CHAPTER 2 BASIC CONCEPTS

In this chapter, we give some definitions, notations, lemmas and some useful results that will be used in the later chapters. Throughout this thesis, we let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}$ be the set of all natural numbers, H be a Hilbert space and E be a Banach space.

2.1 Basic Concepts

Definition 2.1.1. Let X be a nonempty set, and assume that each pair of elements x and y in X can be combined by a process called *addition* to yield an element z in X denoted by $x + y$. Assume also that this operation of addition satisfies the following condition (1)–(4):

(1) $(x + y) + z = x + (y + z)$;

(2) $x + y = y + x$;

(3) there exists a unique element in X, denoted by 0 and called the zero element, or the origin, such that $x + 0 = x$ for all $x \in X$;

(4) each $x \in X$ there corresponds a unique element in X, denoted by $-x$ and called the negative of x, such that $x + (-x) = 0$.

We also assume that each scalar $\alpha \in \mathbb{R}$ and each element x in X can be combined by a process called *scalar multiplication* to yield an element y in X denoted by $y = \alpha x$ satisfying (5)–(8):

(5) $\alpha(\beta x) = (\alpha\beta)x$;

(6) $1 \cdot x = x$;

(7) $(\alpha + \beta)x = \alpha x + \beta x$;

(8) $\alpha(x + y) = \alpha x + \alpha y$.

The system $(X, \cdot, +)$ is called a *linear space* over $\mathbb{R}$ if it satisfies the conditions (1)–(8). A linear space is often called a *vector space*, and its elements are spoken as vectors.

Definition 2.1.2. Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$, satisfying the following conditions for all x, y and z in X:

(A1) $d(x, y) = 0 \Longleftrightarrow x = y$;

(A2) $d(x, y) = d(y, x)$;

(A2) $d(x, y) \leq d(x, z) + d(z, y)$. The conditions (A1)-(A3) are usually called the *metric axioms.*

The function d assigns to each pair (x, y) of element of X a nonnegative real number $d(x, y)$, which does not on the order of the elements; $d(x, y)$ is called the *distance* between x and y. The set X together with a metric, denoted by (X, d), is called a *metric space.*

Definition 2.1.3. Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{R}$ is said to be *a norm on* X if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in X$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$;

(4) $\|\alpha x\| = |\alpha| \|x\|, \forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x, y) = \|x - y\|, \quad (x, y \in X).$$

A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space.*

Definition 2.1.4. A normed space $(X, \|\cdot\|)$ is called strictly convex if for all $x, y \in X$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1$, $\forall \lambda \in (0, 1)$.

Definition 2.1.5. Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim\limits_{n \to \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim\limits_{n \to \infty} x_n = x$ or $x_n \to x$ to mean that x is the limit of the sequence $\{x_n\}$.

Definition 2.1.6. A sequence $\{x_n\}$ in a normed spaces is said to *converge weakly* to some vector x if $\lim_{n \to \infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converges weakly to x.

Definition 2.1.7. Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$.

Theorem 2.1.8. [9] *Let $\{x_n\}$ be a sequence of a normed space $(X, \|\cdot\|)$, $x \in X$ and let $x_n \to x$ if and only if, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging to x.*

Definition 2.1.9. A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.1.10. A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$.

Lemma 2.1.11. [10] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Definition 2.1.12. Let F and X be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \to X$ is called *a linear operator* if $T(x+y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \to \mathbb{K}$ is called *a linear functional on* F if T is a linear operator.

Definition 2.1.13. Let F and X be normed spaces over the field $\mathbb{K}$ and $T : X \to F$ a linear operator. T is said to be *bounded* on X if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Definition 2.1.14. Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

Definition 2.1.15. A subset C of a normed linear space X is said to be *convex subset in* X if $\lambda x + (1-\lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0,1]$.

Definition 2.1.16. The real-value function of two variables $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called *inner product* on a real vector space X if for any $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$ the following conditions are satisfied:

(1) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;

(2) $\langle x, y\rangle = \langle y, x\rangle$;

(3) $\langle x, x\rangle \geq 0$ for each $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

A *real inner product space* is a real vector space equipped with an inner product.

Definition 2.1.17. A *Hilbert spaces* is an inner product space which is complete under the norm induced by its inner product.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Lemma 2.1.18. [9]*(The Schwarz inequality) If x and y are any two vector in an inner product space X, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Remark 2.1.19. In a Hilbert space H, weak convergence is defined by $\lim_{n\to\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

Remark 2.1.20. If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Definition 2.1.21. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set $\{x \in C : f(x) \leq a\}$ is closed.

Lemma 2.1.22. [9] *Let X be an inner product space and $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x$. Then following inequality holds:*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Lemma 2.1.23. [11] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \ \ n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.