\to\infty}\psi_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}\psi_n = \infty.$$

Step 1: Compute

$$z_n = (1-\psi_n)(x_n + \theta_n(x_n - x_{n-1})), \tag{3.17}$$

where $\{\theta_n\}$ is a sequence such that $0 \leq \theta_n \leq \hat{\theta}_n$ and

$$\hat{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.}\end{cases}$$

Step 2: Compute

$$s_n = J_{\eta_n}^{\mathcal{B}}(\mathcal{I} - \eta_n\mathcal{K})z_n. \tag{3.18}$$

If $s_n = z_n$, then stop and s_n is a solution of (1.1). Else, go to **Step 3.**

Step 3: Compute

$$x_{n+1} = \psi_n f(x) + (1-\psi_n)(s_n - \eta_n(\mathcal{K}s_n - \mathcal{K}z_n)) \tag{3.19}$$

where the sizes are adaptively updated as follows:

$$\eta_{n+1} = \begin{cases} \min\left\{\frac{\mu\|s_n - z_n\|}{\|\mathcal{K}s_n - \mathcal{K}z_n\|}, \eta_n + \varphi_n\right\}, & \text{if } \mathcal{K}s_n \neq \mathcal{K}z_n, \\ \eta_n + \varphi_n, & \text{otherwise.}\end{cases} \tag{3.20}$$

Set $n := n + 1$ and go back to Step 1.

Theorem 3.5. *Let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a mapping that satisfies Assumption 1 and a sequence $\{x_n\}$ generated by Algorithm 2. Then $\{x_n\}$ converges strongly to z, where $z = P_\Omega \circ f(z)$.*

Proof. Let

$$t_n = s_n - \eta_n(\mathcal{K}s_n - \mathcal{K}z_n).$$

Claim 1. $\{x_n\}$ is bounded. Indeed, let $z \in \Omega$. Using the same arguments as in the proof of Lemma 3.1, we can show that

$$\|t_n - z\|^2$$

$$\leq \|z_n - z\|^2 - \left(1 - \mu^2 \frac{\eta_n^2}{\eta_{n+1}^2}\right) \|z_n - s_n\|^2. \tag{3.21}$$

From $\lim_{n\to\infty} \eta_n$ exists and $\mu \in (0,1)$, it follows that $\lim_{n\to\infty} \left(1 - \mu^2 \frac{\eta_n^2}{\eta_{n+1}^2}\right) = 1 - \mu > 0$. This mean that

$$\|t_n - z\| \leq \|z_n - z\|. \tag{3.22}$$

From the definition of $\{z_n\}$, we obtain

$$\begin{aligned}
&\|z_n - z\| \\
&= \|(1-\psi_n)(x_n + \theta_n(x_n - x_{n-1})) - z\| \\
&= \|(1-\psi_n)(x_n - z) \\
&\quad + (1-\psi_n)\theta_n(x_n - x_{n-1}) - \psi_n z\| \\
&\leq (1-\psi_n)\|x_n - z\| \\
&\quad + (1-\psi_n)\theta_n\|x_n - x_{n-1}\| + \psi_n\|z\|.
\end{aligned}$$

Let

$$\mathcal{M}_1 = (1-\psi_n)\frac{\theta_n}{\psi_n}\|x_n - x_{n-1}\| + \|z\|.$$

The above expression obtained from the following inequality

$$\lim_{n\to\infty} \frac{\theta_n}{\psi_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\psi_n} = 0. \tag{3.23}$$

Hence,

$$\|z_n - z\| \leq (1-\psi_n)\|x_n - z\| + \psi_n \mathcal{M}_1. \tag{3.24}$$

Therefore, combining (3.22) and (3.24), we have

$$\begin{aligned}
\|t_n - z\| &\leq \|z_n - z\| \\
&\leq (1-\psi_n)\|x_n - z\| + \psi_n\mathcal{M}_1 \\
&\leq \|x_n - z\| + \psi_n\mathcal{M}_1.
\end{aligned} \tag{3.25}$$

From the definition of $\{x_n\}$, we obtain

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&= \|\psi_n f(x_n) + (1-\psi_n)t_n - z\| \\
&= \|\psi_n(f(x_n) - z) + (1-\psi_n)(t_n - z)\| \\
&\leq \psi_n\|f(x_n) - z\| + (1-\psi_n)\|t_n - z\| \\
&\leq \psi_n\|f(x_n) - f(z)\| + \psi_n\|f(z) - z\| \\
&\quad + (1-\psi_n)\|t_n - z\| \\
&\leq \psi_n\delta\|x_n - z\| + \psi_n\|f(z) - z\| \\
&\quad + (1-\psi_n)\|t_n - z\|.
\end{aligned} \tag{3.26}$$

Substituting (3.24) into (3.26), we obtain

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&\leq \psi_n\delta\|x_n - z\| + \psi_n\|f(z) - z\| \\
&\quad + (1-\psi_n)\|x_n - z\| + (1-\psi_n)\psi_n\mathcal{M}_1 \\
&\leq (1-(1-\delta)\psi_n)\|x_n - z\| + \psi_n\mathcal{M}_1 \\
&\quad + \psi_n\|f(z) - z\| \\
&= (1-(1-\delta)\psi_n)\|x_n - z\| \\
&\quad + (1-\delta)\psi_n\frac{\mathcal{M}_1 + \|f(z) - z\|}{1-\delta} \\
&\leq \max\left\{\|x_n - z\|, \frac{\mathcal{M}_1 + \|f(z) - z\|}{1-\delta}\right\} \\
&\quad \vdots \\
&\leq \max\left\{\|x_0 - p\|, \frac{\mathcal{M}_1 + \|f(p) - p\|}{1-\delta}\right\}.
\end{aligned}$$

This mean that $\{x_n\}$ is bounded.

Claim 2.

$$\begin{aligned}
&(1-\psi_n)(1-\mu^2\frac{\eta_n^2}{\eta_{n+1}^2})\|s_n - z_n\|^2 \\
&\quad \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \psi_n\mathcal{M}_4.
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \qquad (3.27)\\
&\leq \psi_n\|f(x_n) - z\|^2 + (1-\psi_n)\|t_n - z\|^2 \\
&\leq \psi_n(\|f(x_n) - f(z)\| + \|f(z) - z\|)^2 \\
&\quad + (1-\psi_n)\|t_n - z\|^2 \\
&\leq \psi_n(\kappa\|x_n - z\| + \|f(z) - z\|)^2 \\
&\quad + (1-\psi_n)\|t_n - z\|^2 \\
&\leq \psi_n(\|x_n - z\| + \|f(z) - z\|)^2
\end{aligned}$$

$$
\begin{aligned}
&+(1-\psi_n)\|t_n-z\|^2\\
&=\psi_n\|x_n-z\|^2+\psi_n(2\|x_n-z\|\cdot\|f(z)-z\|\\
&\quad+\|f(z)-z\|^2)+(1-\psi_n)\|t_n-z\|^2\\
&\le\psi_n\|x_n-z\|^2+(1-\psi_n)\|t_n-z\|^2+\psi_n\mathcal{M}_2 \qquad (3.28)
\end{aligned}
$$

for some $\mathcal{M}_2>0$. By Lemma 3.3, we have

$$
\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\le\|z_n-z\|^2-\left(1-\mu^2\frac{\eta_n^2}{\eta_{n+1}^2}\right)\|z_n-s_n\|^2. \qquad (3.29)
\end{aligned}
$$

Substituting (3.29) into (3.28), we obtain

$$
\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\le\psi_n\|x_n-z\|^2+(1-\psi_n)\|z_n-z\|^2\\
&\quad-(1-\psi_n)\left(1-\mu^2\frac{\eta_n^2}{\eta_{n+1}^2}\right)\|s_n-z_n\|^2\\
&\quad+\psi_n\mathcal{M}_2, \qquad (3.30)
\end{aligned}
$$

which implies from (3.25) that

$$
\begin{aligned}
&\|z_n-z\|^2\\
&\le(\|x_n-z\|+\psi_n\mathcal{M}_1)^2\\
&=\|x_n-z\|^2+\psi_n(2\mathcal{M}_1\|x_n-z\|+\psi_n\mathcal{M}_1^2)\\
&\le\|x_n-z\|^2+\psi_n\mathcal{M}_3, \qquad (3.31)
\end{aligned}
$$

for some $\mathcal{M}_3>0$. Combining (3.30) and (3.31), we obtain

$$
\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\le\psi_n\|x_n-z\|^2+(1-\psi_n)\|x_n-z\|^2+\psi_n\mathcal{M}_3\\
&\quad-(1-\psi_n)\left(1-\mu^2\frac{\eta_n^2}{\eta_{n+1}^2}\right)\|s_n-z_n\|^2\\
&\quad+\psi_n\mathcal{M}_2\\
&=\|x_n-z\|^2+\psi_n\mathcal{M}_3\\
&\quad-(1-\psi_n)\left(1-\mu^2\frac{\eta_n^2}{\eta_{n+1}^2}\right)\|s_n-z_n\|^2\\
&\quad+\psi_n\mathcal{M}_2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&(1-\psi_n)\left(1-\mu^2\frac{\eta_n^2}{\eta_{n+1}^2}\right)\|s_n-z_n\|^2\\
&\le\|x_n-z\|^2-\|x_{n+1}-z\|^2+\psi_n\mathcal{M}_4,
\end{aligned}
$$

where $\mathcal{M}_4:=\mathcal{M}_2+\mathcal{M}_3$.

Claim 3.

$$
\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\le(1-(1-\delta)\psi_n)\|x_n-z\|^2\\
&\quad+(1-\delta)\psi_n\left[\frac{\theta_n}{\psi_n(1-\delta)}\|x_n-x_{n-1}\|^2\right.\\
&\quad\left.+\frac{\psi_n\mathcal{M}_5+2\langle f(z)-z,x_{n+1}-z\rangle}{1-\delta}\right],
\end{aligned}
$$

for some $\mathcal{M}_5>0$. Indeed, using Lemma 2.9, we have

$$
\begin{aligned}
&\|x_{n+1}-z\|^2\\
&=\|\psi_nf(x_n)+(1-\psi_n)t_n-z\|^2\\
&=\|\psi_n(f(x_n)-f(z))+(1-\psi_n)(t_n-z)\\
&\quad+\psi_n(f(z)-z)\|^2\\
&\le\|\psi_n(f(x_n)-f(z))+(1-\psi_n)(t_n-z)\|^2\\
&\quad+2\psi_n\langle f(z)-z,x_{n+1}-z\rangle\\
&\le\psi_n\|f(x_n)-f(z)\|^2+(1-\psi_n)\|t_n-z\|^2\\
&\quad+2\psi_n\langle f(z)-z,x_{n+1}-z\rangle\\
&\le\psi_n\delta^2\|x_n-z\|^2+(1-\psi_n)\|t_n-z\|^2\\
&\quad+2\psi_n\langle f(z)-z,x_{n+1}-z\rangle\\
&\le\psi_n\delta\|x_n-z\|^2+(1-\psi_n)\|t_n-z\|^2\\
&\quad+2\psi_n\langle f(z)-z,x_{n+1}-z\rangle\\
&\le\psi_n\delta\|x_n-z\|^2+(1-\psi_n)\|z_n-z\|^2\\
&\quad+2\psi_n\langle f(z)-z,x_{n+1}-z\rangle. \qquad (3.32)
\end{aligned}
$$

From (3.24), we have

$$
\begin{aligned}
&\|z_n-z\|^2\\
&=\|(1-\psi_n)(x_n+\theta_n(x_n-x_{n-1}))-z\|^2\\
&=\|(1-\psi_n)(x_n-z)\\
&\quad+(1-\psi_n)\theta_n(x_n-x_{n-1})-\psi_nz\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \|(1-\psi_n)(x_n - z) \\
&\quad + (1-\psi_n)\theta_n(x_n - x_{n-1})\|^2 \\
&\quad - 2\psi\langle(1-\psi_n)(x_n - z) \\
&\quad + (1-\psi_n)\theta_n(x_n - x_{n-1}), z\rangle + \psi_n^2\|z\|^2 \\
&\leq \|(1-\psi_n)(x_n - z) \\
&\quad + (1-\psi_n)\theta_n(x_n - x_{n-1})\|^2 + \psi_n^2\|z\|^2 \\
&= (1-\psi_n)\|x_n - z\|^2 + \psi_n^2\|z\|^2 \\
&\quad + (1-\psi_n)\theta_n\|x_n - x_{n-1}\| \\
&\quad - (1-\psi_n)(1-\psi_n)\theta_n\|x_{n-1} - z\|^2 \\
&\leq \|x_n - z\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + \psi_n^2\|z\|^2 \\
&= \|x_n - z\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + \psi_n^2\mathcal{M}_5,
\end{aligned} \tag{3.33}
$$

for some $\mathcal{M}_5 > 0$. From (3.32) and (3.33) we get

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq (1-(1-\delta)\psi_n)\|x_n - z\|^2 + \theta_n\|x_n - x_{n-1}\|^2 \\
&\quad + \psi_n^2\mathcal{M}_5 + 2\psi_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq (1-(1-\delta)\psi_n)\|x_n - z\|^2 \\
&\quad + (1-\delta)\psi_n\left[\frac{\theta_n}{\psi_n(1-\delta)}\|x_n - x_{n-1}\|^2\right. \\
&\quad \left. + \frac{\psi_n\mathcal{M}_5 + 2\langle f(z) - z, x_{n+1} - z\rangle}{1-\delta}\right].
\end{aligned}
$$

Claim 4. $\{\|x_n - z\|^2\}$ converges to zero. Indeed, by Lemma 2.11 it suffices to show that $\limsup_{k\to\infty}\langle f(z) - z, x_{n_k+1} - z\rangle \leq 0$ for every subsequence $\{\|x_{n_k} - z\|\}$ of $\{\|x_n - z\|\}$ satisfying

$$
\liminf_{k\to\infty}(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|) \geq 0.
$$

For this, suppose that $\{\|x_{n_k} - z\|\}$ is a subsequence of $\{\|x_n - z\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|) \geq 0$. Then

$$
\begin{aligned}
&\liminf_{k\to\infty}(\|x_{n_k+1} - z\|^2 - \|x_{n_k} - z\|^2) \\
&= \liminf_{k\to\infty}[(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|) \\
&\quad \times (\|x_{n_k+1} - z\| + \|x_{n_k} - z\|)] \geq 0.
\end{aligned}
$$

By **Claim 2** we obtain

$$
\begin{aligned}
&\limsup_{k\to\infty}(1-\psi_{n_k})\left(1 - \mu^2\frac{\eta_{n_k}^2}{\eta_{n_k+1}^2}\right)\|s_{n_k} - z_{n_k}\|^2 \\
&\leq \limsup_{k\to\infty}[\|x_{n_k} - z\|^2 - \|x_{n_k+1} - z\|^2 \\
&\quad + \psi_{n_k}\mathcal{M}_4] \\
&\leq \limsup_{k\to\infty}[\|x_{n_k} - z\|^2 - \|x_{n_k+1} - z\|^2] \\
&\quad + \limsup_{k\to\infty}\theta_{n_k}\mathcal{M}_4 \\
&= -\liminf_{k\to\infty}[\|x_{n_k+1} - z\|^2 - \|x_{n_k} - z\|^2] \\
&\leq 0.
\end{aligned}
$$

This implies that

$$
\lim_{k\to\infty}\|s_{n_k} - z_{n_k}\| = 0. \tag{3.34}
$$

Now, we show that

$$
\|x_{n_k+1} - x_{n_k}\| \to 0 \text{ as } n \to \infty. \tag{3.35}
$$

Indeed, from (3.34), it follows that

$$
\|t_{n_k} - z_{n_k}\| \tag{3.36}
$$

$$
\begin{aligned}
&= \|s_{n_k} - \eta_{n_k}(\mathcal{K}s_{n_k} - \mathcal{K}z_{n_k}) - z_{n_k}\| \\
&\leq \|s_{n_k} - z_{n_k}\| + \eta_{n_k}\|\mathcal{K}s_{n_k} - \mathcal{K}z_{n_k}\| \\
&\leq \left(1 - \mu^2\frac{\eta_{n_k}^2}{\eta_{n_k+1}^2}\right)\|s_{n_k} - z_{n_k}\|^2 \to 0.
\end{aligned} \tag{3.37}
$$

Moreover, we have

$$
\|x_{n_k+1} - t_{n_k}\| = \theta_{n_k}\|t_{n_k} - f(x_{n_k})\| \to 0, \tag{3.38}
$$

and

$$
\begin{aligned}
&\|z_{n_k} - x_{n_k}\| \\
&= \|x_n + \theta_n(x_n - x_{n-1}) \\
&\quad - \psi_n[x_n + \theta_n(x_n - x_{n-1})] - x_n\| \\
&\leq \theta_n\|x_n - x_{n-1}\| + \psi_n\|x_n\|
\end{aligned}
$$

$$+ \theta_n \psi_n \|x_n - x_{n-1}\|$$
$$= \psi_n \frac{\theta_n}{\psi_n} \|x_n - x_{n-1}\| + \psi_n \|x_n\|$$
$$+ \psi_n^2 \frac{\theta_n}{\psi_n} \|x_n - x_{n-1}\| \to 0. \qquad (3.39)$$

From (3.36), (3.38) and (3.39), we get

$$\|x_{n_k+1} - x_{n_k}\|$$
$$\leq \|x_{n_k+1} - t_{n_k}\| + \|t_{n_k} - z_{n_k}\|$$
$$+ \|z_{n_k} - x_{n_k}\| \to 0.$$

Since the sequence $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z^* \in \mathcal{H}$, such that

$$\limsup_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle$$
$$= \lim_{j\to\infty} \langle f(z) - z, x_{n_{k_j}} - z\rangle$$
$$= \langle f(z) - x, z^* - z\rangle. \qquad (3.40)$$

From (3.34) and Lemma 3.3, we have $z^* \in \Omega$ and, from (3.40) and the definition of $z = P_\Omega \circ f(z)$, we have

$$\limsup_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle$$
$$= \langle f(z) - z, z^* - z\rangle \leq 0. \qquad (3.41)$$

Combining (3.35) and (3.41), we have

$$\limsup_{k\to\infty} \langle f(z) - z, x_{n_k+1} - z\rangle$$
$$\leq \limsup_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle$$
$$= \langle f(z) - z, z^* - z\rangle \leq 0. \qquad (3.42)$$

Hence, by (3.42), $\lim_{n\to\infty} \frac{\theta_n}{\psi_n} \|x_n - x_{n-1}\| = 0$, **Claim 3** and Lemma 2.11, we have $\lim_{n\to\infty} \|x_n - z\| = 0$. That is the desired result. □

4. Convex minimization problem

Let $\mathcal{S} : \mathcal{H} \to \mathbb{R}$ be a convex function and $\mathcal{T} : \mathcal{H} \to \mathbb{R}$ be a convex, lower semi-continuous and non-smooth function. We consider the following convex minimization problem:

$$\min_{x\in\mathcal{H}} \mathcal{S}(x) + \mathcal{T}(x). \qquad (4.1)$$

By Fermat's rule, we know that the above problem is equivalent to the problem of finding $x \in \mathcal{H}$ such that

$$0 \in \nabla\mathcal{S}(x) + \partial\mathcal{T}(x), \qquad (4.2)$$

where $\nabla\mathcal{S}$ is the gradient of $\mathcal{S}$ and $\partial\mathcal{T}$ is the sub-differential of $\mathcal{T}$ defined by

$$\partial\mathcal{T}(x) = \{z \in \mathcal{H} : \mathcal{T}(x) \leq \langle z, x-y\rangle + \mathcal{T}(y)\}.$$

for all $y \in \mathcal{H}$. In this situation, we assume that $\mathcal{S}$ is a convex and differentiable function with its gradient $\nabla\mathcal{S}$ is $\mathcal{L}$-Lipschitz continuous. Further, $\nabla\mathcal{S}$ is cocoercive with a constant $1/\mathcal{L}$ (see [34]). This implies that $\nabla\mathcal{S}$ is monotone and Lipschitz continuous. Moreover, $\partial\mathcal{T}$ is maximal monotone (see [35]). In this point of view, we set $\mathcal{K} = \nabla\mathcal{S}$ and $\mathcal{B} = \partial\mathcal{T}$, then we obtain the following results regarding the problem (4.1).

Assumption 2.

(A1) The function $\mathcal{S} : \mathcal{H} \to \mathbb{R}$ is convex and differentiable and its gradient $\nabla\mathcal{S}$ is $\mathcal{L}$-Lipschitz continuous and $\mathcal{T} : \mathcal{H} \to \mathbb{R}$ is convex and lower semi-continuous which $\mathcal{S} + \mathcal{T}$ attains a minimizer.

Algorithm 3. Adaptive Tseng's type method for convex minimization problem
initialization: Given $x_0, x_1 \in \mathcal{H}$, $\varepsilon_n \in (0,1)$, $\mu \in (0,1)$, $\alpha > 0$, $\eta_1 > 0$ and select a nonnegative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{\infty} \varphi_n < \infty$. Moreover, select

a sequence $\{\psi_n\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \psi_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \psi_n = \infty.$$

Step 1: Compute

$$z_n = (1-\psi_n)(x_n + \theta_n(x_n - x_{n-1})),$$

where $\{\theta_n\}$ is a sequence such that
$0 \le \theta_n \le \hat{\theta}_n$ and

$$\hat{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\varepsilon_n}{\|x_n-x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$s_n = J_{\eta_n}^{\partial\mathcal{T}}(\mathcal{I} - \eta_n \nabla \mathcal{S}) z_n.$$

If $s_n = z_n$, then stop and s_n is a solution of (4.1). Else, go to **Step 3.**
Step 3: Compute

$$x_{n+1} = s_n - \eta_n(\nabla\mathcal{S}s_n - \nabla\mathcal{S}z_n)$$

where the sizes are adaptively updated as follows:

$$\eta_{n+1} = \begin{cases} \min\left\{\frac{\mu\|s_n - z_n\|}{\|\nabla\mathcal{S}s_n - \nabla\mathcal{S}z_n\|}, \eta_n + \varphi_n\right\}, & \text{if } \nabla\mathcal{S}s_n \neq \nabla\mathcal{S}z_n, \\ \eta_n + \varphi_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go back to Step 1.

Theorem 4.1. *Let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a mapping that satisfies Assumption 2 and a sequence $\{x_n\}$ generated by Algorithm 3. Then $\{x_n\}$ converges weakly to a minimizer of $\mathcal{S} + \mathcal{T}$.*

Algorithm 4. Adaptive Tseng's type method for inclusion problem
initialization: Given $x_0, x_1 \in \mathcal{H}$, $\varepsilon_n \in (0,1)$, $\mu \in (0,1)$, $\alpha > 0$, $\eta_1 > 0$ and select a nonnegative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{\infty} \varphi_n < \infty$. Moreover, select a sequence $\{\psi_n\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \psi_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \psi_n = \infty.$$

Step 1: Compute

$$z_n = (1-\psi_n)(x_n + \theta_n(x_n - x_{n-1})),$$

where $\{\theta_n\}$ is a sequence such that
$0 \le \theta_n \le \hat{\theta}_n$ and

$$\hat{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\varepsilon_n}{\|x_n-x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$s_n = J_{\eta_n}^{\partial\mathcal{T}}(\mathcal{I} - \eta_n \nabla \mathcal{S}) z_n.$$

If $s_n = z_n$, then stop and s_n is a solution of (4.1). Else, go to **Step 3.**
Step 3: Compute

$$x_{n+1} = \psi_n f(x_n) + (1-\psi_n)(s_n - \eta_n(\nabla\mathcal{S}s_n - \nabla\mathcal{S}z_n))$$

where the sizes are adaptively updated as follows:

$$\eta_{n+1} = \begin{cases} \min\left\{\frac{\mu\|s_n - z_n\|}{\|\nabla\mathcal{S}s_n - \nabla\mathcal{S}z_n\|}, \eta_n + \varphi_n\right\}, & \text{if } \nabla\mathcal{S}s_n \neq \nabla\mathcal{S}z_n, \\ \eta_n + \varphi_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go back to Step 1.

Theorem 4.2. *Let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a mapping that satisfies Assumption 2 and a sequence $\{x_n\}$ generated by Algorithm 4. Then $\{x_n\}$ converges strongly to a minimizer of $\mathcal{S} + \mathcal{T}$.*

5. Numerical experiments

Example 5.1. Let $\mathcal{H} = [0,4]$. Define the mappings $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ and $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ by the following:

$$\mathcal{K}x = \frac{1}{4}(x-2) \quad \text{and} \quad \mathcal{B}x = 4(x+\frac{1}{2}).$$

We see that the proposed mappings satisfy the assumptions in Theorem 3.4 and Theorem 3.5. For each $\eta > 0$, we obtain that $J_{\eta}^{\mathcal{B}}(\mathcal{I} - \eta\mathcal{K})x = \dfrac{4-\eta}{4+16\eta}x$. In these experiments, we compare our Algorithm 1 and Algorithm 2 with Algorithm (1.6) of Tseng [24]. The parameters are chosen as follows:

- Algorithm 1: $\psi = \frac{1}{(10000(n+1))^2}$, $\varepsilon_n = \psi_n^2$, $\alpha = 3$, $\eta_1 = 0.09$, $\varphi_n = \frac{1}{(n+1)^2}$ and $\mu = 0.6$;
- Algorithm 2: $f(x) = \frac{x}{99}$, $\psi = \frac{1}{(10000(n+1))^2}$, $\varepsilon_n = \psi_n^2$, $\alpha = 3$, $\eta_1 = 0.09$, $\varphi_n = \frac{1}{(n+1)^2}$ and $\mu = 0.6$;
- Algorithm (1.6): $\eta = 0.09$.

We perform numerical experiments with four different point of view x_1 cases and use the stopping criterion $\|x_{n+1} - x_n\| \le 10^{-10}$. The numerical results are summarized in Table 1.

Example 5.2. Consider the minimization problem:

$$\min_{x\in\mathbb{R}^3} \|x\|_1 + \|x\|_2^2 + (-2,1,4)x + 9,$$

where $x = (u_1,u_2,u_3)^T \in \mathbb{R}^3$. Let $\mathcal{S}(x) = \|x\|_2^2 + (-2,1,4)x + 9$ and $\mathcal{T}(x) = \|x\|_1$. So, we have $\nabla\mathcal{S}(x) = 2x + (2,1,4)^T$. It is easy to check that $\mathcal{S}$ is a convex and differentiable function and its gradient $\nabla\mathcal{S}$ is Lipschitz continuous with $\mathcal{L} = 2$. Moreover, $\mathcal{T}$ is a convex and lower semi-continuous function but not differentiable on $\mathbb{R}^3$. From [36], we know that

$$\begin{aligned} J_{\eta}^{\partial\mathcal{T}}(x) &= (I + \eta\partial\mathcal{T})^{-1}(x) \\ &= (\max\{|u_1| - \eta, 0\}sgn(u_1), \\ &\quad \max\{|u_2| - \eta, 0\}sgn(u_2), \\ &\quad \max\{|u_3| - \eta, 0\}sgn(u_3))^T \end{aligned}$$

for $\eta > 0$. In these experiments, we compare our Algorithm 3 and Algorithm 4 with Algorithm (1.6) of Tseng [24] in case $\mathcal{K} = \nabla\mathcal{S}$ and $\mathcal{B} = \partial\mathcal{T}$. The parameters are chosen as follows:

- Algorithm 3: $\psi = \frac{1}{(10000(n+1))^2}$, $\varepsilon_n = \psi_n^2$, $\alpha = 3$, $\eta_1 = 0.49$, $\varphi_n = \frac{1}{(n+1)^2}$ and $\mu = 0.5$;
- Algorithm 4: $f(x) = \frac{x}{2}$, $\psi = \frac{1}{(10000(n+1))^2}$, $\varepsilon_n = \psi_n^2$, $\alpha = 3$, $\eta_1 = 0.49$, $\varphi_n = \frac{1}{(n+1)^2}$ and $\mu = 0.5$;
- Algorithm (1.6): $\eta = 0.49$.

We perform numerical experiments with four different point of view x_1 cases and use the stopping criterion $\|x_{n+1} - x_n\| \le 10^{-10}$. The numerical results are summarized in Table 2.

Example 5.3. In signal processing, compressed sensing can be represented as the under-determinated linear equation system shown below:

$$b = \mathcal{F}x + c, \tag{5.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $b \in \mathbb{R}^M$ is the observed or measured data with noisy c and $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear operator. It is known that to solve (5.1) can be seen as solving the LASSO problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|\mathcal{F}x - b\|_2^2 + \eta\|x\|_1, \tag{5.2}$$

where $\eta > 0$. Next, let $\mathcal{K} = \nabla\mathcal{S}$ be the gradient of $\mathcal{T}$ and $\mathcal{B} = \partial\mathcal{T}$ the sub-differential of $\mathcal{T}$, where $\mathcal{S}(x) = \dfrac{1}{2}\|\mathcal{F}x - b\|_2^2$ and $\mathcal{T}(x) = \eta\|x\|_1$. Then $\nabla\mathcal{S}(x) = \mathcal{F}^T(\mathcal{F}x - b)$ and $\partial\mathcal{T}(x) = \partial(\eta\|x\|_1)$. It is known that

Table 1. Numerical results for Example 5.1.

$x_0 = x_1$	Algorithm 1		Algorithm 2		Algorithm (1.6)	
	iter.	time	iter.	time	iter.	time
0.5	25	0.01	25	0.01	68	0.02
1.5	25	0.01	25	0.01	70	0.02
2.5	25	0.01	25	0.01	72	0.02
3.5	26	0.01	26	0.01	73	0.02

Table 2. Numerical results for Example 5.2.

$x_0 = x_1$	Algorithm 3		Algorithm 4		Algorithm (1.6)	
	iter.	time	iter.	time	iter.	time
$(2, 1, 3)^T$	82	0.03	82	0.03	1046	0.11
$(2, -5, 4)^T$	85	0.03	85	0.04	1069	0.11
$(-150, 150, 100)^T$	95	0.03	95	0.04	1235	0.18
$(-3000, -5000, -700)^T$	108	0.03	108	0.04	1405	0.21

$\mathcal{K}$ is $\|\mathcal{F}\|_2^2$-Lipschitz continuous and monotone. Moreover, $\mathcal{B}$ is maximal monotone (see [35]).

The sparse vector $x \in \mathbb{R}^N$ is created in this experiment from a uniform distribution with m nonzero entries in the interval $[-1, 1]$. A normal distribution with mean zero and one invariance yields the matrix $\mathcal{F} \in \mathbb{R}^{M\times N}$. The mean squared error (MSE) is used to determine the restoration accuracy:

$$\text{MSE} = \frac{1}{N}\|x_n - x\|_2^2 < 10^{-4}, \qquad (5.3)$$

where x is the original signal.

We compare our Algorithm 3 and Algorithm 4 with Algorithm (1.6) of Tseng [24] in case $\mathcal{K} = \nabla\mathcal{S}$ and $\mathcal{B} = \partial\mathcal{T}$. The parameters are chosen as follows:

- Algorithm 3: $\psi = \frac{1}{(10000(n+1))^2}$, $\varepsilon_n = \psi_n^2$, $\alpha = 3$, $\eta_1 = 0.09$, $\varphi_n = \frac{1}{(n+1)^2}$ and $\mu = 0.6$;
- Algorithm 4: $f(x) = \frac{x}{2}$, $\psi = \frac{1}{(10000(n+1))^2}$, $\varepsilon_n = \psi_n^2$, $\alpha = 3$, $\eta_1 = 0.09$, $\varphi_n = \frac{1}{(n+1)^2}$ and $\mu = 0.6$;
- Algorithm (1.6): $\eta = \dfrac{0.04}{\|\mathcal{F}\|_2^2}$.

The starting stances x_1 of all methods are chosen at random in $\mathbb{R}^N$. The numerical test is done using the following two cases:
Case I: $N = 513$, $M = 256$ and $m = 20$.
Case II: $N = 1024$, $M = 512$ and $m = 50$.
The numerical results for all tests show in Figs. 1, 2, 3 and 4.

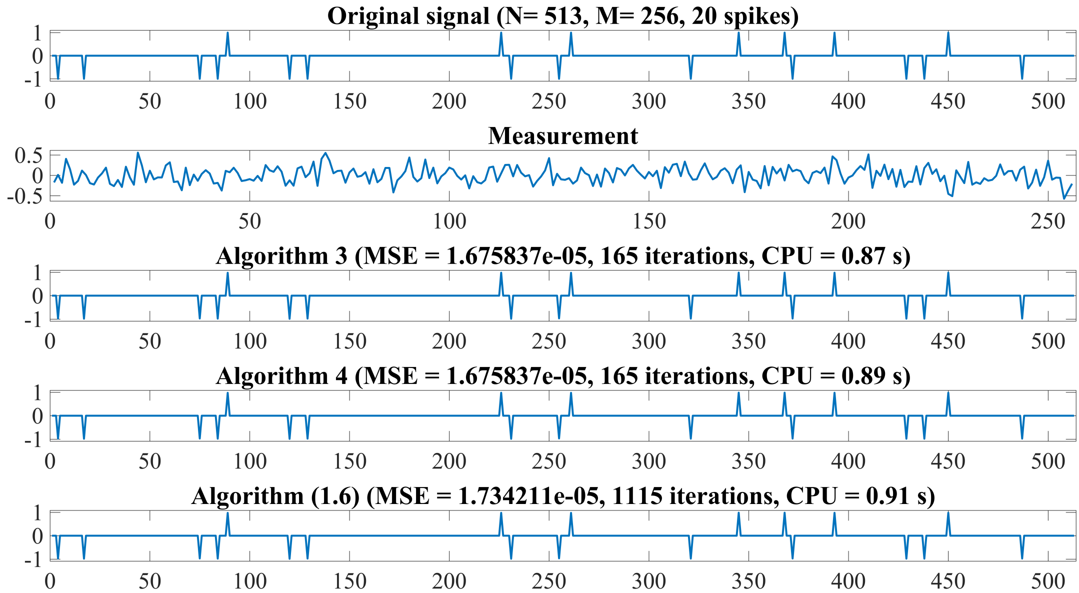

Fig. 1. Comparison of recovered signal by using different algorithms in Case I.

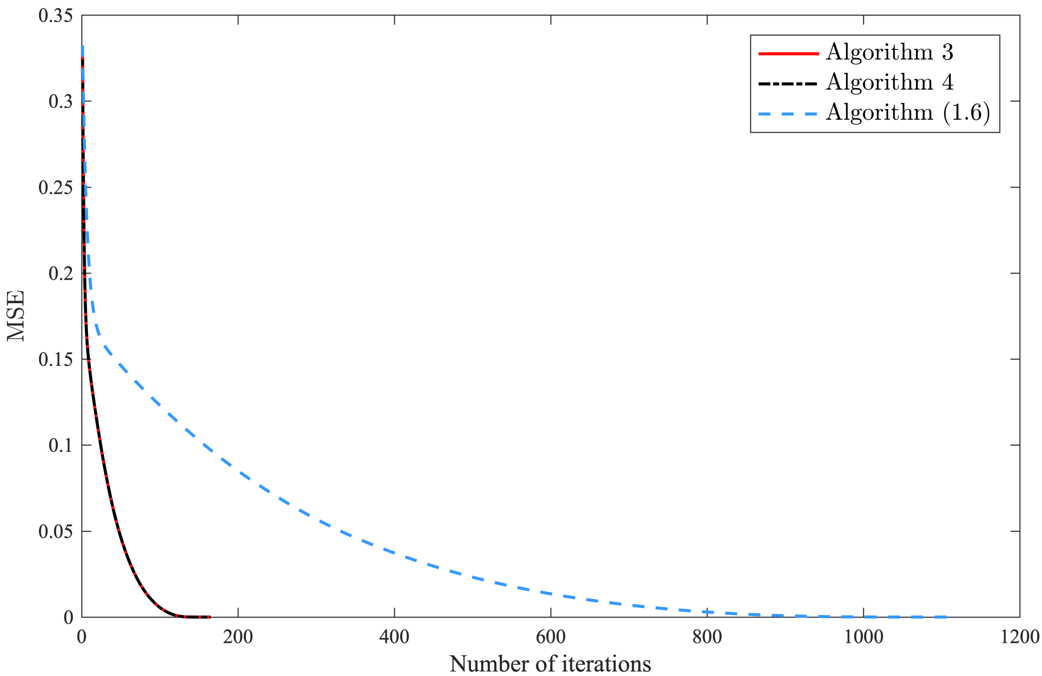

Fig. 2. The plotting of MSE versus number of iterations in Case I.

6. Conclusions

In this paper, we combined inertial and viscosity techniques to propose a modified Tseng's method for solving monotone inclusion problems in real Hilbert spaces. Furthermore, we established the weakly and

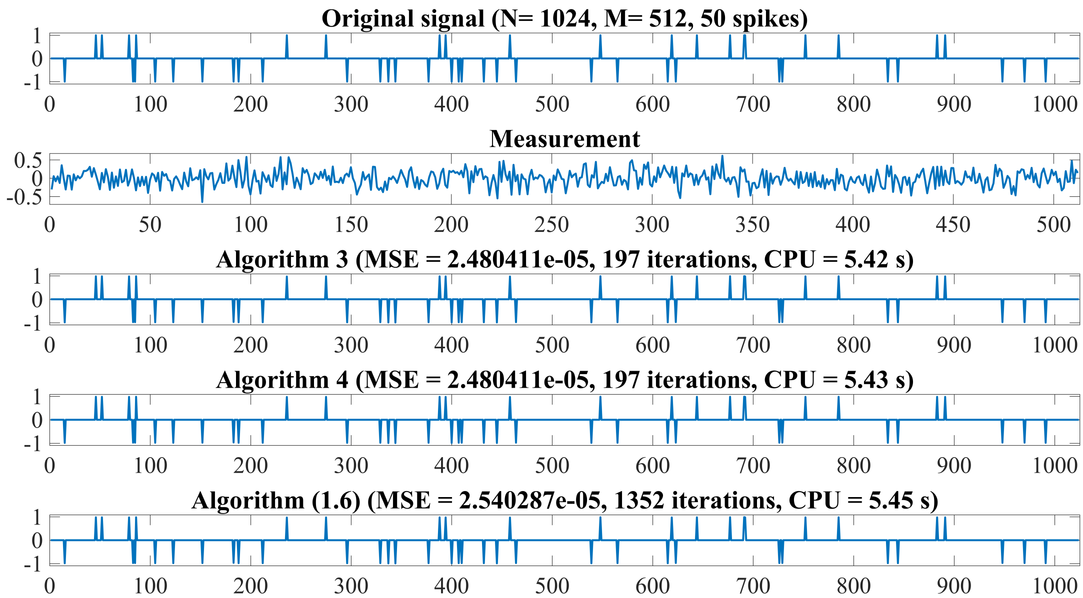

Fig. 3. Comparison of recovered signal by using different algorithms in Case II.

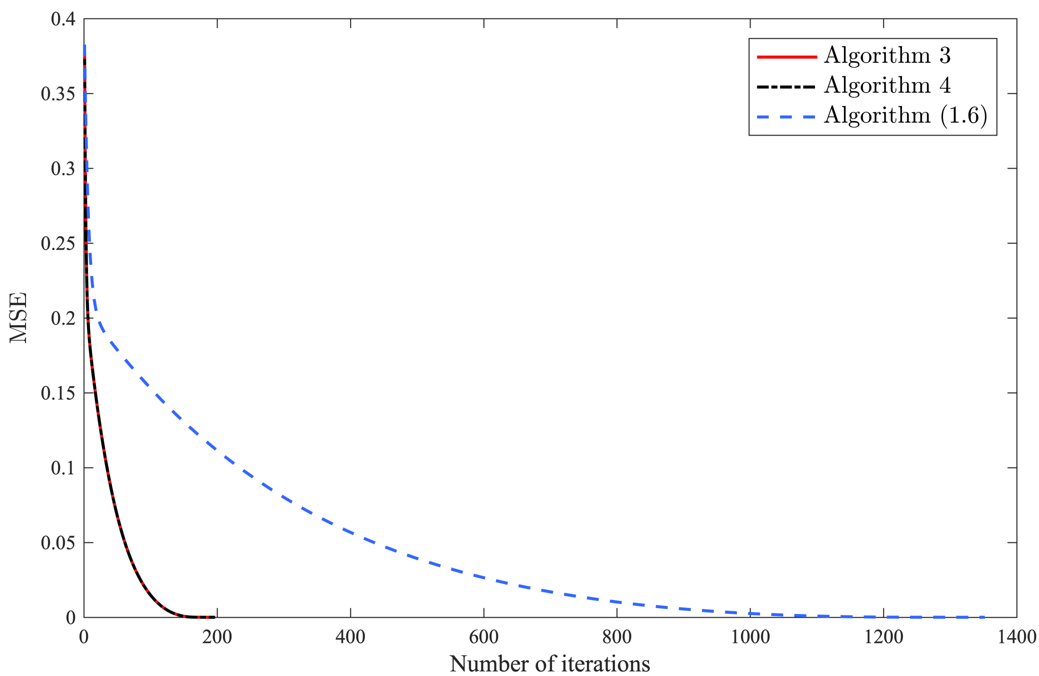

Fig. 4. The plotting of MSE versus number of iterations in Case II.

strongly convergence theorems. Finally, we compared our results with Algorithm (1.6) of Tseng [24] in the convergence rate and applied to signal processing by modified the algorithm 1 and the algorithm 2 in case $\mathcal{K} = \nabla\mathcal{S}$ and $\mathcal{B} = \partial\mathcal{T}$, as in Example 5.1 and Example 5.2, we knew that our algorithm is more efficient than Algorithm (1.6) of Tseng (see Figs. 1, 2, 3 and 4).

Acknowledgement

This project was supported by the Research and Development Institute, Rambhai Barni Rajabhat University (Grant no.2212/2564).

References

[1] Censor Y, Bortfeld T, Martin B, Trofimov A. A unified approach for inversion problems in intensity-modulated radiation therapy. Phys. Med. Biol. 2006; 51:2353-2365.

[2] Censor Y, Elfving T. A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms. 1994; 8:221-239.

[3] Cholamjiak P, Shehu Y. Inertial forward–backward splitting method in Banach spaces with application to compressed sensing. Appl. Math. 2019;64:409-435.

[4] Combettes PL, Wajs VR. Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. 2005;4:1168-1200.

[5] Duchi J, Shalev-Shwartz S, Singer Y, Chandra T. Efficient projections onto the l_1-ball for learning in high dimensions. In Proceedings of the 25th International Conference on Machine Learning, Helsinki, Finland, 5-8 July 2008.

[6] Gibali A, Thong DV. Tseng type methods for solving inclusion problems and its applications. Calcolo. 2018;55:49.

[7] Kitkuan D, Kumam P, Martínez-Moreno J. Sitthithakerngkiet K. Inertial viscosity forward-backward splitting algorithm for monotone inclusions and its application to image restoration problems. Int. J. Comput. Math. 2020;97:482-497.

[8] Kitkuan D, Muangchoo K. Inertial Relaxed CQ Algorithm with an Application to Signal Processing. Thai Journal of Mathematics. 2020;18(3):1091-1103.

[9] Suparatulatorn R, Khemphet A, Charoensawan P, Suantai S, Phudolsitthiphat N. Generalized self-adaptive algorithm for solving split common fixed point problem and its application to image restoration problem. Int. J. Comput. Math. 2019;1-15.

[10] Suparatulatorn R, Charoensawan P, Poochinapan K. Inertial self-adaptive algorithm for solving split feasible problems with applications to image restoration. Math. Methods Appl. Sci. 2019;42:7268-7284.

[11] Yang J, Cholamjiak P, Sunthrayuth P. Modified Tseng's splitting algorithms for the sum of two monotone operators in Banach spaces. AIMS Mathematics. 2021;6(5):4873-4900.

[12] Suparatulatorn R, Cholamjiak W, Gibali A, Mouktonglang T. A parallel Tseng's splitting method for solving common variational inclusion applied to signal recovery problems. Adv. Differ. Equ. 2021;2021:492.

[13] Padcharoen A, Sukprasert P. Nonlinear operators as concerns convex programming and applied to signal processing. Mathematics. 2019;7(9):866.

[14] Cholamjiak W, Dutta H. Viscosity modification with parallel inertial two steps forward-backward splitting methods for inclusion problems applied to signal recovery. Chaos, Solitons & Fractals. 2022;157:111858.

[15] Adamu A, Kitkuan D, Padcharoen A, Chidume CE, Kumam P. Inertial viscosity-typeiterative method for solving inclusion problems with applications. Math. Comput. Simul. 2022;194:445-459.

[16] Lions PL, Mercier B. Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. 1979;16:964-979.

[17] Moudafi A, Oliny M. Convergence of a splitting inertial proximal method for monotone operators. J. Comput. Appl. Math. 2003;155:447-454.

[18] Lorenz DA, Pock T. An inertial forward-backward algorithm for monotone inclusions. J. Math. Imaging Vis. 2015;51:311-325.

[19] Izuchukwu C, Okeke CC, Isiogugu FO. Viscosity iterative technique for split variational inclusion problem and fixed point problem between Hilbert space and Banach space, J. Fixed Point Theory Appl. 2018;20:1-25.

[20] Kitkuan D, Kumam P, Martínez-Moreno J. Generalized Halpern-type forward-backward splitting methods for convex minimization problems with application to image restoration problems. Optimization. 2020;69(7-8):1557-1581.

[21] Padcharoen A, Kitkuan D, Kumam W, Kumam P. Tseng methods with inertial for solving inclusion problems and application to image deblurring and image recovery problems. Comput. Math. Method. 2021;3:e1088.

[22] Attouch H, Peypouquet J, Redont P. Backward-forward algorithms for structured monotone inclusions in Hilbert spaces. J. Math. Anal. Appl. 2018;457:1095-1117.

[23] Dadashi V, Postolache M. Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators. Arab. J. Math. 2020;9:89-99.

[24] Tseng P. A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control Optim. 2000;38:431-446.

[25] Shehu Y. Convergence results of forward-backward algorithms for sum of monotone operators in Banach spaces, Results Math. 2019;74:138.

[26] Bauschke HH, Combettes PL. Convex Analysis and Monotone Operator Theory in Hilbert Spaces; Springer: New York, NY, USA, 2011; Volume 408.

[27] Guo H, He H, Chen R. Convergence theorems for the split equality variational inclusion problem and fixed point problem in Hilbert spaces. Fixed Point Theory and Applications. 2015:223

[28] Byrne C, Censor Y, Gibali A, Reich S. Weak and strong convergence of algorithms for the split common null point problem. J. Nonlinear Convex Anal. 2012;13:759-775.

[29] Brezis H. Ope Rateurs Maximaux Monotones Et Semi-Groupes De Contractions Dans Les Espaces De Hilbert; Elsevier: North Holland, The Netherlands, 1973.

[30] Liu H, Yang J. Weak convergence of iterative methods for solving quasimonotone variational inequalities. Comput Optim Appl. 2020;77(2):491-508.

[31] Oyewole OK, Abass HA, Mewomo OT. A strong convergence algorithm for a fixed point constrained split null point problem, Rend. Circ. Mat. Palermo II; 2020;70:389-408.

[32] Tan KK, Xu HK. Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 1993;178:301-308.

[33] Saejung, S, Yotkaew P. Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal. 2012;75:742-750.

[34] Peypouquet J. Convex Optimization in Normed Spaces: Theory, Methods and Examples, Springer Briefs in Optimization, 2015.

[35] Rockafellar RT. On the maximal monotonicity of subdifferential mappings, Pac. J. Math. 1970;33:209-216.

[36] Hale ET, Yin W, Zhang Y. A fixed-point continuation method for l_1-regularized minimization with applications to compressed sensing, CAAM Technical Report TR07-07, 2007.