APPENDIX A

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Research Article

A Shrinking Projection Method for Generalized Mixed Equilibrium Problems, Variational Inclusion Problems and a Finite Family of Quasi-Nonexpansive Mappings

Wiyada Kumam,[1,2] Chaichana Jaiboon,[3] Poom Kumam,[2,4] and Akarate Singta[1]

[1] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Thanyaburi, Pathumthani 12110, Thailand*
[2] *Centre of Excellence in Mathematics CHE, Si Ayutthaya Road, Bangkok 10140, Thailand*
[3] *Department of Mathematics, Faculty of Liberal Arts, Rajamangala University of Technology Rattanakosin (RMUTR), Bangkok 10100, Thailand*
[4] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand*

Correspondence should be addressed to Chaichana Jaiboon, chaichana.j@rmutr.ac.th

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The purpose of this paper is to consider a shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of a finite family of quasi-nonexpansive mappings, and the set of solutions of variational inclusion problems. Then, we prove a strong convergence theorem of the iterative sequence generated by the shrinking projection method under some suitable conditions in a real Hilbert space. Our results improve and extend recent results announced by Peng et al. (2008), Takahashi et al. (2008), S.Takahashi and W. Takahashi (2008), and many others.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively.

Recall that the following definitions.

(1) A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

(2) A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C,\ p \in F(T). \tag{1.2}$$

We denote $F(T) = \{x \in C : Tx = x\}$ be the set of fixed points of T.

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ to a set-valued mapping. The *variational inclusion problem* is to find $\hat{x} \in H$ such that

$$\theta \in B(\hat{x}) + M(\hat{x}), \tag{1.3}$$

where θ is the zero vector in H. The set of solutions of problem (1.3) is denoted by $I(B, M)$.

Definition 1.1. A mapping $B : H \to H$ is said to be a *ξ-inverse-strongly monotone* if there exists a constant $\xi > 0$ with the property

$$\langle Bx - By, x - y\rangle \geq \xi\|Bx - By\|^2, \quad \forall x, y \in C. \tag{1.4}$$

Remark 1.2. It is obvious that any ξ-inverse-strongly monotone mapping B is monotone and $1/\xi$-Lipschitz continuous. It is easy to see that if any λ constant is in $(0, 2\xi]$, then the mapping $I - \lambda B$ is nonexpansive, where I is the identity mapping on H.

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in M(x)$, and $g \in M(y)$ implying $\langle x - y, f - g\rangle \geq 0$. A monotone mapping; M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$.

Definition 1.3. Let $M : H \to 2^H$ be a set-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(\hat{x}) = (I + \lambda M)^{-1}(\hat{x}), \quad \hat{x} \in H \tag{1.5}$$

is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

Remark 1.4. (R1) The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive for all $\lambda > 0$, that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \quad \forall x, y \in H,\ \forall \lambda > 0. \tag{1.6}$$

(R2) The resolvent operator $J_{M,\lambda}$ is 1-inverse strongly monotone; see [1], that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle J_{M,\lambda}(x) - J_{M,\lambda}(y), x - y\rangle, \quad \forall x, y \in H. \tag{1.7}$$

(R3) The solution of problem (1.3) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$; see also [2], that is,

$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \tag{1.8}$$

(R4) If $0 < \lambda \leq 2\xi$, then the mapping $J_{M,\lambda}(I - \lambda B) : H \to H$ is nonexpansive.

(R5) $I(B, M)$ is closed and convex.

Let $A : C \to H$ be a nonlinear mapping, let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a real-valued function and F a bifunction from $C \times C$ to $\mathbb{R}$. We consider the following *generalized mixed equilibrium problem.*

Finding $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.9}$$

The set of such $x \in C$ is denoted by $\mathrm{GMEP}(F, \varphi, A)$, that is,

$$\mathrm{GMEP}(F, \varphi, A) = \{x \in C : F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C\}. \tag{1.10}$$

It is easy to see that x is solution of problem (1.9) implies that $x \in \operatorname{dom} \varphi = \{x \in C : \varphi(x) < +\infty\}$.

(i) In the case of $A \equiv 0$(:the zero mapping), then the generalized mixed equilibrium problem (1.9) is reduced to *the mixed equilibrium problem.* Finding $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.11}$$

The set of solution of (1.11) is denoted by $\mathrm{MEP}(F, \varphi)$.

(ii) In the case of $\varphi \equiv 0$, then the generalized mixed equilibrium problem (1.9) is reduced to *the generalized equilibrium problem.* Finding $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.12}$$

The set of solution of (1.12) is denoted by $\mathrm{GEP}(F, A)$.

(iii) In the case of $A \equiv 0$(:the zero mapping) and $\varphi \equiv 0$, then the generalized mixed equilibrium problem (1.9) is reduced to *the equilibrium problem.* Finding $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.13}$$

The set of solution of (1.13) is denoted by $\mathrm{EP}(F)$.

(iv) In the case of $F \equiv 0$, $\varphi \equiv 0$ and $A \equiv B$ then the generalized mixed equilibrium problem (1.9) is reduced to *the variational inequality problem*. Finding $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.14}$$

The set of solution of (1.14) is denoted by $\mathrm{VI}(C, B)$.

The generalized mixed equilibrium problem include fixed point problems, optimization problems, variational inequalities problems, Nash equilibrium problems, noncooperative games, economics and the equilibrium problems as special cases (see, e.g., [3–8]). Some methods have been proposed to solve the generalized mixed equilibrium problems, generalized equilibrium problems and equilibrium problems; see, for instance, [9–22].

In 2007, Takahashi et al. [23] proved the following strong convergence theorem for a nonexpansive mapping by using the shrinking projection method in mathematical programming. For a $C_1 = C$ and $x_1 = P_{C_1}x_0$, they defined a sequence as follows

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{aligned} \tag{1.15}$$

where $0 \leq \alpha_n < a < 1$. They proved that the sequence $\{x_n\}$ generated by (1.15) converges weakly to $z \in F(T)$, where $z = P_{F(T)}x_0$.

In 2008, S. Takahashi and W. Takahashi [24] introduced the following iterative scheme for finding a common element of the set of solutions of mixed equilibrium problems and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with arbitrary $x_1 = u \in C$, define sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by

$$\begin{aligned} F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u, u_n - x_n \rangle \geq 0, \quad \forall\, y \in C \\ y_n = \alpha_n u + (1 - \alpha_n)u_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)Ty_n. \end{aligned} \tag{1.16}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$,$\{\beta_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.16) converges strongly to $z = P_{F(T)\cap \mathrm{GEP}(F,A)}u$.

In 2008, Zhang et al. [25] introduced the following new iterative scheme for finding a common element of the set of solutions to the problem (1.3) and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Starting with an arbitrary $x_1 = x \in H$, define sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda Bx_n), \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n)Ty_n, \quad \forall n \geq 0, \end{aligned} \tag{1.17}$$

where $J_{M,\lambda} = (I + \lambda M)^{-1}$ is the resolvent operator associated with M and a positive number λ, $\{\alpha_n\}$ is a sequence in the interval $[0, 1]$.

In 2008, Peng et al. [26] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the set of solutions to the problem (1.3), the set of solutions of an equilibrium problems and the set of fixed points of nonexpansive mappings in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J_{M,\lambda}(u_n - \lambda B u_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T y_n, \quad \forall n \geq 0. \end{gathered} \tag{1.18}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.18) converges strongly to $z = P_{F(T)\cap \mathrm{EP}(F)\cap I(B,M)} f(z)$.

In 2010, Katchang and Kumam [27] introduced an iterative scheme for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of the variational inclusions with set-valued maximal monotone mappings, and inverse strongly monotone mappings and the set of fixed points of a finite family of nonexpansive mappings in a real Hilbert space.

In this paper, motivated and inspired by the previously mentioned results, we introduce an iterative scheme by the shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of variational inclusion problems in a real Hilbert space. Then, we prove a strong convergence theorem of the iterative sequence generated by the proposed shrinking projection method under some suitable conditions. The results obtained in this paper extend and improve several recent results in this area.

2. Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y\in C}\|x - y\|. \tag{2.1}$$

We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.1. *For a given $x \in H$ and $z \in C$,*

$$z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{2.3}$$

Lemma 2.2 (see [1]). *Let $M : H \longrightarrow 2^H$ be a maximal monotone mapping and let $B : H \longrightarrow H$ be a Lipshitz continuous mapping. Then the mapping $S = M + B : H \longrightarrow 2^H$ is a maximal monotone mapping.*

Lemma 2.3 (see [28]). *Let C be a closed convex subset of H and let $\{x_n\}$ be a bounded sequence in H. Assume that*

(1) *the weak ω-limit set $\omega_w(x_n) \subset C$,*

(2) *for each $z \in C$, $\lim_{n\to\infty}\|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.4 (see [29]). *Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\| \tag{2.4}$$

holds for each $y \in H$ with $y \neq x$.

Lemma 2.5 (see [30]). *Each Hilbert space H satisfies the Kadec-Klee property, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \longrightarrow \|x\|$ together imply $\|x_n - x\| \longrightarrow 0$.*

For solving the generalized equilibrium problems, let us give the following assumptions for F, φ, and the set C:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $y \in C, x \longmapsto F(x,y)$ is weakly upper semicontinuous;

(A4) for each $x \in C, y \longmapsto F(x,y)$ is convex and lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \cap \operatorname{dom}\varphi$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle \geq 0; \tag{2.5}$$

(B2) C is bounded set.

Lemma 2.6 (see [31]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $\varphi : H \longrightarrow R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap \operatorname{dom} \neq \emptyset$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}. \tag{2.6}$$

Assume that either $(B1)$ *or* $(B2)$ *holds. Then, the following conclusions hold:*

(1) *for each* $x \in H$, $T_r(x) \neq \emptyset$;

(2) T_r *is single-valued;*

(3) T_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle; \tag{2.7}$$

(4) $F(T_r) = GEP(F, \varphi)$;

(5) $GEP(F, \varphi)$ *is closed and convex.*

Remark 2.7. Replacing x with $x - rAx \in H$ in (2.5), then there exists $z \in C \setminus D_x$, such that

$$F(z, y_x) + \langle Ax, y_x - z\rangle + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.8}$$

3. Main Results

In this section, we will introduce an iterative scheme by using shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of variational inclusion problems in a real Hilbert space.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let $\gamma_1, \ldots, \gamma_N$ be real numbers such that $0 \leq \gamma_i \leq 1$ for every $i = 1, \ldots, N$. We define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_1 &= \gamma_1 T_1 + (1 - \gamma_1) I, \\
U_2 &= \gamma_2 T_2 U_1 + (1 - \gamma_2) I, \\
U_3 &= \gamma_3 T_3 U_2 + (1 - \gamma_3) I, \\
&\vdots \\
U_{N-1} &= \gamma_{N-1} T_{N-1} U_{N-2} + (1 - \gamma_{N-1}) I, \\
K = U_N &= \gamma_N T_N U_{N-1} + (1 - \gamma_N) I.
\end{aligned} \tag{3.1}$$

Such a mapping K is called the *K-mapping* generated by $T_1, \ldots, T_N$ and $\gamma_1, \ldots, \gamma_N$; see [32].

We have the following crucial Lemma 3.1 and Lemma 3.2 concerning K-mapping which can be found in [14]. Now we only need the following similar version in Hilbert spaces.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings and L_i-Lipschitz mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\gamma_1, \ldots, \gamma_N$ be real numbers such that $0 < \gamma_i < 1$ for every $i = 1, \ldots, N-1$, $0 < \gamma_N \leq 1$ and*

$\sum_{i=1}^{N} = 1$. *Let K be the K-mapping generated by $T_1, \dots, T_N$ and $\gamma_1, \dots, \gamma_N$. Then, the followings hold:*

(1) *K is quasi-nonexpansive and Lipschitz,*

(2) $F(K) = \bigcap_{i=1}^{N} F(T_i)$.

Lemma 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{N}$ be a finite family of quasi-nonexpansive mappings and L_i-Lipschitz mappings of C into itself and $\{\gamma_{n,i}\}_{i=1}^{N}$ sequences in $[0,1]$ such that $\gamma_{n,i} \to \gamma_i$, as $n \to \infty$, $(i = 1,2,\dots,N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_2, \dots, T_N$ and $\gamma_1, \gamma_2, \dots, \gamma_N$, and $T_1, T_2, \dots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \dots, \gamma_{n,N}$, respectively. Then, for every $x \in C$, we have $\lim_{n\to\infty}\|K_n x - Kx\| = 0$.*

Now we study the strong convergence theorem concerning the shrinking projection method.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let $\{T_i\}_{i=1}^{N}$ be a finite family of quasi-nonexpansive and L_i-Lipschitz mappings of C into itself, and let A be a β-inverse-strongly monotone mapping of C into H, let B a ξ-inverse-strongly monotone mapping of C into H and $M : H \to 2^H$ be a maximal monotone mapping. Assume that*

$$\Theta := \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{GMEP}(F, \varphi, A) \cap I(B, M) \neq \emptyset. \tag{3.2}$$

Let K_n be the K-mapping generated by $T_1, T_2, \dots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \dots, \gamma_{n,N}$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ and let

$$\begin{gathered}
F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = J_{M,\delta_n}(u_n - \delta_n B u_n), \\
v_n = J_{M,\lambda_n}(y_n - \lambda_n B y_n), \\
z_n = \alpha_n x_n + (1 - \alpha_n) K_n v_n, \\
C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N},
\end{gathered} \tag{3.3}$$

where $\{\alpha_n\} \subset (0,1)$ satisfy the following conditions:

(i) *$\{\alpha_n\} \subset [0, e]$ for some e with $0 \leq e < 1$;*

(ii) *$\{\delta_n\}, \{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\xi$;*

(iii) *$\{r_n\} \subset [c, d]$ for some c, d with $0 < c < d < 2\beta$.*

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta x_0$.

Proof. In the light of the definition of the resolvent, u_n can be rewritten as $u_n = T_{r_n}(x_n - r_nAx_n)$. Let $p \in \Theta := \cap_{i=1}^{N} F(T_i) \cap \mathrm{GMEP}(F,\varphi,A) \cap I(B,M)$, and using the fact $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.6, A is an β-inverse-strongly monotone and that $p = T_{r_n}(p - r_nAp)$, where $\{r_n\} \subset [c,d]$ for some c,d with $0 < c < d < 2\beta$, we can write

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \\
&\le \|(x_n - r_nAx_n) - (p - r_nAp)\|^2 \\
&= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - 2r_n\beta\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\beta)\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{3.4}$$

Next, we will divide the proof into six steps.

Step 1. We first show that $\{x_n\}$ is well defined and C_n is closed and convex for any $n \in \mathbb{N}$.

From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \ge 1$. Next, we show that C_{k+1} is closed and convex for some k. For any $p \in C_k$, we obtain that

$$\|z_k - p\| \le \|x_k - p\| \tag{3.5}$$

is equivalent to

$$\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - p\rangle \le 0. \tag{3.6}$$

Thus C_{k+1} is closed and convex. Then, C_n is closed and convex for any $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Step 2. Next, we show by induction that $\Theta \subset C_n$ for each $n \ge 1$.

Taking $p \in \Theta$ and by condition (ii), we get that $p = J_{M,\delta_k}(p - \delta_k Bp) = J_{M,\lambda_k}(p - \lambda_k Bp)$ is nonexpansive for all $n \geq 1$. From the assumption, we see that $\Theta \subset C = C_1$. Suppose $\Theta \subset C_k$ for some $k \geq 1$. For any $p \in \Theta = C_k$, we have

$$\begin{aligned}
\|y_k - p\| &= \|J_{M,\delta_k}(u_k - \delta_k Bu_k) - J_{M,\delta_k}(p - \delta_k Bp)\| \\
&\leq \|(u_k - \delta_k Bu_k) - (p - \delta_k Bp)\| \\
&= \|(I - \delta_k B)u_k - (I - \delta_k B)p\| \\
&\leq \|u_k - p\| \leq \|x_k - p\|, \\
\|v_k - p\| &= \|J_{M,\lambda_k}(y_k - \lambda_k By_k) - J_{M,\lambda_k}(p - \lambda_k Bp)\| \\
&\leq \|(y_k - \lambda_k By_k) - (p - \lambda_k Bp)\| \\
&= \|(I - \lambda_k B)y_k - (I - \lambda_k B)p\| \\
&\leq \|y_k - p\| \leq \|x_k - p\|.
\end{aligned} \tag{3.7}$$

Thus, we have

$$\begin{aligned}
\|z_k - p\| &= \|\alpha_k(x_k - p) + (1 - \alpha_k)(K_k v_k - p)\| \\
&\leq \alpha_k\|x_k - p\| + (1 - \alpha_k)\|v_k - p\| \\
&\leq \alpha_k\|x_k - p\| + (1 - \alpha_k)\|x_k - p\| = \|x_k - p\|.
\end{aligned} \tag{3.8}$$

It follows that $p \in C_{k+1}$. This implies that $\Theta \subset C_n$ for each $n \geq 1$.

Step 3. Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0 \tag{3.9}$$

for each $y \in C_n$. Using $\Theta \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - p\rangle \geq 0, \quad \forall p \in \Theta,\ n \in \mathbb{N}. \tag{3.10}$$

So, for $p \in \Theta$, we have

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - p\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\
&= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - p\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|.
\end{aligned} \tag{3.11}$$

This implies that

$$\|x_0 - x_n\| \le \|x_0 - p\|, \quad \forall p \in \Theta,\ n \in \mathbb{N}. \tag{3.12}$$

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \ge 0. \tag{3.13}$$

From (3.13), we have, for $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\langle x_0 - x_n, x_0 - x_n \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|. \end{aligned} \tag{3.14}$$

It follows that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|. \tag{3.15}$$

Thus the sequence $\{\|x_n - x_0\|\}$ is a bounded and nonincreasing sequence, so $\lim_{n\to\infty}\|x_n - x_0\|$ exists, that is,

$$\lim_{n\to\infty} \|x_n - x_0\| = m. \tag{3.16}$$

Indeed, from (3.13), we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0 \rangle + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned} \tag{3.17}$$

From (3.16), we obtain

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.18}$$

Since $x_{n+1} = P_{C_{n+1}} \in C_{n+1} \subset C_n$, we have

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_n - x_{n+1}\|. \tag{3.19}$$

By (3.18), we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.20}$$

Step 4. Next, we show that $\lim_{n\to\infty}\|K_n v_n - v_n\| = 0$.

For any given $p \in \Theta$, $\lambda_n \in (0, 2\xi]$. It is easy to see that $p = J_{M,\lambda_n}(p - \lambda_n Bp)$. As $p - \lambda_n Bp$ is nonexpansive, we have

$$\begin{aligned}
\|v_n - p\|^2 &= \left\|J_{M,\lambda_n}(y_n - \lambda_n B y_n) - J_{M,\lambda_n}(p - \lambda_n Bp)\right\|^2 \\
&\leq \left\|(y_n - \lambda_n B y_n) - (p - \lambda_n Bp)\right\|^2 \\
&= \left\|(y_n - p) - \lambda_n (B y_n - Bp)\right\|^2 \\
&= \left\|y_n - p\right\|^2 - 2\lambda_n \langle y_n - p, B y_n - Bp\rangle + \lambda_n^2 \left\|B y_n - Bp\right\|^2 \\
&\leq \left\|x_n - p\right\|^2 - 2\lambda_n \langle y_n - p, B y_n - Bp\rangle + \lambda_n^2 \left\|B y_n - Bp\right\|^2 \\
&\leq \left\|x_n - p\right\|^2 + \lambda_n(\lambda_n - 2\xi)\left\|B y_n - Bp\right\|^2.
\end{aligned} \tag{3.21}$$

Similarly, we can prove that

$$\left\|y_n - p\right\|^2 \leq \left\|x_n - p\right\|^2 + \delta_n(\delta_n - 2\xi)\left\|B u_n - Bp\right\|^2. \tag{3.22}$$

Observe that

$$\begin{aligned}
\left\|z_n - p\right\|^2 &\leq \alpha_n \left\|x_n - p\right\|^2 + (1 - \alpha_n)\left\|K_n v_n - p\right\|^2 \\
&\leq \alpha_n \left\|x_n - p\right\|^2 + (1 - \alpha_n)\left\|v_n - p\right\|^2.
\end{aligned} \tag{3.23}$$

Substituting (3.21) into (3.23), and using conditions (i) and (ii), we have

$$\begin{aligned}
\left\|z_n - p\right\|^2 &\leq \alpha_n \left\|x_n - p\right\|^2 + (1 - \alpha_n)\left\|v_n - p\right\|^2 \\
&\leq \alpha_n \left\|x_n - p\right\|^2 + (1 - \alpha_n)\left\{\left\|x_n - p\right\|^2 + \lambda_n(\lambda_n - 2\xi)\left\|B y_n - Bp\right\|^2\right\} \\
&= \left\|x_n - p\right\|^2 + (1 - \alpha_n)\lambda_n(\lambda_n - 2\xi)\left\|B y_n - Bp\right\|^2.
\end{aligned} \tag{3.24}$$

It follows that

$$
\begin{aligned}
(1-e)a(2\xi-b)\|By_n-Bp\|^2 &\le (1-\alpha_n)\lambda_n(2\xi-\lambda_n)\|By_n-Bp\|^2\\
&\le \|x_n-p\|^2-\|z_n-p\|^2\\
&= (\|x_n-p\|-\|z_n-p\|)(\|x_n-p\|+\|z_n-p\|)\\
&\le \|x_n-z_n\|(\|x_n-p\|+\|z_n-p\|).
\end{aligned}
\tag{3.25}
$$

Since $\lim_{n\to\infty}\|x_n-z_n\|=0$, we obtain

$$
\lim_{n\to\infty}\|By_n-Bp\|=0. \tag{3.26}
$$

Since the resolvent operator J_{M,λ_n} is 1-inverse strongly monotone, we obtain

$$
\begin{aligned}
\|v_n-p\|^2 &= \|J_{M,\lambda_n}(y_n-\lambda_n By_n)-J_{M,\lambda_n}(p-\lambda_n Bp)\|^2\\
&= \|J_{M,\lambda_n}(I-\lambda_n B)y_n-J_{M,\lambda_n}(I-\lambda_n B)p\|^2\\
&\le \langle (I-\lambda_n B)y_n-(I-\lambda_n B)p, v_n-p\rangle\\
&= \frac{1}{2}\Big\{\|(I-\lambda_n B)y_n-(I-\lambda_n B)p\|^2+\|v_n-p\|^2\\
&\quad -\|(I-\lambda_n B)y_n-(I-\lambda_n B)p-(v_n-p)\|^2\Big\}\\
&\le \frac{1}{2}\Big\{\|y_n-p\|^2+\|v_n-p\|^2-\|(y_n-v_n)-\lambda_n(By_n-Bp)\|^2\Big\}\\
&\le \frac{1}{2}\Big\{\|x_n-p\|^2+\|v_n-p\|^2-\|y_n-v_n\|^2\\
&\quad -\lambda_n^2\|By_n-Bp\|^2+2\lambda_n\langle y_n-v_n, By_n-Bp\rangle\Big\},
\end{aligned}
\tag{3.27}
$$

which yields that

$$
\|v_n-p\|^2 \le \|x_n-p\|^2-\|y_n-v_n\|^2+2\lambda_n\|y_n-v_n\|\|By_n-Bp\|. \tag{3.28}
$$

Similarly, we obtain

$$
\|y_n-p\|^2 \le \|x_n-p\|^2-\|u_n-y_n\|^2+2\delta_n\|u_n-y_n\|\|Bu_n-Bp\|. \tag{3.29}
$$

Substituting (3.28) into (3.23), and using condition (i), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{\|x_n - p\|^2 - \|y_n - v_n\|^2 + 2\lambda_n \|y_n - v_n\| \|By_n - Bp\|\right\} \\ &= \|x_n - p\|^2 - (1-\alpha_n)\|y_n - v_n\|^2 + 2(1-\alpha_n)\lambda_n \|y_n - v_n\| \|By_n - Bp\|.\end{aligned} \tag{3.30}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\|y_n - v_n\|^2 &\le \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1-\alpha_n)\lambda_n \|y_n - v_n\| \|By_n - Bp\| \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1-\alpha_n)\lambda_n \|y_n - v_n\| \|By_n - Bp\|.\end{aligned} \tag{3.31}$$

Applying $\|x_n - z_n\| \to 0$ and $\|By_n - Bp\| \to 0$ as $n \to \infty$ to the last inequality, we get

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{3.32}$$

Note that

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|K_n v_n - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2.\end{aligned} \tag{3.33}$$

Substituting (3.22) into (3.33), and using conditions (i) and (ii), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{\|x_n - p\|^2 + \delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2\right\} \\ &= \|x_n - p\|^2 + (1-\alpha_n)\delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2.\end{aligned} \tag{3.34}$$

It follows that

$$\begin{aligned}(1-e)a(2\xi - b)\|Bu_n - Bp\|^2 &\le (1-\alpha_n)\delta_n(2\xi - \delta_n)\|Bu_n - Bp\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).\end{aligned} \tag{3.35}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we obtain

$$\lim_{n\to\infty}\|Bu_n - Bp\| = 0. \tag{3.36}$$

Substituting (3.29) into (3.33), and using conditions (i) and (ii), we have

$$\begin{aligned}
\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\left\{\|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\delta_n\|u_n - y_n\|\|Bu_n - Bp\|\right\} \\
&= \|x_n - p\|^2 - (1-\alpha_n)\|u_n - y_n\|^2 + 2(1-\alpha_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\|.
\end{aligned} \tag{3.37}$$

It follows that

$$\begin{aligned}
(1-\alpha_n)\|u_n - y_n\|^2 &\le \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1-\alpha_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\| \\
&\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1-\alpha_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\|.
\end{aligned} \tag{3.38}$$

Applying $\|x_n - z_n\| \to 0$ and $\|Bu_n - Bp\| \to 0$ as $n \to \infty$ to the last inequality, we get

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.39}$$

From (3.32) and (3.39), we have

$$\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{3.40}$$

From (3.33), (3.4), and condition (iii), we have

$$\begin{aligned}
\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\left\{\|x_n - p\|^2 + r_n(r_n - 2\beta)\|Ax_n - Ap\|^2\right\} \\
&= \|x_n - p\|^2 + (1-\alpha_n)r_n(r_n - 2\beta)\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.41}$$

It follows that

$$\begin{aligned}
(1-e)c(2\beta - d)\|Ax_n - Ap\|^2 &\le (1-\alpha_n)r_n(2\beta - r_n)\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - \|z_n - v\|^2 \\
&\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned} \tag{3.42}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we obtain

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.43}$$

On the other hand, in the light of Lemma 2.6(3), T_{r_n} is firmly nonexpansvie, so we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \\
&\le \langle T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp), u_n - p\rangle \\
&= \langle x_n - r_nAx_n - (p - r_n\mathrm{A}p), u_n - p\rangle \\
&= \frac{1}{2}\Big\{\|(x_n - r_nAx_n) - (p - r_nAp)\|^2 + \|u_n - p\|^2 \\
&\qquad -\|(x_n - r_nAx_n) - (p - r_nAp) - (u_n - p)\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - p\|^2 - r_n(Ax_n - Ap)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\
&\qquad +2r_n\langle x_n - u_n, Ax_n - Ap\rangle - r_n^2\|Ax_n - Ap\|^2\Big\},
\end{aligned} \tag{3.44}$$

which implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|. \tag{3.45}$$

Using (3.41) again and (3.45), we have

$$\begin{aligned}
\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big\{\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|\Big\} \\
&= \|x_n - p\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 + 2(1 - \alpha_n)r_n\|x_n - u_n\|\|Ax_n - Ap\|.
\end{aligned} \tag{3.46}$$

It follows from the condition (i) that

$$\begin{aligned}
(1 - e)\|x_n - u_n\|^2 &\le (1 - \alpha_n)\|x_n - u_n\|^2 \\
&\le \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1 - \alpha_n)r_n\|x_n - u_n\|\|Ax_n - Ap\| \\
&\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1 - \alpha_n)r_n\|x_n - u_n\|\|Ax_n - Ap\|.
\end{aligned} \tag{3.47}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and $\lim_{n\to\infty}\|Ax_n - Ap\| = 0$, it is implied that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.48}$$

From (3.39) and (3.48), we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.49}$$

By (3.3), we get

$$z_n - x_n = (1-\alpha_n)(K_n v_n - x_n). \tag{3.50}$$

Since $\{\alpha_n\} \subset [0,e]$ for some e with $0 \le e < 1$, and $\|x_n - z_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$, we also have

$$\lim_{n\to\infty}\|K_n v_n - x_n\| = 0. \tag{3.51}$$

From (3.40) and (3.48), we have

$$\lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.52}$$

Furthermore, by the triangular inequality, we also have

$$\|K_n v_n - v_n\| \le \|K_n v_n - x_n\| + \|x_n - v_n\|. \tag{3.53}$$

Applying (3.51) and (3.52), we obtain

$$\lim_{n\to\infty}\|K_n v_n - v_n\| = 0. \tag{3.54}$$

Let K be the mapping defined by (3.1). Since $\{v_n\}$ is bounded, applying Lemma 3.2 and (3.54), we have

$$\|Kv_n - v_n\| \le \|Kv_n - K_n v_n\| + \|K_n v_n - v_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.55}$$

Step 5. Next, we show that $q \in \Theta := \cap_{i=1}^{N} F(T_i) \cap \mathrm{GMEP}(F,\varphi,A) \cap I(B,M)$.

Since $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ which converges weakly to q. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup q$. Since $v_{n_i} \subset C$ and C is closed and convex, C is weakly closed and hence $q \in C$. From $\|Kv_n - v_n\| \to 0$, we obtain $Kv_{n_i} \rightharpoonup q$.

(a) First, we prove that $q \in I(B,M)$.

We observe that B is a $1/\xi$-Lipschitz monotone mapping and $D(B) = H$. From Lemma 2.2, we know that $M + B$ is maximal monotone. Let $(v,g) \in G(M+B)$, that is, $g - Bv \in M(v)$. Since $v_{n_i} = J_{M,\lambda_{n_i}}(y_{n_i} - \lambda_{n_i} B y_{n_i})$, we have

$$y_{n_i} - \lambda_{n_i} B y_{n_i} \in (I + \lambda_{n_i} M)(v_{n_i}), \tag{3.56}$$

that is,

$$\frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} B y_{n_i}) \in M(v_{n_i}). \tag{3.57}$$

By virtue of the maximal monotonicity of $M + B$, we have

$$\left\langle v - v_{n_i}, g - Bv - \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} B y_{n_i}) \right\rangle \geq 0, \tag{3.58}$$

and so

$$\begin{aligned} \langle v - v_{n_i}, g\rangle &\geq \left\langle v - v_{n_i}, Bv + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} B y_{n_i}) \right\rangle \\ &= \left\langle v - v_{n_i}, Bv - Bz_n + Bv_{n_i} - By_{n_i} + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) \right\rangle \\ &\geq 0 + \langle v - v_{n_i}, Bv_{n_i} - By_{n_i}\rangle + \left\langle v - v_{n_i}, \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) \right\rangle. \end{aligned} \tag{3.59}$$

It follows from $\|y_n - v_n\| \to 0$, $\|By_n - Bv_n\| \to 0$ and $v_{n_i} \rightharpoonup q$ that

$$\lim_{n\to\infty} \langle v - v_{n_i}, g\rangle = \langle v - q, g\rangle \geq 0. \tag{3.60}$$

It follows from the maximal monotonicity of $M + B$ that $\theta \in (M + B)(q)$, that is, $q \in I(B, M)$.

(b) Next, we show that $q \in \mathrm{GMEP}(F, \varphi, A)$. Since $u_n = T_{r_n}(x_n - r_n A x_n) \in \operatorname{dom} \varphi$, we have

$$F(u_n, y) + \langle Ax_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.61}$$

From (A2), we also have

$$\langle Ax_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C. \tag{3.62}$$

And hence

$$\langle Ax_{n_i}, y - u_{n_i}\rangle + \varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.63}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, we have $y_t \in C$. So, from (3.63), we have

$$\begin{aligned}
\langle y_t - u_{n_i}, Ay_t \rangle &\geq \langle y_t - u_{n_i}, Ay_t \rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle y_t - u_{n_i}, Ax_{n_i} \rangle \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + F(y_t, u_{n_i}) \\
&= \langle y_t - u_{n_i}, Ay_t - Au_{n_i} \rangle + \langle y_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle - \varphi(y_t) + \varphi(u_{n_i}) \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + F(y_t, u_{n_i}).
\end{aligned} \tag{3.64}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of A, we have $\langle y_t - u_{n_i}, Ay_t - Au_{n_i} \rangle \geq 0$. So, from (A5), the weakly lower semicontinuity of φ, $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup q$, we have

$$\langle y_t - q, Ay_t \rangle \geq -\varphi(y_t) + \varphi(q) + F(y_t, q) \tag{3.65}$$

as $i \to \infty$. From (A1), (A4) and (3.65), we also get

$$\begin{aligned}
0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq tF(y_t, y) + (1-t)F(y_t, q) + t\varphi(y) - (1-t)\varphi(q) - \varphi(y_t) \\
&= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[F(y_t, q) + \varphi(q) - \varphi(y_t)] \\
&\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t - q, Ay_t \rangle \\
&= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)t\langle y - q, Ay_t \rangle, \\
0 &\leq F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - q, Ay_t \rangle.
\end{aligned} \tag{3.66}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(q, y) + \varphi(y) - \varphi(q) + \langle y - q, Aq \rangle \geq 0. \tag{3.67}$$

This implies that $q \in \mathrm{GMEP}(F, \varphi, A)$.

(c) Now, we prove that $q \in F(K) = \bigcap_{i=1}^{N} F(T_i)$.

Assume $q \notin F(K)$. Since $\|x_n - v_n\| \to 0$ and we know that $v_{n_i} \rightharpoonup q$ $(i \to \infty)$ and $q \neq Kq$, it follows by the Opial's condition (Lemma 2.4) that

$$\begin{aligned}\liminf_{i\to\infty}\|v_{n_i} - q\| &< \liminf_{i\to\infty}\|v_{n_i} - Kq\| \\ &\leq \liminf_{i\to\infty}(\|v_{n_i} - Kv_{n_i}\| + \|Kv_{n_i} - Kq\|) \\ &< \liminf_{i\to\infty}\|v_{n_i} - q\|,\end{aligned} \tag{3.68}$$

which is a contradiction. Thus, we get $q \in F(\mathrm{K}) = \bigcap_{i=1}^{N} F(T_i)$.

The conclusion is $q \in \Theta := \cap_{i=1}^{N} F(T_i) \cap \mathrm{GMEP}(F, \varphi, A) \cap I(B, M)$.

Step 6. Finally, we show that $x_n \to z$ and $u_n \to z$, where $z = P_\Theta x_0$.

Since Θ is nonempty closed convex subset of H, there exists a unique $z' \in \Theta$ such that $z' = P_\Theta x_0$. Since $z' \in \Theta \subset C_n$ and $x_n = P_{C_n} x_0$, we have

$$\|x_0 - x_n\| = \|x_0 - P_{C_n} x_0\| \leq \|x_0 - z'\| \tag{3.69}$$

for all $n \in \mathbb{N}$. From (3.69), $\{x_n\}$ is bounded, so $\omega_w(x_n) \neq \emptyset$. By the weak lower semicontinuity of the norm, we have

$$\|x_0 - z\| \leq \liminf_{i\to\infty}\|x_0 - x_{n_i}\| \leq \|x_0 - z'\|. \tag{3.70}$$

However, Since $z \in \omega_w(x_n) \subset \Theta$, we have

$$\|x_0 - z'\| \leq \|x_0 - P_{C_n} x_0\| \leq \|x_0 - z\|. \tag{3.71}$$

Using (3.69) and (3.70), we obtain $z' = z$. Thus $\omega_w(x_n) = \{z\}$ and $x_n \rightharpoonup z$. So, we have

$$\|x_0 - z'\| \leq \|x_0 - z\| \leq \liminf_{n\to\infty}\|x_0 - x_n\| \leq \limsup_{n\to\infty}\|x_0 - x_n\| \leq \|x_0 - z'\|. \tag{3.72}$$

Thus, we obtain that

$$\|x_0 - z\| = \lim_{n\to\infty}\|x_0 - x_n\| = \|x_0 - z'\|. \tag{3.73}$$

From $x_n \rightharpoonup z$, we obtain $(x_0 - x_n) \rightharpoonup (x_0 - z)$. Using the Kadec-Klee property (Lemma 2.5) of H, we obtain that

$$\|x_n - z\| = \|(x_n - x_0) - (z - x_0)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.74}$$

and hence $x_n \to z$ in norm. Finally, noticing $\|u_n - z\| = \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(z - r_n A z)\| \leq \|x_n - z\|$. we also conclude that $u_n \to z$ in norm. This completes the proof. □

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself, let A be a β-inverse-strongly monotone mapping of C into H, let B be a ξ-inverse-strongly monotone mapping of C into H and $M : H \to 2^H$ a maximal monotone mapping. Assume that*

$$\Theta := \bigcap_{i=1}^{N} F(T_i) \cap GMEP(F, \varphi, A) \cap I(B, M) \neq \emptyset. \tag{3.75}$$

Let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,N}$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by (3.3) satisfying the following conditions in Theorem 3.3. Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta x_0$.

From Theorem 3.3, we can obtain the following results.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let $\{T_i\}_{i=1}^{N}$ be a finite family of quasi-nonexpansive and L_i-Lipschitz mappings of C into itself, let A be a β-inverse-strongly monotone mapping of C into H and let B be a ξ-inverse-strongly monotone mapping of C into H. Assume that*

$$\Theta := \bigcap_{i=1}^{N} F(T_i) \cap GMEP(F, \varphi, A) \cap VI(C, B) \neq \emptyset. \tag{3.76}$$

Let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,N}$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$, $u_n \in C$ and let

$$\begin{aligned}
F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,\\
y_n = P_C(u_n - \delta_n B u_n),\\
v_n = P_C(y_n - \lambda_n B y_n),\\
z_n = \alpha_n x_n + (1 - \alpha_n) K_n v_n,\\
C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\},\\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N},
\end{aligned} \tag{3.77}$$

where $\{\alpha_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, e]$ *for some e with $0 \leq e < 1$;*

(ii) $\{\delta_n\}$, $\{\lambda_n\} \subset [a, b]$ *for some a, b with $0 < a < b < 2\xi$;*

(iii) $\{r_n\} \subset [c, d]$ *for some c, d with $0 < c < d < 2\beta$.*

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta x_0$.

Proof. In Theorem 3.3 take $M = \partial\delta_C : H \to 2^H$, where $\delta_C : 0 \to [0, \infty]$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{3.78}$$

Then the variational inclusion problem (1.3) is equivalent to variational inequality problem (1.14), that is, to find $\hat{x} \in C$ such that

$$\langle B\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \tag{3.79}$$

Again, since $M = \delta_C$, then

$$J_{M,\lambda_n} = J_{M,\delta_n} = P_C, \tag{3.80}$$

and so we have

$$\begin{aligned} y_n &= P_C(u_n - \delta_n B u_n) = J_{M,\delta_n}(P_C(u_n - \delta_n B u_n)), \\ v_n &= P_C(y_n - \lambda_n B y_n) = J_{M,\lambda_n}(P_C(y_n - \lambda_n B y_n)). \end{aligned} \tag{3.81}$$

We can obtain the desired conclusion from Theorem 3.3 immediately. □

Next, we consider another class of important nonlinear mappings: strict pseudocontractions.

Definition 3.6. A mapping $S : C \to C$ is called *strictly pseudocontraction* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{3.82}$$

If $\kappa = 0$, then S is nonexpansive.

In this case, let $S : C \to C$ a κ-strictly pseudocontraction. Putting $B = I - S : C \to H$, then B is a $(1 - \kappa)/2$-inverse-strongly monotone mapping. In fact, from (3.82) we have

$$\|(I - B)x - (I - B)y\|^2 \leq \|x - y\|^2 + \kappa\|Bx - By\|^2, \quad \forall x, y \in C. \tag{3.83}$$

Observe that

$$\|(I - B)x - (I - B)y\|^2 = \|x - y\|^2 - 2\langle x - y, Bx - By \rangle + \|Bx - By\|^2, \quad \forall x, y \in C. \tag{3.84}$$

Hence, we obtain

$$\langle x - y, Bx - By \rangle \geq \frac{1-\kappa}{2} \|Bx - By\|^2, \quad \forall x, y \in C. \tag{3.85}$$

This shows that B is $(1-\kappa)/2$-inverse-strongly monotone mapping.

Now, we get the following result.

Theorem 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive and L_i-Lipschitz mappings of C into itself, let S_A be a κ_β-strictly pseudocontraction mapping of C into C and let S_B be a κ_ξ-strictly pseudocontraction mapping of C into C. Assume that*

$$\Theta := \bigcap_{i=1}^N F(T_i) \cap GMEP(F, \varphi, I - S_A) \cap F(S_B) \neq \emptyset. \tag{3.86}$$

Let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,N}$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, $C_1 = C, x_1 = P_{C_1}x_0$, $u_n \in C$ and let

$$\begin{aligned}
F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle (I - S_A)x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,& \\
y_n = (1-\delta_n)u_n + \delta_n S_B u_n,& \\
v_n = (1-\lambda_n)y_n + \lambda_n S_B y_n,& \\
z_n = \alpha_n x_n + (1-\alpha_n)K_n v_n,& \\
C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\},& \\
x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N},&
\end{aligned} \tag{3.87}$$

where $\{\alpha_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, e]$ *for some e with $0 \leq e < 1$;*

(ii) $\{\delta_n\}, \{\lambda_n\} \subset [a, b]$ *for some a, b with $0 < a < b < 1 - \kappa_\xi$;*

(iii) $\{r_n\} \subset [c, d]$ *for some c, d with $0 < c < d < 1 - \kappa_\beta$.*

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_\Theta x_0$.

Proof. Taking $A = I - S_A$ and $B = I - S_B$, respectively. Then we see that A is $(1-\kappa_\beta)/2$-inverse-strongly monotone and B is $(1-\kappa_\xi)/2$-inverse-strongly monotone, respectively. We have $F(S_B) = \mathrm{VI}(C, B)$ and

$$\begin{aligned}
y_n &= P_C(u_n - \delta_n B u_n) = P_C((1-\delta_n)u_n + \delta_n S_B u_n) = (1-\delta_n)u_n + \delta_n S_B u_n \in C, \\
v_n &= P_C(y_n - \lambda_n B y_n) = P_C((1-\lambda_n)y_n + \lambda_n S_B y_n) = (1-\lambda_n)y_n + \lambda_n S_B y_n \in C.
\end{aligned} \tag{3.88}$$

By using Theorem 3.5, it is easy to obtain the desired conclusion. □

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