THE UPPER BOUNDS OF THE RUIN PROBABILITY FOR AN INSURANCE DISCRETE-TIME RISK MODEL WITH PROPORTIONAL REINSURANCE AND INVESTMENT

Apichart Luesamai

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ABSTRACT

Title of Dissertation	The Upper Bounds of the Ruin Probability for an Insurance Discrete-Time Risk Model with Proportional Reinsurance and Investment
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In this study, the two upper bounds of the ruin probability for discrete time risk model derived by adding two controlled factors to the classical discrete time risk model: proportional reinsurance and investment are proposed. These upper bounds are derived using an inductive method and rely on a recursive form of the finite time and/or an integral equation of ultimate (infinite time) of ruin probability which is also derived in this study. Both of the upper bounds are formulated by the assumption that the retention level of reinsurance and the amount of stock investment during each time period are controlled as constant values. The first upper bound can be used with the finite time ruin probability and the ultimate ruin probability under the condition that the value of the adjustment coefficient can be found. The second upper bound is formulated by a using new worse than used distribution. This upper bound can only be used with the finite time ruin probability, and its value can be found even though the value of the adjustment coefficient does not exist. However, this upper bound has limitations on the total claims amount which the total claims amount in each time period must come from the summation of independent and identically distributed (i.i.d.) claim amounts, and the number of claims is also i.i.d. in each time period.

Two numerical examples are used to consider the characteristics of the derived upper bounds. In the first example, the total claims amount is assumed to follow an exponential distribution from which the value of the adjustment coefficient can be found to show the first derived upper bound. In the other example, the claim amounts

are set as a Pareto distribution, from which the adjustment coefficient cannot be found and is used to show some of the characteristics of the second upper bound. Moreover, real-life motor insurance claims data that fits a log-normal distribution is used to show the application of the derived upper bounds. Under the different agreements of the three aforementioned situations, it was found that the values of the two upper bounds derived in this study responded to the two additional controlled factors in the proposed risk model in the same direction.

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CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Insurance companies have been established on the concept of risk avoidance from the risk-bearer. Under an insurance contract known as the *insurance policy* between the company known as *insurer* and policyholder also known as *insured*, the insurers accept to provide a part or all of the insured's losses, referred to as a *claim*, upon the occurrence of specified event covered by the policy. In return for this protection the insured agrees to pay the prescribed sum of money known as a *premium* (Promislow, 2011: 3-4).

The uncertainty of both the claim amount and occurrence add to a complicated situation for the insurer when evaluating the *surplus*, which is the amount of money that remains after all liabilities have been met. To ensure that the company does not go bankrupt, the insurer needs to have efficient tools for analyzing the surplus, and the *ruin probability* is one of several models an insurer can use to analyze the surplus (Challa, 2012: 1). This model informs us of the chances of a business to become *ruined*, a business being in state of ruin if the surplus is negative (Tse, 2009: 144). The ruin probability is usually considered from a *surplus model*, also known as a *reserve*, or *risk model*, and basically depends on the initial capital, inflow, and outflow of the business. A surplus model can be considered over time, thus insurance companies will evaluate a suitable surplus as frequently as is required, this approach is known as a *continuous time risk model*. Some insurance companies may take equal intervals of time, such as for each month, each quarter, or each year considering the surplus, giving rise to *discrete time risk models* (Bowers, Gerber, Hickman, Jones and Nesbitt, 1997; Kaas, Goovaerts, Dhaene and Denuit, 2008; Tse, 2009).

Research concerning the ruin probability and the surplus of insurance companies over continuous time has been undertaken for nearly a century. Particularly over the last decade, much advancement in continuous time risk models has been established due to the fact that insurance companies can purchase reinsurance, invest in the stock market, obtain dividends, and partake in other transactions. However, studies that have concentrated on discrete time risk models are relatively few, and one of the crucial drawbacks is that differentiation cannot be used to solve this problem. Undoubtedly, continuous time models are powerful when describing the practical world, but discrete time risk models are suitable in some situations. For example, when considering risk models with reinsurance, companies can only change the level or type of reinsurance once each year (Jasiulewicz and Kordecki, 2015; Lin, Dongjin and Yanru, 2015).

In this study, the ruin probability under a discrete time risk model controlled by reinsurance and investment is presented. However, obtaining an explicit solution for the ruin probability is actually a difficult task. One alternative method commonly used is deriving the bounds for the ruin probability (Diasparra and Romera, 2009; Lin et al., 2015), thus the focus in this study is on the upper bound for the ruin probability.

1.2 The Objectives of the Study

The objectives of this study are as follows:

1) To propose a recursive formula for the ruin probability of a discrete time risk model controlled by reinsurance and investment.

2) To propose the upper bounds of the ruin probability.

3) To show the application of the proposed method to real-life data.

1.3 The Scope of the Study

This study of ruin probability focuses on a discrete time risk model controlled by reinsurance and investment. The restrictions are that only proportional reinsurance is considered, while two assets are allowed for investment: bonds with a finite countable number of possible interest rate values following a time-homogeneous Markov chain and stocks with a return on investment driven by discrete time.

1.4 The Usefulness of the Study

This study may benefit insurance companies by giving them a more accurate picture of the risks of losses and increasing the chance of making a profit due to recognizing the financial risks from the ruin probability by assimilating the reinsurance level and the investment amount.

Furthermore, it may spark new concepts, theories, and a deeper interest in this field.

CHAPTER 2

LITERATURE REVIEW

In this chapter, an introduction to both classical continuous and discrete time risk models, which constitute the foundation for developing other models including the proposed model presented in this study is offered.

Besides, this chapter also contains ruin probability methods and their upper bounds under the aforementioned two risk models. Furthermore, a review of the upper bounds of ruin probability under various risk models (both continuous and discrete time) related to the proposed risk model in this study is presented.

2.1 The Classical Risk Model and the Ruin Probability

2.1.1 The Classical Risk Model or the Cramér-Lundberg Model

The foundations of modern risk theory were laid out by Fillip Lundberg in 1903. One of Lundberg's major contributions was the introduction of a simple model capable of describing the dynamics of a homogeneous insurance portfolio. In 1930, Lundberg's model was extensively developed by the famous probabilistic actuary Harald Cramér. Thus, the resulting model is called the Cramér-Lundberg model or the classical risk model. This model is given by (Kaas et al., 2008: 87-91)

$$
\begin{aligned}
U_t &= u + X_t - S_t, \\
&= u + ct - \sum_{i=1}^{N(t)} Y_i
\end{aligned}
\tag{2.1}
$$

where

1) U_t is the insurer's surplus at time $t > 0$,

2) $u = U_0$ is the initial surplus or surplus at time 0,

3) X_t is the premium income for the time interval (0,*t*]. Because the premium income from the insurance contract is spread over a period of time, there are two assumptions on premium income. First, it is continuous overtime. Second, in any time interval the premium income is proportional to the interval length. The result of this assumption is $X_t = ct$, where constant $c > 0$ is called the premium rate, and

4) $S_t = \sum_{i=1}^{N(t)} Y_i$ is the aggregated claims at time t. The claim amount Y_i arrives at time T_i. The sequence $\{Y_i,\ i = 1,2,3,...\}$ constitutes an independent and identically distributed (i.i.d.) sequence of non-negative random variables with common distribution function $P(y) = \Pr(Y_i \le y); y \ge 0$. Moreover, sequence $\{Y_i,\ i = 1,2,3,...\}$ is also mutually independent of sequence $\{T_i,\ i = 1,2,3,...\}$. The number of claims $N(t)$ is defined as $N(t) = max\{i \mid i \ge 1; T_i \le t, \text{ and } t > 0\}$. At $t = 0$, $N(t) = 0$ and $N(t)$ is assumed to be distributed as Poisson with parameter λt. Furthermore, if $N(t) = 0$, $S_t = 0$.

By considering model (2.1) at the claim arrival-time, i.e. $t = T_i,\ i = 1,2,3,\ldots$, we can write

$$U_{T_i} = u + cT_i - S_{T_i}. \tag{2.2}$$

Since $N(T_m) = max\{i \mid i \ge 1, T_i \le T_m\} = m$, for all $i = 1,2,3,...,m$. Let $Z_i = T_i - T_{i-1}$, $i = 1,2,3,\ldots$, then model (2.2) can be written in the form

$$\begin{aligned} U_{T_m} &= u + cT_m - S_{T_m} \\ &= u + cT_m + cT_{m-1} - cT_{m-1} - S_{T_{m-1}} - Y_m \\ &= \left\{u + cT_{m-1} - S_{T_{m-1}}\right\} + c\left\{T_m - T_{m-1}\right\} - Y_m \\ &= U_{T_{m-1}} + cZ_m - Y_m. \end{aligned} \tag{2.3}$$

The recursive form of the risk model in Equation (2.3) is useful for defining the discrete time risk model, which is explained later on.

2.1.2 The Ruin Probability and the Upper Bound of the Ruin Probability

The ruin probability is the probability that the insurer's surplus falls below zero at some time in the future (Dickson, 2005: 129).

Let $T = \min\{t \mid t > 0, U_t < 0\}$ denote the time of ruin (the first time that surplus becomes negative).

The ruin probability at infinite time, also known as the ultimate ruin probability, is defined as (Bowers et.al, 1977: 400)

$$\psi(u) = \Pr(T < \infty \mid U_0 = u), \tag{2.4}$$

while the finite time ruin probability can be written as

$$\psi_t(u) = \Pr(T < t \mid U_0 = u), \tag{2.5}$$

which is the probability that the insurer's surplus falls below zero for finite time interval $(0, t]$.

In general, obtaining an exact expression for ultimate ruin probability, $\psi(u)$, and finite time ruin probability $\psi_t(u)$ is quite challenging. The analysis commonly used in ruin theory is to derive inequalities for the ruin probability. In the classical risk model, the Lundberg inequality (upper bound) is the well-known upper bound for ruin probabilities (Cai and Dickson, 2004: 4).

Lundberg's upper bound is exponential bound for the ultimate ruin probability of the classical risk model as long as the moment generating function (m.g.f.) of the claim amount distribution exists. An important quantity for obtaining Lundberg's upper bound is the "adjustment coefficient" defined as follows.

Definition 1. The Adjustment Coefficient, represented by R_0, for the classical risk model in Equation (2.1) is the smallest positive value of real variable R that satisfies equation (Bowers et al., 1997: 410)

$$M_{S_t - ct}(R) = E[e^{R(S_t - ct)}] = e^{-Rct} M_{S_t}(R) = 1. \tag{2.6}$$

Remark 1.

1) From $M_{S_t}(R) = M_{N(t)}\left[\log M_{Y_i}(R)\right]$ (Bowers et. al, 1997: 369) and $N(t) \sim Poi(\lambda t)$, the m.g.f. $M_{N(t)}(R) = \exp\left[\lambda t\left(e^R - 1\right)\right]$ and the m.g.f. of an aggregate claim $M_{S_t}(R) = \exp\left[\lambda t\left(e^{\log M_{Y_i}(R)} - 1\right)\right]$. Thus, by replacing $M_{S_t}(R)$ in Equation (2.6), we obtain the following equations:

$$e^{-Rct} \exp\left[\lambda t\left(e^{\log M_{Y_i}(R)} - 1\right)\right] = 1,$$

$$\lambda t\left(e^{\log M_{Y_i}(R)} - 1\right) = Rct,$$

$$\lambda M_{Y_i}(R) - \lambda - Rc = 0. \tag{2.7}$$

Therefore, the adjustment coefficient R_0 in Definition 1 can be obtained from Equation (2.7).

2) From $E(S_t) = E\left[N(t)\right]E(Y_i) = \lambda t E(Y_i)$ and by assuming that the constant premium rate c is over the expected claim per unit time, $\lambda E(Y_i)$, then $c = (1+\theta)\lambda E(Y_i)$, where $0 \le \theta \le 1$, is called the safety loading factor (Bowers et. al, 1997: 410). Thus, by replacing c, Equation (2.7) can be rewritten as

$$M_{Y_i}(R) = 1 + (1+\theta)\mu R, \tag{2.8}$$

where $\mu = E(Y_i)$.

To show that the adjustment coefficient exists, Dickson (2005) showed by letting

$L(R) = \lambda M_{Y_i}(R) - \lambda - Rc$, which is on the left-hand side of Equation (2.7), and $L(0) = 0$, then we can take derivative $L(R)$ with respect to R as

$L'(R) = \lambda M'_{Y_i}(R) - c = \lambda E\left[Y_i e^{RY_i}\right] - c$ and

$L'(0) = \lambda E[Y_i] - c < 0$ (from $c = (1+\theta)\lambda\,\mathrm{E}(Y_i)$ in Remark 1.(2)),

$L''(R) = \lambda E\left[Y_i^2 e^{RY_i}\right]$,

$L''(0) = \lambda E\left[Y_i^2\right] > 0$.

Furthermore, suppose γ is a positive real value such that the largest open interval $(-\infty, \gamma)$ where the m.g.f. of $P(y)$ exists, and $M_{Y_i}(R) = E\left(e^{RY_i}\right)$ tends toward $+\infty$ as R tends toward γ. Clearly if $\gamma < \infty$, then $\lim_{R \to \gamma^-} L(R) = \infty$. This conclusion includes the case of $\gamma = \infty$.

Since all claim amount are positive, there exists a positive number ε and a probability q such that

$\Pr(Y_i > \varepsilon) = q > 0$,

so $\quad M_{Y_i}(R) = \int_0^\infty e^{Ry} dP(y) \geq \int_\varepsilon^\infty e^{Ry} dP(y) \geq e^{r\varepsilon} q$.

Hence,

$$\lim_{R\to\infty} L(R) = \lim_{R\to\infty} \left\{\lambda M_{Y_i}(R) - \lambda - Rc\right\} \geq \lim_{R\to\infty} \left\{\lambda e^{Ry} q - \lambda - Rc\right\} = \infty.$$

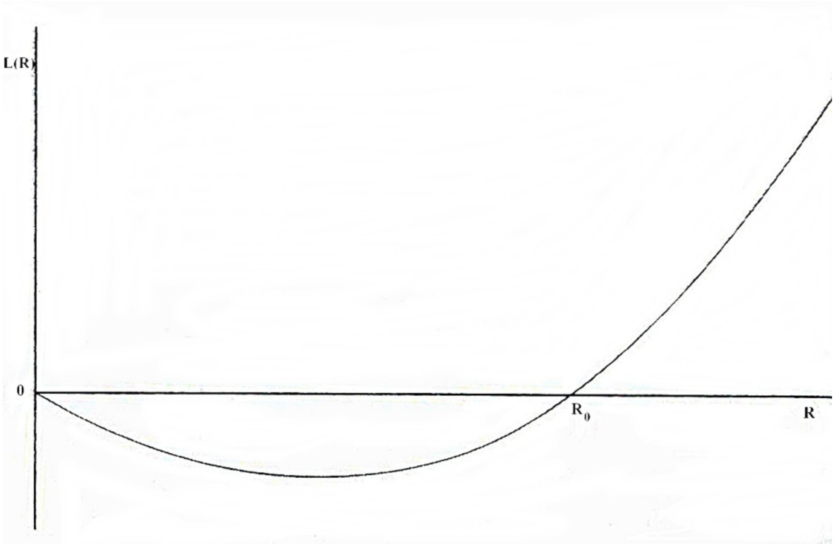

Figure 2.1 The Function $L(R)$ Giving the Definition of the Adjustment Coefficient for the Classical Risk Model.

Source: Dickson, 2005: 130.

As illustrated in Figure 2.1, the graph of $L(R)$ (see Figure 2.1) is upwardly concave and tends toward $+\infty$ as R tends toward a positive real value γ. Thus, adjustment coefficient R_0 is a unique positive root of Equation (2.6).

By relying on the adjustment coefficient, Lundberg's upper bound can be ascertained as follows.

Theorem 1. (Lundberg's Upper Bound). The ultimate ruin probability $\psi(u)$ of the classical risk model in Equation (2.1) given initial surplus u satisfying the following inequality can be written as

$$\psi(u) \leq \exp(-R_0 u), \tag{2.9}$$

where R_0 is the adjustment coefficient satisfying Equation (2.6).

The proof of Theorem 1 can be found in Appendix B.

2.2 A Review of the Upper Bound for the Ruin Probability in General Claim Size Distributions

As shown in Theorem 1, Lundberg's upper bound can be used for the probability of ultimate ruin in the classical risk model when the m.g.f. of the claim amount random variable exists. However, in many practical distributions, the m.g.f. does not exist, and so the Lundberg inequality is not available in these cases (Cai and Wu, 1996; Cai and Garrido, 1999). Many researchers have derived upper bounds of ruin probability that can be applied to more general claim amount distributions (e.g. Dickson, 1994; Willmot, 1994; Kalashnikov, 1999).

From the classical risk model in Equation (2.1), Dickson (1994) showed that if claim amount Y_i is i.i.d. with mean μ and common distribution function $P(y)$, and $P(0)=0$, then an alternative definition of R_0 as in Equation (2.8) is a unique positive number satisfying

$$\int_0^\infty \exp\{Ry\} b(y)\, dy = 1+\theta , \tag{2.10}$$

where $b(y)=\frac{1}{\mu}\{1-P(y)\}$ and θ is the safety loading factor. After that, Dickson truncated the condition in Equation (2.10) to give

$$\int_0^t \exp\{K_t y\} b(y)\, dy = 1+\theta . \tag{2.11}$$

Thus, the unique positive solution of K_t in Equation (2.11) is used as a condition for deriving the upper bound of the ultimate ruin probability for general claim amount distributions.

Willmot (1994) derived an upper bound for the tail of total claims amount distribution by using a class of distribution called new worse than used (NWU). The distribution function $B(x)$ of the non-negative random variables is NWU if

$\overline{B}(x)\overline{B}(y) \le \overline{B}(x+y)$ for $x \ge 0, y \ge 0$ and where $\overline{B}(x) = 1 - B(x)$. The author defined the number of claims as N with $p_m = \Pr(N = m)$ and $a_m = \sum_{k=m+1}^{\infty} p_k$, where $m = 0,1,2,\ldots$. In addition, N is independent of the i.i.d claim amount Y_i, $\mathrm{i} = 1,2,3,\ldots$ with common distribution function $P(y) = \Pr(Y_i \le y)$. The total claim is $S = \sum_{i=1}^{N} Y_i$ with $G(s) = \Pr(S \le s)$ and $\overline{G}(s) = 1 - G(s)$. Further assumptions for deriving the upper bound are that there exists positive number $\phi < 1$ such that $a_{m+1} \le \phi a_m$, $m = 0,1,2,\ldots$ and there exists non-negative function $B(x); x \ge 0$ such that $\overline{B}(x)\overline{B}(y) \le \overline{B}(x+y)$, for $x \ge 0, y \ge 0$, satisfies $\int_0^{\infty} \left\{\overline{B}(y)\right\}^{-1} dP(y) \le \phi^{-1}$ and $P(x) \le c(x)\overline{B}(x)\int_x^{\infty} \left\{\overline{B}(y)\right\}^{-1} dP(y)$; $x \ge 0$ where $c(x)$ is a non-decreasing function for $x \ge 0$. Thus, the upper bound for the tail of total claim $\overline{G}(s)$ is $\overline{G}(s) \le \phi^{-1}(1 - p_0)c(x)\overline{B}(x)$; $x \ge 0$. In this study, the other upper bounds for the tail of the total claim are derived from a variation of a subclass of the NWU distribution. Besides, previous results have been applied to find the upper bound of the ultimate ruin probability under the classical risk model in Equation (2.1).

Kalashnikov (1996) rewrote the classical risk model in Equation (2.1) in geometric sum form, as the sum of i.i.d. random variables, with the number of summands being a random variable also having a geometric distribution. Subsequently, he derived the upper bound of a geometric sum risk model, the results of which were used as a two-sided bound of the ruin probability.

Cai and Wu (1997) improved the Lundberg bound in Equation (2.9) by using an NWU distribution based on the renewal theory and the result from Willmot (1994). Furthermore, the lower bound of ruin probability was improved based on a class of distribution called the new better than used (NBU). The distribution function $B(x)$ of a non-negative random variable is NBU if $\overline{B}(x)\overline{B}(y) \geq \overline{B}(x+y)$, for $x \geq 0, y \geq 0$, where $\overline{B}(x) = 1 - B(x)$.

2.3 The Classical Discrete Time Risk Model and the Ruin Probability

2.3.1 The Classical Discrete Time Risk Model

The classical discrete time risk/surplus model is defined by considering the values of classical risk model U_t at only integer values of time t. Traditionally, this sequence of random variables is denoted by $\{U_n |$ the time $n = 1, 2, 3, ...\}$. Therefore the classical discrete time risk model is given by (Bowers et al., 1997: 401)

$$U_n = u + cn - \sum_{i=1}^{n} Y_i \, , \tag{2.12}$$

where

1) U_n is the insurer's surplus at the end of time period $n = 1, 2, 3, \ldots$,

2) $u = U_0$ is the initial surplus or surplus at time 0,

3) c is the constant premium income per unit time period, and

4) Y_i is total claim amount in the ith time period where $\{Y_i,\ i = 1, 2, ..., n\}$ is an i.i.d. sequence of non-negative random variables.

Model (2.12) can be written in the form

$$\begin{aligned} U_n &= u + cn - \sum_{i=1}^{n} Y_i \\ &= u + c(n-1) + c - \sum_{i=1}^{n-1} Y_i - Y_n \\ &= u + c(n-1) - \sum_{i=1}^{n-1} Y_i + c - Y_n \\ &= U_{n-1} + c - Y_n \end{aligned} \tag{2.13}$$

This form of classical discrete time risk model is used to develop the new risk model of this study, which is presented in the next chapter.

2.3.2 The Ruin Probability and the Upper Bound of the Ruin Probability

The ruin probability for a classical discrete time risk model can be defined as follows.

Let T (time of ruin) be the first time that the surplus becomes negative, then it is defined as (Bowers et al., 1997: 401)

$$T = \min\left\{k \mid k = 1, 2, 3...., U_k < 0\right\}.$$

The ultimate ruin probability can be written as

$$\begin{aligned}\psi(u) &= \Pr\left(T < \infty \mid U_0 = u\right),\\ &= \Pr\left(U_k < 0 \text{ for some } k = 1, 2, 3, ... \mid U_0 = u\right),\\ &= \Pr\left\{\bigcup_{k=1}^{\infty}\left(U_k < 0\right) \mid U_0 = u\right\}.\end{aligned} \tag{2.14}$$

Mean while, the probability that ruin will occur before a time n is defined as

$$\begin{aligned}\psi_n(u) &= \Pr\left(T < n \mid U_0 = u\right),\\ &= \Pr\left(U_k < 0 \text{ for some } 1 \le k \le n \mid U_0 = u\right),\\ &= \Pr\left\{\bigcup_{k=1}^{n}\left(U_k < 0\right) \mid U_0 = u\right\}.\end{aligned} \tag{2.15}$$

From (2.14) and (2.15), the ruin probabilities are the cumulative probability, then

$$\psi_1(u) \le \psi_2(u) \le \psi_3(u) \le ... \tag{2.16}$$

and

$$\lim_{n \to \infty} \psi_n(u) = \psi(u). \tag{2.17}$$

Obtaining the exact expressions for the ultimate ruin probability $\psi(u)$ and finite time ruin probability $\psi_n(u)$ under a discrete time risk model is challenging in the same way as for a continuous time risk model. Therefore, the upper bound of the ruin probability is still an option when studying ruin probability. Lundberg's upper bound is likewise a well-known upper bound with which to carry out the study (Cai and Dickson, 2002: 4), although obtaining it depends upon an important quantity: the adjustment coefficient. For a discrete time risk model, this quantity is defined as follows.

Definition 2. The Adjustment Coefficient, denoted by R_0, for the classical discrete time risk model in Equation (2.12) is the smallest positive value of real variable R satisfying the equation

$$M_{Y_i-c}(R) = E\left[e^{R(Y_i-c)}\right] = e^{-Rc}M_{Y_i}(R) = 1 \qquad (2.18)$$

(Bowers et al., 1997: 401).

To show that the adjustment coefficient exists, Tse (2009) let $L(R) = E\left[e^{R(Y_i-c)}\right]$, which is a term in Equation (2.18), and consequently, $L(0) = 1$. Subsequently, take derivative $L(R)$ with respect to R to obtain

$L'(R) = E\left[(Y-c)e^{R(Y_i-c)}\right]$,

$L'(0) = E(Y_i - c) < 0$ (under the assumption that $c > E(Y_i)$),

$L''(R) = E\left[(Y_i-c)^2 e^{R(Y_i-c)}\right] > 0$, and

$L''(0) = E\left[(Y_i-c)^2\right] > 0$.

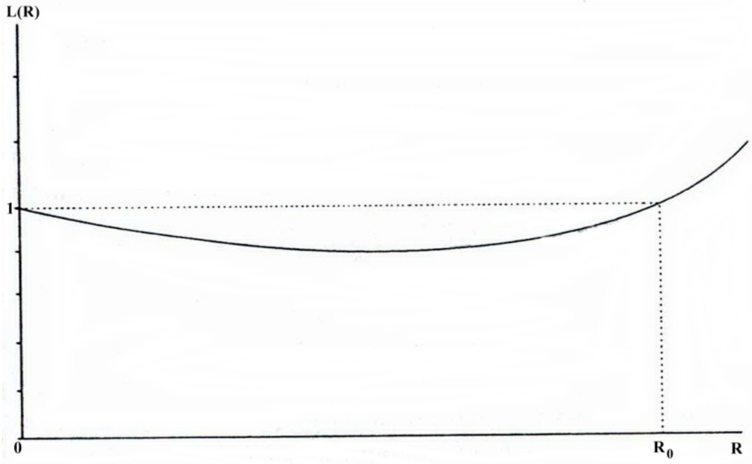

Figure 2.2 Function $L(R)$ Giving the Definition of the Adjustment Coefficient for the Classical Discrete Time Risk Model.

Source: Dickson, 2005: 121.

Thus, the graph of $L(R)$ is upwardly concave (see Figure 2.2). Furthermore, suppose that $\varepsilon > c$ such that $\Pr(Y_i \geq \varepsilon) > 0$, then $L(R) \geq e^{R(\varepsilon - c)} \Pr(Y_i \geq \varepsilon)$, i.e. $\lim_{R \to \infty} L(R) = \infty$. Hence, unique positive value R_0 satisfies Equation (2.18)

The Lundberg upper bound for a discrete time risk model can be obtained by using the adjustment coefficient as follows.

Theorem 2. **(Lundberg's Upper Bound).** The ultimate ruin probability $\psi(u)$ of the classical discrete time risk model in Equation (2.22) given initial surplus u satisfies the following inequality

$$\psi(u) \leq \exp(-R_0 u), \tag{2.19}$$

where R_0 is the adjustment coefficient satisfying Equation (2.18).

The proof of Theorem 2 can be found in Appendix B.

2.4 Reinsurance

Reinsurance is the mechanism that insurance companies use to transfer part or all of the risk to a second insurance carrier, the reinsurer. Basically, if a particular risk is too high for an insurance company or if the loss potential of the entire portfolio is too heavy, the insurance company can purchase reinsurance treaties. The form of the reinsurance treaties depends upon the manner by which the risk is shared between the insurer and the reinsurer. Reinsurance treaties are classified into proportional and non-proportional types (Booth, et al., 2005).

Proportional reinsurance is a common form of reinsurance for claims of moderate size, and requires the reinsurer to cover the fraction of each claim equal to the fraction of total premiums that the reinsurer receives from the insurer. The principal types of proportional reinsurance cover are quota share and surplus. Under quota share reinsurance, both claims and premiums are in the same proportions at fixed percentages. In a surplus treaty, the reinsurer agrees to accept a particular risk of the sum insured in excess of the direct retention limit set by the ceding company (Goovaerts and Vyncke, 2006)

Non-proportional reinsurance differs from proportional reinsurance in that the insurer and reinsurer do not share the amount of insurance coverage, premium and claim in the same proportion. Under a non-proportional agreement, the reinsurer only pays the insurer when the claim has exceeded a predetermined limit sometimes referred to as the excess point or retention (Outreville, 1997). The traditional forms of non-proportional reinsurance cover are known as excess of loss and stop loss. Excess of loss reinsurance covers claims which exceed a predetermined limit, while, under stop loss reinsurance, the reinsurer pays claims that exceed a specified percentage of the claim amount incurred during a specified period. However, it does not cover individual claims, rather the total percentage or amount of claims incurred by an insurer (Life Office Management Association, 2000; Booth et al., 2005).

Table 2.1 Percentage of Investment Assets of Thai Insurance Companies for the Years 2010-2014.

	Investment assets (%)									
	Non-life insurance companies					Life insurance companies				
	2010	2011	2012	2013	2014	2010	2011	2012	2013	2014
Bonds	35.17	27.89	25.93	25.26	22.54	63.73	61.87	59.48	59.14	57.78
Notes	5.53	11.81	1.99	0.82	0.85	6.52	9.66	10.19	9.02	6.99
Stocks	23.73	24.99	26.77	24.27	27.33	9.51	7.39	8.28	7.28	7.33
Debentures	8.71	7.74	7.85	8.22	9.06	10.84	11.98	12.87	14.35	17.26
Investment Units	7.54	5.87	4.64	4.24	4.39	1.04	0.86	1.23	1.89	2.25
Cash and Deposits	14.97	18.47	29.84	33.51	32.85	0.95	1.92	2.17	2.29	2.84
Loans	2.84	2.32	1.62	1.3	1.18	5.96	5.37	4.96	4.92	4.62
others	1.51	0.91	1.36	2.38	1.8	1.45	0.95	0.82	1.11	0.93
Total	100	100	100	100	100	100	100	100	100	100

Source: Office of insurance commission, 2017.

2.5 Investment

Investment is the current commitment of resources for a period of time in the expectation of receiving future resources that will compensate the investor for 1) the time the resources are committed, 2) the expected rate of inflation, and 3) the risk (the uncertainty of future payments). The investor is trading a known (or reasonably certain) amount of resources (e.g. money) today for expected future resources (e.g. a lump sum of cash or an income stream) that will be greater than the current outlay (Reilly and Norton, 2006: 5)

Investment is one of the activities carried out by insurance companies as investment income is significant for them. This income contributes to earnings and so affects the pricing of insurance policies (Nissim, 2010). There are many types of asset in the investment portfolios of insurance companies such as bonds, stock, mortgages, and real estate. Table 2.1 contains details of the investment assets of Thai insurance companies.

2.6 A Review of the Upper Bound for Ruin Probability under the Generalized Classical Discrete Time Risk Model

Cai (2002) generalized the classical risk model by adding the interest rate during each period to it. The rate of interest I_n is assumed to consist of i.i.d. nonnegative random variables. In addition, I_n is assumed to be independent of Y_n (the total claim amount in time period n) and X_n (the premium at time period n) for $n = 1, 2, 3, \ldots$ In this article, two different risk models are introduced under the differences of time to receive the premium: beginning and end of each time period. The upper bounds of the ruin probability for both risk models were derived by using two methods. First, the upper bounds were derived using NWU and NBU distributions and second, the upper bounds were derived using a recursive renewal technique. The numerical results showed that the upper bounds derived using the second were tighter than the first.

The effects of interest rates and the time of receiving the premium on a risk model by considering the ruin probability were considered by Cai (2002) and continued by Cai and Dickson (2004). Differently, i.i.d. nonnegative random variables I_n were assumed to follow a Markov chain and I_n took a finite number of possible values. Moreover, recursive forms and integral equations for the ruin probabilities were provided. Afterward, the upper bounds for ruin probabilities were presented using two approaches: inductive and Martingale. The numerical results suggest that the upper bounds derived by the inductive approach were tighter than those obtained by the Martingale method.

Cai and Dickson (2004) studied the effect of only the interest rate on the risk model while assuming the premium will be received at the end of the time period. The i.i.d. random variable I_n is still assumed to follow a Markov chain and I_n takes a finite number of possible values. However, the value of I_n was studied in 2 cases: only positive and positive or negative. From the results of the first case with recursive and integral equations, the upper bound of the ruin probability using either the inductive or Martingale approaches; was similar. In the second case (positive or negative values of I_n), the asymptotic formulas for the ruin probabilities were derived as initial surplus $u \to \infty$.

Diasparra and Romera (2009) generalized the risk model proposed by Cai and Dickson (2004) by adding a controller: proportional reinsurance. The risk model is controlled by choosing retention level $b_n \in (0,1]$ during the nth period. In this article, the retention level is restricted as a stationary (i.e. $b_n = b$ for all $n \geq 1$) i.i.d. nonnegative random variable I_n that is assumed to follow a Markov chain and takes a finite number of possible values. The upper bounds for the ruin probability are derived from the recursive and integral equations of ruin probabilities by either the inductive or Martingale approach. Corresponding with Cai and Dickson (2004), the numerical results suggest that the upper bounds derived by the inductive approach were tighter than those obtained by the Martingale approach.

Following the ideas of Cai and Dickson (2004) and Diasparra and Romera (2009), the article by Jasiulewicz and Kordecki (2015) presented the ruin probability for the generalized risk process with proportional reinsurance and investment surplus according to a random interest rate which follows a time-homogeneous Markov chain, and the recursive form and integral equations of the ruin probability are illustrated. The upper bound of the ruin probability is derived using the Lundberg adjustment coefficient which exists only for a light-tailed distribution of claims. Therefore, the asymptotic formulae of the ruin probabilities are derived for heavy-tailed distributions of claims as the initial surplus $u \to \infty$.

Lin, Dongjin and Yanru (2015) added a different controller from the work of Diasparra and Romera (2009) in which the return from the risky investment is added to the risk model studied by Cai and Dickson (2004). The authors showed that stationary policies of investment are appropriate for minimizing the upper bound of the ruin probability presented in a recursive form with integral equations. The upper bound of the ruin probability was once again derived by two approaches: inductive and Martingale.

CHAPTER 3

THE PROPOSED RISK MODEL

The main point in this chapter is to present the new discrete time risk model to which two controllers: proportional reinsurance and investment are added. Furthermore, in the rest of the chapter, another form of the proposed risk model is presented for studying the ruin probability mentioned in the next chapter.

3.1 The Risk Model and its Description

From the classical discrete time risk model (Equation 2.13)

$$U_n = U_{n-1} + c - Y_n \quad ; \; n = 1, 2, 3, \ldots \tag{3.1}$$

where U_n denotes the insurer's surplus at the end of time period n (i.e. from time $n-1$ to n) with initial constant $U_0 = u$, c is the constant premium income per unit time, and Y_n is the total claims amount during period n. We assume that this sequence consists of i.i.d. random variables with common distribution function $P(y) = \Pr(Y_n \le y); y \ge 0$. In this study, the proposed risk model is formulated by adding proportional reinsurance and investment on the right hand side of Equation (3.1).

Under proportional reinsurance contracts, the reinsurer agrees to cover a fraction of each claim equal to the fraction of premiums that it receives from the insurer. Throughout this study, $b_n \in (0,1]$ is defined as the retention level of a reinsurance contract for time period n. This means that the insurer pays $b_n Y_n$ of total claim amount Y_n while the reinsurer is liable for $(1-b_n)Y_n$. Similarly, the reinsurer receives $(1-b_n)c$ of the constant premium c while the insurer retains $b_n c$, and if the retention level $b_n = 1$, this means that there is no reinsurance. Let $h(b_n, Y_n)$ denote the fraction of the total claim amount Y_n paid by the insurer, $0 < h(b_n, Y_n) \le Y_n$, with $G(y_b) = \Pr\left[h(b_n, Y_n) \le y_b\right], y_b \ge 0$, then $h(b_n, Y_n)$ can be evaluated by $h(b_n, Y_n) = b_n Y_n$ (this is the case throughout this dissertation). By the expected value principle with safety loading factor $\theta > 0$, the premium constant is calculated as $c = (1+\theta)E(Y_n)$ and paid at the end of every time period unit $(n-1, n]$. Let δ be the safety loading factor added by the reinsurer and c_{re} be the premium constant for the reinsurer. Thus, by the expected value principle, the constant premium for reinsurer is given by

$$\begin{aligned}
c_{re} &= (1+\delta)E\left[Y_n - h(b_n, Y_n)\right], \\
&= (1+\delta)E\left[Y_n - b_n Y_n\right], \\
&= (1+\delta)E\left[(1-b_n)Y_n\right], \\
&= (1+\delta)(1-b_n)E(Y_n). \qquad (3.2)
\end{aligned}$$

Next, the constant premium which is retained by the insurer in a unit period denoted by $c(b_n)$, when $0 \le c(b_n) \le c$, can be calculated as

$$\begin{aligned}
\mathrm{c}(\mathrm{b_n}) &= c - c_{re} \\
&= \left[(1+\theta) - (1+\delta)(1-b_n)\right]E(Y_n). \qquad (3.3)
\end{aligned}$$

For the effect of an investment on a risk model, we assume that the insurer can invest in two assets. One is a bond with a known interest rate at the initial time (I_0); the interest rate at time n $(I_n, n = 1, 2, 3...)$ has a finite countable number (d_n) of possible values ($I_n = i_k$, where $k \in 1, 2, 3, ..., d_n$), and we assume that $d_n = d$ for all n throughout this dissertation. In addition, I_n is assumed to follow a time-homogeneous

Markov chain, i.e. both the transition probabilities and the time are independent, and are denoted by

$$\Pr\left\{I_n = i_c \mid I_{n-1} = i_b, \ldots, I_1 = i_a, I_0 = i_s\right\} = \Pr\left\{I_n = i_c \mid I_{n-1} = i_b\right\},$$
$$= p_{bc}, \tag{3.4}$$

where i_s is assumed to be the known value of I_0; and $i_a,\ i_b,\ i_c$ are the possible values of $I_1,\ I_{n-1},\ I_n$, respectively, for which $a,\ b,\ c \in \{1, 2, \ldots, d_n\}$ and $\sum_{c=1}^{d_n} p_{bc} = 1$ for all $a,\ b,\ c \in \{1, 2, \ldots, d_n\}$.

Thus, the transition probability matrix of I_n can be written as

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1d_n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2d_n} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3d_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{d_n 1} & p_{d_n 2} & p_{d_n 3} & \cdots & p_{d_n d_n} \end{bmatrix}$$

The other investment asset is a stock with simple net return R_n and the price of one share of stock S_n at time n is defined as

$$S_n = S_{n\text{-}1}\left(1 + R_n\right) = S_{n\text{-}1} W_n \ ; n = 1, 2, 3, \ldots \tag{3.5}$$

A standard assumption on the stock market is $1 + R_n = W_n > 0$, which is called the gross return. Throughout this study, W_n is assumed to be a sequence of i.i.d. nonnegative random variables with distribution functions $F(w) = \Pr\left(W_n \le w\right),\ w \ge 0.$

Base on the risk model in Equation (3.1), if the insurer at the beginning of n^{th} period signs a reinsurance contract with retention level b_n, invests $p_n U_{n-1}$ of the surplus in the stock, and invests the retain surplus $\left(1 - p_n\right) U_{n-1}$ in the bond, where p_n

is the proportion of investment in nth period, $0 \le p_n \le 1$, then the surplus at the end of the nth period becomes

$$\begin{aligned}
U_n &= p_n U_{n-1} W_n + (1-p_n) U_{n-1}(1+I_n) + c(b_n) - h(b_n, Y_n), \\
&= p_n U_{n-1} W_n + U_{n-1}(1+I_n) - p_n U_{n-1}(1+I_n) + c(b_n) - h(b_n, Y_n), \\
&= U_{n-1}(1+I_n) + p_n U_{n-1}\{W_n - (1+I_n)\} + c(b_n) - h(b_n, Y_n), \\
&= U_{n-1}(1+I_n) + p_n U_{n-1}\{1 - (1+I_n)/W_n\} W_n + c(b_n) - h(b_n, Y_n), \\
&= U_{n-1}(1+I_n) + \alpha_n W_n + c(b_n) - h(b_n, Y_n),
\end{aligned} \tag{3.6}$$

where $\alpha_n = p_n U_{n-1}\{1-(1+I_n)/W_n\}$. (3.7)

The quantity of α_n in Equation (3.6) can be considered as the amount of money which the insurers invests in the stock at the beginning of the nth period, but the value of I_n and W_n are unknown at that time. Thus, we identify the amount of α_n using the information from $\{I_j \text{ and } W_j : j = 0,1,2,...,n-1\}$.

From Equation (3.7), consider that $\alpha_n \ge 0$ if and only if

$$p_n U_{n-1}\{1-(1+I_n)/W_n\} \ge 0,$$

$$p_n U_{n-1} \ge p_n U_{n-1}(1+I_n)/W_n,$$

$$W_n \ge (1+I_n),$$

$$1+R_n \ge 1+I_n,$$

and $R_n \ge I_n$. (3.8)

The meaning of Equation (3.8) is that the insurer will decide to invest in stock ($\alpha_n >$ 0) if he thinks that its simple net return R_n is greater than or equal to the bond interest rate I_n.

3.2 The Other Form of the Proposed Risk Model

From the proposed risk model in Equation (3.6), if we replace the values of n as $n=1,2,..,m$, then the output from this action is another form of the previous model written as

$$U_1 = U_0\left(1+I_1\right)+\alpha_1 W_1 + c\left(b_1\right) - h\left(b_1, Y_1\right),$$

$$\begin{aligned}
U_2 &= U_1\left(1+I_2\right)+\alpha_2 W_2 + c\left(b_2\right) - h\left(b_2, Y_2\right), \\
&= U_0 \prod_{j=1}^{2}\left(1+I_j\right) + \alpha_1 W_1\left(1+I_2\right) + c\left(b_1\right)\left(1+I_2\right) - h\left(b_1, Y_1\right)\left(1+I_2\right) \\
&\quad + \ \alpha_2 W_2 + c\left(b_2\right) - h\left(b_2, Y_2\right),
\end{aligned}$$

$$\begin{aligned}
U_3 &= U_2\left(1+I_3\right)+\alpha_3 W_3 + c\left(b_3\right) - h\left(b_3, Y_3\right), \\
&= U_0 \prod_{j=1}^{3}\left(1+I_j\right) + \alpha_1 W_1 \prod_{k=2}^{3}\left(1+I_k\right) + c\left(b_1\right)\prod_{k=2}^{3}\left(1+I_k\right) - h\left(b_1, Y_1\right)\prod_{k=2}^{3}\left(1+I_k\right) \\
&\quad + \ \alpha_2 W_2\left(1+I_3\right) + c\left(b_2\right)\left(1+I_3\right) - h\left(b_2, Y_2\right)\left(1+I_3\right) \\
&\quad + \ \alpha_3 W_3 + c\left(b_3\right) - h\left(b_3, Y_3\right),
\end{aligned}$$

.

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.

$$U_m = U_0 \prod_{j=1}^{m}\left(1+I_j\right) + \sum_{j=1}^{m}\left[\left(\alpha_j W_j + c\left(b_j\right) - h\left(b_j, Y_j\right)\right)\prod_{k=j+1}^{m}\left(1+I_k\right)\right].$$

Therefore, the other form of U_n is

$$U_n = U_0 \prod_{j=1}^{n}\left(1+I_j\right) + \sum_{j=1}^{n}\left[\left(\alpha_j W_j + c\left(b_j\right) - h\left(b_j, Y_j\right)\right)\prod_{k=j+1}^{n}\left(1+I_k\right)\right] \quad ; n=1,2,3,\ldots \tag{3.9}$$

Remark 1. In the case where the value of *k* is greater than *n*, I_k does not exist in $\prod_{k=j+1}^{n}\left(1+I_k\right)$, thus we assume $I_k = 0$, i.e. $\prod_{k=j+1}^{n}\left(1+I_k\right) = \prod_{k=j+1}^{n}\left(1\right) = 1$.

CHAPTER 4

THE RUIN PROBABILITY FOR THE PROPOSED RISK MODEL

This chapter is divided into two parts. The first part introduces the ruin probability of the proposed risk model presented in a recursive form as well as the integral equation, both in the case of finite time and infinite time. The second part presents two upper bounds of ruin probability which were developed previously. Both upper bounds are created under the assumption that the retention level of reinsurance and the amount of stock investment in each time period are controlled as constant values. The first upper bound of ruin probability will be developed both finite and infinite time under the additional assumption which the m.g.f. of the claim amounts exist. The second upper bound of ruin probability is specially developed in case of finite time by using an NWU distribution. For those which are infinite times cannot find the closed forms.

4.1 Ruin Probability

With the same principle as for the ruin probability defined in Equations (2.14) and (2.15), those of the proposed risk model defined in Equation (3.9) given the initial values $U_0 = u$ and $I_0 = i_s$ are as follows.

The ruin probability for finite time is given by

$$\begin{aligned}\psi_n\left(u,i_s\right) &= \Pr\left\{\bigcup_{k=1}^{n}\left(U_k<0\right) \mid U_0=u, I_0=i_s\right\}\\ &= \Pr\left\{\begin{array}{l}\bigcup_{k=1}^{n}\left[\begin{array}{l}U_0\prod_{j=1}^{k}\left(1+I_j\right)\\ +\sum_{j=1}^{k}\left(\left(\alpha_j W_j + c\left(b_j\right)-h\left(b_j,Y_j\right)\right)\prod_{m=j+1}^{k}\left(1+I_m\right)\right)<0\end{array}\right]\\ \mid U_0=u, I_0=i_s\end{array}\right\}, \quad (4.1)\end{aligned}$$

and the ultimate ruin probability can be written as

$$\psi\left(u,i_s\right)=\Pr\left\{\bigcup_{k=1}^{\infty}\left(U_k<0\right)\mid U_0=u,I_0=i_s\right\}$$

$$=\Pr\left\{\begin{array}{l}\bigcup\limits_{k=1}^{\infty}\left[\begin{array}{l}U_0\prod\limits_{j=1}^{k}\left(1+I_j\right)\\ +\sum\limits_{j=1}^{k}\left(\left(\alpha_jW_j+c\left(b_j\right)-h\left(b_j,Y_j\right)\right)\prod\limits_{m=j+1}^{k}\left(1+I_m\right)\right)<0\end{array}\right]\\ \mid U_0=u,I_0=i_s\end{array}\right\}. \quad (4.2)$$

The ruin probability of the proposed risk model could be written in a recursive form in the case of finite time and also written as an integral equation in the case of infinite time. The aforementioned results are shown in the following theorem.

Theorem 3. The recursive form of the finite time ruin probability and the integral equations for the ultimate ruin probability under the proposed risk model as in Equation (3.6) are given as follows.

The recursive form of the finite time ruin probability is

$$\psi_{n+1}(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\int_0^{\pi}\psi_n\left(u\left(1+i_t\right)+\alpha_1w-z\left(y_b\right),i_t\right)dG\left(y_b\right)dF\left(w\right)$$

$$+\sum_{t=0}^{d}p_{st}\int_0^{\infty}\overline{G}\left(\pi\right)dF\left(w\right), \quad (4.3)$$

where $\overline{G}\left(\pi\right)=1-G\left(\pi\right)=\Pr\left(h\left(b_1,Y_1\right)\geq\pi\right)$ and $\pi=u\left(1+i_t\right)+\alpha_1w+c\left(b_1\right)$.

The integral equation of the ultimate ruin probability is

$$\psi(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\int_0^{\pi}\psi\left(u\left(1+i_t\right)+\alpha_1w-z\left(y_b\right),i_t\right)dG\left(y_b\right)dF\left(w\right)$$

$$+\sum_{t=0}^{d}p_{st}\int_0^{\infty}\overline{G}\left(\pi\right)dF\left(w\right), \quad (4.4)$$

and the ruin probability in first time period is

$$\psi_1(u,i_s) = \sum_{t=0}^{d} p_{st} \int_0^\infty \bar{G}(\pi)\, dF(w). \tag{4.5}$$

Proof.

Let $Z_n = z\left[h(b_n,Y_n)\right] = h(b_n,Y_n) - c(b_n)$, $n = 1,2,3,\ldots$ and suppose that $I_1 = i_t$, $t \in \{0,1,2,\ldots,d_n\}$, $W_1 = w$, $w \geq 0$, $h(b_1,Y_1) = y_b$, and $y_b \geq 0$. Thus, $Z_1 = z(y_b)$ $= h(b_1,Y_1) - c(b_1)$.

Consider from Equation (3.6) that

$$\begin{aligned} U_1 &= U_0(1+I_1) + \alpha_1 W_1 + c(b_1) - h(b_1,Y_1) \\ &= U_0(1+I_1) + \alpha_1 W_1 - \left[h(b_1,Y_1) - c(b_1)\right] \\ &= u(1+i_t) + \alpha_1 w - z(y_b) \\ &= h - z(y_b), \end{aligned} \tag{4.6}$$

where $h = u(1+i_t) + \alpha_1 w$.

Thus, if $z(y_b) > h$, then

$$\Pr\left\{U_1 < 0 \mid W_1 = w,\ h(b_1,Y_1) = y_b,\ I_1 = i_t,\ I_0 = i_s, U_0 = u\right\} = 1,$$

implying that for $z(y_b) > h$,

$$\Pr\left\{\bigcup_{k=1}^{n+1} (U_k < 0) \mid W_1 = w, h(b_1,Y_1) = y_b, I_1 = i_t, I_0 = i_s, U_0 = u\right\} = 1. \tag{4.7}$$

Meanwhile, if $0 \leq z(y_b) \leq h$, then

$$\Pr\left\{U_1 < 0 \mid W_1 = w, h(b_1,Y_1) = y_b, I_1 = i_t, I_0 = i_s, U_0 = u\right\} = 0,$$

implying that for $0 \le z\left(y_b\right) \le h$,

$$\Pr\left\{\bigcup_{k=1}^{n+1}\left(U_k < 0\right) | W_1 = w,\ h\left(b_1, Y_1\right) = y_b, I_1 = i_t, I_0 = i_s, U_0 = u\right\}$$

$$= \Pr\left\{\bigcup_{k=2}^{n+1}\left(U_k < 0\right) | W_1 = w, h\left(b_1, Y_1\right) = y_b, I_1 = i_t, I_0 = i_s, U_0 = u\right\} \text{ ; by (4.7)}$$

$$= \Pr\left\{\begin{array}{l}\bigcup_{k=2}^{n+1}\left\{\left[\left(h - z\left(y_b\right)\right)\prod_{j=1}^{k}\left(1 + I_j\right) + \sum_{j=1}^{k}\left(\left(\alpha_j W_j - Z_j\right)\prod_{m=j+1}^{k}\left(1 + I_m\right)\right)\right] < 0\right\} \\ | U_1 = h - z\left(y_b\right), I_1 = i_t\end{array}\right\}$$

$$= \Pr\left\{\begin{array}{l}\bigcup_{r=1}^{n}\left\{\left[\left(h - z\left(y_b\right)\right)\prod_{j=1}^{r}\left(1 + I_j\right) + \sum_{j=1}^{r}\left(\left(\alpha_j W_j - Z_j\right)\prod_{m=j+1}^{r}\left(1 + I_m\right)\right)\right] < 0\right\} \\ | U_0 = h - z\left(y_b\right), I_0 = i_t\end{array}\right\}$$

$$= \psi_n\left(h - z\left(y_b\right), i_t\right) \qquad \text{; by (3.11)}$$

$$= \psi_n\left(u\left(1 + i_t\right) + \alpha_1 w - z\left(y_b\right), i_t\right). \text{ ; } h = u\left(1 + i_t\right) + \alpha_1 w. \tag{4.8}$$

Consider $\psi_{n+1}\left(u, i_s\right)$ from Equation (3.11) is as follows:

$$\psi_{n+1}\left(u, i_s\right) = \Pr\left\{\bigcup_{k=1}^{n+1}\left(U_k < 0\right) | U_0 = u, I_0 = i_s\right\},$$

then by analogy, as $\Pr\left(x_i, y_i\right) = \sum_{y_i}\Pr\left(x_i, y_i | Y = y_i\right)$, thus we can rewrite $\psi_{n+1}\left(u, i_s\right)$ as

$$\psi_{n+1}\left(u, i_s\right) = \sum_{t=0}^{d} p_{st}\int_0^{\infty}\int_0^{\infty}\Pr\left\{\begin{array}{l}\bigcup_{k=1}^{n+1}\left(U_k < 0\right) | U_0 = u, I_0 = i_s, I_1 = i_t, \\ h\left(b_1, Y_1\right) = y_b, W_1 = w\end{array}\right\} dG\left(y_b\right) dF\left(w\right)$$

$$= \sum_{t=0}^{d} p_{st}\int_0^{\infty}\int_0^{\infty}\Pr\left\{\bigcup_{k=1}^{n+1}\left(U_k < 0\right) | \beta\right\} dG\left(y_b\right) dF\left(w\right), \tag{4.9}$$

where $\beta = \left\{U_0 = u, I_0 = i_s, I_1 = i_t, h\left(b_1, Y_1\right) = y, W_1 = w\right\}$.

From Equation (4.4), consider that ruin will occur in the first period if $z\left(y_b\right) > h$ or $h\left(b_1, Y_1\right) > u\left(1+i_t\right) + \alpha_1 w + c\left(b_1\right)$ and will occur in another period if $h\left(b_1, Y_1\right) \le u\left(1+i_t\right) + \alpha_1 w + c\left(b_1\right)$. Since $h\left(b_1, Y_1\right) = y_b$ is defined at the beginning of the proof, we now define $u\left(1+i_t\right) + \alpha_1 w + c\left(b_1\right) = \pi$ for the short term. In order to use Equations (4.7) and (4.8) to derive the recursive form, we need to rewrite $\psi_{n+1}\left(u, i_s\right)$ in Equation (4.9) as

$$
\begin{aligned}
\psi_{n+1}(u, i_s) &= \sum_{t=0}^{d} p_{st} \int_0^{\infty} \left\{ \begin{array}{l} \int_0^{\pi} \Pr\left[\bigcup_{k=1}^{n+1} \left(U_k < 0\right) \mid \beta \right] dG\left(y_b\right) \\ + \int_{\pi}^{\infty} \Pr\left[\bigcup_{k=1}^{n+1} \left(U_k < 0\right) \mid \beta \right] dG\left(y_b\right) \end{array} \right\} dF\left(w\right) \\
&= \sum_{t=0}^{d} p_{st} \int_0^{\infty} \int_0^{\pi} \psi_n\left(u\left(1+i_t\right) + \alpha_1 w - z\left(y_b\right), i_t\right) dG\left(y_b\right) dF\left(w\right) \\
&\quad + \sum_{t=0}^{d} p_{st} \int_0^{\infty} \int_{\pi}^{\infty} dG\left(y_b\right) dF\left(w\right).
\end{aligned}
$$

Since $\bar{G}\left(\pi\right) = \Pr\left(h\left(b_1, Y_1\right) \ge \pi\right) = \int_{\pi}^{\infty} dG\left(y_b\right)$, then we can rewrite $\psi_{n+1}(u, i_s)$ as

$$
\begin{aligned}
\psi_{n+1}(u, i_s) &= \sum_{t=0}^{d} p_{st} \int_0^{\infty} \int_0^{\pi} \psi_n\left(u\left(1+i_t\right) + \alpha_1 w - z\left(y_b\right), i_t\right) dG\left(y_b\right) dF\left(w\right) \\
&\quad + \sum_{t=0}^{d} p_{st} \int_0^{\infty} \bar{G}\left(\pi\right) dF\left(w\right).
\end{aligned} \tag{4.10}
$$

By using Equation (2.17) and the Lebesgue dominated convergence theorem, the result of taking $n \to \infty$ in Equation (4.10) becomes

$$
\begin{aligned}
\psi(u, i_s) &= \lim_{n \to \infty} \psi_{n+1}(u, i_s) \\
&= \sum_{t=0}^{d} p_{st} \int_0^{\infty} \int_0^{\pi} \psi\left(u\left(1+i_t\right) + \alpha_1 w - z\left(y_b\right), i_t\right) dG\left(y_b\right) dF\left(w\right)
\end{aligned}
$$

$$+\sum_{t=0}^{d} p_{st} \int_{0}^{\infty} \bar{G}(\pi)\, dF(w).$$

Furthermore, following on from Equation (4.5), we can obtain

$$\begin{aligned}
\psi_1(u,i_s) &= \Pr\{U_1 < 0 \mid U_0 = u, I_0 = i_s\} \\
&= \Pr\{h - z(y_b) < 0 \mid U_0 = u, I_0 = i_s\} \quad \text{; by (4.6)} \\
&= \Pr\{Z_1 > h \mid U_0 = u, I_0 = i_s\} \qquad ;\ Z_1 = z(y_b) \\
&= \Pr\{Z_1 > u(1+i_t) + \alpha_1 w \mid U_0 = u, I_0 = i_s\} \qquad ;\ h = u(1+i_t) + \alpha_1 w \\
&= \Pr\{h(b_1,Y_1) - c(b_1) > u(1+i_t) + \alpha_1 w \mid U_0 = u, I_0 = i_s\} \\
&\qquad ;\ Z_1 = h(b_1,Y_1) - c(b_1) \\
&= \Pr\{h(b_1,Y_1) > u(1+i_t) + \alpha_1 w + c(b_1) \mid U_0 = u, I_0 = i_s\} \\
&= \sum_{t=0}^{d} p_{st} \int_{0}^{\infty} \Pr\{h(b_1,Y_1) > u(1+i_t) + \alpha_1 w + c(b_1) \\
&\qquad \mid U_0 = u, I_0 = i_s, I_1 = i_t, W_1 = w\}\, dF(w) \\
&= \sum_{t=0}^{d} p_{st} \int_{0}^{\infty} \Pr\{h(b_1,Y_1) > \pi \mid U_0 = u, I_0 = i_s, I_1 = i_t, W_1 = w\}\, dF(w) \\
&\qquad ;\ \pi = u(1+i_t) + \alpha_1 w + c(b_1) \\
&= \sum_{t=0}^{d} p_{st} \int_{0}^{\infty} \bar{G}(\pi)\, dF(w).
\end{aligned}$$

□

4.2 The Upper Bounds of the Ruin Probabilities

In this section, two upper bounds of the ruin probabilities (as in Theorem 3) are derived by the inductive method and under the assumption that the retention level of reinsurance and the amount of stock investment in each time period are controlled to be constant values. The first upper bound is derived based on the condition that the adjustment coefficient (as in Definition 2) exists, while the second upper bound is derived based on an NWU distribution.

Theorem 4. Under the assumptions that the retention level of reinsurance and the amount of stock investment in each time period are controlled to be constant values, i.e. $b_n = b$ and $\alpha_n = \alpha$, for $n = 1, 2, 3, \ldots$, and the adjustment coefficient as in Equation (2.18) exists, the upper bounds of finite time ruin probability $\psi_{n+1}(u, i_s)$ and ultimate ruin probability $\psi(u, i_s)$ of the proposed risk model are given by

$$\psi_{n+1}(u,i_s) \le \beta_0 E\left(e^{-R_0\left[u\left(1+I_1\right)+\alpha W_1\right]} \mid I_0 = i_s\right), \tag{4.11}$$

where

$$\beta_0^{-1} = \inf_{m\ge 0} \frac{\int_m^\infty e^{R_0 y_b} dG\left(y_b\right)}{e^{R_0 m}\bar{G}\left(m\right)} \qquad \text{, for all } m \ge 0 .$$

and $$\psi(u,i_s) = \lim_{n\to\infty} \psi_{n+1}(u,i_s)$$

$$\le \beta_0 E\left(e^{-R_0\left[u\left(1+I_1\right)+\alpha W_1\right]} \mid I_0 = i_s\right). \tag{4.12}$$

Proof.

Let $\bar{G}\left(m\right) = 1 - G\left(m\right) = \Pr\left[h\left(b_1, Y_1\right) > m\right]$, for all $m \ge 0$, then $\bar{G}\left(m\right)$ can be rewritten as

$$\bar{G}(m) = \left(\frac{\int_m^{\infty} e^{R_0 y_b} dG(y_b)}{e^{R_0 m} \bar{G}(m)} \right)^{-1} e^{-R_0 m} \int_m^{\infty} e^{R_0 y_b} dG(y_b)$$

$$\le \beta_0 e^{-R_0 m} \int_m^{\infty} e^{R_0 y_b} dG(y_b) \qquad ; \text{where } \beta_0^{-1} = \inf_{m \ge 0} \frac{\int_m^{\infty} e^{R_0 y_b} dG(y_b)}{e^{R_0 m} \bar{G}(m)}$$

$$\le \beta_0 e^{-R_0 m} \int_{-\infty}^{\infty} e^{R_0 y_b} dG(y_b)$$

$$= \beta_0 e^{-R_0 m} E\left(e^{R_0 h(b_1, Y_1)} \right). \tag{4.13}$$

From Equation (4.5) in Theorem 3, we can obtain

$$\psi_1(u, i_s) = \sum_{t=0}^{d} p_{st} \int_0^{\infty} \bar{G}(\pi) dF(w),$$

where $\pi = u(1 + i_t) + \alpha_1 w + c(b_1)$. Subsequently,

$$\psi_1(u, i_s) \le \sum_{t=0}^{d} p_{st} \int_0^{\infty} \beta_0 e^{-R_0 \{u(1+i_t) + \alpha_1 w + c(b_1)\}} E\left(e^{R_0 h(b_1, Y_1)} \right) dF(w)$$

(from Equation (4.13))

$$= \beta_0 E\left(e^{R_0 h(b_1, Y_1)} \right) \sum_{t=0}^{d} p_{st} \int_0^{\infty} e^{-R_0 \{u(1+i_t) + \alpha_1 w + c(b_1)\}} dF(w)$$

$$= \beta_0 E\left(e^{R_0 h(b_1, Y_1)} \right) E\left(e^{-R_0 \{u(1+I_1) + \alpha_1 W_1 + c(b_1)\}} \mid I_0 = i_s \right)$$

$$= \beta_0 E\left(e^{-R_0 \{c(b_1) - h(b_1, Y_1)\}} \right) E\left(e^{-R_0 \{u(1+I_1) + \alpha_1 W_1\}} \mid I_0 = i_s \right)$$

$$= \beta_0 E\left(e^{-R_0 \{u(1+I_1) + \alpha_1 W_1\}} \mid I_0 = i_s \right), \tag{4.14}$$

where $E\left(e^{-R_0 \{c(b_1) - h(b_1, Y_1)\}} \right) = 1$ (Diasparra and Romera, 2009, p. 102).

By using the inductive method, we get

$$\psi_n(u,i_s) \le \beta_0 E\left(e^{-R_0\left\{u(1+I_1)+\alpha_1 W_1\right\}} \mid I_0 = i_s\right). \tag{4.15}$$

Replace u and i_s by $u\left(1+i_t\right)+\alpha_1 w - z\left(y_b\right)$ and i_t in Equation (4.15), and consider Equation (4.3) in that $u\left(1+i_t\right)+\alpha_1 w - z\left(y_b\right) > 0$ when $u\left(1+i_t\right)+\alpha_1 w > z\left(y_b\right)$, then

$$\begin{aligned}\psi_n\left(u(1+i_t)+\alpha_1 w - z(y_b), i_t\right) &\le \beta_0 E\left(e^{-R_0\left\{\left[u\left(1+i_t\right)+\alpha_1 w - z\left(y_b\right)\right](1+I_1)+\alpha_1 W_1\right\}} \mid I_0 = i_t\right)\\ &\le \beta_0 e^{-R_0\left[u\left(1+i_t\right)+\alpha_1 w - z\left(y_b\right)\right]}.\end{aligned} \tag{4.16}$$

From Equation (4.3) in Theorem 3, we can write

$$\begin{aligned}\psi_{n+1}(u,i_s) = &\sum_{t=0}^{d} p_{st} \int_0^\infty \int_0^\pi \psi_n\left(u\left(1+i_t\right)+\alpha_1 w - z\left(y_b\right), i_t\right) dG\left(y_b\right) dF\left(w\right)\\ &+\sum_{t=0}^{d} p_{st} \int_0^\infty \bar{G}\left(\pi\right) dF\left(w\right),\end{aligned} \tag{4.17}$$

and by replacing Equations (4.13) and, (4.16) in Equation (4.17), we can achieve

$$\begin{aligned}\psi_{n+1}\left(u,i_s\right) \le &\sum_{t=0}^{d} p_{st} \int_0^\infty \int_0^\pi \beta_0 e^{-R_0\left[u\left(1+i_t\right)+\alpha_1 w - z\left(y_b\right)\right]} dG\left(y_b\right) dF\left(w\right)\\ &+ \sum_{t=0}^{d} p_{st} \int_0^\infty \beta_0 e^{-R_0\left[u\left(1+i_t\right)+\alpha_1 w + c\left(b_1\right)\right]} \int_\pi^\infty e^{R_0 y_b} dG\left(y_b\right) dF\left(w\right)\\ = &\sum_{t=0}^{d} p_{st} \int_0^\infty \beta_0 e^{-R_0\left[u\left(1+i_t\right)+\alpha_1 w + c\left(b_1\right)\right]} \int_0^\infty e^{R_0 y_b} dG\left(y_b\right) dF\left(w\right)\\ = &\sum_{t=0}^{d} p_{st} \int_0^\infty \beta_0 e^{-R_0\left[u\left(1+i_t\right)+\alpha_1 w + c\left(b_1\right)\right]} E\left(e^{R_0 h\left(b_1, Y_1\right)}\right) dF\left(w\right)\\ = &\beta_0 E\left(e^{-R_0\left[c\left(b_1\right)-h\left(b_1,Y_1\right)\right]}\right) E\left(e^{-R_0\left[u\left(1+I_1\right)+\alpha_1 W_1\right]} \mid I_0 = i_s\right)\\ = &\beta_0 E\left(e^{-R_0\left[u\left(1+I_1\right)+\alpha_1 W_1\right]} \mid I_0 = i_s\right) \qquad \text{(see Equation (4.14))}\\ = &\beta_0 E\left(e^{-R_0\left[u\left(1+I_1\right)+\alpha W_1\right]} \mid I_0 = i_s\right).\end{aligned}$$

and $$\psi(u,i_s) = \lim_{n\to\infty} \psi_{n+1}(u,i_s)$$
$$\leq \beta_0 E\left(e^{-R_0\left[u(1+I_1)+\alpha W_1\right]} \mid I_0 = i_s\right).$$

□

It is not difficult to calculate the results for the upper bound in Theorem 4, but it cannot be applied for all claims distribution because many practical distributions (especially heavy-tailed distributions such as Pareto and Weibull), the m.g.f. does not exist (i.e. the adjustment coefficient does not exist). Thus, the upper bound of the ruin probability in Theorem 4 cannot be applied for these distributions. Hence, in the next theorem, the upper bound of the ruin probability is applied to the claims distribution whether the m.g.f. exists or not. In this dissertation, it is derived base on the NWU distribution but only restricts the results where Y_n is a summation of the i.i.d. claim amounts in order to use the outcome of Willmot (1994) to support this procedure. Therefore, the additional assumptions for the next theorem are as follows.

Let $B(x)$ be the distribution function of a non-negative random variable and $\overline{B}(x) = 1 - B(x)$, then $B(x)$ is the NWU if $\overline{B}(x)\overline{B}(y) \leq \overline{B}(x+y)$, for $x \geq 0, y \geq 0$ (Willmot, 1994)

Let $Y_n = \sum_{i=1}^{N_n} V_{ni}$; $n = 1,2,3,\ldots$, and $i = 1,2,3,\ldots,N_n$, (4.18)

where

V_{ni} is the i$^{\text{th}}$ claim amount occurring during time period n (i.e. from $n-1$ to n) which is assumed to be an i.i.d. sequence with common distribution function $O(v) = \Pr(V_{ni} \leq v)$, $v \geq 0$; and

N_n is the number of claims occurring during time period n, which is assumed to be an i.i.d. sequence with

$$j_{nm} = \Pr(N_n = m)\,;\; m = 0,1,2,... \tag{4.19}$$

and

$$a_{nm} = \sum_{k=m+1}^{\infty} j_{nk}\,;\; m = 0,1,2,... \tag{4.20}$$

Suppose there exist positive numbers $0 < \phi_n < 1$ (see Willmot and Lin (1994) for more details) such that

$$a_{n(m+1)} \le \phi_n a_{nm}\,;\; m = 0,1,2,...\,, \tag{4.21}$$

and since the sequence of N_n, $n = 1,2,3,...$ is assumed to be i.i.d., then the values of j_{nm}, a_{nm}, and ϕ_n as defined in Equations (4.19) - (4.21) are constant for all values of n. To make this easier, we define

$$j_m = j_{nm} = \Pr(N_n = m)\,;\; m = 0,1,2,... \tag{4.22}$$

and

$$a_m = a_{nm} = \sum_{k=m+1}^{\infty} j_{nk} = \sum_{k=m+1}^{\infty} j_k\,;\; m = 0,1,2,... \tag{4.23}$$

Suppose there exist positive numbers $0 < \phi_n = \phi < 1$ such that

$$a_{m+1} \le \phi a_m\,;\; m = 0,1,2,... \tag{4.24}$$

From the afore mentioned additional assumptions, Willmot (1994) showed us that if the non-negative, non-increasing function $B(x)$, $x \ge 0$ exists (which is NWU) such that

$$\int_0^{\infty} \left\{\overline{B}(v)\right\}^{-1} dO(v) \le \phi^{-1} \tag{4.25}$$

and

$$\overline{O}(y) = \int_y^\infty dO(v) \le \overline{B}(y)\int_y^\infty \left\{\overline{B}(v)\right\}^{-1} dO(v)\,;\;\; y \ge 0\,, \tag{4.26}$$

then the upper bound for $\overline{P}(y) = 1 - P(y)$, where $P(y) = \Pr(Y_n \le y)$; $y \ge 0$, and the common distribution of the total claims Y_n, can be written as

$$\overline{P}(y) \le \phi^{-1}(1 - j_0)\overline{B}(y), \tag{4.27}$$

where j_0 is defined as in Equation (4.22).

Since the total claims amount $Y_n = \sum_{i=1}^{N_n} V_{ni}$; $n = 1, 2, 3, \ldots$; $i = 1, 2, 3, \ldots, N_n$ is assumed and the fraction of the total claim amount paid by the insurer when the company signed the reinsurance contract is $h(b_n, Y_n) = b_n Y_n$; $n = 1, 2, 3, \ldots$, then $h(b_n, Y_n)$ can be rewritten as

$$h(b_n, Y_n) = \sum_{i=1}^{N_n} b_n V_{ni} = \sum_{i=1}^{N_n} h(b_n, V_{ni})\,, \tag{4.28}$$

where $h(b_n, V_{ni})$ is a fraction of the claims amount paid by the insurer which is i.i.d. and the common distribution function is assumed to be $Q(v_b) = \Pr\left\{h(b_n, V_{ni}) \le v_b\right\}$, $v_b \ge 0$.

Similarly to Equations (4.25) - (4.27), if the non-negative, non-increasing function $D(x), x \ge 0$ (which is NWU) exists such that

$$\int_0^\infty \left\{\overline{D}(v_b)\right\}^{-1} dQ(v_b) \le \phi^{-1} \tag{4.29}$$

and in addition,

$$\overline{Q}(y_b) = \int_{y_b}^{\infty} dQ(v_b) \le \overline{D}(y_b) \int_{y_b}^{\infty} \left\{\overline{D}(v_b)\right\}^{-1} dQ(v_b) \quad ; y_b \ge 0, \tag{4.30}$$

then the upper bound for $\overline{G}(y_b) = 1 - G(y_b)$, where $G(y_b) = \Pr\left[h(b_n, Y_n) \le y_b\right]$ and $y_b \ge 0$ is the common distribution of claim $h(b_n, Y_n)$, can be expressed as

$$\overline{G}(y_b) \le \phi^{-1}(1 - j_0)\overline{D}(y_b). \tag{4.31}$$

The next theorem is derived from the previous information.

Theorem 5. Let the total claims amount Y_n, $n = 1, 2, 3, ...$ satisfy Equation (4.18) and the quantity $0 < \phi_n = \phi < 1$ satisfy Equation (4.24), and suppose there exists non-negative and non-increasing function $\overline{D}(x) = 1 - D(x)$ for $x \ge 0$, in which $D(x)$ is the NWU, $\overline{D}(0) = 1$, and $\overline{D}(x)$ satisfies Equations (4.29) and (4.30). Thus, the upper bound of the finite time ruin probability (as in Equation (4.3)) under the assumptions that the retention level of reinsurance and the amount of stock investment in each time period are controlled to be constant values, i.e. $b_n = b$ and $\alpha_n = \alpha$, for $n = 1, 2, 3, ...$ is given as

$$\psi_{n+1}(u, i_s) \le \phi^{-1}(1 - j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^n E\left[\overline{D}(\pi) \mid I_0 = i_s\right], \tag{4.32}$$

where (as before) $\pi = u(1 + i_t) + \alpha_1 w + c(b_1)$.

Proof

From Equation (4.5) in Theorem 3, we obtain

$$\psi_1(u, i_s) = \sum_{t=0}^{d} p_{st} \int_0^{\infty} \overline{G}(\pi) dF(w),$$

where $\pi = u(1 + i_t) + \alpha_1 w + c(b_1)$.

We can rewrite $\psi_1(u,i_s)$ using Equation (4.31) as

$$\psi_1(u,i_s) \le \sum_{t=0}^{d} p_{st} \int_0^\infty \left[\phi^{-1}(1-j_0)\overline{D}(\pi)\right] dF(w)$$

$$= \phi^{-1}(1-j_0) \sum_{t=0}^{d} p_{st} \int_0^\infty \left[\overline{D}(\pi)\right] dF(w)$$

$$= \phi^{-1}(1-j_0) \sum_{t=0}^{d} p_{st} \int_0^\infty \left[\overline{D}\left\{u(1+i_t)+\alpha_1 w + c(b_1)\right\}\right] dF(w)$$

$$= \phi^{-1}(1-j_0) E\left[\overline{D}\left\{u(1+i_t)+\alpha_1 w + c(b_1)\right\} \mid I_0 = i_s\right]$$

$$= \phi^{-1}(1-j_0) E\left[\overline{D}(\pi) \mid I_0 = i_s\right]$$

$$= \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(0)} E\left[\overline{D}(\pi) \mid I_0 = i_s\right].$$

By using the inductive method, we arrive at

$$\psi_n(u,i_s) \le \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} E\left[\overline{D}(\pi) \mid I_0 = i_s\right]$$

$$= \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} E\left[\overline{D}\{u(1+i_t)+\alpha_1 w + c(b_1)\} \mid I_0 = i_s\right] \tag{4.33}$$

If we replace u and i_s by $\pi - y_b$ and i_t, we can rewrite $\psi_n(u,i_s)$ in Equation (4.33) as

$$\psi_n(\pi - y, i_t) \le \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} E\left[\overline{D}\{(\pi - y_b)(1+i_t)+\alpha_1 w + c(b_1)\} \mid I_0 = i_t\right]$$

$$\le \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} E\left[\overline{D}(\pi - y_b) \mid I_0 = i_t\right]$$

$$= \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} \overline{D}(\pi - y_b). \tag{4.34}$$

From Equation (4.3) in Theorem 1, we obtain

$$\psi_{n+1}(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^\infty\left\{\int_0^\pi\psi_n\left(u\left(1+i_t\right)+\alpha_1 w-z\left(y_b\right),i_t\right)dG\left(y_b\right)+\bar{G}\left(\pi\right)\right\}dF\left(w\right)$$

(4.35)

By considering $u\left(1+i_t\right)+\alpha_1 w-z\left(y_b\right)$ in Equation (4.35) as

$$\begin{aligned}u\left(1+i_t\right)+\alpha_1 w-z\left(y_b\right)&=u\left(1+i_t\right)+\alpha_1 w+c\left(b_1\right)-h\left(b_1,Y_1\right)\\&=\pi-y_b,\end{aligned} \tag{4.36}$$

and by replacing Equation (4.36) in Equation (4.35), we can achieve

$$\begin{aligned}\psi_{n+1}(u,i_s)&=\sum_{t=0}^{d}p_{st}\int_0^\infty\left\{\int_0^\pi\psi_n\left(\pi-y_b,i_t\right)dG\left(y_b\right)+\bar{G}\left(\pi\right)\right\}dF\left(w\right)\\
&\le\sum_{t=0}^{d}p_{st}\int_0^\infty\left\{\begin{array}{l}\int_0^\pi\left\{\phi^{-1}\left(1-j_0\right)\left[E\left\{\bar{D}\left(y_b\right)\right\}^{-1}\right]^{(n-1)}\bar{D}\left(\pi-y_b\right)\right\}dG\left(y_b\right)\\+\phi^{-1}\left(1-j_0\right)\bar{D}\left(\pi\right)\end{array}\right\}dF\left(w\right)\\
&\le\sum_{t=0}^{d}p_{st}\int_0^\infty\left\{\begin{array}{l}\int_0^\pi\left\{\phi^{-1}\left(1-j_0\right)\left[E\left\{\bar{D}\left(y_b\right)\right\}^{-1}\right]^{(n-1)}\bar{D}\left(\pi-y_b\right)\right\}dG\left(y_b\right)\\+\phi^{-1}\left(1-j_0\right)\bar{D}\left(\pi-y_b\right)\end{array}\right\}dF\left(w\right)\\
&\le\sum_{t=0}^{d}p_{st}\int_0^\infty\left\{\begin{array}{l}\int_0^\pi\left\{\phi^{-1}\left(1-j_0\right)\left[E\left\{\bar{D}\left(y_b\right)\right\}^{-1}\right]^{(n-1)}\bar{D}\left(\pi-y_b\right)\right\}dG\left(y_b\right)\\+\phi^{-1}\left(1-j_0\right)\int_\pi^\infty\bar{D}\left(\pi-y_b\right)dG\left(y_b\right)\end{array}\right\}dF\left(w\right)\\
&\le\sum_{t=0}^{d}p_{st}\int_0^\infty\left\{\begin{array}{l}\phi^{-1}\left(1-j_0\right)\left[E\left\{\bar{D}\left(y_b\right)\right\}^{-1}\right]^{(n-1)}\bar{D}\left(\pi\right)\int_0^\pi\left\{\bar{D}\left(y_b\right)\right\}^{-1}dG\left(y_b\right)\\+\ \phi^{-1}\left(1-j_0\right)\bar{D}\left(\pi\right)\int_\pi^\infty\left\{\bar{D}\left(y_b\right)\right\}^{-1}dG\left(y_b\right)\end{array}\right\}dF\left(w\right)\\
&\le\sum_{t=0}^{d}p_{st}\int_0^\infty\left[\phi^{-1}\left(1-j_0\right)\left[E\left\{\bar{D}\left(y_b\right)\right\}^{-1}\right]^{(n-1)}\bar{D}\left(\pi\right)\int_0^\infty\left\{\bar{D}\left(y_b\right)\right\}^{-1}dG\left(y_b\right)\right]dF\left(w\right)\end{aligned}$$

$$= \sum_{t=0}^{d} p_{st} \int_{0}^{\infty} \left\{ \phi^{-1}\left(1 - j_0\right)\left[E\left\{ \overline{D}\left(y_b\right) \right\}^{-1} \right]^n \overline{D}\left(\pi\right) \right\} dF\left(w\right)$$

$$= \phi^{-1}\left(1 - j_0\right)\left[E\left\{ \overline{D}\left(y_b\right) \right\}^{-1} \right]^n \sum_{t=0}^{d} p_{st} \int_{0}^{\infty} \left\{ \overline{D}\left(\pi\right) \right\} dF\left(w\right)$$

$$= \phi^{-1}\left(1 - j_0\right)\left[E\left\{ \overline{D}\left(y_b\right) \right\}^{-1} \right]^n E\left[\overline{D}\left(\pi\right) \mid I_0 = i_s \right].$$

□

CHAPTER 5

NUMERICAL COMPUTATIONS

Two numerical examples are presented in the first part of this chapter in order to show the characteristics of the 2 upper bounds of the ruin probability derived in the previous chapter. The latter part of this chapter shows the application of the derived upper bounds for real-life data.

5.1 Numerical Examples

The two examples are presented under the assumptions that the retention level of reinsurance and the amount of stock investment in each time period of the insurer are controlled to be constant values. These are the main assumptions for the derived upper bounds in this dissertation.

In Example 1, the total claims amount in each time period is assumed to be an i.i.d. exponential distribution. The adjustment coefficient can be obtained from this distribution in order to show the characteristics of the upper bound derived in the theorem 4 in the previous chapter.

In Example 2, each claim amount is assumed to be an i.i.d. Pareto distribution, from which the adjustment coefficient cannot be found in order to show the characteristics of the upper bound derived in Theorem 5 in the previous chapter.

Example 1. We suppose that total claims amount $Y_n \sim \exp\left(\frac{1}{9}\right)$ in time periods $n = 1, 2, 3, ...$ and that the insurance company has the chance to invest its financial surplus in both the bond and stock markets. The bond interest rates during time periods $n = 1, 2, 3, ...$ are $I_n \in \{0.02,\ 0.03,\ 0.05\}$, respectively. We also assume that the bond interest rates follow a time-homogeneous Markov chain with transition probability matrix

$$\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

Gross stock return (W_n) is assumed to be $W_n = e^{\left(\mu - \frac{\sigma^2}{2}\right) + \sigma K_n}$ with $K_n \sim N(0,1)$ for time periods $n = 1, 2, 3, \ldots$ which is the one of commonly used functions explaining the behavior of stock return (Lin, Dongjin, and Yanru, 2015: 812). In this case, $\mu = 0.7$ and $\sigma = 0.5$ are also assumed. The safety loading factor given by the insurer and the reinsurer are 10 and 12%, respectively.

Due to the total claims amount $Y_n \sim \exp\left(\frac{1}{9}\right)$ being assumed, the distribution function of Y_n is $P(y) = \Pr(Y_n < y) = 1 - e^{-\frac{y}{9}}$, which leads to the distribution function of a fraction of total claims amount $h(b_n, Y_n) = b_n Y_n$, where $b_n = b$, $b \in (0,1]$ can be written as $G(y_b) = \Pr[h(b, Y_n) \le y_b] = 1 - e^{-\left(\frac{y_b}{9b}\right)}$. Thus, the adjustment coefficient can be obtained from this distribution. Therefore, we show the upper bound for the finite time and the ultimate ruin probability values derived in Theorem 4.

To show (in Table 5.1) how the factors in the proposed risk model affect the values of the upper bound of the ruin probability, we set the initial value of the factors additional to the assumptions at the beginning of this example as follows:

1) The initial surplus U_0 is set at 50, 100, or 500.

2) The initial value of interest rate I_0 is set at 0.02 or 0.05.

3) The retention level of reinsurance b_n is assumed (as for the derived upper bound) to be a constant value and $b_n = b$ in each time period $n = 1, 2, 3, \ldots$. Here $b =$ 0.2, 0.6, or 1.0.

4) The amount of stock investment α_n is also assumed (as for the derived upper bound) to be a constant value, and $\alpha_n = \alpha$ in each time period $n = 1, 2, 3, \ldots$. We now obtain four values of α by Equation (3.7 based on the other assumption that the proportion of stock investment p_n is constant, and $p_n = p$ is assumed to be 0, 0.25, 0.75, or 1. The values of I_n and R_n in Equation (3.7) are assumed to be I_0 and 10% respectively.

The steps for getting the upper bound of the finite time ruin probability and the ultimate ruin probability calculations are as follows:

1) Calculate the R_0 value using the formula in Equation (2.18)

2) Calculate the B_0^{-1} value using the formula in Equation (4.12)

3) Define the distribution function of W_1 using the data from the set up assumption. In this example, the distribution function of W_1 is

$$F(w) = \Pr(W_n \le w) = \Phi\left(\frac{\ln w - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma}\right)$$, where Φ is the standard normal distribution function, $w \ge 0$, $\mu = 0.7$, and $\sigma = 0.5$.

4) Calculate the upper bound of the ruin value using Equation (4.11).

Table 5.1 The Proposed Upper Bounds for the Ruin Probability Compares with Lundberg's Upper Bound.

U_0	α	b	The proposed upper bounds $I_0 = 0.02$	The proposed upper bounds $I_0 = 0.05$	Lundberg's upper bounds
50	0.00	0.2	0.0944	0.0923	0.3762
		0.6	0.1936	0.1909	
		1.0	0.3022	0.2993	
	0.91	0.2	0.0871	0.0852	
		0.6	0.1838	0.1812	
		1.0	0.2916	0.2888	
	2.73	0.2	0.0745	0.0729	
		0.6	0.1659	0.1635	
		1.0	0.2718	0.2692	
	3.63	0.2	0.0692	0.0676	
		0.6	0.1579	0.1556	
		1.0	0.2627	0.2601	
100	0.00	0.2	0.0097	0.0093	0.1415
		0.6	0.0444	0.0431	
		1.0	0.1108	0.1087	
	1.80	0.2	0.0083	0.0079	
		0.6	0.0400	0.0389	
		1.0	0.1033	0.1013	
	5.45	0.2	0.0061	0.0059	
		0.6	0.0328	0.0319	
		1.0	0.0900	0.0883	
	7.27	0.2	0.0053	0.0051	
		0.6	0.0298	0.0290	
		1.0	0.0841	0.0825	
500	0.00	0.2	0.0000	0.0000	0.0001
		0.6	0.0000	0.0000	
		1	0.0000	0.0000	
	9.09	0.2	0.0000	0.0000	
		0.6	0.0000	0.0000	
		1.0	0.0000	0.0000	
	27.27	0.2	0.0000	0.0000	
		0.6	0.0000	0.0000	
		1.0	0.0000	0.0000	
	36.36	0.2	0.0000	0.0000	
		0.6	0.0000	0.0000	
		1.0	0.0000	0.0000	

The results from Table 5.1 show that the upper bound value decreased when either initial surplus U_0 or the investment value in the stock α increased whereas it increased when the reinsurance contract retention level b increased. In addition, the results show that the upper bound value from Theorem 4 was sharper than the well-known Lundberg upper bound.

Example 2. Here, it is assumed that the total claims amount Y_n; $n=1,2,3,...$ is a summation of i.i.d. claim amounts V_{ni}, $V_{ni} \sim pareto(1.5, 0.5)$; $i=1,2,3,...,N_n$. The number of claims N_n during time period n is an i.i.d. Poisson distribution with mean $\lambda = 3$. It is also assumed that the bond interest rates during time periods $n=1,2,3,...$ are $I_n \in \{0.02,\ 0.03,\ 0.05\}$ with initial value $I_0 = 0.02$. Furthermore, they follow a time-homogeneous Markov chain with transition probability matrix

$$\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

Gross stock return (W_n) is assumed, as before, to be $W_n = e^{\left(\mu - \frac{\sigma^2}{2}\right) + \sigma K_n}$ with $K_n \sim N(0,1)$ for time periods $n=1,2,3,...$ and $\mu = 0.7$, $\sigma = 0.5$ are also assumed. The safety loading factors given by the insurer and the reinsurer are 10% and 12%, respectively. There exists $\bar{D}(x) = (1+kx)^{-1}$, which is NWU.

Since claim amount $V_{ni} \sim pareto(1.5, 0.5)$ is assumed, the distribution function of V_{ni} is given by

$$O(v) = \Pr(V_{ni} \le v) = 1 - \left(\frac{\beta}{v}\right)^{\alpha}, \text{ where } v \ge \beta = 0.5 \text{ and } \alpha = 1.5.$$

Thus, the distribution function of the fraction of the i[th] claim amount that occurs during time period n, $h(b_n, V_{ni}) = b_n V_{ni}$, where $b_n = b$, $b \in (0,1]$, can be expressed as

$$Q(v_b) = \Pr\left[h(b, V_{ni}) \le v_b\right] = 1 - \left(\frac{b\beta}{v_b}\right)^{\alpha}, \text{ where } v_b \ge \beta.$$

The m.g.f. of the Pareto distribution does not exist as in the case of the m.g.f. of $h(b_n, Y_n) = \sum_{i=1}^{N_n} h(b_n, V_{ni})$; therefore, we cannot find the adjustment coefficient value R_0. In this example, the upper bounds for the finite time ruin probabilities derived in Theorem 5 are shown.

To portray (in Table 5.2) how the factors in the proposed risk model affect the upper bound of the finite time ruin probability, we also set the initial value of the factors additional to the assumptions on the beginning of this example as follows:

1) The time period n is obtained for period 1, 2, 3, 4, and 5.

2) The initial surplus U_0 is set at 50, 100, or 500.

3) The retention level of reinsurance b_n, which is assumed (as the derived upper bound) to be a constant value, where $b_n = b$ in each time period $n = 1, 2, 3, \ldots$ is set at 0.2, 0.6, or 1.0.

4) The amount of stock investment, α_n, which is also assumed (as the derived upper bound) to be a constant value, and $\alpha_n = \alpha$ in each time period $n = 1, 2, 3, \ldots$. Four values of α by Equation (3.7) are obtained based on the other assumption that the proportion of stock investment p_n is a constant value and $p_n = p$: 0, 0.25, 0.75, and 1. The values of I_n and R_n in Equation (3.7) are assumed to be I_0 and 10% respectively

The steps in the upper bound of the finite time ruin probability value calculation are as follows:

1) Define the probability function for the number of claims N_n during the time period and calculate p_0.

2) Calculate the value of ϕ using Equation (4.24).

3) Find the k value in the function of $\bar{D}(x) = (1 + kx)^{-1}$, thereby making Equations (4.29) and (4.30) real.

4) Define the distribution function of W_1 using the information from the assumptions.

5) Calculate the upper bound of the finite time ruin probability using Equation (4.32).

Table 5.2 The Upper Bounds of the Finite Time Ruin Probability (Time Period n = 1,2,3,4, and 5).

U_0	α	b	Upper bounds $n=1$	$n=2$	$n=3$	$n=4$	$n=5$
50	0.00 (p_n= 0.00)	0.2	0.0342	0.0533	0.0831	0.1296	0.2021
		0.6	0.0940	0.1466	0.2286	0.3565	0.5560
		1	0.1410	0.2200	0.3430	0.5349	0.8343
	0.91 (p_n= 0.25)	0.2	0.0330	0.0515	0.0803	0.1253	0.1954
		0.6	0.0911	0.1421	0.2216	0.3456	0.5390
		1	0.1372	0.2139	0.3336	0.5202	0.8113
	2.73 (p_n= 0.75)	0.2	0.0310	0.0484	0.0755	0.1177	0.1835
		0.6	0.0860	0.1341	0.2092	0.3262	0.5087
		1	0.1301	0.2030	0.3165	0.4936	0.7698
	3.63 (p_n= 1.00)	0.2	0.0301	0.0470	0.0733	0.1143	0.1783
		0.6	0.0837	0.1306	0.2036	0.3176	0.4952
		1	0.1270	0.1980	0.3088	0.4816	0.7511
100	0.00 (p_n= 0.00)	0.2	0.0174	0.0271	0.0422	0.0659	0.1028
		0.6	0.0498	0.0777	0.1212	0.1890	0.2947
		1	0.0784	0.1223	0.1908	0.2975	0.4640
	1.80 (p_n= 0.25)	0.2	0.0168	0.0262	0.0408	0.0637	0.0993
		0.6	0.0482	0.0752	0.1173	0.1830	0.2854
		1	0.0761	0.1186	0.1850	0.2886	0.4500
	5.45 (p_n= 0.75)	0.2	0.0157	0.0246	0.0383	0.0598	0.0932
		0.6	0.0454	0.0708	0.1104	0.1721	0.2684
		1	0.0718	0.1119	0.1746	0.2723	0.4246
	7.27 (p_n= 1.00)	0.2	0.0153	0.0238	0.0372	0.0580	0.0905
		0.6	0.0441	0.0688	0.1072	0.1673	0.2608
		1	0.0699	0.1089	0.1699	0.2650	0.4132
500	0.00 (p_n= 0.00)	0.2	0.0035	0.0055	0.0086	0.0134	0.0208
		0.6	0.0105	0.0163	0.0255	0.0397	0.0619
		1	0.0172	0.0269	0.0419	0.0654	0.1020
	9.09 (p_n= 0.25)	0.2	0.0034	0.0053	0.0083	0.0129	0.0201
		0.6	0.0101	0.0158	0.0246	0.0384	0.0598
		1	0.0167	0.0260	0.0405	0.0632	0.0986
	27.27 (p_n= 0.75)	0.2	0.0032	0.0050	0.0078	0.0121	0.0189
		0.6	0.0095	0.0148	0.0231	0.0360	0.0561
		1	0.0156	0.0244	0.0380	0.0593	0.0925
	36.36 (p_n= 1.00)	0.2	0.0031	0.0048	0.0075	0.0117	0.0183
		0.6	0.0092	0.0144	0.0224	0.0349	0.0545
		1	0.0152	0.0237	0.0369	0.0576	0.0899

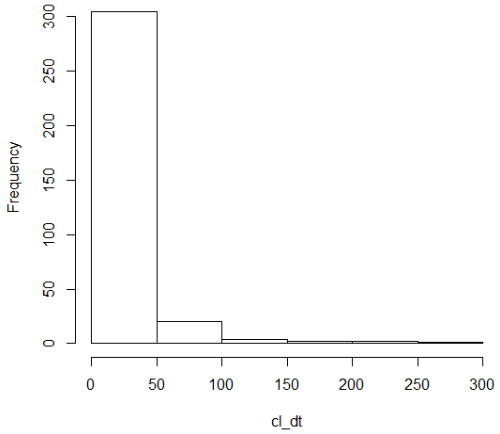
The results in Table 5.2 show that the upper bound for the finite time ruin probability increases as the number of time periods (n) increases corresponding with Equation (2.16), whereby the values for the finite time ruin probability do not decrease as n increases. The effects of variations in the other factors, i.e. the initial surplus (U_0), the stock investment value (α), and the reinsurance contract retention level (b) on the upper bound values of finite time ruin probability were the same as in Example 1.

Remark 3. The values of the upper bound from Theorem 5 depend on not only the change of factors in the risk model mentioned in Example 5.2, but also the function (which is NWU) selected. Based on running the results from the data in Table 5.2 and the resembling data, we found that the upper bound from Theorem 5 is appropriate when the initial surplus is sufficiently large. In the other cases, overestimated values of the upper bound may lead to a misunderstanding of the risk level, which would make insurers more wary of the risk than is necessary.

5.2 A Real-Life Data Example

The data for 334 real-life motor insurance claims from a broker occurring in a single year are used to analyze the upper bound for the ruin probability. Figure 5.1 shows that the minimum and maximum values of real-life claims dataset are 0.6063 and 284.764 respectively. The mean and standard deviation of this dataset are 21.0704 and 31.5294. Histogram and skewness's value (4.4317) support conclusion that the distribution of this dataset is the right-skewness distribution.

The real-life claims dataset was fitted to many right-skewness distributions by using R programming. Figure 5.2 shows that lognormal distribution is closest to the reference line. Thus, characteristic of this real-life claim dataset is explained by lognormal distribution with log data parameters $\hat{\mu} = 2.4671$ and $\hat{\sigma} = 1.0394$ which are estimated by maximum likelihood method by using R programming. The m.g.f. of the lognormal distribution is infinite at any positive number, thus the upper bound of the ruin probability in Theorem 5 is appropriate in this situation.

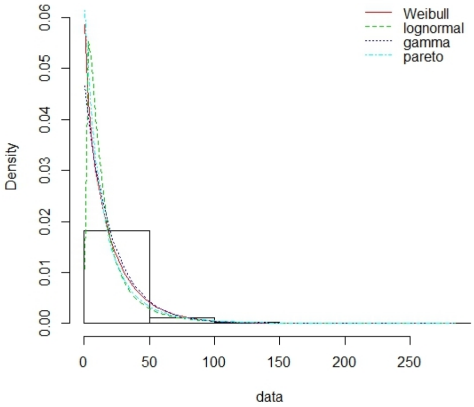


a) Histogram

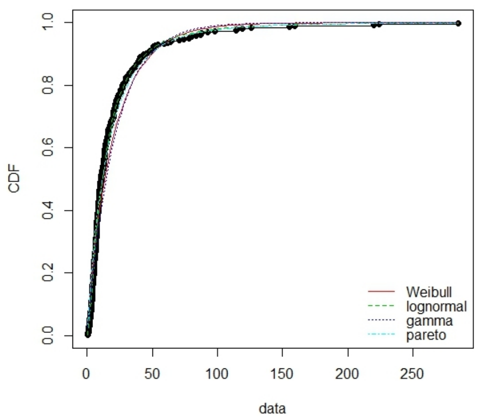
Minimum:	0.60633
Maximum:	284.764
median:	10.67222
mean:	21.07041
Standard deviation:	31.52941
Skewness:	4.431711
Kurtosis:	28.94935

b) Summary statistics

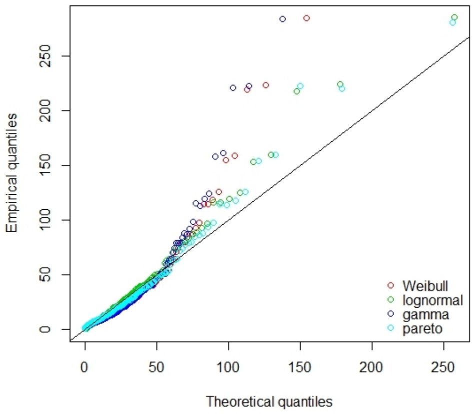
Figure 5.1 The Descriptive Statistics for Real-Life Motor Insurance Claims Data.



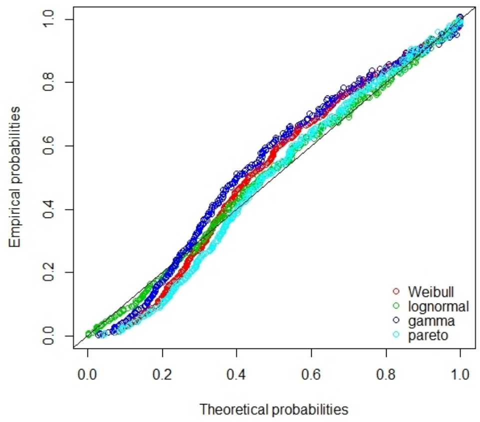
a) Histogram and theoretical densities



b) Empirical and theoretical CDFs



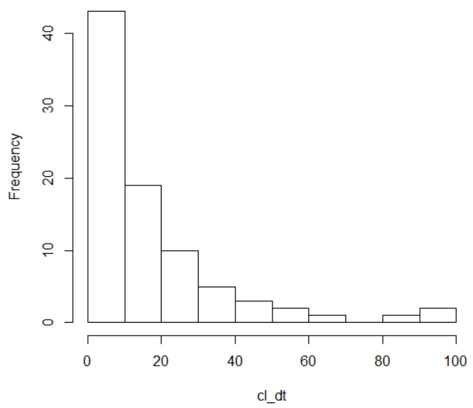
c) Q-Q plot



d) P-P plot

Figure 5.2 Goodness of Fit Graphs for the Real-Life Motor Insurance Claims Data**.**

The dataset from the fourth quarter was used to find each upper bound of the ruin probability for the next 4 quarters. Figure 5.3 show that the minimum and maximum values of this dataset are 1.1889 and 98.0688 respectively. The mean and standard deviation of this dataset are 17.7649 and 19.6756. Histogram and skewness's value (2.3562) show that the distribution of this dataset is still the right-skewness distribution. Figure 5.4 shows that lognormal distribution is still closest to the reference line. Thus, characteristic of this fourth quarter dataset is still explained by lognormal distribution with log data parameters $\hat{\mu} = 2.4171$ and $\hat{\sigma} = 0.9547$ estimated by maximum likelihood method by using R programming.



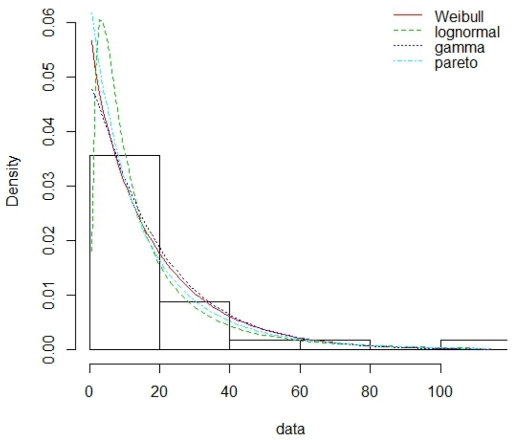
a) Histogram

```
min:                 1.18889
max:                98.06877
median:             10.28151
mean:               17.76487
standard deviation: 19.67565
skewness:            2.35617
kurtosis:            9.03775
```

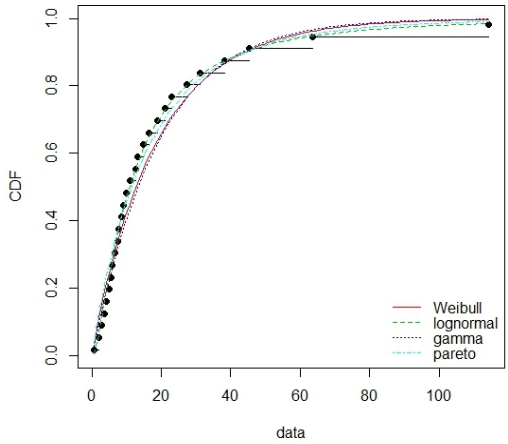
b) Summary statistics

Figure 5.3 The Descriptive Statistics for the Fourth Quarter of Real-Life Motor Insurance Claims Data**.**

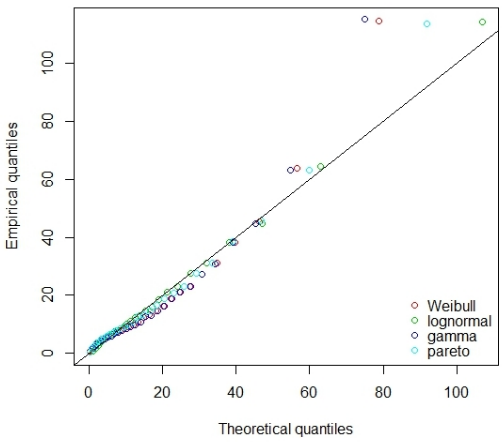
The number of claims occurring in each quarter was assumed to be i.i.d. with a Poisson distribution for which the mean was estimated as the average value of the 4 quarters of real-life claims data (the result was 83.25). The other factors for finding the upper bound of the ruin probability for this broker were an initial bond interest rate at 0.03 (based on Example 1), and $\overline{D}(x) = (1+kx)^{-1}$, $x \geq 0$ (which is NWU) was selected. The initial surplus was assumed to be 50 and 500 million baht, which is around 10 and 100 times the total value of the 334 claims in 1 year. The dealer was assumed to invest in 2 categories: bonds and stocks and similarly, the retention level of reinsurance were set to show the behavior of upper bounds for each set of values for these variables.



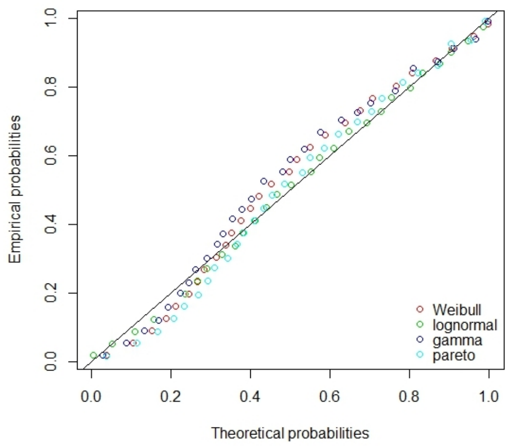
a) Histogram and theoretical densities



b) Empirical and theoretical CDFs



c) Q-Q plot



d) P-P plot

Figure 5.4 Goodness of Fit Graphs for the Fourth Quarter with the Real-Life Motor Insurance Claims Data.

Table 5.3 The Upper Bounds of the Finite Time Ruin Probability for Real-Life Motor Insurance Claims Data with Various Sets of Parameter Values for the Next 4 Quarters (Time Period $n =$ 1, 2, 3, 4).

U_0 (Million bath)	α	b	Upper bounds $n=1$	$n=2$	$n=3$	$n=4$
50	688.073 (p_n = 0.25)	0.6	0.1167	0.1167	0.1167	0.1167
		1	0.2662	0.2662	0.2663	0.2663
	2,064.220 (p_n = 0.75)	0.6	0.1098	0.1098	0.1098	0.1098
		1	0.2532	0.2532	0.2532	0.2533
500	6,880.733 (p_n = 0.25)	0.6	0.0132	0.0132	0.0132	0.0132
		1	0.0359	0.0359	0.0359	0.0359
	20,642.201 (p_n = 0.75)	0.6	0.0123	0.0123	0.0123	0.0123
		1	0.0336	0.0336	0.0336	0.0336

The results in Table 5.3 show that the variation of each factor affected the value of the upper bound in the same way as the results in Table 5.2, in that the upper bound values for the finite time ruin probability increased as the number of time periods (n) or reinsurance retention level (b) increased, and the upper bound values for finite time ruin probability decreased as the initial surplus (U_0) or the stock investment value (α) increased.

CHAPTER 6

CONCLUSIONS

6.1 Conclusions

Two upper bounds for the ruin probabilities of the proposed discrete time risk model are derived in this study, both of which were created using the inductive method and it is assumed that the retention level of reinsurance and the amount of stock investment in each time period are controlled to be constant values. The first derived upper bound can be used with both finite time and ultimate (Infinite Time) ruin probabilities under the condition that the m.g.f. of the distribution of total claims amount must exist (i.e. the value of Lundberg's coefficient must be found). This upper bound can be viewed as an extension of the ideas of Diasparra and Romera (2009) and Jasiulewicz and Kordecki (2015) by adding investment in stocks and shares to their risk models. The second upper bound is formed by using an NWU distribution, as per the idea of Willmot (1994). This upper bound can be used with the finite time ruin probability only and it is able to find the values even without the m.g.f. of the total claims amount value. However, the second upper bound still has an additional condition on the total claims amount, i.e, the total claims amount in each time period must occur according to the sum of the i.i.d. claim amounts and the number of claims occurring in each period of time must be an i.i.d. random variable.

The proposed discrete time risk model is different from other studies because two controlling factors: proportional reinsurance and investment are added to the classical discrete time risk model. In terms of the second controlling factor (investment), insurers are allowed to invest in two assets: bonds with a finite countable number of possible values of interest rate following a time-homogeneous Markov chain and stocks with returns on investment driven by discrete time interval. The ruin probability created from proposed discrete time risk model is presented in a recursive form in the case of finite time and as an integral equation in the case of infinite time. To calculate the values of the ruin probabilities based on the derived formula is quite challenging or even impossible, but the form in the present study is useful for deriving the upper bound of ruin probability.

For the cases of the two presented numerical examples, assumptions are raised for calculating the values of the upper bounds in order to consider the characteristics the latter derived in this study. In Example 1, the total claims amount is assumed to be an exponential distribution that is able to find the adjustment coefficient value in order to show the upper bound created in Theorem 4. In the second example, the claim amounts are assumed to follow a Pareto distribution for which we cannot find the adjustment coefficient in order to show the upper bound derived in Theorem 5. The output from the two numerical examples shows that the upper bounds responded to the changes in the two additional controlled factors inserted in the proposed risk model in the same direction; in other words, when the insurer increases the retention level of reinsurance, the upper bound values increase. On the other hand, increasing value of the second controlling factor, investment, was found to reduce the values of the upper bound. Besides the effects of the two controlling factors, when an insurance company increases its initial surplus, this will lower the values of the upper bound of the ruin probability. Furthermore, the application of the proposed upper bounds to real-life claims data estimated using a lognormal distribution supports the conclusions of the numerical study.

6.2 Future Studies

This study may be extended in the future as follows:

1) Find the optimal values of the retention level of reinsurance and the amount of money invested in the stock in each time period.

2) Remove some of the restrictions on the conditions used in this study, such as setting the retention level of reinsurance and the amount of money invested in the stock in each time period so they are not constant values.

3) Generalize the risk model by adding another factor such as dividends.

BIBLIOGRAPHY

Apostol, T. M. 1974. **Mathematical Analysis**. Cambridge: Pearson.

Booth, P. et al. 2005. **Modern Actuarial Theory and Practice**. New York: Chapman & Hall/CRC.

Bowers Jr., N. L.; Gerber, H. U.; Hickman, J. C.; Jones, D. A. and Nesbitt, C J. 1997. **Actuarial Mathematics**. Schaumburg, IL: The Society of Actuaries.

Cai, J. 2002. Discrete Time Risk Models under Rates of Interest. **Probability in the Engineering and Information Sciences**. 16 (July): 309-324**.**

Cai, J. and Dickson, D. C. M. 2004. Ruin Probabilities with a Markov Chain Interest Model. **Insurance: Mathematics and Economics**. 35 (June): 513-525.

Cai, J. and Garrido, J. 1999. A Unified Approach to the Study of Tail Probabilities of Compound Distributions. **Journal of Applied Probability**. 36 (December): 1058-1073.

Cai, J. and Wu, Y. 1997. Some Improvements on the Lundberg Bound for the Ruin Probability. **Statistics and Probability Letters**. 33 (May): 395-403.

Challa, A. 2012. **Insurance Models and Risk-function Premium Principle**. Retrieved June 26, 2017 from https://www2.warwick.ac.uk/fac/cross_fac/complexity/study/emmcs/outcomes/studentprojects/challa_m2.pdf.

Diasparra, M. A. and Romera, R. 2009. Bounds for the Ruin Probability of a Discrete-time Risk Process. **Journal of Applied Probability**. 46 (March): 99-112.

Dickson, D. C. M. 1994. An Upper Bound for the Probability of Ultimate Ruin. **Scandinavian Actuarial Journal**. 2 (July): 131-138.

Dickson, D. C. M. 2005. **Insurance Risk and Ruin**. New York: Cambridge University Press.

Goovaerts, M. J. and Vyncke, D. 2006. Reinsurance. In **Encyclopedia of Actuarial Science**. Vol 2. Chichester: Wiley. Pp. 1153-1160.

Jasiulewicz, H. and Kordecki, W. 2015. Ruin Probability of a Discrete-time Risk Process with Proportional Reinsurance and Investment for Exponential and Pareto Distribution. **Operations Research and Decisions**. 25 (March): 17-38.

Kaas, R.; Goovaerts, M.; Dhaene, J. and Denuit, M. 2008. **Modern Actuarial Risk Theory Using R**. New York: Springer-Verlag Berlin Heidelberg.

Kalashnikov, V. 1999. Bounds for Ruin Probabilities in the Presence of Large Claims and Their Comparison. **North American Actuarial Journal**. 3 (February): 116-128.

Life Office Management Association. 2000. **Insurance Company Operations.** Vancouver, Canada: LOMA Education and Training.

Lin, X.; Dongjin, Z. and Yanru, Z. 2015. Minimizing Upper Bound of Ruin Probability under Discrete Risk Model with Markov Chain Interest Rate. **Communications in Statistics – Theory and Methods**. 44 (April): 810-822.

Lin, X. S. 2006. **Introductory Stochastic Analysis for Finance and Insurance**. Chichester, UK: Wiley.

Liu, C. S. and Yang, H. 2013. Optimal Investment for an Insurer to Minimize its Probability of Ruin. **North American Actuarial Journal**. 8 (February): 11-31.

Mennis, E. A. and Ryals, S. D. 1992. Insurance Company Investment Management Handbook. New York: New York Institute of Finance.

Nissim, D. 2010. **Analysis and Valuation of Insurance Companies**. New York: Center for Excellence in Accounting & Security Analysis.

Office of Insurance Commission. 2013. **Comparative Investment Assets of Life and Non-Life Insurance Business**. Retrieved June 10, 2017 from http://www1.oic.or.th/th/statistics/yearly.php.

Outreville, J. F. 1998. Theory and Practice of Insurance. New York: Springer.

Promislow, S. D. 2011. **Fundamentals of Actuarial Mathematics**. Chichester, UK: Wiley.

Reilly, F. K. and Nortor, E. A. 2006. **Cram101 Textbook Outlines to Accompany Investments**. Ventura, CA: Academic Internet Publishers.

Tapiero, C. S. 1998. **Applied Stochastic Models and Control for Finance and Insurance**. Dordrecht, the Netherlands: Kluwer Academic Publishers.

Tse, Y. K. 2009. **Nonlife Actuarial Models**. New York: Cambridge University Press.

Vassiliou, P. C. G. 2010. **Discrete-Time Asset Pricing Models in Applied Stochastic Finance**. Chicherter, UK: Wiley.

Willmot, G. E. 1994. Refinements and Distributional Generalizations of Lundberg's Inequality. **Insurance: Mathematics and Economics**. 15 (October): 49-63.

Willmot, G. E. and Lin, X. 1994. Lundberg Bounds on the Tails of Compound Distributions. **Journal of Applied Probability**. 31 (September): 743-756.

Willmot, G. E. and Lin, X. 1996. Bounds on the Tails of Convolutions of Compound Distributions. **Insurance: Mathematics and Economics**. 18 (May): 29-33.

APPENDICES

APPENDIX A

The Lebesgue Dominated Convergence Theorem

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The Lebesgue Dominated Convergence Theorem

Theorem. Let $\{f_n\}$ be a sequence of Lebesgue-integrable functions on an interval I, and assume that

1) $\{f_n\}$ converges almost everywhere on I to a limit function f, and

2) there is a nonnegative function g in $L(I)$ such that, for all $n \geq 1$, $|f_n(x)| \leq g(x)$ almost everywhere on I.

Subsequently, the limit function $f \in L(I)$, the sequence $\left\{\int_I f_n\right\}$ converges, and

$$\int_I f = \lim_{n\to\infty} \int_I f_n .$$

APPENDIX B

The Proofs of Lundberg's Upper Bound

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The Proofs of Lundberg's Upper Bound

The Proof of Theorem 1

Consider Equations (2.4) and (2.5) in that $\lim_{t\to\infty}\psi_t(u)=\psi(u)$ (Tse, 2009). Thus, it is sufficient to prove that $\psi_t(u)\le\exp(-R_0u)$. In addition, consider the classic risk model in Equation (2.1) and the finite time ruin probability in Equation (2.5) such that if ruin occurs, it must occur at once for the time at which claim amount Y_i arrives $T_i \le t$. Therefore, the finite time ruin probability in Equation (2.5) can be defined as follows (Dickson, 2005):

$$\psi_t(u)=\Pr\left[U_{T_i}<0 \text{ for some } 0<T_i\le t \mid U_0=u\right].$$

Consider the ruin probability when the first claim Y_1 occurs as

$$\begin{aligned}\psi_{T_1}(u)&=\Pr\left[U_{T_1}<0 \mid U_0=u\right],\\ &=\int_0^\infty \Pr\left[U_{T_1}<0 \mid T_1=t,U_0=u\right]\Pr\left[T_1=t\right]dt.\end{aligned}$$

Since T_1 is the time until the first claim occurs, then the distribution of T_1 is exponential with parameter λ (Dickson, 2005). Thus,

$$\begin{aligned}\psi_{T_1}(u)&=\int_0^\infty \Pr\left[U_{T_1}<0 \mid T_1=t,U_0=u\right]\lambda e^{-\lambda t}dt,\\ &=\int_0^\infty \Pr\left[u+ct-Y_1<0 \mid T_1=t,U_0=u\right]\lambda e^{-\lambda t}dt. \qquad \text{(B.1)}\end{aligned}$$

From $P(y) = \Pr(Y_i \leq y)$; $y \geq 0$, then consider from Equation (B.1) that

$$\Pr\left[u + ct - Y_1 < 0\right] = \Pr\left[Y_1 > u + ct\right] = \int_{u+ct}^{\infty} dP(y) . \tag{B.2}$$

Since $u + ct - y \leq 0$, then $\exp\left[-R_0\left(u + ct - y\right)\right] \geq 1$. (B.3)

From Equation (B.3), we can write Equation (B.2) as

$$\begin{aligned}
\Pr\left[u + ct - Y_1 < 0\right] &\leq \int_{u+ct}^{\infty} \exp\left[-R_0\left(u + ct - y\right)\right] dP(y) , \\
&= \exp\left[-R_0\left(u + ct\right)\right] \int_{u+ct}^{\infty} \exp\left[R_0 y\right] dP(y) , \\
&\leq \exp\left[-R_0\left(u + ct\right)\right] \int_{0}^{\infty} \exp\left[R_0 y\right] dP(y) , \\
&= \exp\left[-R_0\left(u + ct\right)\right] M_{Y_1}(R_0) , \qquad \text{(B.4)}
\end{aligned}$$

Replace Equation (B.4) in Equation (B.1) such that

$$\begin{aligned}
\psi_{T_1}(u) &\leq \int_{0}^{\infty} \exp\left[-R_0\left(u + ct\right)\right] M_{Y_1}(R_0) \lambda e^{-\lambda t} dt , \\
&= \exp\left[-R_0 u\right] M_{Y_1}(R_0) \lambda \int_{0}^{\infty} \exp\left[-t\left(cR_0 + \lambda\right)\right] dt , \\
&= \exp\left[-R_0 u\right] M_{Y_1}(R_0) \lambda \left. \frac{\exp\left[-t\left(cR_0 + \lambda\right)\right]}{-\left[cR_0 + \lambda\right]} \right|_{0}^{\infty} , \\
&= \frac{\exp\left[-R_0 u\right] M_{Y_1}(R_0) \lambda}{\left[cR_0 + \lambda\right]} . \qquad \text{(B.5)}
\end{aligned}$$

From Remark 1, replace $c = (1 + \theta)\mu\lambda$ and $M_{Y_1}(R_0) = 1 + (1 + \theta)\mu R_0$ in Equation (B.5), resulting in

$$\psi_{T_1}(u) \le \frac{\exp\left[-R_0 u\right] M_{Y_1}(R_0)\lambda}{\left[(1+\theta)\mu\lambda R_0 + \lambda\right]},$$

$$= \frac{\exp\left[-R_0 u\right]\left[1+(1+\theta)\mu R_0\right]\lambda}{\left[(1+\theta)\mu\lambda R_0 + \lambda\right]},$$

$$= \exp\left[-R_0 u\right]. \tag{B.6}$$

By induction, we suppose that

$$\psi_{T_n}(u) \le \exp\left[-R_0 u\right]. \tag{B.7}$$

Consider the ruin probability within the (n+1)$^{\text{th}}$ claim.

Suppose the first claim occurs at time $t > 0$ and that the amount of this claim is y, thus

$$\begin{aligned}
\psi_{T_{n+1}}(u) &= \Pr\left[U_{T_i} < 0 \text{ for some } i \le n+1 \,|\, U_0 = u\right], \\
&= \int_0^\infty \int_0^\infty \Pr\left[U_{T_1} < 0 \,|\, T_1 = t, Y_1 = y, U_0 = u\right] \Pr\left[Y_1 = y\right] \Pr\left[T_1 = t\right] dy dt, \\
&= \int_0^\infty \int_0^\infty \Pr\begin{bmatrix} U_{T_i} < 0 \text{ for some } i \le n+1 \\ | \, T_1 = t, Y_1 = y, U_0 = u \end{bmatrix} dP(y)\lambda e^{-\lambda t} dt.
\end{aligned} \tag{B.8}$$

Since $U_{T_i} = u + cT_i - y$, and if ruin occurs at the $(n+1)^{\text{th}}$ claim time, then it occurs at the first time or the n^{th} claim time. Hence,

$$\Pr\left[U_{T_i} < 0 \text{ for some } i \le n+1 \,|\, T_1 = t, Y_1 = y, U_0 = u\right]$$

$$= \begin{cases} 1 & \text{if } u + ct - y < 0, \\ \psi_{T_n}(u + ct - y) & \text{if } u + ct - y \ge 0. \end{cases} \tag{B.9}$$

By putting Equation (B.9) in Equation (B.8), we obtain

$$\psi_{T_{n+1}}(u) = \int_0^\infty \int_{u+ct}^\infty 1 \, dP(y)\lambda e^{-\lambda t} dt + \int_0^\infty \int_0^{u+ct} \psi_{T_n}(u + ct - y) \, dP(y)\lambda e^{-\lambda t} dt.$$

Using the Equations (B.3) and (B.7) at the first and second terms respectively, we can achieve

$$
\begin{aligned}
\psi_{T_{n+1}}(u) &\le \int_0^{\infty}\int_{u+ct}^{\infty} \exp\left[-R_0\left(u+ct-y\right)\right] dP(y)\lambda e^{-\lambda t}dt \\
&\quad + \int_0^{\infty}\int_0^{u+ct} \exp\left[-R_0\left(u+ct-y\right)\right] dP(y)\lambda e^{-\lambda t}dt, \\
&\le \int_0^{\infty}\int_0^{\infty} \exp\left[-R_0\left(u+ct-y\right)\right] dP(y)\lambda e^{-\lambda t}dt\,, \\
&= \int_0^{\infty} \exp\left[-R_0\left(u+ct\right)\right] M_{Y_1}(R_0)\lambda e^{-\lambda t}dt \qquad \text{, from (B.4)} \\
&= \exp\left[-R_0 u\right] \qquad \text{, from (B.6))} \qquad\qquad \text{(B.10)}
\end{aligned}
$$

□

The proof of Theorem 2

From $\lim_{n\to\infty} \psi_n(u,i_s) = \psi(u,i_s)$ in Equation (2.17), it is sufficient to prove that $\psi_n(u,i_s) \le \exp(-R_0 u)$.

From Equation (2.15), we obtain

$$\psi_n(u) = \Pr\left\{ \bigcup_{k=1}^{n} (U_k < 0) \,|\, U_0 = u \right\}.$$

Consider that the ruin probability the 1st claim Y_1 occurs as

$$\begin{aligned}
\psi_1(u) &= \Pr\{U_1 < 0 \,|\, U_0 = u\}, \\
&= \Pr\{u + c - Y_1 < 0\}, \\
&= \Pr\{Y_1 > u + c\}, \\
&= \int_{u+c}^{\infty} dP(y). && \text{(B.11)}
\end{aligned}$$

Since $u + c - y \le 0$, then $\exp\left[-R_0(u+c-y)\right] \ge 1$ (B.12)

From Equations (B.11) and (B.12), we get

$$\begin{aligned}
\psi_1(u) &\le \int_{u+c}^{\infty} \exp\left[-R_0(u+c-y)\right] dP(y), \\
&\le \int_{0}^{\infty} \exp\left[-R_0(u+c-y)\right] dP(y), \\
&= \exp\left[-R_0 u\right] \int_{0}^{\infty} \exp\left[R_0(y-c)\right] dP(y), \\
&= \exp\left[-R_0 u\right] E\left[\exp\{R_0(y-c)\}\right], \\
&= \exp\left[-R_0 u\right]. && \text{(from Equation (2.18))}
\end{aligned}$$

By the induction approach, we suppose that

$$\psi_n(u) \le \exp\left[-R_0 u\right]. \tag{B.13}$$

Consider the ruin probability for the (n+1)th claim.

From Equation (2.15), we can write

$$\psi_{n+1}(u) = \Pr\left\{\bigcup_{k=1}^{n+1}\left(U_k < 0\right) | U_0 = u\right\},$$

$$= \int_0^\infty \Pr\left[\bigcup_{k=1}^{n+1}\left(U_k < 0\right) | Y_1 = y, U_0 = u\right] dy. \tag{B.14}$$

Since $U_n = u + cn - \sum_{i=1}^{n} Y_i$, and if ruin occur at the $(n+1)^{\text{th}}$ claim time, it occurs at the first time or the n^{th} claim time. Hence,

$$\Pr\left\{\bigcup_{k=1}^{n+1}\left(U_k < 0\right) | U_0 = u\right\} = \begin{cases} 1 & \text{if } u + c - y < 0, \\ \psi_n\left(u + c - y\right) & \text{if } u + c - y \ge 0. \end{cases} \tag{B.15}$$

From Equation (B.15), we can rewrite Equation (B.14) such that

$$\psi_{n+1}(u) = \int_0^\infty \Pr\left[\bigcup_{k=1}^{n+1}\left(U_k < 0\right) | Y_1 = y, U_0 = u\right] dP(y),$$

$$= \int_{u+c}^{\infty} 1 dP(y) + \int_0^{u+c} \psi_n\left(u + c - y\right) dP(y).$$

Replace Equations (B.12) and (B.13) at the first and second terms, respectively, to give

$$\psi_{n+1}(u) \le \int_{u+c}^{\infty} \exp\left[-R_0\left(u + c - y\right)\right] dP(y) + \int_0^{u+c} \exp\left[-R_0\left(u + c - y\right)\right] dP(y),$$

$$= \int_0^\infty \exp\left[-R_0\left(u + c - y\right)\right] dP(y),$$

$$= \exp\left[-R_0 u\right] \int_0^\infty \exp\left[R_0\left(y - c\right)\right] dP(y),$$

$$= \exp\left[-R_0 u\right] E\left[\exp\left\{R_0\left(y-c\right)\right\}\right],$$

$$= \exp\left[-R_0 u\right]. \qquad \text{(from Equation (2.18))}$$

□

BIOGRAPHY

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