Songklanakarin J. Sci. Technol.
44 (4), 987–999, Jul. – Aug. 2022

http://www.sjst.psu.ac.th

Original Article

Bipolar picture fuzzy sets and relations with applications

Waheed Ahmad Khan[1*], Khurram Faiz[1], and Abdelghani Taouti[2]

[1] *Department of Mathematics, University of Education Lahore, Attock Campus, Pakistan*

[2] *ETS-Maths and NS Engineering Division, Higher Colleges of Technology, Sharjah, United Arab Emirates*

Received: 4 November 2021; Revised: 27 January 2022; Accepted: 15 March 2022

Abstract

The notions of both the bipolar fuzzy sets and picture fuzzy sets have been studied by many authors, the bipolar picture fuzzy set is the nice combination of these two notions. Basically, the concepts we present in our study are the direct extensions of both the bipolar fuzzy sets and picture fuzzy sets. In this study, we add few more operations and results in the theory of the bipolar picture fuzzy sets. We also initiate the notion of bipolar picture fuzzy relations along with their applications. We present numerous basic operations along with the algebraic sums, bounded sums, algebraic product, bounded difference on bipolar picture fuzzy sets. Different types of distances between two bipolar picture fuzzy sets are also addressed. We provide the application of bipolar picture fuzzy sets towards decision making theory along with its algorithm. Afterward, we introduce different types of bipolar picture fuzzy relations like bipolar picture fuzzy reflexive, symmetric and transitive relations. Subsequently, we introduce the concepts of the bipolar picture fuzzy equivalence relation and partition. We also produce numerous interesting results based on these relations. Finally, we establish the criteria for the detection of covid-19 at the base of bipolar picture fuzzy relations.

Keywords: bipolar picture fuzzy sets, ring sum of bipolar picture fuzzy sets, bipolar picture fuzzy equivalence relation, bipolar picture fuzzy partition, distances between bipolar picture fuzzy sets

1. Introduction

Zadeh (Zadeh, 1965) introduced the notion of fuzzy sets (FSs) in 1965 and most of the theories ware given fuzziness and that introduced the uncertainty situation. After this, fuzzy sets have been widely used in different fields of life such as medical, management sciences, social sciences etc. Zadeh (Zadeh, 1971) also introduced the notions of fuzzy pre-ordering, fuzzy ordering and similarity relation. Moreover, fuzzy linear ordering was also established and numerous characteristics of all types of fuzzy ordering were investigated in Zadeh (1971). The term fuzzy set was further generalized by allocating the membership value in [-1,1] instead of [0, 1] and was termed bipolar fuzzy sets (BPFSs) (Zhang, 1998). BPFSs were proven a useful tool for modelling the problems existing in the multiagent decision analysis. Extension of BPFSs termed bipolar complex fuzzy sets and its application were introduced in Alkour, Massa'deh, and Ali (2020). In which ranges of values are extended to $[0,1]e^{i\alpha[0,1]}$ and $[-1,0]e^{i\alpha[-1,0]}$ for both positive and negative membership functions, respectively, which was the replacement for [-1,0] x [0,1] in the BPFSs. Recently, the term Complex bipolar fuzzy sets and its application towards transport flow was discussed in Gulistan, Yaqoob, Elmoasry, and Alebraheem (2021). Similarly, the other generalization of BPFSs named bipolar Pythagorean fuzzy sets and its applications towards decision making theory were explored in Mandal (2021). On the other hands, different types of relations on BPFSs were discussed in Gerstenkorn and Ma'nko (1995). Basically, the term bipolar fuzzy relation (BPFR) is the direct extension of fuzzy relation. BPFRs were also given a name bi-fuzzy relations (Gerstenkorn & Ma'nko, 1995). The term equivalent of bipolar fuzzy relations along with some new types of BPFRs were explored in Duzdziak and Pekala, (2010).

*Corresponding author
Email address: sirwakhan2003@yahoo.com

Another generalization of the FSs termed intuitionistic fuzzy sets (IFSs) were introduced by Atanassov (1986). Numerous operations and characterizations were also discussed. Ezhilmaran *et al*. (Ezhilmaran & Sankar, 2015) introduced the concepts of bipolar intuitionistic fuzzy sets (BPIFSs) and bipolar intuitionistic fuzzy relations with some basic operations. Numerous intuitionistic fuzzy relations and their connections with the complementary relations were explored in Bustince and Burillo (1996). Direct extension of both the FSs and IFSs termed picture fuzzy sets (PFSs) was introduced by B. C. Cong (2014). Various basic operations were also defined and applied to PFSs. Numerous algebraic properties were discussed in Silambarasan (2021). Different types of picture fuzzy relations and their compositions were also addressed. Afterward, Chunxin *et al*. (Bo & Zhang, 2017) introduced a few new operations on picture fuzzy relations and discussed their important properties. Ashraf, Mahmood, Abdullah, and Khan (2019) introduced diverse approaches to solve multi-criteria group decision making problems in the frame of the picture fuzzy environment. Similarly, the concepts of similarity measure between PFSs based on relationship matrix has been proposed and applied towards multi-attribute decision making theory (Luo, Zhang & Fu, 2021). The notions of spherical fuzzy sets (SFSs) and T-spherical fuzzy sets (T-SFSs) have been introduced in Mahmood, Ullah, Khan, and Jan (2019). These are the generalizations of FS, IFS and PFS. On the other hand, a bipolar soft sets and T-bipolar soft sets along with its applications towards decision-making problems have been explored in Mehmood, (2020).

In the above paragraphs we have seen that BPFSs and PFSs are the more generalized forms of the FSs. The specific study of the bipolar picture fuzzy operators towards distance measures based on the bipolar picture fuzzy sets were described in Riaz, Garg, Farid, and Chinram (2021). However, in this manuscript, we introduce and discuss different types of relations based on the bipolar fuzzy sets (BPPFSs). Thus, our discussions about bipolar picture fuzzy sets (BPPFSs) are totally the set theoretic based. Moreover, we present the fresh concepts related with the BPPFRs. In our discussions, firstly, we introduce and apply some basic operations such as containment, equality, union, intersection, complement and *mth* power of bipolar picture fuzzy sets. In addition, some algebraic operations like algebraic and bounded sums, bounded difference and algebraic product are also studied. We also present some discussions about different types of distances between two BPPFSs. Subsequently, we present an application of BPPFSs towards decision making theory. In section 3, we introduce several relations on BPPFSs. We also discuss some properties of bipolar picture fuzzy equivalence relations. Subsequently, we present few characterizations of bipolar picture fuzzy partitions. At the end of section 3, we provide the criteria for the detection of covid-19 by using the bipolar picture fuzzy relations. We also observe that the results and applications produced novel.

2. Preliminaries

Throughout this section by X, we mean a universe of discourse.

Definition 1. (Zadeh, 1965) A FS S defined on X is represented by the collection

$$S=\{(x,\alpha_S(x)):x\in X,\alpha_S(x)\in[0,1]\}.$$

Definition 2. (Zimmermann, 2011) The cartesian product of the FSs $S_1,S_2,\ldots,\ S_n$ on $X_1,X2,\ldots,X_n$ is the FS on the product $X_1\times X_2\times\ldots\times X_n$ having a membership function $\mu_{(S_1\times\ldots\times S_n)}(x)=\{\min(\alpha_{S_i}(x_i)):x=(x_1,\ldots,x_n),x_i\in X_i\}$.

Definition 3. (Zimmermann, 2011) The mth power of any FS S of X is again a FS whose membership function is given by $\alpha_{S^m}(x)=\{[\alpha_S(x)]^m:x\in X\}$.

Definition 4. (Zimmermann, 2011) Let S and T be the two FSs. Then

(*i*) The algebraic sum of two FSs represented by $R=S+T$ is the fuzzy set $R=\{(x,\alpha_{S+T}(x)):x\in X\}$, where $\alpha_{S+T}(x)=\alpha_S(x)+\alpha_T(x)-\alpha_S(x)\cdot\alpha_T(x)$.

(*ii*) The bounded sum of two FSs $R=S\oplus T$ is the FS $R=\{(x,\alpha_{S\oplus T}(x)):x\in X\}$, where $\alpha_{S\oplus T}(x)=\min\{1,\alpha_S(x)+\alpha_T(x)\}$.

(*iii*) The bounded difference of two FSs $R=S\Theta T$ is the fuzzy set $R=\{(x,\alpha_{S\Theta T}(x)):x\in X\}$, where $\alpha_{S\Theta T}(x)=\max\{0,\alpha_S(x)+\alpha_T(x)-1\}$.

(*iv*) The algebraic product of two FSs $\mathrm{R}=\mathrm{S}\cdot\mathrm{T}$ is the fuzzy set $R=\{(x,\alpha_S(x)\cdot\alpha_T(x)):x\in X\}$.

Definition 5. (Zhang, 1998) A BPFS is the pair (α_P,α_N), where $\alpha^P:X\to[0,1]$ and $\alpha^N:X\to[0,1]$ represents any mappings.

Definition 6. (Azhagappan & Kamaraj, 2016) Bipolar fuzzy empty set (resp., the bipolar fuzzy whole set) represented by $0_S=(0_S^P,0_S^N)$ (resp., $1_S=(1_S^P,1_S^N)$) is a BPFS in X defined by $0_S^P=0=0_S^N$ (resp., $1_S^P=1$ and $1_S^N=-1$, for each $x\in X$.

Definition 7. (Samanta & Pal, 2014) For any two BPFs $S=(\alpha_S^P,\alpha_S^N)$ and $T=(\alpha_T^P,\alpha_T^N)$, we have

$$(S \cap T)(x) = ((\alpha_S^P(x) \wedge \alpha_T^P(x)), (\alpha_S^N(x) \vee \alpha_T^N(x)))$$
$$(S \cup T)(x) = ((\alpha_S^P(x) \vee \alpha_T^P(x)), (\alpha_S^N(x) \wedge \alpha_T^N(x))).$$

Definition 8. (Zhang, 1998) A mapping $S = (\alpha_S^P, \alpha_S^N) : X \times X \to [-1,0] \times [0,1]$ is the bipolar fuzzy relation (BPFR) on *X*, where $\alpha_S^P(x, y) \in [0,1]$ and $\alpha_S^N(x, y) \in [-1,0]$.

Definition 9. (Zhang, 1998) The empty BPFR (resp., the whole BPFR) is the BPFR on *X* which may be described by $\alpha_S^P(x, y) = 0 = \alpha_S^N(x, y)$ (resp., $\alpha_S^P(x, y) = 1$ and $\alpha_S^N(x, y) = -1$) for each $x, y \in X$.

Definition 10. (Atanassov, 1986) An IFS *S* in *X* is the collection $S = \{x, \alpha_S(x), \beta_S(x)) x \in X\}$, where $\alpha_S : X \to [0,1]$, $\beta_S : X \to [0,1]$ are the degrees of membership and the degree of non-membership of $x \in X$, respectively, while for each $x \in X$, $0 \le \alpha_S(x) + \beta_S(x) \le 1$.

Definition 11. (Ezhilmaran & Sankar, 2015) A BPIFS is the set $S = \{(x, \alpha_S^P(x), \alpha_S^N(x), \beta_S^P(x), \beta_S^N(x)) : x \in X\}$, where $\alpha_S^P : X \to [0,1]$, $\alpha_S^N : X \to [-1,0]$, $\beta_S^P : X \to [0,1]$ and $\beta_S^N : X \to [-1,0]$ are the mappings satisfying $0 \le \alpha_S^P(x) + \beta_P^S(x) \le 1$, $-1 \le \alpha_S^P(x) + \beta_P^S(x) \le 0$.

Definition 12. (Ezhilmaran & Sankar, 2015) A mapping $S = (\alpha_S^P, \alpha_S^N, \beta_S^P, \beta_S^N) : X \times X \to [-1,0] \times [0,1] \times [-1,0] \times [0,1]$ is the bipolar intuitionistic fuzzy relation on *X*, where $\alpha_S^P(x, y) \in [0,1]$, $\alpha_S^N(x, y) \in [0,1]$, $\beta_S^P(x, y) \in [0,1]$, and $\beta_S^N(x, y) \in [0,1]$.

Definition 13. (Coung, 2014) A PFS *S* on *X* is the collection $S = \{(x, \alpha_S(x), \gamma_S(x), \beta_S(x)) : x \in X\}$, where $\alpha_S(x) \in [0,1]$ is the degree of positive membership of *x* in *S*, and $\gamma_S(x) \in [0,1]$ represents the degree of neutral membership of *x* in *S* and $\beta_S(x) \in [0,1]$ the degree of negative membership of *x* in *S*, while α_S, γ_S and β_S satisfying $\alpha_S(x) + \gamma_S(x) + \beta_S(x) \le 1$, for all $x \in X$.

3. Bipolar Picture Fuzzy Sets (BPPFSs)

The term bipolar picture fuzzy sets (BPPFSs) is the direct generalization of both the BPFSs and PFSs. In this section, we present the theory and an application of BPPFSs. We introduce several basics and algebraic operations on BPPFSs and demonstrate all of these operations through Example 1. At the end of this section, we provide an application of a BPPFSs towards decision-making theory.

We begin our discussion with the definition of BPPFSs.

Definition 14. Let *X* be a nonempty set. A bipolar picture fuzzy set (BPPFS) on *X* is the collection $S = \{(x, \alpha_S^P(x), \alpha_S^N(x), \gamma_S^P(x), \gamma_S^N(x), \beta_S^P(x), \beta_S^N(x)) : x \in X\}$, where $\alpha_S^P : X \to [0,1]$, $\alpha_S^N : X \to [-1,0]$, $\gamma_S^P : X \to [0,1]$, $\gamma_S^N : X \to [-1,0]$, $\beta_S^P : X \to [0,1]$ and $\beta_S^N : X \to [-1,0]$ are the mappings with $0 \le \alpha_S^P(x) + \gamma_S^P(x) + \beta_S^P(x) \le 1$, $-1 \le \alpha_S^N(x) + \gamma_S^N(x) + \beta_S^N(x) \le 0$.

For each *x* in *X*, $\alpha_S^P(x)$ stands for the positive membership degree, $\beta_S^P(x)$ for the positive non-membership degree and $\gamma_S^P(x)$ for the positive neutral degree. Alternatively, $\alpha_S^N(x)$ represents the negative membership degree, $\beta_S^P(x)$ is the negative non-membership degree and $\gamma_S^P(x)$ is a negative neutral degree. On the other hand, if $\alpha_S^P(x) \ne 0$ while all other mappings are mapped to zero then it means that *x* has only a positive membership property of the BPPFS. Similarly, if $\alpha_S^N(x) \ne 0$ while all other mappings matched to zero (or equal to zero) then it reflects that *x* has only the negative membership property of a BPPFS. Also, if $\gamma_S^P(x) \ne 0$ and remaining mappings are mapped to zero then it reflects that *x* has only the positive neutral property of a BPPFS. By $\gamma_S^N(x) \ne 0$ and the other mapping goes to zero then we mean that *x* has only the negative neutral property of a BPPFS. However, if $\beta_S^P(x) \ne 0$ while all other mapping matched to zero then it implies that *x* has only the positive nonmembership property of a BPPFS. Finally, if $\beta_S^P(x) \ne 0$ while remaining are zero then it implies that *x* has only the negative nonmembership property in a BPPFS.

Now we introduce few basic operations like containment, equality, union, intersection and complement on a BPFS. We also apply these operations in Example 1.

Definition 15. Let S and T be the two BPPFSs. Then
1. $S \subseteq T$ if and only if $\alpha_S^P(x) \le \alpha_T^P(x),\ \alpha_S^N(x) \ge \alpha_T^N(x),\ \gamma_S^P(x) \le \gamma_T^P(x),\ \ \gamma_S^N(x) \ge \alpha_T^N(x)$ and $\beta_S^P(x) \ge \beta_T^P(x),$ $\beta_S^N(x) \le \beta_T^N(x),$ for all $x \in X.$
2. $S = T$ if and only if $S \subseteq T$ and $T \subseteq S.$
3. $S \bigcup T = \{\alpha_S^P(x) \vee \alpha_T^P(x), \alpha_S^N(x) \wedge \alpha_T^N(x), \gamma_S^P(x) \wedge \gamma_T^P(x), \gamma_S^N(x) \vee \gamma_T^N(x),\ \beta_S^T(x) \wedge \beta_T^P(x), \beta_S^N(x) \vee \beta_T^N(x) : x \in X\}$
4. $S \bigcap T = \{\alpha_S^P(x) \wedge \alpha_T^P(x), \alpha_S^N(x) \vee \alpha_T^N(x), \gamma_S^P(x) \wedge \gamma_T^P(x), \gamma_S^N(x) \vee \gamma_T^N(x),\ \beta_S^T(x) \vee \beta_T^P(x), \beta_S^N(x) \wedge \beta_T^N(x) : x \in X\}$
5. $S^c = \{(\beta_S^P(x), \beta_S^N(x), \gamma_S^P(x), \gamma_S^N(x), \alpha_S^P(x), \alpha_S^N(x)) : x \in X\}.$

Cartesian product and *m*th power of BPPFSs can be defined as follows.

Definition 16. Let $S_1,...,S_n$ be the BPPFSs in $X_1,...,X_n$. Then their cartesian product is a BPPFS in $X_1 \times ... \times X_n$ having membership function $\mu_{(S_1 \times .. \times S_n)}(x) = \{\min(\alpha_{S_i}^P(x_i)), \max(\alpha_{S_i}^N(x_i)), \min(\gamma_{S_i}^P(x_i)), \max(\gamma_{S_i}^N(x_i)), \max(\beta_{S_i}^P(x_i)),$ $\min(\beta_{S_i}^N(x_i)) : x = (x_1,...,x_n), x_i \in X_i\}.$

Definition 17. The *m*th power of a BPPFS S is a BPPFS with the membership function $\alpha_{S^m}^P(x) = [\alpha_S^P(x)]^m,\ \alpha_{S^m}^N(x) = [\alpha_S^N(x)]^m,$ $\gamma_{S^m}^P(x) = [\gamma_S^P(x)]^m,\ \gamma_{S^m}^N(x) = [\gamma_S^N(x)]^m,\ \beta_{S^m}^P(x) = [\beta_S^P(x)]^m,\ \beta_{S^m}^N(x) = [\beta_S^N(x)]^m,\ x \in X.$

Now we define several algebraic operations like algebraic sum, bounded sums, bounded difference, algebraic product etc. We also provide few results relevant to these operations. Moreover, we apply these operations in Example 1.

Definition 18. The algebraic sum of two BPPFSs $R = S + T$ is defined by

$$R = \{(x, \alpha_{S+T}^P(x), \alpha_{S+T}^N(x), \gamma_{S+T}^P(x), \gamma_{S+T}^N(x), \beta_{S+T}^P(x), \beta_{S+T}^N(x)) : x \in X\},\ \text{where}$$

$$\alpha_{S+T}^P(x) = (\alpha_S^P(x) + \alpha_T^P(x) - \alpha_S^P(x) \cdot \alpha_T^P(x))/2,\ \alpha_{S+T}^N(x) = (\alpha_S^N(x) + \alpha_T^N(x) - \alpha_S^N(x) \cdot \alpha_T^N(x))/2,$$

$$\gamma_{S+T}^P(x) = (\gamma_S^P(x) + \gamma_T^P(x) - \gamma_S^P(x) \cdot \gamma_T^P(x))/2,\ \gamma_{S+T}^N(x) = (\gamma_S^N(x) + \gamma_T^N(x) - \gamma_S^N(x) \cdot \gamma_T^N(x))/2,$$

$$\beta_{S+T}^P(x) = (\beta_S^P(x) + \beta_T^P(x) - \beta_S^P(x) \cdot \beta_T^P(x))/2,\ \beta_{S+T}^N(x) = (\beta_S^N(x) + \beta_T^N(x) - \beta_S^N(x) \cdot \beta_T^N(x))/2.$$

Theorem 1. The algebraic sum of two BPPFSs S and T is a BPPFS.

Proof. Let $S = \{x, \alpha_S^P(x), \alpha_S^N(x), \gamma_S^P(x), \gamma_S^N(x), \beta_S^P(x), \beta_S^N(x)\}$ and $T = \{x, \alpha_T^P(x),\ \alpha_T^P(x), \alpha_T^N(x), \gamma_T^P(x), \gamma_T^N(x), \beta_T^P(x),$ $\beta_T^N(x)\}$ be two bipolar picture fuzzy sets of X. We show that $S + T = R = \{(x, \alpha_R^P(x), \alpha_R^N(x), \gamma_R^P(x), \gamma_R^N(x), \beta_R^P(x),$ $\beta_R^N(x)) : x \in X\}$ is a BPPFS. Since elements of algebraic sum set generates R in the following way.

$$\alpha_R^P(x) = (\alpha_S^P(x) + \alpha_T^P(x) - \alpha_S^P(x) \cdot \alpha_T^P(x))/2,\ \alpha_R^N(x) = (\alpha_S^N(x) + \alpha_T^N(x) - \alpha_S^N(x) \cdot \alpha_T^N(x))/2,$$

$$\gamma_R^P(x) = (\gamma_S^P(x) + \gamma_T^P(x) - \gamma_S^P(x) \cdot \gamma_T^P(x))/2,\ \gamma_R^N(x) = (\gamma_S^N(x) + \gamma_T^N(x) - \gamma_S^N(x) \cdot \gamma_T^N(x))/2,$$

$$\beta_R^P(x) = (\beta_S^P(x) + \beta_T^P(x) - \beta_S^P(x) \cdot \beta_T^P(x))/2,\ \beta_R^N(x) = (\beta_S^N(x) + \beta_T^N(x) - \beta_S^N(x) \cdot \beta_T^N(x))/2.$$

We know that $R = \{(x, \alpha_{S+T}^P(x), \alpha_{S+T}^N(x), \gamma_{S+T}^P(x), \gamma_{S+T}^N(x), \beta_{S+T}^P(x), \beta_{S+T}^N(x)) : x \in X\}$. Hence $R = \{(x, \alpha_R^P(x), \alpha_R^N(x), \gamma_R^P(x),$ $\gamma_R^N(x), \beta_R^P(x), \beta_R^N(x)) : x \in X\}$ is another BPPFS. Which completes the proof.

Definition 19. The bounded sum of two BPPFSs S and T denoted by $S \oplus T$ is defined as

$$R = \{(x, \alpha_{S \oplus T}^P(x), \alpha_{S \oplus T}^N(x), \gamma_{S \oplus T}^P(x), \gamma_{S \oplus T}^N(x), \beta_{S \oplus T}^P(x), \beta_{S \oplus T}^N(x)) : x \in X\},\ \text{where}$$

$$\alpha_{S \oplus T}^P(x) = (\min\{1, \alpha_S^P(x) + \alpha_T^P(x)\})/2,\ \alpha_{S \oplus T}^N(x) = (\max\{-1, \alpha_S^N(x) + \alpha_T^N(x)\})/2,$$

$$\gamma_{S \oplus T}^P(x) = (\min\{1, \gamma_S^P(x) + \gamma_T^P(x)\})/2,\ \gamma_{S \oplus T}^N(x) = (\max\{-1, \gamma_S^N(x) + \gamma_T^N(x)\})/2,$$

$$\beta_{S \oplus T}^P(x) = (\min\{1, \beta_S^P(x) + \beta_T^P(x)\})/2,\ \beta_{S \oplus T}^N(x) = (\max\{-1, \beta_S^N(x) + \beta_T^N(x)\})/2.$$

Theorem 2. The bounded sum $R = S \oplus T$ of two BPPFSs S and T is a BPPFS.

Proof. Let $S = \{x, \alpha_S^P(x), \alpha_S^N(x), \gamma_S^P(x), \gamma_S^N(x), \beta_S^P(x), \beta_S^N(x)\}$ and $T = \{x, \alpha_T^P(x)\ , \alpha_T^P(x), \alpha_T^N(x), \gamma_T^P(x), \gamma_T^N(x), \beta_T^P(x),$ $\beta_T^N(x)\}$ be two BPPFSs of X. We have to prove that $R = \{(x, \alpha_R^P(x), \alpha_R^N(x), \gamma_R^P(x), \gamma_R^N(x), \beta_R^P(x), \beta_R^N(x)) : x \in X\}$. We assume that $R = S \oplus T$ then the elements of bounded sum set generating R are in the following ways.

$$\alpha_R^P(x)=\alpha_{S\oplus T}^P(x)=(\min\{1,\alpha_S^P(x)+\alpha_T^P(x)\})/2$$
$$\alpha_R^N(x)=\alpha_{S\oplus T}^N(x)=(\max\{-1,\alpha_S^N(x)+\alpha_T^N(x)\})/2$$
$$\gamma_R^P(x)=\gamma_{S\oplus T}^P(x)=(\min\{1,\gamma_S^P(x)+\gamma_T^P(x)\})/2$$
$$\gamma_R^N(x)=\gamma_{S\oplus T}^N(x)=(\max\{-1,\gamma_S^N(x)+\gamma_T^N(x)\})/2$$
$$\beta_R^P(x)=\beta_{S\oplus T}^P(x)=(\min\{1,\beta_S^P(x)+\beta_T^P(x)\})/2$$
$$\beta_R^N(x)=\beta_{S\oplus T}^N(x)=(\max\{-1,\beta_S^N(x)+\beta_T^N(x)\})/2.$$

We know that $R=\{(x,\alpha_{S\oplus T}^P(x),\alpha_{S\oplus T}^N(x),\gamma_{S\oplus T}^P(x),\gamma_{S\oplus T}^N(x),\beta_{S\oplus T}^P(x),\beta_{S\oplus T}^N(x)):x\in X\}$. Now, it becomes another BPPFS $R=\{(x,\alpha_R^P(x),\alpha_R^N(x),\gamma_R^P(x),\gamma_R^N(x),\beta_R^P(x),\ \beta_R^N(x)):x\in X\}$, which is bounded sum of two BPPFSs.

Definition 20. The bounded difference $S\Theta T$ is defined by

$$R=\{(x,\alpha_{S\Theta T}^P(x),\alpha_{S\Theta T}^N(x),\gamma_{S\Theta T}^P(x),\gamma_{S\Theta T}^N(x),\beta_{S\Theta T}^P(x),\beta_{S\Theta T}^N(x)):x\in X\}, \text{ where}$$
$$\alpha_{S\Theta T}^P(x)=\max\{0,\alpha_S^P(x)+\alpha_T^P(x)-1\},\ \alpha_{S\Theta T}^N(x)=\min\{0,\alpha_S^N(x)+\alpha_T^N(x)+1\},$$
$$\gamma_{S\Theta T}^P(x)=\max\{0,\gamma_S^P(x)+\gamma_T^P(x)-1\},\ \gamma_{S\Theta T}^N(x)=\min\{0,\gamma_S^N(x)+\gamma_T^N(x)+1\},$$
$$\beta_{S\Theta T}^P(x)=\max\{0,\beta_S^P(x)+\beta_T^P(x)-1\},\ \beta_{S\Theta T}^N(x)=\min\{0,\beta_S^N(x)+\beta_T^N(x)+1\}.$$

Theorem 3. Let *S* and *T* be two BPPFSs. Then their bounded difference $R=\ S\Theta T$ is also a BPPFS.

Proof. Let $S=\{x,\alpha_S^P(x),\alpha_S^N(x),\gamma_S^P(x),\gamma_S^N(x),\beta_S^P(x),\beta_S^N(x)\}$ and $T=\{x,\alpha_T^P(x)\ ,\alpha_T^P(x),\alpha_T^N(x),\gamma_T^P(x),\gamma_T^N(x),\beta_T^P(x),$ $\beta_T^N(x)\}$ be two bipolar picture fuzzy sets of *X*. We have to prove that $R=\{x,\alpha_R^P(x),\alpha_R^N(x),\gamma_R^P(x),\gamma_R^N(x),\beta_R^P(x),$ $\beta_R^N(x):x\in X\}$. We assume that $R=\ S\Theta T$, then the elements of bounded difference set generates *R* are

$$\alpha_R^P(x)=\alpha_{S\Theta T}^P(x)=\max\{0,\alpha_S^P(x)+\alpha_T^P(x)-1\}\ \alpha_R^N(x)=\alpha_{S\Theta T}^N(x)=\min\{0,\alpha_S^N(x)+\alpha_T^N(x)+1\}$$
$$\gamma_R^P(x)=\gamma_{S\Theta T}^P(x)=\max\{0,\gamma_S^P(x)+\gamma_T^P(x)-1\}\ \gamma_R^N(x)=\gamma_{S\Theta T}^N(x)=\min\{0,\gamma_S^N(x)+\gamma_T^N(x)+1\}$$
$$\beta_R^P(x)=\beta_{S\Theta T}^P(x)=\max\{0,\beta_S^P(x)+\beta_T^P(x)-1\}\ \beta_R^N(x)=\beta_{S\Theta T}^N(x)=\min\{0,\beta_S^N(x)+\beta_T^N(x)+1\}.$$

We know that $R=\{(x,\alpha_{S\Theta T}^P(x),\alpha_{S\Theta T}^N(x),\gamma_{S\Theta T}^P(x),\gamma_{S\Theta T}^N(x),\beta_{S\Theta T}^P(x),\beta_{S\Theta T}^N(x)):x\in X\}$. Now it becomes another BPPFS $R=\{x,\alpha_R^P(x),\alpha_R^N(x),\gamma_R^P(x),\gamma_R^N(x),\beta_R^P(x),\beta_R^N(x):x\in X\}$, which is bounded difference of two BPPFSs.

Definition 21. The algebraic product $R=S\cdot T$ is defined as

$$R=S\cdot T=\{\alpha_S^P(x)\cdot\alpha_T^P(x),-1\cdot\alpha_S^N(x)\cdot\alpha_T^N(x),\gamma_S^P(x)\cdot\gamma_T^P(x),-1\cdot\gamma_S^N(x)\cdot\gamma_T^N(x),\beta_S^P(x)\cdot\ \beta_T^P(x),\ -1\cdot\beta_S^N(x)\cdot\beta_T^N(x)\}.$$

Theorem 4. Let *S* and *T* be two BPPFSs. Then their algebraic product $R=S\cdot T$ is a BPPFS.

Proof. Let $S=\{x,\alpha_S^P(x),\alpha_S^N(x),\gamma_S^P(x),\gamma_S^N(x),\beta_S^P(x),\beta_S^N(x)\}$ and $T=\{x,\alpha_T^P(x)\ ,\alpha_T^P(x),\alpha_T^N(x),\gamma_T^P(x),\gamma_T^N(x),\beta_T^P(x),\beta_T^N(x)\}$ be two bipolar picture fuzzy sets of *X*. We have to show $R=\{x,\alpha_R^P(x),\alpha_R^N(x),\gamma_R^P(x),\gamma_R^N(x),\beta_R^P(x),\beta_R^N(x)\}$ We assume that $R=S\cdot T$, then elements of bounded difference set generates *R* by following way.

$$\alpha_R^P(x)=\alpha_S^P(x)\cdot\alpha_T^P(x),\ \alpha_R^N(x)=-1\cdot\alpha_S^N(x)\cdot\alpha_T^N(x),$$
$$\gamma_R^P(x)=\gamma_S^P(x)\cdot\gamma_T^P(x),\ \gamma_R^N(x)=-1\cdot\gamma_S^N(x)\cdot\gamma_T^N(x),$$
$$\beta_R^P(x)=\beta_S^P(x)\cdot\beta_T^P(x),\ \beta_R^N(x)=-1\cdot\beta_S^N(x)\cdot\beta_T^N(x).$$

We know algebraic product of two BPPFSs is that $R=\{(x,\alpha_{S\cdot T}^P(x),\alpha_{S\cdot T}^N(x),\gamma_{S\cdot T}^P(x),\ \gamma_{S\cdot T}^N(x),\beta_{S\cdot T}^P(x),\beta_{S\cdot T}^N(x)):x\in X\}$. Now, its becomes an other BPPFS $R=\{(x,\alpha_R^P(x),\ \alpha_R^N(x),\gamma_R^P(x),\gamma_R^N(x),\beta_R^P(x),\beta_R^N(x)):x\in X\}$, which is algebraic product of two BPPFS.

Example 1. Let us consider BPPFSs S_1 and S_2 on *X*. The full description of BPPFS S_1 is $S_1=\{x_1,\alpha_{S_1}^P(x_1),\alpha_{S_1}^N(x_1),\gamma_{S_1}^P(x_1),\gamma_{S_1}^N(x_1),\beta_{S_1}^P(x_1),\beta_{S_1}^N(x_1)\}$ and S_2 is $S_2=\{x_2,\alpha_{S_2}^P(x_2),\alpha_{S_2}^N(x_2),\gamma_{S_2}^P(x_2),\gamma_{S_2}^N(x_2),\beta_{S_2}^P(x_2),\beta_{S_2}^N(x_2)\}$. Let $S_1=\{(0.3,-0.2,0.4,\ -0.3,0.2,-0.2)\}$ and $S_2=\{(0.2,-0.1,0.2,-0.3,0.5,-0.4)\}$ be the corresponding values for X_1 and X_2, respectively. Here, $S_1\le S_2$ because $0.2\le 0.3,$ $-0.1\ge -0.2,\ 0.2\le 0.4,\ -0.3\ge -0.3,$ and $0.5\ge 0.2,\ -0.4\le -0.2.$ Then

1. The union $S_1 \cup S_2 = \{(0.3, -0.1, 0.4, -0.3, 0.2, -0.4)\}$.
2. The intersection $S_1 \cap S_2 = \{(0.2, -0.2, 0.4, -0.3, 0.5, -0.2)\}$.
3. The complement of S_1 is $\mathrm{co}(S_1) = \{0.2, -0.2, 0.4, -0.3, 0.3, -0.2\}$.
4. The complement of S_2 is $\mathrm{co}(S_2) = \{0.5, -0.4, 0.2, -0.3, 0.5, -0.2\}$.
5. The cartesian product $\mu_{S_1 \times S_2}(x) = \{0.2. -0.2, 0.2, -0.3, 0.5, -0.2\}$.
6. The *mth* power of BPPFS for $m = 3$ is $\alpha^P_{S_1^3}(x) = 0.027$, $\alpha^N_{S_1^3}(x) = -0.008$, $\gamma^P_{S_1^3}(x) = 0.064$, $\gamma^N_{S_1^3}(x) = -0.027$, $\beta^P_{S_1^3}(x) = 0.008$, $\beta^N_{S_1^3}(x) = -0.0008$.
7. The algebraic sum $R = S + T$ is defined as $R = \{0.22, -0.14, 0.26, -0.255, 0.3, -0.26\}$.
8. The bounded sum $R = S \oplus T$ is defined as $R = \{0.25, -0.15, 0.3, -0.3, 0.35, -0.3\}$.
9. The bounded difference $R = S \oplus T$ is defined as $R = \{0, -0, 0, -0, 0, -0\}$.
10. The algebraic product $R = S \cdot T$ is defined as $R = \{0.06, -0.02, 0.08, -0.09, 0.1, -0.08\}$.

3.1 Distances between bipolar picture fuzzy sets

The normalized hamming distance and normalized Euclidean distance between two BPPFSs are defined as follows. Distances between two BPPFSs S and T in $X = \{x_1, x_2, \ldots, x_n\}$ may be defined as

• Normalized Hamming distance $d_P(S,T)$

$$d_P(S,T) = \frac{1}{n}\sum_{i=1}^{n} (|\alpha^P_S(x_i) - \alpha^P_T(x_i)| + |\alpha^N_S(x_i) - \alpha^N_T(x_i)| + |\gamma^P_S(x_i) - \gamma^P_T(x_i)| + |\gamma^N_S(x_i) - \gamma^N_T(x_i)| + |\beta^P_S(x_i) - \beta^P_T(x_i)| + |\beta^N_S(x_i) - \beta^N_T(x_i)|)$$

• Normalized Euclidean distance $e_P(S,T)$ is $e_P(S,T) = \sqrt{K}$, where

$$K = \frac{1}{n}\sum_{i=1}^{n} ([\alpha^P_S(x_i) - \alpha^P_T(x_i)]^2 + [\alpha^N_S(x_i) - \alpha^N_T(x_i)]^2 + [\gamma^P_S(x_i) - \gamma^P_T(x_i)]^2 + [\gamma^P_S(x_i) - \gamma^P_T(x_i)]^2 + [\beta^P_S(x_i) - \beta^P_T(x_i)]^2 + [\beta^N_S(x_i) - \beta^N_T(x_i)]^2)$$

Proposition 1.

1. $d_P(S_1,S_3) \le d_P(S_1,S_2) + d_P(S_2,S_3)$ and similarly $e_P(S_1,S_3) \le e_P(S_1,S_2) + e_P(S_2,S_3)$.
2. $d_P(S_i,S_j) = d_P(S_j,S_i)$ and similarly $e_P(S_i,S_j) = e_P(S_j,S_i)$
3. $d_P(S_i,S_i) = 0$ and same on $e_P(S_i,S_i) = 0$

We provide an example in support of the above proposition.

Example 2. Let S_1, S_2 and S_3 be the BPPFSs in $X = \{x_1, x_2, x_3\}$. Their complete descriptions are as follows:

$$S_1 = \{x_1, \alpha^P_{S_1}(x_1), \alpha^N_{S_1}(x_1), \gamma^P_{S_1}(x_1), \gamma^N_{S_1}(x_1), \beta^P_{S_1}(x_1), \beta^N_{S_1}(x_1)\}$$
$$S_2 = \{x_2, \alpha^P_{S_2}(x_2), \alpha^N_{S_2}(x_2), \gamma^P_{S_2}(x_2), \gamma^N_{S_2}(x_2), \beta^P_{S_2}(x_2), \beta^N_{S_2}(x_2)\}$$
$$S_3 = \{x_3, \alpha^P_{S_3}(x_3), \alpha^N_{S_3}(x_3), \gamma^P_{S_3}(x_3), \gamma^N_{S_3}(x_3), \beta^P_{S_3}(x_3), \beta^N_{S_3}(x_3)\}$$

Let us take

$S_1 = \{(0.3, -0.2, 0.4, -0.3, 0.1, -0.4), (0.1, -0.5, 0.5, -0.2, 0.2, -0.1), (0.6, -0.3, 0.1, -0.2, 0.1, -0.3)\}$, $S_2 = \{(0.2, -0.1, 0.5, -0.3, 0.2, -0.2), (0.3, -0.3, 0.2, -0.3, 0.1, -0.4), (0.3, -0.2, 0.4, -0.4, 0.2, -0.3)\}$, $S_3 = \{(0.1, -0.3, 0.6, -0.2, 0.2, -0.4), (0.3, -0.1, 0.2, -0.4, 0.4, -0.3), (0.5, -0.1, 0.1, -0.3, 0.3, -0.3)\}$.

Then, $d_P(S_1,S_2) = 0.766667$, $d_P(S_2,S_3) = 0.73333$, $d_P(S_1,S_3) = 0.93333$ and $e_P(S_1,S_2) = \sqrt{0.2}$, $e_P(S_2,S_3) = \sqrt{0.42/3}$, $e_P(S_1,S_3) = \sqrt{0.62/3}$.

3.2 Application

Let S be a finite alternatives set $S = \{S_1, \ldots, S_n\}$. Assume the basic evaluations of alternatives according to the criterion is a BPPFS E on S. Let $E = \{e(S_1), \ldots, e(S_n)\}$, where for each i, $e(S_i) = (\alpha^P(S_i), \alpha^N(S_i), \gamma^P(S_i), \gamma^N(S_i), \beta^P(S_i), \beta^P(S_i))$, $0 \le \alpha^P(S_i), \gamma^P(S_i), \beta^P(S_i), \le 1$, $-1 \le \alpha^N(S_i), \gamma^N(S_i), \beta^N(S_i), \le 0$, $\alpha^P(S_i) + \gamma^P(S_i) + \beta^P(S_i) \le 1$, $\alpha^N(S_i) + \gamma^N(S_i) + \beta^N(S_i) \le -1$. Now we can make the decision by grading the alternatives set and then we choose the best possible solution. For this, the following algorithm works.

Algorithm:

Step 1. First, we define suitable score functions on S for all $i's$ as follows.

$$h_1(S_i) = \gamma^P(S_i) + \beta^P(S_i) - \alpha^P(S_i)$$
$$h_2(S_i) = \gamma^N(S_i) + \beta^N(S_i) - \alpha^N(S_i)$$
$$h_3(S_i) = \alpha^P(S_i) + \beta^P(S_i) - \gamma^P(S_i)$$
$$h_4(S_i) = \alpha^N(S_i) + \beta^N(S_i) - \gamma^N(S_i)$$

$$h_5(S_i) = \alpha^P(S_i) + \gamma^P(S_i) - \beta^P(S_i)$$
$$h_6(S_i) = \alpha^N(S_i) + \gamma^N(S_i) - \beta^N(S_i).$$

Step 2. Defining order $\gtrsim_1$ on S at the base of $S_i \gtrsim_1 S_k \Leftrightarrow \boldsymbol{h}_1(S_i) \geq \boldsymbol{h}_1(S_k)$,

$$\gtrsim_2 \text{ on } S \text{ while } S_i \gtrsim_2 S_k \Leftrightarrow h_2(S_i) \geq h_2(S_k)$$
$$\gtrsim_3 \text{ on } S \text{ while } S_i \gtrsim_3 S_k \Leftrightarrow h_3(S_i) \geq h_3(S_k)$$
$$\gtrsim_4 \text{ on } S \text{ while } S_i \gtrsim_4 S_k \Leftrightarrow h_4(S_i) \geq h_4(S_k)$$
$$\gtrsim_5 \text{ on } S \text{ while } S_i \gtrsim_5 S_k \Leftrightarrow h_5(S_i) \geq h_5(S_k)$$
$$\gtrsim_6 \text{ on } S \text{ while } S_i \gtrsim_6 S_k \Leftrightarrow h_6(S_i) \geq h_6(S_k)$$

Step 3. Define aggregation order $\gtrsim_*$ on S by utilizing $\geqslant_1, \geqslant_2, \geqslant_3, \geqslant_4, \geqslant_5, \geqslant_6$, and then grading the alternative set and select the solution.

No doubt, the above algorithm in the setting of bipolar picture fuzzy sets is the refinement of the algorithm generated in the picture fuzzy set scenario.

3.3 Comparison between the proposed and existing algorithms

Now we present the comparative analysis of the terms and characteristics of the proposed algorithm and the existing algorithm. There are various objectives to present this algorithm; some features are described below:

1. The first objective to present the modification of the algorithm presented in Cuong (2014). Through bipolar picture fuzzy sets we have included the bipolarity which was not included in the existing algorithm.

2. By comparing the algorithm with the algorithm presented in Cuong (2014), we find that it is strong, valid and superior to the existing one. In general, the comparison analysis of BPFS with the existing models is expressed in Table. 1

3. The third objective is to characterize the connection of BPFS to the decision making problems. Finally, one can easily find that our presented algorithm givens better results than any other existing algorithm while dealing with the decision making problems in the fields of medical, business, artificial intelligence and engineering etc.

Table 1. Comparison of the proposed and the existing algorithms

Set theoretic models	Satisfaction grad	Abstinence grade	Dissatisfaction grade	Bipolarity
PFS	✓	✓	✓	×
Proposed BPPFS	✓	✓	✓	✓

4. Bipolar Picture Fuzzy Relations (BPPFRs)

In this section, we introduce several types of relations on BPPFSs and present some interesting results related to these relations. Finally, we present the application of these relations.

Definition 23. A mapping $S = (\alpha_S^P, \alpha_S^N, \gamma_S^P, \gamma_S^N, \beta_S^P, \beta_S^N): X \times X \to [-1,0] \times [0,1] \times [-1,0] \times [0,1] \times [-1,0] \times [0,1]$ on a non-empty set X is a bipolar picture fuzzy relation on X, where $\alpha_S^P(x,y) \in [0,1]$, $\alpha_S^N(x,y) \in [-1,0]$, $\gamma_S^P(x,y) \in [0,1]$, $\gamma_S^N(x,y) \in [-1,0]$, $\beta_S^P(x,y) \in [0,1]$ and $\beta_S^N(x,y) \in [-1,0]$.

Particularly, BPPFR on X is a BPPFR from X to X.

Definition 24. A BPPFR on X is said to be an empty BPPFR (resp., the whole BPPFR) if it satisfies $\alpha_S^P(x,y) = 0 = \alpha_S^N(x,y)$ (resp., $\alpha_S^P(x,y) = 1$ and $\alpha_S^N(x,y) = -1$) $\gamma_S^P(x,y) = 0 = \gamma_S^N(x,y)$ (resp., $\gamma_S^P(x,y) = 1$ and $\gamma_S^N(x,y) = -1$) $\beta_S^P(x,y) = 0 = \beta_S^N(x,y)$ (resp., $\beta_S^P(x,y) = 1$ and $\beta_S^N(x,y) = -1$) for all $(x,y) \in X \times X$.

Hereafter, BPPFR(X) (resp., BPPFR$(X \times Y)$) will represent the set of all BPPFRs on X (resp., from X to Y).

Obviously, if $S = \{\alpha_S^P(x),\ \alpha_S^N(x),\ \gamma_S^P(x),\ \gamma_S^N(x),\ \beta_S^P(x),\ \beta_S^N(x)\} \in BPPFR(X)$, then $\alpha_S^P(x), -\alpha_S^N(x), \gamma_S^P(x), -\gamma_S^N(x), \beta_S^P(x), \beta_S^N(x)$ are the bipolar picture fuzzy relations on X, where $(-\alpha_S^{-1})(x,y) = -\alpha_S^N(x,y)$, $-\gamma_S^{-1}(x,y) = -\gamma_S^N(x,y)$, $(-\beta_S^{-1})(x,y) = -\beta_S^N(x,y)$, for all $(x,y) \in X \times X$.

Definition 25. Let S be a BPPFR defined on $X \times Y$. Then

(i) $S^{-1} = \{(\alpha_S^{-1})^P, (\alpha_S^{-1})^N, (\gamma_S^{-1})^P, (\gamma_S^{-1})^N, (\beta_S^{-1})^P, (\beta_S^{-1})_S^N\}$ is a BPPFR from Y to X, an inverse of the relation S, such that for every $(y,x) \in Y \times X$, $S^{-1}(x,y) = S(y,x)$, we have

$$(\alpha_S^{-1})^P(y,x) = \alpha_S^P(x,y), \qquad (\alpha_S^{-1})^N(y,x) = \alpha_S^N(x,y),$$
$$(\gamma_S^{-1})^P(y,x) = \gamma_S^P(x,y), \qquad (\gamma_S^{-1})^N(y,x) = \gamma_S^N(x,y),$$
$$(\beta_S^{-1})^P(y,x) = \beta_S^P(x,y), \qquad (\beta_S^{-1})^N(y,x) = \beta_S^N(x,y).$$

(ii) $S^c = \{(\alpha_S^c)^P, (\alpha_S^c)^N, (\gamma_S^c)^P, (\gamma_S^c)^N, (\beta_S^c)^P, (\beta_S^c)_S^N\}$ is a BPPFR from X to Y, complement of S, and is defined as: for every $(x,y) \in X \times Y$, $(\alpha_S^c)^P(x,y) = 1 - \alpha_S^P(x,y)$, $(\alpha_S^c)^N(x,y) = -1 - \alpha_S^N(x,y)$, $(\gamma_S^c)^P(x,y) = 1 - \gamma_S^P(x,y)$, $(\gamma_S^c)^N(x,y) = -1 - \gamma_S^N(x,y)$, $(\beta_S^c)^P(x,y) = 1 - \beta_S^P(x,y)$, $(\beta_S^c)^N(x,y) = -1 - \beta_S^N(x,y)$.
(iii) $S \cup T = \{(\alpha_S^P(x,y) \vee \alpha_T^P(x,y)), (\alpha_S^N(x,y) \wedge \alpha_T^N((x,y)), (\gamma_S^P(x,y) \wedge \gamma_T^P(x,y)), (\gamma_S^N(x,y) \vee \gamma_T^N(x,y)), (\beta_S^P(x,y) \wedge \beta_T^P(x,y)), (\beta_S^N(x,y) \vee \beta_T^N(x,y))): (x,y) \in X \times Y\}$.
(iv) $S \cap T = \{((\alpha_S^P(x,y) \wedge \alpha_T^P(x,y)), (\alpha_S^N(x,y) \vee \alpha_T^N(x,y)), (\gamma_S^P(x,y) \wedge \gamma_S^P(x,y)), (\gamma_S^N(x,y) \vee \gamma_T^N(x,y)), (\beta_S^P(x,y) \vee \beta_T^P(x,y)), (\beta_S^N(x,y) \wedge \beta_T^N(x,y))) : (x,y) \in X \times Y\}$.

Proposition 2. Let S, T, R be the BPPFR on $X \times Y$. Then
1. $(S^c)^{-1} = (S^{-1})^c$
2. $(S^{-1})^{-1} = S$
3. $S \subset S \cup T$ and $T \subset S \cup T$
4. $S \cap T \subset S$ and $S \cap T \subset T$
5. $S \subset T \Rightarrow S^{-1} \subset T^{-1}$
6. if $S \subset T$ and $T \subset R$, then $S \cup T \subset R$
7. if $S \subset T$ and $T \subset R$, then $R \subset S \cup T$
8. if $S \subset T$, then $S \cup T \subset T$ and $S \cap T \subset S$
9. $(S \cup T)^{-1} = S^{-1} \cup T^{-1}$, $(S \cap T)^{-1} = S^{-1} \cap T^{-1}$

Proof. 1. Assume that $(x,y) \in X \times Y$. Then
$$\begin{aligned}[(\alpha_S^c)^{-1}]^P(x,y) &= (\alpha_S^c)^P(y,x) = 1 - \alpha_S^P(y,x)\\ &= 1 - (\alpha_S^{-1})^P(x,y)\\ &= [(\alpha_S^{-1})^c]^P(x,y)\end{aligned}$$
Similarly, $[(\alpha_S^c)^{-1}]^N(x,y) = [(\alpha_S^{-1})^c]^N(x,y)$, $[(\gamma_S^c)^{-1}]^P(x,y) = [(\gamma_S^{-1})^c]^P(x,y)$, $[(\gamma_S^c)^{-1}]^N(x,y) = [(\gamma_S^{-1})^c]^N(x,y)$, $[(\beta_S^c)^{-1}]^P(x,y) = [(\beta_S^{-1})^c]^P(x,y)$ and $[(\beta_S^c)^{-1}]^N(x,y) = [(\beta_S^{-1})^c]^N(x,y)$.
2. Easy to prove by using definition 3.
3. For any $(x,y) \in X \times Y$, we have
$S \cup T = \{(\alpha_S^P(x,y) \vee \alpha_T^P(x,y)), (\alpha_S^N(x,y) \wedge \alpha_T^N((x,y)), (\gamma_S^P(x,y) \wedge \gamma_T^P(x,y)), (\gamma_S^N(x,y) \vee \gamma_T^N(x,y)), (\beta_S^P(x,y) \wedge \beta_T^P(x,y)), (\beta_S^N(x,y) \vee \beta_T^N(x,y))): (x,y) \in X \times Y\}$ and $S = \{\alpha_S^P(x,y), \alpha_S^N(x,y), \gamma_S^P(x,y), \gamma_S^N(x,y), \beta_S^P(x,y), \beta_S^N(x,y)\}$. Then
$$(\alpha_S^P(x,y) \vee \alpha_T^P(x,y)) \geq \alpha_S^P(x,y), (\alpha_S^N(x,y) \wedge \alpha_T^N(x,y)) \leq \alpha_S^N((x,y, (\gamma_S^P(x,y) \wedge \gamma_T^P(x,y)) \geq \gamma_S^P(x,y),$$
$$\left(\gamma_S^N(x,y) \vee \gamma_T^N(x,y)\right) \leq \gamma_S^N(x,y), \left(\beta_S^P(x,y) \wedge \beta_T^P(x,y)\right) \leq \beta_S^P(x,y), (\beta_S^N(x,y) \vee \beta_T^N(x,y))) \geq \beta_S^N(x,y)$$
Thus, $S \subset S \cup T$. Similarly, we have $T \subset S \cup T$.
Proofs of the remaining parts (4, 5, 6, 7, 8 &9) of this theorem are the direct consequences of the Definitions [15 & 25].

Definition 26. The composition of any two BPPFRs S and T defined on $X \times Y$ and $Y \times Z$, respectively, is $S \circ T = \{\alpha_{S\circ T}^P, \alpha_{S\circ T}^N, \gamma_{S\circ T}^P, \gamma_{S\circ T}^N, \beta_{S\circ T}^P, \beta_{S\circ T}^N\}$ is a BPPFR from X to Z and is defined as
$$\begin{aligned}\alpha_{S\circ T}^P(x,z) &= \alpha_{S^P\circ T^P}(x,z) = \bigvee_{y\in\mu} [\alpha_S^P(x,y) \wedge \alpha_T^P(y,z)],\\ \alpha_{S\circ T}^N(x,z) &= \alpha_{S^N\circ T^N}(x,z) = \bigwedge_{y\in\mu} [\alpha_S^N(x,y) \vee \alpha_T^N(y,z)]\\ \gamma_{S\circ T}^P(x,z) &= \gamma_{S^P\circ T^P}(x,z) = \bigvee_{y\in\mu} [\gamma_S^P(x,y) \wedge \gamma_T^P(y,z)]\\ \gamma_{S\circ T}^N(x,z) &= \gamma_{S^N\circ T^N}(x,z) = \bigwedge_{y\in\mu} [\gamma_S^N(x,y) \vee \gamma_T^N(y,z)]\\ \beta_{S\circ T}^P(x,z) &= \beta_{S^P\circ T^P}(x,z) = \bigwedge_{y\in\mu} [\beta_S^P(x,y) \vee \beta_T^P(y,z)]\\ \beta_{S\circ T}^N(x,z) &= \beta_{S^N\circ T^N}(x,z) = \bigvee_{y\in\mu} [\beta_S^N(x,y) \wedge \beta_T^N(y,z)]\end{aligned}$$
for all $(x,z) \in X \times Z$.

It is easy to observe that $(-\alpha_{S^N} \circ -\alpha_{T^P})(x,y) = \vee_{y\in\mu} [-\alpha_S^N(x,y) \wedge -\alpha_T^N(x,y)]$, $(-\gamma_{S^N} \circ -\gamma_{T^P})(x,y) = \vee_{y\in\mu} [-\gamma_S^N(x,y) \wedge -\gamma_T^N(x,y)](-\beta_{S^N} \circ -\beta_{T^P})(x,y) = \wedge_{y\in\mu} [-\beta_S^N(x,y) \vee -\beta_T^N(x,y)]$. It is easy to prove the below proposition.

Proposition 3. Let X, Y, Z and W be the non-empty sets. Then
1. $S \circ (T \circ R) = (S \circ T) \circ R$, where $S, T\&R$ be the BPPFRs on $(X \times Y)$, $(Y \times Z)$ and $(Z \times W)$, respectively.
2. $S \circ (T \cup R) = S \circ T \cup S \circ R$, $S \circ (T \cap R) = S \circ T \cap S \circ R$, where $T\&R, S$ be the BPPFRs on the $(X \times Y)$ and $(Y \times Z)$, respectively.
3. If $R \subset T$, then $S \circ R \subset S \circ T$, where $T\&R, S$ be the BPPFRs on the $(X \times Y)$ and $(Y \times Z)$, respectively.
4. $(S \circ T)^{-1} = S^{-1} \circ T^{-1}$, where $S\&T$ be the BPPFRs on $(X \times Y)$ and $(Y \times Z)$.

Proof.
1. The proof is straight forward.
2. The proof is straight forward.

3. Let $R,T \in BPPFRs(X \times Y)$ and $S \in BPPFRs(Y \times Z)$. Suppose $R \subset T$ and let $(x,z) \in X \times Z$. Then

$$\alpha^P_{S\circ R}(x,z) = \vee_{y\in\mu}[\alpha^P_S(x,y) \wedge \alpha^P_R(y,z)], \leq \vee_{y\in\mu}[\alpha^P_S(x,y) \wedge \alpha^P_T(y,z)],$$
$$[since\ R \subset T, \alpha^P_T(y,z) \subset \alpha^P_T(y,z)] = \alpha^P_{S\circ T}(x,z)$$

Similarly, we can prove that $\alpha^N_{S\circ R}(x,z) \leq \alpha^N_{S\circ T}(x,z)$, $\gamma^P_{S\circ R}(x,z) \leq \gamma^P_{S\circ T}(x,z)$, $\gamma^N_{S\circ R}(x,z) \leq \gamma^N_{S\circ T}(x,z)$, $\beta^P_{S\circ R}(x,z) \leq \beta^P_{S\circ T}(x,z)$, $\beta^N_{S\circ R}(x,z) \leq \beta^N_{S\circ T}(x,z)$. Furthermore, the proof of the second part is similar to the first part. Thus, the result holds.

Let $S \in (X \times Y)$ and $T \in (Y \times Z)$ and $(x,z) \in (X \times Z)$

$$\alpha^P_{(S\circ T)^{-1}}(z,x) = \alpha^P_{S\circ T}(x,z) = \vee_{y\in\mu}[\alpha^P_S(x,y) \wedge \alpha^P_T(y,z)]$$
$$= \vee_{y\in\mu}\left[\alpha^P_{T^{-1}}(z,y) \wedge \alpha^P_{S^{-1}}(y,x)\right] = \alpha^P_{S^{-1}\circ T^{-1}}(z,x)$$

Similarly, we can see that $\alpha^N_{(S\circ T)^{-1}}(z,x) = \alpha^N_{S^{-1}\circ T^{-1}}(z,x)$, $\gamma^P_{(S\circ T)^{-1}}(z,x) = \gamma^P_{S^{-1}\circ T^{-1}}(z,x)$, $\gamma^N_{(S\circ T)^{-1}}(z,x) = \gamma^N_{S^{-1}\circ T^{-1}}(z,x)$, $\beta^P_{(S\circ T)^{-1}}(z,x) = \beta^P_{S^{-1}\circ T^{-1}}(z,x)$, $\beta^N_{(S\circ T)^{-1}}(z,x) = \beta^N_{S^{-1}\circ T^{-1}}(z,x)$. Thus, the result holds.

Remark 1. For any BPPFRs S and T, $S \circ T \neq T \circ S$, in general.

Definition 27. The relation I_X (or simply I) is said to be the BPPFR on X and is defined as

$$\alpha^P_I(x,y) = \gamma^P_I(x,y) = \beta^P_I(x,y) = \begin{cases} 1, & if\ x = y; \\ 0, & if\ x \neq y, \end{cases},$$
$$\alpha^N_I(x,y) = \gamma^N_I(x,y) = \beta^N_I(x,y) = \begin{cases} -1, & if\ x = y; \\ 0, & if\ x \neq y. \end{cases}$$

for every $(x,y) \in X \times X$. Thus, $I = \{\alpha^P_I, \alpha^N_I, \gamma^P_I, \gamma^N_I, \beta^P_I, \beta^N_I\}$.

Clearly, $I = I^{-1}$ and $I^c = (I^c)^{-1}$. Moreover, it is evident that if I_X is the bipolar picture fuzzy identity relation defined on X, then α^P_I, α^N_I, γ^P_I, γ^N_I, β^P_I and β^N_I are picture fuzzy identity relations on X.

Definition 28. A BPPFR S on X is called

1. reflexive, if for every $x \in X$, $S(x,x) = (-1,1)$, i.e.,

$$\alpha^P_S(x,x) = \gamma^P_S(x,x) = \beta^P_S(x,x) = 1, \qquad \alpha^N_S(x,x) = \gamma^N_S(x,x) = \beta^N_S(x,x) = -1,$$

2. anti-reflexive, if for every $x \in X$, $S(x,x) = (0,0)$.

From Definitions [27 & 28], it is evident that S is a bipolar picture fuzzy reflexive relation if and only if $I \subset S$.

Obviously, $S = \{\alpha^P_S, \alpha^N_S, \gamma^P_S, \gamma^N_S, \beta^P_S, \beta^N_S\}$ is a bipolar picture fuzzy reflexive (resp., anti-reflexive) relation on X implies α^P_S, α^N_S, γ^P_S, γ^N_S, β^P_S and β^N_S are picture fuzzy reflexive (resp., anti-reflexive) relations on X. Consequently, the necessary and sufficient condition for S and T are to be the picture fuzzy reflexive (resp., anti-reflexive) relations on X if one of $(-S,T)$ or $(-T,S)$ is a bipolar picture fuzzy reflexive (resp., anti-reflexive) relation defined on X.

Proposition 4. Let $S \in BPPFR(X)$. Then

1. S is reflexive $\Leftrightarrow$ S^{-1} is reflexive.
2. S is reflexive $\Rightarrow$ $S \cup T$ is reflexive, for each $T \in BPPFR(X)$.
3. S is reflexive $\Rightarrow$ $S \cap T$ is reflexive $\Leftrightarrow$ $T \in BPPFR(X)$ is reflexive.

Proposition 5. Let $S \in BPPFR(X)$. Then

1. S is anti-reflexive $\Leftrightarrow$ S^{-1} is anti-reflexive.
2. S is anti-reflexive $\Rightarrow$ $S \cap T$ is anti-reflexive, for each $T \in BPPFR(X)$.
3. S is anti-reflexive $\Rightarrow$ $S \cup T$ is anti-reflexive $\Leftrightarrow$ $T \in BPPFR(X)$ is anti-reflexive.

Proposition 6. If S and T are the two bipolar picture fuzzy reflexive relations defined on a non-empty set X, then their composition is also a reflexive.

Definition 29. Let $S \in BPPFR(X)$. Then

1. S is known as symmetric, if for every $x,y \in X$, $S(x,y) = S(y,x)$ i.e.,

$$\alpha^P_S(x,y) = \alpha^P_S(y,x) \text{ and } \alpha^N_S(x,y) = \alpha^N_S(y,x);$$
$$\gamma^P_S(x,y) = \gamma^P_S(y,x) \text{ and} \gamma^N_S(x,y) = \gamma^N_S(y,x);$$
$$\beta^P_S(x,y) = \beta^P_S(y,x) \text{ and } \beta^N_S(x,y) = \beta^N_S(y,x).$$

2. S is known as anti-symmetric, if for every $x,y \in X$, along $x \neq y$,

$$\alpha^P_S(x,y) \neq \alpha^P_S(y,x) \text{ and } \alpha^N_S(x,y) \neq \alpha^N_S(y,x);$$
$$\gamma^P_S(x,y) \neq \gamma^P_S(y,x) \text{ and } \gamma^N_S(x,y) \neq \gamma^N_S(y,x);$$
$$\beta^P_S(x,y) \neq \beta^P_S(y,x) \text{ and } \beta^N_S(x,y) \neq \beta^N_S(y,x).$$

From Definitions [28 & 29], it is clear that the empty bipolar picture fuzzy relation S_0 is symmetric and anti-reflexive BPPFR. Similarly, the whole bipolar picture fuzzy relation S_1 and bipolar picture fuzzy identity relation I are symmetric and reflexive BPPFRs, and I^c is an anti-reflexive BPPFR.

The below results are the immediate consequences of the Definitions [25 & 29].

Proposition 7. Let S be a BPPFR on X. Then, S is symmetric if and only if $S = S^{-1}$.

Proposition 8. Let S and T be the two bipolar picture fuzzy symmetric relations on X. Then, their union and intersection are also symmetric.

Remark 2. If S and T are bipolar picture fuzzy symmetric relations on X then their composition may not be symmetric.

Corollary 1. If S is a bipolar picture fuzzy symmetric relation on X, then $S^n = R \circ R \circ \ldots R \circ$ (n-times)is also a bipolar picture fuzzy symmetric relation on X, where n is a positive integer.

Definition 30. A bipolar picture fuzzy relation S defined on X is said to be a transitive if $S \circ S \subset S$, i.e., $S^2 \subset S$.

Clearly, if $S = \{\alpha_S^P, \alpha_S^N, \gamma_S^P, \gamma_S^N, \beta_S^P, \beta_S^N\}$ is a bipolar picture fuzzy transitive relation on X, then α_S^P, α_S^N, γ_S^P, γ_S^N, β_S^P and β_S^N are picture fuzzy transitive relations on X. Thus, S and T are picture fuzzy transitive relations on X iff $(-S, T)$ and $(-T, S)$ are bipolar picture fuzzy transitive relations on X.

Proposition 9. Let S be a BPPFR on X and S is transitive $\Rightarrow$ S^{-1} is also.

Proposition 10. Let S be a BPPFR on X and S is transitive $\Rightarrow$ so is S^2.

Proposition 11. Let S and T be BPPFRs on the st X and both are transitive $\Rightarrow$ $S \cap T$ is transitive.

Remark 3. In general, Union of two bipolar picture fuzzy transitive relations is not a bipolar picture fuzzy transitive relation.

Definition 31. An $R \in BPPFR(X)$ is called a similarity (or equivalence) relation on X, if it is reflexive, symmetric and transitive. We denote the set of all equivalence relations on X by BPPFERs(X).

We can easily see that $S = \{\alpha_S^P, \alpha_S^N, \gamma_S^P, \gamma_S^N, \beta_S^P, \beta_S^N\}$ is a bipolar picture fuzzy equivalence relation on X, then α_S^P, α_S^N, γ_S^P, γ_S^N, β_S^P and β_S^N are picture fuzzy equivalence relations on X. Furthermore, S and T are picture fuzzy equivalence relations on X iff $(-S, T)$ and $(-T, S)$ are picture bipolar fuzzy equivalence relations on X.

The following propositions are the immediate consequence of the propositions [3, 7 & 10].

Proposition 12. Let $(S_j)_{j\in J}$ be the subset of BPPFR on X. Then, $\cap S_j \in BPPFE(X)$.

Proposition 13. Let S be the BPPFR on X. Then $S = S \circ S$.

Definition 32. Let S be the BPPFR on X. Then, S is known as normal, if

$$\bigvee_{x\in X} \alpha_S^P(x) = 1, \bigwedge_{x\in X} \alpha_S^N(x) = -1; \bigvee_{x\in X} \gamma_S^P(x) = 1,$$
$$\bigwedge_{x\in X} \gamma_S^N(x) = -1; \bigwedge_{x\in X} \beta_S^P(x) = 1, \bigvee_{x\in X} \beta_S^N(x) = -1.$$

Definition 33. Let S be the BPPFR on X and $x \in X$. Then, the bipolar picture fuzzy equivalence class of x by S, denoted by S_x, is a BPPFS in X defined as:

$$S_x = \{\alpha_{S_x}^P, \alpha_{S_x}^N, \gamma_{S_x}^P, \gamma_{S_x}^N, \beta_{S_x}^P, \beta_{S_x}^N\}$$

where $\alpha_{S_x}^P, \gamma_{S_x}^P, \beta_{S_x}^P : X \to [0,1]$; $\alpha_{S_x}^N, \gamma_{S_x}^N, \beta_{S_x}^N : X \to [-1,0]$ are mappings defined as: for each $y \in X$,

$$\alpha_{S_x}^P(y) = \alpha_S^P(x,y), \alpha_{S_x}^N(y) = \alpha_S^N(x,y), \gamma_{S_x}^P(y) = \gamma_S^P(x,y),$$
$$\gamma_{S_x}^N(y) = \gamma_S^N(x,y), \beta_{S_x}^P(y) = \beta_S^P(x,y), \beta_{S_x}^N(y) = \beta_S^N(x,y).$$

Henceforth, X/S will represent the set of all bipolar picture fuzzy equivalence classes defined of S, which is also the bipolar picture fuzzy quotient set of X by S.

Proposition 14. Let $S \in BPPFE(X)$, and let $x, y \in X$. Then

1. S_x is normal; in fact $S_x \neq 0$,
2. $S_x \cap S_y = 0 \Leftrightarrow S(x, y) = (0,0)$,
3. $S_x = S_y \Leftrightarrow S(x, y) = (-1,1)$,
4. $\cup_{x\in X} S_x = 1$.

Definition 34. Let $\sum = (S_j)_{j\in J}$ be a subset of BPPFR on $X \Rightarrow \sum$ is said to be a bipolar picture fuzzy partition of X, if it satisfies

1. S_j is normal, for every $j \in J$,
2. either $S_j = S_k$ or $S_j \neq S_k$, for any $j, k \in J$,
3. $\cup_{j\in J} S_j = 1$

The following is the immediate result of proposition 13 and definition 34.

Corollary 2. Let S be a BPPFR on $X \Rightarrow X/S$ is a bipolar picture fuzzy partition of X.

Proposition 15. Let $\sum$ be a bipolar fuzzy partition of X. We define $S(\sum) = \{\alpha^P_{S(\sum)}, \alpha^N_{S(\sum)}, \gamma^P_{S(\sum)}, \gamma^N_{S(\sum)}, \beta^P_{S(\sum)}, \beta^N_{S(\sum)}\}$ as: for each $(x, y) \in X \times X$,

$$\alpha^P_{S(\sum)}(x,y) = \vee_{S\in\sum} [\alpha^P_S(x) \wedge \alpha^P_S(y)], \quad \alpha^N_{S(\sum)}(x,y) = \wedge_{S\in\sum} [\alpha^N_S(x) \vee \alpha^N_S(y)]$$
$$\gamma^P_{S(\sum)}(x,y) = \vee_{S\in\sum} [\gamma^P_S(x) \wedge \gamma^P_S(y)], \quad \gamma^N_{S(\sum)}(x,y) = \wedge_{S\in\sum} [\gamma^N_S(x) \vee \gamma^N_S(y)]$$
$$\beta^P_{S(\sum)}(x,y) = \wedge_{S\in\sum} [\beta^P_S(x) \vee \beta^P_S(y)], \quad \beta^N_{S(\sum)}(x,y) = \vee_{S\in\sum} [\beta^N_S(x) \wedge \beta^N_S(y)]$$

where $\alpha^P_{S(\sum)}, \gamma^P_{S(\sum)}, \beta^P_{S(\sum)}: X \times X \to [0,1]$, $\alpha^N_{S(\sum)}, \gamma^N_{S(\sum)}, \beta^N_{S(\sum)}: X \times X \to [-1,0]$ are mappings. Then, $S(\sum) \in BPPFE(X)$.

Proposition 16. Let S and T be the BPPFRs on X. Then $S \subset T \Leftrightarrow S_x \subset T_x$, for every $x \in X$.

Proposition 17. Let S and T be the BPPFRs on X. Then, $T \circ S \in BPPFE(X) \Leftrightarrow T \circ S = S \circ T$.

Proposition 18. Let S and T may be the BPPFRs on X and $S \cup T = T \circ S \Rightarrow S \cup T \in BPPFE(X)$.

4.1 Application

Let us consider four patients of covid-19 s_1, s_2, s_3 and s_4. Their symptoms are temperature, dry cough and tiredness. Let the set of patients is given by the set $S = \{s_1, s_2, s_3, s_4\}$ and the set of symptoms is denoted by $T = \{\text{temperature, dry cough, tiredness}\}$. The BPPFR $R \in BPPFS(S \times T)$ is hypothetical given in Table 2.

Table 2. R is a bipolar picture fuzzy relation between the set of patients S and the set of symptoms T.

R	Temperature	Dry cough	Tiredness
s_1	{0.3, -0.2, 0.05, -0.4, 0.6, -0.3}	{0.4, -0.3, 0.15, -0.14, 0.4, -0.4}	{0.2, -0.3, 0.15, -0.4, 0.4, -0.2}
s_2	{0.2, -0.3, 0.15, -0.3, 0.5, -0.2}	{0.2, -0.4, 0.05, -0.2, 0.5, -0.2}	{0.1, -0.1, 0.2, -0.15, 0.2, -0.3}
s_3	{ 0.1, -0.4, 0.20, -0.2 0.4, -0.1}	{0.3, -0.2, 0.30 -0.3, 0.3, -0.3}	{0.3, -0.2, 0.3, -0.2, 0.3, -0.1}
s_4	{0.2, -0.5, 0.30, -0.1, 0.3, -0.2}	{0.4, -0.1, 0.10, -0.4, 0.3, -0.1}	{0.4, -0.4, 0.1, -0.3, 0.1, -0.25}

Let the set of different covid tests be $Q = \{1st, 2nd, 3rd\}$. The BPPFR $P \in BPPFS(T \times Q)$ is given in Table 3.

Table 3. P is a bipolar picture fuzzy relation between the set of symptoms T and the set of different covid test Q.

P	1st	2nd	3rd
Temperature	{0.4, -0.1, 0.5, -0.3, 0.03, -0.2}	{0.3, -0.3, 0.1, -0.14, 0.6, -0.4}	{0.1, -0.2, 0.2, -0.4, 0.3, -0.1}
Drycough	{0.3, -0.3, 0.4, -0.1, 0.2, -0.4}	{0.2, -0.2, 0.5, -0.1, 0.1, -0.3}	{0.2, -0.3,0.3, -0.1, 0.4, -0.4}
Tiredness	{0.2, -0.4, 0.04, -0.3, 0.6, -0.3}	{0.1, -0.1, 0.2 -0.3, 0.2, -0.4}	{0.3, -0.4, 0.2, -0.3, 0.2, -0.3}

The composed relation $O = Q \circ S$ is given in Table 4.

Table 4. O is a bipolar picture fuzzy relation between the set of patients S and the set of different covid test terms Q.

O	1st	2nd	3rd
s_1	{0.4, -0.4, 0.1, -0.1, 0.3, -0.2}	{0.4, -0.1, 0.10, -0.4, 0.4, -0.4}	{0.4, -0.4, 0.1, -0.1 0.2, -0.3}
s_2	{0.2, -0.5, 0.2, -0.2, 0.4, -0.1}	{0.3, -0.2, 0.25, -0.3, 0.2, -0.2}	{0.3, -0.1, 0.1, -0.4, 0.1, -0.1}
s_3	{0.1, -0.2, 0.2, -0.3, 0.5, -0.3}	{0.2, -0.3, 0.40, -0.1, 0.3, -0.3}	{0.1, -0.3, 0.3, -0.3, 0.3, -0.2}
s_4	{0.3, -0.1, 0.3, -0.4, 0.2, -0.4}	{0.1, -0.4, 0.20, -0.2, 0.1, -0.1}	{0.2, -0.4, 0.2, -0.2, 0.5, -0.4}

The correspondence between patients s and the covid test q is expressed as a double triplet containing $\alpha_O^P(s,q)$, $\alpha_O^N(s,q)$, $\gamma_O^P(s,q)$, $\gamma_O^N(s,q)$, $\beta_O^P(s,q)$, $\beta_O^N(s,q)$.

For each $(s,q) \in S \times Q$. We calculate $P_O(s,q)$ as $P_O(s,q) = P_O^P(s,q) - P_O^P(s,q)$,

$P_O^P(s,q) = \alpha_O^P(s,q) - \beta_O^P(s,q) \cdot \pi_O^P(s,q)$, where $\pi_O^P(s,q) = 1 - [\alpha_O^P(s,q) + \gamma_O^P(s,q) + \beta_O^P(s,q)]$, $P_O^N(s,q) = \alpha_O^N(s,q) - \beta_O^N(s,q) \cdot \pi_O^N(s,q)$, where $\pi_O^N(s,q) = -1 - [\alpha_O^N(s,q) + \gamma_O^N(s,q) + \beta_O^N(s,q)]$.

Table 5. P_O, The correspondence between patients s and the covid test q.

O	1st	2nd	3rd
s_1	0.8	0.5	0.34
s_2	0.64	0.51	0.19
s_3	0.26	0.59	0.35
s_4	0.4	0.47	0.43

It can easily be seen that if $\alpha_O^P(s,q) + \gamma_O^P(s,q) + \beta_O^P(s,q) = 1$, then $P_O^P(s,q) = \alpha_O^P(s,q)$, and if $\alpha_O^N(s,q) + \gamma_O^N(s,q) + \beta_O^N(s,q) = -1$ then $P_O^N(s,q) = \alpha_O^N(s,q)$.

Now, if $P_O(s,q) > 0.5$ then the covid test q of patient s is negative otherwise positive. So, from Table 5, it is obvious that the 1st covid test of the patients s_1 and s_2 is negative, the 2nd covid test of patients s_1, s_2 and s_3 is also negative, while the 3rd covid test of all patients is positive.

5. Conclusions and Future Studies

In this manuscript, some new algebraic operations, distances, and relations have been added in the theory of the bipolar picture fuzzy sets (BPPFSs). We have introduced the normalized hamming distance and normalized Euclidean distance between two BPPFSs. Throughout, applications of the newly established terms have been presented in this paper. In future study, we can shift all of the results produced in this article towards the thoery of spherical fuzzy sets by introducing the term spherical bipolar picture fuzzy sets.

References

Alkouri, A. U. M., Massa′deh, M. O., & Ali, M. (2020). On bipolar complex fuzzy sets and its application. *Journal of Intelligent and Fuzzy Systems*, *1*, 383–397.

Ashraf, S., Mahmood, T., Abdullah, S., & Khan, Q. (2019). Different approaches to multi-criteria group decision making problems for picture fuzzy environment. *Bulletin of the Brazilian Mathematical Society, New Series*, *50*(2), 373-397.

Atanassov, K. T. (1986) Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *1*(20), 87–96.

Azhagappan, M., & Kamaraj, M. (2016). Notes on bipolar valued fuzzy RW-closaed and bipolar valued fuzzy RW-open sets in bipolar valued fuzzy topological spaces. *International Journal of Mathematical Archive*, *7*(3), 30–36.

Bo, C., & Zhang, X. (2017). New operations of picture fuzzy relations and fuzzy comprehensive evaluation. *Symmetry*, *9*(11), 268.

Bustince, H., & Burillo, P. (1996). Structures on intuitionistic fuzzy relations. *Fuzzy Sets and Systems*, *78*(*3*), 293–303.

Cuong, B. C. (2014). Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, *30*(4), 409–420.

Dudziak, U., & Pekala, B. (2010). Equivalent bipolar fuzzy relations. *Fuzzy Sets and Systems*, *161*(2), 234–253.

Ezhilmaran, D., & Sankar, K. (2015). Morphism of bipolar intuitionistic fuzzy graphs. *Journal of Discrete Mathematical Sciences and Cryptography*, *18*(*5*), 605–621.

Gerstenkorn, T., & Ma′nko, J. (1995). Bifuzzy probabilistic sets. *Fuzzy Sets and Systems*, *71*(2), 207–214.

Gulistan, M., Yaqoob, N., Elmoasry, A., & Alebraheem, J. (2021). Complex bipolar fuzzy sets: an application in a transport's company. *Journal of Intelligent and Fuzzy Systems*, *Preprint*, 1-17.

Luo, M., Zhang, Y., & Fu, L. (2021). A new similarity measure for picture fuzzy sets and its application to multi-attribute decision making. *Informatica*, 1–22.

Mandal, W. A. (2021). Bipolar pythagorean fuzzy sets and their application in Multi-attribute decision making problems. *Annals of Data Science*, 1–33.

Mahmood, T., Ullah, K., Khan, Q., & Jan, N. (2019). An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, *31*(11), 7041-7053.

Mahmood, T. (2020). A novel approach towards bipolar soft sets and their applications. *Journal of Mathematics*, 2020.

Riaz, M., Garg, H., Farid, H. M. A., & Chinram, R. (2021). Multi-criteria decision making based on bipolar picture fuzzy operators and new distance measures. *Computer Modeling in Engineering and Sciences*, 127(2), 771-800.

Samanta, S., & Pal, M. (2014). Some more results on bipolar fuzzy sets and bipolar fuzzy intersection graphs. *The Journal of Fuzzy Mathematics*, *22*(*2*), 253–262.

Silambarasan, I. (2021). Some algebraic properties of picture fuzzy sets. *Bulletin of International Mathematical Virtual Institute*, *11*(*3*), 429–442.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(*3*), 338–353.

Zadeh, L. A. (1971). Similarity relations and fuzzy orderings. *Information Sciences*, *3*(*2*), 177–200.

Zhang, W-R. (1998). (Yin) (Yang) bipolar fuzzy sets. *In 1998 IEEE International Conference on Fuzzy Systems Proceedings*. *IEEE World Congress on Computational Intelligence* (*Cat. No.98CH36228*), *1*, 835–840.

Zimmermann, H. J. (2011). *Fuzzy set theory and its applications*. NY: Springer Science and Business Media.