CHAPTER 3 PRELIMINARIES AND THEORY

3.1 Some Definitions

Throughout this paper, we assume that H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively.

Recall that the following definitions.

(1) A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

(2) A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \ \forall x \in C \text{ and } p \in F(T).$$

We denote $F(T) = \{x \in C : Tx = x\}$ be the set of fixed points of T.

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. The *variational inclusion problem* is to find $\hat{x} \in H$ such that

$$\theta \in B(\hat{x}) + M(\hat{x}), \tag{3.1.1}$$

where θ is the zero vecter in H. The set of solutions of problem (3.1.1) is denoted by $I(B, M)$.

Definition 3.1.1. A mapping $B : H \to H$ is said to be a *ξ-inverse-strongly monotone* if there exists a constant $\xi > 0$ with the property

$$\langle Bx - By, x - y \rangle \geq \xi \|Bx - By\|^2, \ \ \forall x, y \in C.$$

Remark 3.1.2. It is obvious that any ξ-inverse-strongly monotone mapping B is monotone and $\frac{1}{\xi}$-Lipschitz continuous. It is easy to see that for any λ constant is in $(0, 2\xi]$, then the mapping $I - \lambda B$ is nonexpansive, where I is the identity mapping on H.

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in M(x)$, and $g \in M(y)$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$.

Definition 3.1.3. Let $M : H \to 2^H$ be a set-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(\hat{x}) = (I + \lambda M)^{-1}(\hat{x}), \quad \hat{x} \in H \tag{3.1.2}$$

is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

Remark 3.1.4. (R1) The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive for all $\lambda > 0$, that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \quad \forall x, y \in H \quad \text{and} \quad \forall \lambda > 0.$$

(R2) The resolvent operator $J_{M,\lambda}$ is 1-inverse strongly monotone; see([59]), that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \left\langle J_{M,\lambda}(x) - J_{M,\lambda}(y), x - y \right\rangle, \quad \forall x, y \in H.$$

(R3) The solution of problem (3.1.1) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$; see also ([74]), that is,
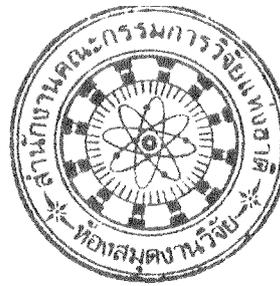
$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0.$$

(R4) If $0 < \lambda \leq 2\alpha$, then the mapping $J_{M,\lambda}(I - \lambda B) : H \to H$ is nonexpansive.

(R5) $I(B, M)$ is closed and convex.

3.2 Generalized mixed equilibrium problems

Let $A : C \to H$ be a nonlinear mapping, let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a real-valued function and F be a bifunction from $C \times C$ to $\mathbb{R}$. We consider the following *generalized mixed equilibrium problem.*

Finding $x \in C$ such that

$$F(x,y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{3.2.1}$$

The set of such $x \in C$ is denoted by $GMEP(F, \varphi, A)$, that is,

$$GMEP(F, \varphi, A) = \{x \in C : F(x,y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}.$$

It is easy to see that x is solution of problem (3.2.1) implies that $x \in \operatorname{dom} \varphi = \{x \in C : \varphi(x) < +\infty\}$.

1. In the case of $A \equiv 0$(:the zero mapping), then the generalized mixed equilibrium problem (3.2.1) is reduced to *the mixed equilibrium problem*. Finding $x \in C$ such that

$$F(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{3.2.2}$$

The set of solution of (3.2.2) is denoted by $MEP(F, \varphi)$.

2. In the case of $\varphi \equiv 0$, then the generalized mixed equilibrium problem (3.2.1) is reduced to *the generalized equilibrium problem*. Finding $x \in C$ such that

$$F(x,y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{3.2.3}$$

The set of solution of (3.2.3) is denoted by $GEP(F, A)$.

3. In the case of $A \equiv 0$(:the zero mapping) and $\varphi \equiv 0$, then the generalized mixed equilibrium problem (3.2.1) is reduced to *the equilibrium problem*. Finding $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{3.2.4}$$

The set of solution of (3.2.4) is denoted by $EP(F)$.

4. In the case of $F \equiv 0$, $\varphi \equiv 0$ and $A \equiv B$ then the generalized mixed equilibrium problem (3.2.1) is reduced to *the variational inequality problem*.

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{3.2.5}$$

The set of solution of (3.2.5) is denoted by $VI(C, B)$.

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

We recall some lemmas which will be needed in the rest of this paper.

Lemma 3.2.1. *For a given* $x \in H$ *and* $z \in C$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

It is well known that P_C *is a firmly nonexpansive mapping of* H *onto* C *and satisfies*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \tag{3.2.6}$$

Lemma 3.2.2. *[59] Let* $M : H \to 2^H$ *be a maximal monotone mapping and let* $B : H \to H$ *be a Lipshitz continuous mapping. Then the mapping* $S = M + B : H \to 2^H$ *is a maximal monotone mapping.*

Lemma 3.2.3. [56] *Let* C *be a closed convex subset of* H *and let* $\{x_n\}$ *be a bounded sequence in* H. *Assume that*

(1) the weak ω-*limit set* $\omega_w(x_n) \subset C$,

(2) for each $z \in C$, $\lim_{n \to \infty} \|x_n - z\|$ *exists.*

Then $\{x_n\}$ *is weakly convergent to a point in* C.

Lemma 3.2.4. [76] *Each Hilbert space* H *satisfies Opial's condition, that is, for any sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$, *the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

hold for each $y \in H$ *with* $y \neq x$.

Lemma 3.2.5. [81] *Each Hilbert space* H *satisfies the Kadec-Klee property, that is, for any sequence* $\{x_n\}$ *with* $x_n \rightharpoonup x$ *and* $\|x_n\| \to \|x\|$ *together imply* $\|x_n - x\| \to 0$.

For solving the generalized equilibrium problems, let us give the following assumptions for F, φ, and the set C:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $y \in C, x \mapsto F(x,y)$ is weakly upper semicontinuous;

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \cap dom\varphi$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x \rangle \geq 0; \qquad (3.2.7)$$

(B2) C is bounded set.

Lemma 3.2.6. [80] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $\varphi : H \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap dom \neq \emptyset$. For $r > 0$ and $x \in H$, defined a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}. \qquad (3.2.8)$$

Assume that either $(B1)$ or $(B2)$ holds. Then, the following conclusions hold:

(1) for each $x \in H$, $T_r(x) \neq \emptyset$;

(2) T_r is single-valued;

(3) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle;$$

(4) $F(T_r) = GEP(F, \varphi)$;

(5) $GEP(F, \varphi)$ is closed and convex.

Remark 3.2.7. Replacing x with $x - rAx \in H$ in (3.2.7), then there exists $z \in C \setminus D_x$, such that

$$F(z, y_x) + \langle Ax, y_x - z \rangle + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

3.2.1 Basic Concepts in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. We have the following are hold:

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \tag{3.2.9}$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \tag{3.2.10}$$

$$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x \rangle, \tag{3.2.11}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{3.2.12}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Lemma 3.2.8. [12] *Let* $(E, \langle ., . \rangle)$ *be an inner product space. Then for all* $x, y, z \in E$ *and* $\alpha, \beta, \gamma \in [0, 1]$ *with* $\alpha + \beta + \gamma = 1$, *we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 3.2.9. [13] *A Hilbert space* H *satisfies the* **Opial condition** *that is, for any sequence* $\{x_n\}$ *with* $x_n \rightharpoonup x$, *the inequality* $\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$, *holds for every* $y \in H$ *with* $y \neq x$.

Lemma 3.2.10. [14],[15] *A Hilbert space* H *satisfies the* **Kadec-Klee property** *that is, for any sequence* $\{x_n\}$ *with* $x_n \to x$ *and* $\|x_n\| \to \|x\|$ *together imply* $\|x_n - x\| \to 0$.

3.3 Nonlinear Mappings

3.3.1 Nonlinear Mappings in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. Let $T : C \to C$ a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Definition 3.3.1. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{3.3.1}$$

Definition 3.3.2. A mapping $f : C \to C$ is called a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|. \tag{3.3.2}$$

We use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f \mid f : C \to C \ a \ contraction\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C.

Theorem 3.3.3. [9] *(Banach Contraction Mapping Principle) Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point, i.e. there exists a unique $x^* \in X$ such that $Tx^* = x^*$.*

Definition 3.3.4. Let A be a strongly positive bounded linear operator on H if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \tag{3.3.3}$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(S)} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{3.3.4}$$

where A is a nonexpansive mapping and b is a given point in H.

Optimization problem (for short, OP) as the following

$$\min_{x \in F} \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \tag{3.3.5}$$

where $F = \cap_{n=1}^{\infty} C_n$, $C_1, C_2, \cdots$ are infinitely closed convex subsets of H such that $\cap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive linear bounded operator on H and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Lemma 3.3.5. [16] *Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to \mathbb{R} \cup \{\infty\}$ be a proper lower-semicontinuous differentiable convex function. If z is a solution to the minimization problem*

$$g(z) = \inf_{x \in C} g(x),$$

then

$$\left\langle g'(x), x - z \right\rangle \geq 0, \quad x \in C.$$

In particular, if z solves problem OP, then

$$\left\langle u + \left[\gamma f - (I + \mu A)\right]z, x - z \right\rangle \leq 0.$$

Lemma 3.3.6. [17] *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Definition 3.3.7. The *metric (nearest point) projection* P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H; \tag{3.3.6}$$

$$\langle x - P_C x, P_C x - z \rangle \geq 0, \quad \forall z \in C; \tag{3.3.7}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C; \tag{3.3.8}$$

and

$$\|(x - y) - (P_C x - P_C y)\|^2 \geq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{3.3.9}$$

Definition 3.3.8. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C. \tag{3.3.10}$$

Definition 3.3.9. A is called α-*inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C. \tag{3.3.11}$$

Lemma 3.3.10. *Let $A : H \to H$ be a α-inverse-strongly monotone mapping. If $\lambda \le 2\alpha$, for any $\lambda > 0$ and $\alpha > 0$ then $I - \lambda A$ is a nonexpansive mapping from H into itself.*

Proof. *Let $u, v \in H$ and $\lambda > 0$,*

$$
\begin{aligned}
\|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\
&= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle + \lambda^2\|Au - Av\|^2 \\
&\le \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2.
\end{aligned}
$$

$\square$

Definition 3.3.11. A mapping $A : C \to C$ is called *L-Lipschitz-continuous* if there exists a positive real number L such that

$$\|Au - Av\| \le L\|u - v\|, \quad \forall u, v \in C. \tag{3.3.12}$$

Remark 3.3.12. It is easy to see that if A is an α-inverse-strongly monotone mapping of C into H, then A is $\frac{1}{\alpha}$-Lipschitz continuous.

Definition 3.3.13. The mapping $S : C \to C$ is called a *κ-strict pseudo-contraction mapping* if there exists a constant $0 \le \kappa < 1$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{3.3.13}$$

Definition 3.3.14. Let $\eta : C \times C \to H$ and $B : C \to H$ be two mappings. B is said to be:

(1) *monotone* if

$$\langle Bx - By, \eta(x, y)\rangle \ge 0, \quad \forall x, y \in C;$$

(2) *σ-strongly monotone* if there exists a positive real number σ such that

$$\langle Bx - By, \eta(x, y)\rangle \ge \sigma\|x - y\|^2, \quad \forall x, y \in C;$$

(3) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \le L\|x - y\|, \quad \forall x, y \in C.$$

Definition 3.3.15. Let $K : C \to \mathbb{R}$ be a differentiable functional on a convex set C, which is called:

(1) η-convex [18] if

$$K(y) - K(x) \geq \Big\langle K'(x), \eta(y, x) \Big\rangle, \ \forall x, y \in C,$$

where $K'(x)$ is the Fréchet derivative of K at x;

(2) η-strongly convex [19] if there exists a constant $\xi > 0$ such that

$$K(y) - K(x) - \Big\langle K'(x), \eta(y, x) \Big\rangle \geq \frac{\xi}{2} \|x - y\|^2, \ \forall x, y \in C.$$

In particular, if $\eta(x, y) = x - y$ for all $x, y \in C$, then K is said to be *strongly convex*.

Lemma 3.3.16. [20]. *Let $V : C \to H$ be a k-strict pseudo-contraction, then*

(1) the fixed point set $F(V)$ of V is closed convex so that the projection $P_{F(V)}$ is well defined;

(2) define a mapping $T : C \to H$ by

$$Tx = tx + (1 - t)Vx, \ \forall x \in C. \tag{3.3.14}$$

If $t \in [k, 1)$, then T is a nonexpansive mapping such that $F(V) = F(T)$.

Definition 3.3.17. A family of mappings $\{V_i : C \to H\}_{i=1}^{\infty}$ is called *a family of uniformly k-strict pseudo-contractions*, if there exists a constant $k \in [0, 1)$ such that

$$\|V_i x - V_i y\|^2 \leq \|x - y\|^2 + k\|(I - V_i)x - (I - V_i)y\|^2, \quad \forall x, y \in C, \forall i \geq 1.$$

Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly k-strict pseudo-contractions and $\{T_i : C \to C\}_{i=1}^{\infty}$ be the sequence of nonexpansive mappings defined by (3.3.14), i.e.,

$$T_i x = tx + (1 - t)V_i x, \forall x \in C, \forall i \geq 1, t \in [k, 1). \tag{3.3.15}$$

For the infinite family of nonexpansive mapping of $T_1, T_2, ...$ (see [21]), we define

the mapping W_n of C into itself as follows:

$$
\begin{cases}
U_{n,n+1} = I, \\[4pt]
U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\[4pt]
U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\[4pt]
\quad\vdots \\[4pt]
U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\[4pt]
U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\[4pt]
\quad\vdots \\[4pt]
U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\[4pt]
W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I,
\end{cases}
\qquad (3.3.16)
$$

where $T_1, T_2, \dots$ be an infinite family of nonexpansive mappings of C into itself and $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 \le \lambda_n \le 1$ for every $n \in \mathbb{N}$.

Lemma 3.3.18. [21] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \dots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \dots$ be real numbers such that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then,*

(1) W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{n} F(T_i)$, $\forall n \ge 1$;

(2) for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n \to \infty} U_{n,k} x$ exists;

(3) a mapping $W : C \to C$ defined by

$$
Wx := \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x, \forall x \in C \qquad (3.3.17)
$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by $T_1, T_2, \dots$ and $\mu_1, \mu_2, \dots$.

Lemma 3.3.19. [22] *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\cap_{i=1}^{\infty} F(T_i) \ne \emptyset$, $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \le b < 1, \forall i \ge 1$. If D is any bounded subset of C, then*

$$
\lim_{n \to \infty} \sup_{x \in D} \|Wx - W_n x\| = 0.
$$

Definition 3.3.20. A family $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;

(ii) $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$;

(iii) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for all $x \in C, s \mapsto S(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common *fixed points of* $\mathcal{S} = \{S(s) : s \geq 0\}$, i.e., $F(\mathcal{S}) = \cap_{s \geq 0} F(S(s))$. It is known that $F(\mathcal{S})$ is closed and convex.

Lemma 3.3.21. [23] *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, then for any $h \geq 0$,*

$$\lim_{t \to \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)x\,ds - T(h)(\frac{1}{t} \int_0^t T(s)x\,ds) \right\| = 0.$$

Lemma 3.3.22. [24] *Let C be a nonempty bounded closed convex subset of H, $\{x_n\}$ be a sequence in C and $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. If the following conditions are satisfied:*

(i) $x_n \rightharpoonup z$;

(ii) $\limsup_{s \to \infty} \limsup_{n \to \infty} \|S(s)x_n - x_n\| = 0$, then $z \in \mathcal{S}$.

Lemma 3.3.23. [25, Lemma 3.1] *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\{S_n\}$ a sequence of mappings of C into itself. Suppose that*

$$\lim_{k,l \to \infty} \rho_l^k = 0 \tag{3.3.18}$$

where $\rho_l^k = \sup\{\|S_k z - S_l z\| : z \in C\} < \infty, \ \forall k, l \in \mathbb{N}$. Then for each $x \in C$, $\{S_n x\}$ converges strongly to some point of C. Moreover, let S be a mapping from C into itself defined by

$$Sx = \lim_{n \to \infty} S_n x, \ \forall x \in C.$$

Then $\limsup_{n \to \infty} \{\|Sz - S_n z\| : z \in C\} = 0$.

Lemma 3.3.24. [14] (Demi-closedness Principle) Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If S has a fixed point, the $I - S$ is demi-closed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I - S)x_n\}$ converges strongly to some y (for short, $(I - S)x_n \to y$), it follows that $(I - S)x = y$. Here I is the identity operator of H.