CHAPTER 4 THE MAIN RESULTS

4.1 Our Theorems

In this section, we will introduce an iterative scheme by using shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of variational inclusion problems in a real Hilbert space.

Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpanxive mappings of C into itself, and let $\gamma_1, ..., \gamma_N$ be real numbers such that $0 \leq \gamma_i \leq 1$ for every $i = 1, ..., N$. We define a mapping $K : C \rightarrow C$ as follows:

$$\begin{aligned} U_1 &= \gamma_1 T_1 + (1-\gamma_1)I, \\ U_2 &= \gamma_2 T_2 U_1 + (1-\gamma_2)I, \\ U_3 &= \gamma_3 T_3 U_2 + (1-\gamma_3)I, \\ &\cdot \\ &\cdot \\ &\cdot \\ U_{N-1} &= \gamma_{N-1} T_{N-1} U_{N-2} + (1-\gamma_{N-1})I, \\ K &= U_N = \gamma_N T_N U_{N-1} + (1-\gamma_N)I. \end{aligned} \tag{4.1.1}$$

Such a mapping K is called the *K-mapping* generated by $T_1, ..., T_N$ and $\gamma_1, ..., \gamma_N$; see [57].

We have the following crucial Lemma 4.1.1 and Lemma 4.1.2 concerning K-mapping which can be found in [67]. Now we only need the following similar version in Hilbert spaces.

Lemma 4.1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{N}$ be a finite family of quasi-nonexpanxive mappings and L_i-Lipschitz mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $\gamma_1, ..., \gamma_N$ be real numbers such* *that* $0 < \gamma_i < 1$ *for every* $i = 1, ..., N-1$, $0 < \gamma_N \leq 1$ *and* $\sum_{i=1}^{N} = 1$. *Let* K *be the* K*-mapping generated by* $T_1, ..., T_N$ *and* $\gamma_1, ..., \gamma_N$. *Then, the followings hold:*

(1) K *is quasi-nonexpansive and Lipschitz,*

(2) $F(K) = \bigcap_{i=1}^{N} F(T_i)$.

Lemma 4.1.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_i\}_{i=1}^{N}$ *be a finite family of quasi-nonexpanxive mappings and* L_i*-Lipschitz mappings of* C *into itself and* $\{\gamma_{n,i}\}_{i=1}^{N}$ *sequences in [0,1] such that* $\gamma_{n,i} \to \gamma_i$, *as* $n \to \infty$, $(i = 1, 2, ..., N)$. *Moreover, for every* $n \in \mathbb{N}$, *let* K *and* K_n *be the* K*-mappings generated by* $T_1, T_2, ..., T_N$ *and* $\gamma_1, \gamma_2, ..., \gamma_N$, *and* $T_1, T_2, ..., T_N$ *and* $\gamma_{n,1}, \gamma_{n,2}, ..., \gamma_{n,N}$, *respectively. Then, for every* $x \in C$, *we have* $\lim_{n\to\infty} \|K_n x - Kx\| = 0$.

Now we study the strong convergence theorem concerning the shrinking projection method.

Theorem 4.1.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) and let* $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let* $\{T_i\}_{i=1}^{N}$ *a finite family of quasi-nonexpansive and* L_i*-Lipschitz mappings of* C *into itself, let* A *be a* β*-inverse-strongly monotone mapping of* C *into* H, *let* B *be a* ξ*-inverse-strongly monotone mapping of* C *into* H *and* $M : H \to 2^H$ *be a maximal monotone mapping. Assume that*

$$\Theta := \cap_{i=1}^{N} F(T_i) \cap GMEP(F, \varphi, A) \cap I(B, M) \neq \emptyset.$$

Let K_n *be the* K*-mapping generated by* $T_1, T_2, ..., T_N$ *and* $\gamma_{n,1}, \gamma_{n,2}, ..., \gamma_{n,N}$. *Let* $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ *and let*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J_{M,\delta_n}(u_n - \delta_n Bu_n), \\ v_n = J_{M,\lambda_n}(y_n - \lambda_n By_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n)K_n v_n, \\ C_{n+1} = \Big\{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\Big\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}. \end{cases} \tag{4.1.2}$$

where $\{\alpha_n\} \subset (0,1)$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset [0, e]$ *for some* e *with* $0 \leq e < 1$,

(ii) $\{\delta_n\}$, $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < 2\xi$,

(iii) $\{r_n\} \subset [c, d]$ *for some* c, d *with* $0 < c < d < 2\beta$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z = P_\Theta x_0$.

Proof. In the light of the definition of the resolvent, u_n can be rewritten as $u_n = T_{r_n}(x_n - r_n A x_n)$. Let $p \in \Theta := \cap_{i=1}^N F(T_i) \cap GMEP(F, \varphi, A) \cap I(B, M)$ and using the fact $\{T_{r_n}\}$ be a sequence of mappings defined as in Lamma 3.2.6, A be an β-inverse-strongly monotone and that $p = T_{r_n}(p - r_n Ap)$, where $\{r_n\} \subset [c, d]$ for some c, d with $0 < c < d < 2\beta$, we can write

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(p - r_n Ap)\|^2 \\
&\leq \|(x_n - r_n A x_n) - (p - r_n Ap)\|^2 \\
&= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n\beta\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\beta)\|Ax_n - Ap\|^2 \qquad (4.1.3)\\
&\leq \|x_n - p\|^2.
\end{aligned}$$

Next, we will divide the proof into six steps.

Step 1. We first show that $\{x_n\}$ is well defined and C_n is closed and convex for any $n \in \mathbb{N}$.

From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. Next, we show that C_{k+1} is closed and convex for some k. For any $p \in C_k$, we obtain that

$$\|z_k - p\| \leq \|x_k - p\|$$

is equivalent to

$$\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - p\rangle \leq 0. \qquad (4.1.4)$$

Thus C_{k+1} is closed and convex. Then, C_n is closed and convex for any $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined.

Step 2. Next, we show by induction that $\Theta \subset C_n$ for each $n \geq 1$.

Taking $p \in \Theta$ and by condition (ii), we get that $p = J_{M,\delta_k}(p - \delta_k Bp) = J_{M,\lambda_k}(p - \lambda_k Bp)$ is nonexpansive for all $n \geq 1$. From the assumption, we see that $\Theta \subset C = C_1$. Suppose $\Theta \subset C_k$ for some $k \geq 1$. For any $p \in \Theta = C_k$, we have

$$\begin{aligned} \|y_k - p\| &= \|J_{M,\delta_k}(u_k - \delta_k Bu_k) - J_{M,\delta_k}(p - \delta_k Bp)\| \\ &\leq \|(u_k - \delta_k Bu_k) - (p - \delta_k Bp)\| \\ &\leq \|(I - \delta_k B)u_k - (I - \delta_k B)p\| \\ &\leq \|u_k - p\| \leq \|x_k - p\| \end{aligned}$$

and

$$\begin{aligned} \|v_k - p\| &= \|J_{M,\lambda_k}(y_k - \lambda_k By_k) - J_{M,\lambda_k}(p - \lambda_k Bp)\| \\ &\leq \|(y_k - \lambda_k By_k) - (p - \lambda_k Bp)\| \\ &\leq \|(I - \lambda_k B)y_k - (I - \lambda_k B)p\| \\ &\leq \|y_k - p\| \leq \|x_k - p\|. \end{aligned} \tag{4.1.5}$$

Thus, we have

$$\begin{aligned} \|z_k - p\| &= \|\alpha_k(x_k - p) + (1 - \alpha_k)(K_k v_k - p)\| \\ &\leq \alpha_k\|x_k - p\| + (1 - \alpha_k)\|v_k - p\| \\ &\leq \alpha_k\|x_k - p\| + (1 - \alpha_k)\|x_k - p\| = \|x_k - p\|. \end{aligned}$$

It follows that $p \in C_{k+1}$. This implies that $\Theta \subset C_n$ for each $n \geq 1$.

Step 3. Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0$$

for each $y \in C_n$. Using $\Theta \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - p\rangle \geq 0, \quad \forall p \in \Theta \quad \text{and} \quad n \in \mathbb{N}.$$

So, for $p \in \Theta$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - p \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - p \rangle \\ &= -\langle x_0 - x_n, x_0 - x_n \rangle + \langle x_0 - x_n, x_0 - p \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - p\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_0 - p\|, \quad \forall p \in \Theta \quad \text{and} \quad n \in \mathbb{N}.$$

From $x_n = P_{C_n} x_0$, and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{4.1.6}$$

From (4.1.6), we have, for $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\langle x_0 - x_n, x_0 - x_n \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|. \end{aligned}$$

It follows that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

Thus the sequence $\{\|x_n - x_0\|\}$ is a bounded and nonincreasing sequence, so $\lim_{n\to\infty} \|x_n - x_0\|$ exists, that is,

$$\lim_{n\to\infty} \|x_n - x_0\| = m. \tag{4.1.7}$$

Indeed, from (4.1.6), we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0 \rangle + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

From (4.1.7), we obtain

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{4.1.8}$$

Since $x_{n+1} = P_{C_{n+1}} \in C_{n+1} \subset C_n$, we have

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (4.1.8), we obtain

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{4.1.9}$$

Step 4. Next, we show that $\lim_{n\to\infty} \|K_n v_n - v_n\| = 0$.

For any given $p \in \Theta$, $\lambda_n \in (0, 2\xi]$. It is easy to see that $p = J_{M,\lambda_n}(p - \lambda_n Bp)$. As $p - \lambda_n Bp$ is nonexpansive, we have

$$\begin{aligned}
\|v_n - p\|^2 &= \|J_{M,\lambda_n}(y_n - \lambda_n By_n) - J_{M,\lambda_n}(p - \lambda_n Bp)\|^2 \\
&\leq \|(y_n - \lambda_n By_n) - (p - \lambda_n Bp)\|^2 \\
&= \|(y_n - p) - \lambda_n(By_n - Bp)\|^2 \\
&= \|y_n - p\|^2 - 2\lambda_n\langle y_n - p, By_n - Bp\rangle + \lambda_n^2\|By_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 - 2\lambda_n\langle y_n - p, By_n - Bp\rangle + \lambda_n^2\|By_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2. \qquad (4.1.10)
\end{aligned}$$

Similarly, we can prove that

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 + \delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2. \tag{4.1.11}$$

Observe that

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|K_n v_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|v_n - p\|^2. \qquad (4.1.12)
\end{aligned}$$

Substituting (4.1.10) into (4.1.12), and using conditions (i) and (ii), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|v_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big\{\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2\Big\} \\
&= \|x_n - p\|^2 + (1 - \alpha_n)\lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}(1-e)a(2\xi-b)\|By_n-Bp\|^2 &\leq (1-\alpha_n)\lambda_n(2\xi-\lambda_n)\|By_n-Bp\|^2\\ &\leq \|x_n-p\|^2-\|z_n-p\|^2\\ &= (\|x_n-p\|-\|z_n-p\|)(\|x_n-p\|+\|z_n-p\|)\\ &\leq \|x_n-z_n\|(\|x_n-p\|+\|z_n-p\|).\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-z_n\|=0$, we obtain

$$\lim_{n\to\infty}\|By_n-Bp\|=0. \tag{4.1.13}$$

Since the resolvent operator J_{M,λ_n} is 1-inverse strongly monotone, we obtain

$$\begin{aligned}\|v_n-p\|^2 &= \|J_{M,\lambda_n}(y_n-\lambda_n By_n)-J_{M,\lambda_n}(p-\lambda_n Bp)\|^2\\ &= \|J_{M,\lambda_n}(I-\lambda_n B)y_n-J_{M,\lambda_n}(I-\lambda_n B)p\|^2\\ &\leq \Big\langle (I-\lambda_n B)y_n-(I-\lambda_n B)p, v_n-p\Big\rangle\\ &= \frac{1}{2}\Big\{\|(I-\lambda_n B)y_n-(I-\lambda_n B)p\|^2+\|v_n-p\|^2\\ &\quad -\|(I-\lambda_n B)y_n-(I-\lambda_n B)p-(v_n-p)\|^2\Big\}\\ &\leq \frac{1}{2}\Big\{\|y_n-p\|^2+\|v_n-p\|^2-\|(y_n-v_n)-\lambda_n(By_n-Bp)\|^2\Big\}\\ &\leq \frac{1}{2}\Big\{\|x_n-p\|^2+\|v_n-p\|^2-\|y_n-v_n\|^2\\ &\quad -\lambda_n^2\|By_n-Bp\|^2+2\lambda_n\langle y_n-v_n, By_n-Bp\rangle\Big\},\end{aligned}$$

which yields that

$$\|v_n-p\|^2\leq\|x_n-p\|^2-\|y_n-v_n\|^2+2\lambda_n\|y_n-v_n\|\|By_n-Bp\|. \tag{4.1.14}$$

Similarly, we obtain

$$\|y_n-p\|^2\leq\|x_n-p\|^2-\|u_n-y_n\|^2+2\delta_n\|u_n-y_n\|\|Bu_n-Bp\|. \tag{4.1.15}$$

Substituting (4.1.14) into (4.1.12), and using condition (i), we have

$$\begin{aligned}\|z_n-p\|^2 &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2\\ &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\Big\{\|x_n-p\|^2-\|y_n-v_n\|^2+2\lambda_n\|y_n-v_n\|\|By_n-Bp\|\Big\}\\ &= \|x_n-p\|^2-(1-\alpha_n)\|y_n-v_n\|^2+2(1-\alpha_n)\lambda_n\|y_n-v_n\|\|By_n-Bp\|.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\|y_n - v_n\|^2 &\le \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1-\alpha_n)\lambda_n\|y_n - v_n\|\|By_n - Bp\| \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1-\alpha_n)\lambda_n\|y_n - v_n\|\|By_n - Bp\|.\end{aligned}$$

Applying $\|x_n - z_n\| \to 0$ and $\|By_n - Bp\| \to 0$ as $n \to \infty$ to the last inequality, we get

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \qquad (4.1.16)$$

Note that

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|K_n v_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2. \qquad (4.1.17)\end{aligned}$$

Substituting (4.1.11) into (4.1.17), and using conditions (i) and (ii), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 + \delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2\Big\} \\ &= \|x_n - p\|^2 + (1-\alpha_n)\delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}(1-e)a(2\xi - b)\|Bu_n - Bp\|^2 &\le (1-\alpha_n)\delta_n(2\xi - \delta_n)\|Bu_n - Bp\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we obtain

$$\lim_{n\to\infty} \|Bu_n - Bp\| = 0. \qquad (4.1.18)$$

Substituting (4.1.15) into (4.1.17), and using conditions (i) and (ii), we have

$$\begin{aligned}&\|z_n - p\|^2 \\ \le\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ \le\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\delta_n\|u_n - y_n\|\|Bu_n - Bp\|\Big\} \\ =\ & \|x_n - p\|^2 - (1-\alpha_n)\|u_n - y_n\|^2 + 2(1-\alpha_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\|.\end{aligned}$$

It follows that

$$\begin{aligned}
& (1-\alpha_n)\|u_n - y_n\|^2 \qquad (4.1.19)\\
\leq\ & \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1-\alpha_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\| \\
\leq\ & \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1-\alpha_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\|.
\end{aligned}$$

Applying $\|x_n - z_n\| \to 0$ and $\|Bu_n - Bp\| \to 0$ as $n \to \infty$ to the last inequality, we get

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \qquad (4.1.20)$$

From (4.1.16) and (4.1.20), we have

$$\lim_{n\to\infty} \|u_n - v_n\| = 0. \qquad (4.1.21)$$

From (4.1.17), (4.1.3) and condition (iii), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \qquad (4.1.22)\\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 + r_n(r_n - 2\beta)\|Ax_n - Ap\|^2\Big\} \\
&= \|x_n - p\|^2 + (1-\alpha_n)r_n(r_n - 2\beta)\|Ax_n - Ap\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-e)c(2\beta - d)\|Ax_n - Ap\|^2 &\leq (1-\alpha_n)r_n(2\beta - r_n)\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - v\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we obtain

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \qquad (4.1.23)$$

On the other hand, in the light of Lemma 3.2.6(3) T_{r_n} is firmly nonexpansie, so we have

$$
\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \\
&\leq \langle T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp), u_n - p\rangle \\
&= \langle x_n - r_nAx_n - (p - r_nAp), u_n - p\rangle \\
&= \frac{1}{2}\Big\{\|(x_n - r_nAx_n) - (p - r_nAp)\|^2 + \|u_n - p\|^2 \\
&\quad -\|(x_n - r_nAx_n) - (p - r_nAp) - (u_n - p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - p\|^2 - r_n(Ax_n - Ap)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\
&\quad +2r_n\langle x_n - u_n, Ax_n - Ap\rangle - r_n^2\|Ax_n - Ap\|^2\Big\},
\end{aligned}
$$

which implies that

$$
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|. \quad (4.1.24)
$$

Using (4.1.22) again and (4.1.24), we have

$$
\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big\{\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|\Big\} \\
&= \|x_n - p\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 + 2(1 - \alpha_n)r_n\|x_n - u_n\|\|Ax_n - Ap\|. \quad (4.1.25)
\end{aligned}
$$

It follows that and condition (i), we have

$$
\begin{aligned}
&(1 - e)\|x_n - u_n\|^2 \\
\leq\ & (1 - \alpha_n)\|x_n - u_n\|^2 \\
\leq\ & \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1 - \alpha_n)r_n\|x_n - u_n\|\|Ax_n - Ap\| \\
\leq\ & \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1 - \alpha_n)r_n\|x_n - u_n\|\|Ax_n - Ap\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and $\lim_{n\to\infty}\|Ax_n - Ap\| = 0$ implies that

$$
\lim_{n\to\infty}\|x_n - u_n\| = 0. \quad (4.1.26)
$$

From (4.1.20) and (4.1.26), we have

$$
\lim_{n\to\infty}\|x_n - y_n\| = 0. \quad (4.1.27)
$$

By (5.2.1), we get

$$z_n - x_n = (1-\alpha_n)(K_n v_n - x_n).$$

Since $\{\alpha_n\} \subset [0, e]$ for some e with $0 \le e < 1$, and $\|x_n - z_n\| \to \infty$ as $n \to \infty$, we also have

$$\lim_{n\to\infty} \|K_n v_n - x_n\| = 0. \tag{4.1.28}$$

From (4.1.21) and (4.1.26), we have

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{4.1.29}$$

Furthermore, by the triangular inequality, we also have

$$\|K_n v_n - v_n\| \ \le \ \|K_n v_n - x_n\| + \|x_n - v_n\|.$$

Applying (4.1.28) and (4.1.29), we obtain

$$\lim_{n\to\infty} \|K_n v_n - v_n\| = 0. \tag{4.1.30}$$

Let K be the mapping defined by (4.1.1). Since $\{v_n\}$ is bounded, applying Lemma 4.1.2 and (4.1.30), we have

$$\|Kv_n - v_n\| \le \|Kv_n - K_n v_n\| + \|K_n v_n - v_n\| \to 0 \text{ as } n \to \infty.$$

Step 5. Next, we show that $q \in \Theta := \cap_{i=1}^{N} F(T_i) \cap GMEP(F, \varphi, A) \cap I(B, M)$.

Since $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ which converges weakly to q. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup q$. Since $v_{n_i} \subset C$ and C is closed and convex, C is weakly closed and hence $q \in C$. From $\|Kv_n - v_n\| \to 0$, we obtain $Kv_{n_i} \rightharpoonup q$.

(a) First, we prove that $q \in I(B, M)$.

We observe that B is an $1/\xi$-Lipschitz monotone mapping and $D(B) = H$. From Lemma 3.2.2, we know that $M + B$ is maximal monotone. Let $(v, g) \in G(M + B)$, that is, $g - Bv \in M(v)$. Since $v_{n_i} = J_{M,\lambda_{n_i}}(y_{n_i} - \lambda_{n_i} By_{n_i})$, we have

$$y_{n_i} - \lambda_{n_i} By_{n_i} \in (I + \lambda_{n_i} M)(v_{n_i}),$$

that is,

$$\frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} By_{n_i}) \in M(v_{n_i}). \tag{4.1.31}$$

By virtue of the maximal monotonicity of $M+B$, we have

$$\left\langle v - v_{n_i}, g - Bv - \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} B y_{n_i}) \right\rangle \geq 0, \tag{4.1.32}$$

and so

$$\begin{aligned} \left\langle v - v_{n_i}, g \right\rangle &\geq \left\langle v - v_{n_i}, Bv + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} B y_{n_i}) \right\rangle \\ &= \left\langle v - v_{n_i}, Bv - Bz_n + Bv_{n_i} - By_{n_i} + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) \right\rangle \qquad (4.1.33) \\ &\geq 0 + \langle v - v_{n_i}, Bv_{n_i} - By_{n_i} \rangle + \left\langle v - v_{n_i}, \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) \right\rangle. \end{aligned}$$

It follows from $\|y_n - v_n\| \to 0$, $\|By_n - Bv_n\| \to 0$ and $v_{n_i} \rightharpoonup q$ that

$$\lim_{n\to\infty} \langle v - v_{n_i}, g \rangle = \langle v - q, g \rangle \geq 0. \tag{4.1.34}$$

It follows from the maximal monotonicity of $M+B$ that $\theta \in (M+B)(q)$, that is, $q \in I(B, M)$.

(b) Next, we show that $q \in GMEP(F, \varphi, A)$. Since $u_n = T_{r_n}(x_n - r_n A x_n) \in$ dom φ, we have

$$F(u_n, y) + \langle Ax_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

From (A2), we also have

$$\langle Ax_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq F(y, u_n), \quad \forall y \in C.$$

and hence

$$\langle Ax_{n_i}, y - u_{n_i} \rangle + \varphi(y) - \varphi(u_{n_i}) + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{4.1.35}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, we have $y_t \in C$. So, from (4.1.35), we have

$$\begin{aligned} \langle y_t - u_{n_i}, Ay_t \rangle &\geq \langle y_t - u_{n_i}, Ay_t \rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle y_t - u_{n_i}, Ax_{n_i} \rangle - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle + \\ &= \langle y_t - u_{n_i}, Ay_t - Au_{n_i} \rangle + \langle y_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle + F(y_t, u_{n_i}). \end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of A, we have $\langle y_t - u_{n_i}, Ay_t - Au_{n_i} \rangle \geq 0$. So, from (A4),(A5), and the weak lower semicontinuity of $\varphi, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup q$. we have at the limit

$$\langle y_t - q, Ay_t \rangle \geq -\varphi(y_t) + \varphi(q) + F(y_t, q) \tag{4.1.36}$$

as $i \to \infty$. From (A1),(A4) and (4.1.36), we also get

$$\begin{aligned}
0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq tF(y_t, y) + (1-t)F(y_t, q) + t\varphi(y) - (1-t)\varphi(q) - \varphi(y_t) \\
&= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[F(y_t, q) + \varphi(q) - \varphi(y_t)] \\
&\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t - q, Ay_t \rangle \\
&= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)t\langle y - q, Ay_t \rangle, \\
0 &\leq F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - q, Ay_t \rangle.
\end{aligned}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(q, y) + \varphi(y) - \varphi(q) + \langle y - q, Aq \rangle \geq 0.$$

This implies that $q \in GMEP(F, \varphi, A)$.

(c) Now, we prove that $q \in F(K) = \bigcap_{i=1}^{N} F(T_i)$.

Assume $q \notin F(K)$. Since $\|x_n - v_n\| \to 0$, we know that $v_{n_i} \rightharpoonup q$ $(i \to \infty)$ and $q \neq Kq$, it follows by the Opial's condition (Lemma 3.3.5) that

$$\begin{aligned}
\liminf_{i\to\infty} \|v_{n_i} - q\| &< \liminf_{i\to\infty} \|v_{n_i} - Kq\| \\
&\leq \liminf_{i\to\infty} \big(\|v_{n_i} - Kv_{n_i}\| + \|Kv_{n_i} - Kq\|\big) \\
&< \liminf_{i\to\infty} \|v_{n_i} - q\|,
\end{aligned}$$

which is a contradiction. Thus, we get $q \in F(K) = \bigcap_{i=1}^{N} F(T_i)$.

The conclusion is $q \in \Theta := \cap_{i=1}^{N} F(T_i) \cap GMEP(F, \varphi, A) \cap I(B, M)$.

Step 6. Finally, we show that $x_n \to z$ and $u_n \to z$, where $z = P_\Theta x_0$.

Since Θ is nonempty closed convex subset of H, there exists a unique $z' \in \Theta$ such that $z' = P_\Theta x_0$. Since $z' \in \Theta \subset C_n$ and $x_n = P_{C_n} x_0$, we have

$$\|x_0 - x_n\| \leq \|x_0 - P_{C_n} x_0\| \leq \|x_0 - z'\| \tag{4.1.37}$$

for all $n \in \mathbb{N}$. From (4.1.37). $\{x_n\}$ is bounded, so $\omega_w(x_n) \neq \emptyset$. By the weak lower semi-continuity of the norm, we have

$$\|x_0 - z\| \leq \text{liminf}_{i\to\infty} \|x_0 - x_{n_i}\| \leq \|x_0 - z'\|. \tag{4.1.38}$$

However, Since $z \in \omega_w(x_n) \subset \Theta$, we have

$$\|x_0 - z'\| \leq \|x_0 - P_{C_n}x_0\| \leq \|x_0 - z\|.$$

Using (4.1.37) and (4.1.38), we obtain $z' = z$. Thus $\omega_w(x_n) = \{z\}$ and $x_n \rightharpoonup z$. So, we have

$$\|x_0 - z'\| \leq \|x_0 - z\| \leq \text{liminf}_{n\to\infty}\|x_0 - x_n\| \leq \text{limsup}_{n\to\infty}\|x_0 - x_n\| \leq \|x_0 - z'\|.$$

Thus, we obtain that

$$\|x_0 - z\| = \lim_{n\to\infty} \|x_0 - x_n\| = \|x_0 - z'\|.$$

From $x_n \rightharpoonup z$, we obtain $(x_0 - x_n) \rightharpoonup (x_0 - z)$. Using the Kadec-Klee property (Lemma 3.2.5) of H, we obtain that

$$\|x_n - z\| = \|(x_n - x_0) - (z - x_0)\| \to 0 \quad as \quad n \to \infty$$

and hence $x_n \to z$ in norm. Finally, noticing $\|u_n - z\| = \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(z - r_nAz)\| \leq \|x_n - z\|$. We also conclude that $u_n \to z$ in norm. This completes the proof. □

Corollary 4.1.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings of C into itself, let A be a β-inverse-strongly monotone mapping of C into H, let B be a ξ-inverse-strongly monotone mapping of C into H and $M : H \to 2^H$ be a maximal monotone mapping. Assume that*

$$\Theta := \cap_{i=1}^N F(T_i) \cap GMEP(F, \varphi, A) \cap I(B, M) \neq \emptyset.$$

Let K_n be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\gamma_{n,1}, \gamma_{n,2}, ..., \gamma_{n,N}$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by (5.2.1) satisfy the following conditions in Theorem 5.2.1. Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta x_0$.

4.2 Applications

From Theorem 5.2.1, we can obtain the following results:

Theorem 4.2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let $\{T_i\}_{i=1}^N$ a finite family of quasi-nonexpansive and L_i-Lipschitz mappings of C into itself, let A be a β-inverse-strongly monotone mapping of C into H and let B be a ξ-inverse-strongly monotone mapping of C into H. Assume that*

$$\Theta := \cap_{i=1}^N F(T_i) \cap GMEP(F,\varphi,A) \cap VI(C,B) \neq \emptyset.$$

Let K_n be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\gamma_{n,1}, \gamma_{n,2}, ..., \gamma_{n,N}$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ and let

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \delta_n B u_n), \\ v_n = P_C(y_n - \lambda_n B y_n), \\ z_n = \alpha_n x_n + (1-\alpha_n)K_n v_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,e]$ for some e with $0 \leq e < 1$,

(ii) $\{\delta_n\}$, $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\xi$,

(iii) $\{r_n\} \subset [c,d]$ for some c, d with $0 < c < d < 2\beta$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta x_0$.

Proof. In Theorem 5.2.1 take $M = \partial\delta_C : H \to 2^H$, where $\delta_C : 0 \to [0,\infty]$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C; \\ +\infty, & x \notin C. \end{cases}$$

Then the variational inclusion problems (3.1.1) is equivalent to variational inequality problem (3.2.5), that is, to find $\hat{x} \in C$ such that

$$\langle B\hat{x}, y - \hat{x}\rangle \geq 0, \quad \forall y \in C.$$

Again, since $M = \delta_C$, then

$$J_{M,\lambda_n} = J_{M,\delta_n} = P_C,$$

and so we have

$$y_n = P_C(u_n - \delta_n B u_n) = J_{M,\delta_n}(P_C(u_n - \delta_n B u_n))$$

and

$$v_n = P_C(y_n - \lambda_n B y_n) = J_{M,\lambda_n}(P_C(y_n - \lambda_n B y_n)).$$

We can obtain the desired conclusion from Theorem 5.2.1 immediately. □

Next, we consider another class of important nonlinear mapping: strict pseudo-contractions.

Definition 4.2.2. A mapping $S : C \to C$ is called *strictly pseudo-contraction* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{4.2.1}$$

If $\kappa = 0$, then S is nonexpansive.

In this case, let $S : C \to C$ is a κ-strictly pseudo-contraction. Putting $B = I - S : C \to H$, then B is a $\frac{1-\kappa}{2}$-inverse-strongly monotone mapping. In fact, from (4.2.1) we have

$$\|(I - B)x - (I - B)y\|^2 \leq \|x - y\|^2 + \kappa\|Bx - By\|^2, \quad \forall x, y \in C.$$

Observe that

$$\|(I - B)x - (I - B)y\|^2 = \|x - y\|^2 - 2\langle x - y, Bx - By\rangle + \|Bx - By\|^2, \quad \forall x, y \in C.$$

Hence, we obtain

$$\langle x - y, Bx - By\rangle \geq \frac{1-\kappa}{2}\|Bx - By\|^2, \quad \forall x, y \in C. \tag{4.2.2}$$

This shows that B is $\frac{1-\kappa}{2}$-inverse-strongly monotone mapping.

Now, we get the following result.

Theorem 4.2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4)and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let $\{T_i\}_{i=1}^N$ a finite family of quasi-nonexpansive and L_i-Lipschitz mappings of C into itself, let S_A be an κ_β-strictly pseudo-contraction mapping of C into C and let S_B be an κ_ξ-strictly pseudo-contraction mapping of C into C. Assume that*

$$\Theta := \cap_{i=1}^N F(T_i) \cap GMEP(F,\varphi, I - S_A) \cap F(S_B) \neq \emptyset.$$

Let K_n be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\gamma_{n,1}, \gamma_{n,2}, ..., \gamma_{n,N}$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ and let

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle (I - S_A)x_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = (1-\delta_n)u_n + \delta_n S_B u_n, \\ v_n = (1-\lambda_n)y_n + \lambda_n S_B y_n, \\ z_n = \alpha_n x_n + (1-\alpha_n)K_n v_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, e]$ for some e with $0 \leq e < 1$,

(ii) $\{\delta_n\}$, $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 1 - \kappa_\xi$,

(iii) $\{r_n\} \subset [c, d]$ for some c, d with $0 < c < d < 1 - \kappa_\beta$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_\Theta x_0$.

Proof. Taking $A = I - S_A$ and $B = I - S_B$, respectively. Then we see that A is $\frac{1-\kappa_\beta}{2}$-inverse-strongly monotone and B is $\frac{1-\kappa_\xi}{2}$-inverse-strongly monotone, respectively. We have $F(S_B) = VI(C, B)$ and

$$y_n = P_C(u_n - \delta_n B u_n) = P_C((1-\delta_n)u_n + \delta_n S_B u_n) = (1-\delta_n)u_n + \delta_n S_B u_n \in C$$

and

$$v_n = P_C(y_n - \lambda_n B y_n) = P_C((1-\lambda_n)y_n + \lambda_n S_B y_n) = (1-\lambda_n)y_n + \lambda_n S_B y_n \in C.$$

By using Theorem 5.2.3, it is easy to obtain the desired conclusion. □