PROGRESS IN APPLIED SCIENCE AND TECHNOLOGY
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
Prog Appl Sci Tech. 2022; 12(3):7-14
ISSN (Print): 2730-3012
ISSN (Online): 2730-3020
https://ph02.tci-thaijo.org/index.php/past

Research Article

Received: July 25, 2022
Revised: September 26, 2022
Accepted: October 03, 2022

DOI: 10.14456/x0xx00000x

Solutions of the Pell equation $x^2 - Dy^2 = \pm N$

Aruntida Sichiangha and Piyanut Puangjumpa*
Department of Mathematics, Faculty of Science and Technology,
Surindra Rajabhat University, Surin 32000, Thailand
*E-mail: piyanut.pu@srru.ac.th

Abstract

In this paper, we obtained some formulas for the integer solutions of the Pell equation $x^2 - Dy^2 = N$ and the negative Pell equation $x^2 - Dy^2 = -N$ where $D > 1$ is a non-square integer and N is a positive integer.

Keywords: Pell equation, Fundamental solution.

1. Introduction

The equation $x^2 - Dy^2 = N$ is called Pell equation with D, N are integers and x, y are unknowns. If D is negative, it can have only a finite number of solutions. If D is a square number, for $D = k^2$, the equation reduces to $(x - ky)(x + ky) = N$ and again there is only a finite number of solutions. If D is a non-square integer, it can have infinitely many solutions.

Let $\frac{F_n}{L_n}$ denotes the sequence of convergent to the regular continued fraction expansion of $\sqrt{D}$. Then the pair (x_1, y_1) solving Pell equation satisfies $x_1 = F_n$ and $y_1 = L_n$ for some n is positive integer. This pair is called the fundamental solution. Thus, the fundamental solution may be found by performing the continued fraction expansion and testing each successive convergent until a solution to Pell equation is found. Once the fundamental solution is found, all remaining solutions may be calculated algebraically from $\left(x_1 + y_1\sqrt{D}\right)^m = x_m + y_m\sqrt{D}$ expanding the left side, equating coefficients of $\sqrt{D}$ on both sides, and equating the other terms on both sides.

For completeness we recall that there are many papers in which are considered different types of Pell equation. Many authors such as Tekcan (6), Kaplan & Williams (2), Matthews (3) and the others consider some specific Pell equations and their integer solutions. In 2007, Tekcan (7) obtained some formulas for the integer solutions the Pell equation $x^2 - Dy^2 = \pm 4$. In 2008, Shabani (5) proved two conjectures related to Pell equation $x^2 - Dy^2 = \pm 4$. In 2011, Chandoul (1) obtained some formulas for the integer solutions the Pell equation $x^2 - Dy^2 = \pm k^2$. In 2015, Ramya et al. (5) obtained some formulas for the integer solutions the Pell equation $x^2 - Dy^2 = \pm 390625$.

In this paper, we obtain some formulas for the integer solutions of the Pell equation $x^2 - Dy^2 = \pm N$ where $D > 1$ is a non-square integer and N is a positive integer.

2. Preliminaries

In this section, we will prove that (x_{2n+1}, y_{2n+1}) is solution of Pell equation $x^2 - Dy^2 = \pm N$ where $D > 1$ is a non-square integer and N is a positive integer.

Lemma 2.1 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = N$ and let

$$\begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.1}$$

for $n \ge 0$. Then the integer solutions of the Pell equation $x^2 - Dy^2 = N$ are (x_{2n+1}, y_{2n+1}) where

$$(x_{2n+1}, y_{2n+1}) = \left(\frac{u_{2n+1}}{N^n}, \frac{v_{2n+1}}{N^n} \right) \tag{2.2}$$

for u_{2n+1} and v_{2n+1} are multiples of N^n .

Proof. We prove theorem by mathematical induction on n . For $n = 0$, we have from (2.1), $(u_1, v_1) = (x_1, y_1)$ which is the fundamental solution of $x^2 - Dy^2 = N$. With the assumption that the Pell equation $x^2 - Dy^2 = N$ is satisfied for $n \ge 0$, that is

$$x_{2n+1}^2 - Dy_{2n+1}^2 = \frac{u_{2n+1}^2 - Dv_{2n+1}^2}{N^{2n}} = N \tag{2.3}$$

To prove that the Pell equation $x^2 - Dy^2 = N$ is true for $n+1$.

$$\begin{aligned}
\begin{pmatrix} u_{2n+3} \\ v_{2n+3} \end{pmatrix} &= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2} \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2} \begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} \\
&= \begin{pmatrix} \left(x_1^2 + Dy_1^2\right) u_{2n+1} + 2Dx_1y_1v_{2n+1} \\ 2x_1y_1u_{2n+1} + \left(x_1^2 + Dy_1^2\right) v_{2n+1} \end{pmatrix}.
\end{aligned}$$

Next, we will show that (x_{2n+3}, y_{2n+3}) is solution of Pell equation $x^2 - Dy^2 = N$.
Hence, by (2.2) we obtain

$$\begin{aligned}
x_{2n+3}^2 - Dy_{2n+3}^2 &= \left(\frac{u_{2n+3}}{N^{n+1}}\right)^2 - D\left(\frac{v_{2n+3}}{N^{n+1}}\right)^2 \\
&= \frac{1}{N^{2n+2}}\left(u_{2n+3}^2 - Dv_{2n+3}^2\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N^{2n+2}}\Big[\left(\left(x_1^2 + Dy_1^2\right)u_{2n+1} + 2Dx_1y_1v_{2n+1}\right)^2 \\
&\quad - D\left(2x_1y_1u_{2n+1} + \left(x_1^2 + Dy_1^2\right)v_{2n+1}\right)^2\Big] \\
&= \frac{1}{N^{2n+2}}\Big[\left(x_1^2 + Dy_1^2\right)^2 u_{2n+1}^2 \\
&\quad + 4\left(x_1^2 + Dy_1^2\right)Dx_1y_1u_{2n+1}v_{2n+1} + 4D^2x_1^2y_1^2v_{2n+1}^2 \\
&\quad - 4Dx_1^2y_1^2u_{2n+1}^2 - 4\left(x_1^2 + Dy_1^2\right)Dx_1y_1u_{2n+1}v_{2n+1} \\
&\quad - D\left(x_1^2 + Dy_1^2\right)^2 v_{2n+1}^2\Big] \\
&= \frac{1}{N^{2n+2}}\Big[\left(x_1^2 + Dy_1^2\right)^2 u_{2n+1}^2 + 4D^2x_1^2y_1^2v_{2n+1}^2 \\
&\quad - 4Dx_1^2y_1^2u_{2n+1}^2 - D\left(x_1^2 + Dy_1^2\right)^2 v_{2n+1}^2\Big] \\
&= \frac{1}{N^{2n+2}}\Big[\left(\left(x_1^2 + Dy_1^2\right)^2 - 4Dx_1^2y_1^2\right)u_{2n+1}^2 \\
&\quad - \left(\left(x_1^2 + Dy_1^2\right)^2 - 4Dx_1^2y_1^2\right)Dv_{2n+1}^2\Big] \\
&= \frac{\left(\left(x_1^2 + Dy_1^2\right)^2 - 4Dx_1^2y_1^2\right)\left(u_{2n+1}^2 - Dv_{2n+1}^2\right)}{N^{2n+2}} \\
&= \frac{\left(x_1^2 - Dy_1^2\right)^2\left(u_{2n+1}^2 - Dv_{2n+1}^2\right)}{N^{2n+2}}.
\end{aligned}$$

By (2.3) we have $u_{2n+1}^2 - Dv_{2n+1}^2 = N^{2n+1}$ and since (x_1, y_1) is the fundamental solution of the Pell equation $x^2 - Dy^2 = N$. Hence, we conclude that

$$\begin{aligned}
x_{2n+3}^2 - Dy_{2n+3}^2 &= \frac{\left(x_1^2 - Dy_1^2\right)^2\left(u_{2n+1}^2 - Dv_{2n+1}^2\right)}{N^{2n+2}} \\
&= \frac{N^2N^{2n+1}}{N^{2n+2}} = N .
\end{aligned}$$

This complete the proof.

Example 2.1 Consider the Pell equation $x^2 - 3y^2 = 6$. The fundamental solution is $(x_1, y_1) = (3,1)$. By (2.1) we have

$$\begin{pmatrix}u_1\\v_1\end{pmatrix}=\begin{pmatrix}3&3\\1&3\end{pmatrix}^1\begin{pmatrix}1\\0\end{pmatrix}=6^0\begin{pmatrix}2&3\\1&2\end{pmatrix}^0\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$$

$$\begin{pmatrix}u_3\\v_3\end{pmatrix}=\begin{pmatrix}3&3\\1&3\end{pmatrix}^3\begin{pmatrix}1\\0\end{pmatrix}=6^1\begin{pmatrix}2&3\\1&2\end{pmatrix}^1\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}54\\30\end{pmatrix}$$

$$\begin{pmatrix}u_5\\v_5\end{pmatrix}=\begin{pmatrix}3&3\\1&3\end{pmatrix}^5\begin{pmatrix}1\\0\end{pmatrix}=6^2\begin{pmatrix}2&3\\1&2\end{pmatrix}^2\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}1188\\684\end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix}u_{2n+1}\\v_{2n+1}\end{pmatrix}=\begin{pmatrix}3&3\\1&3\end{pmatrix}^{2n+1}\begin{pmatrix}1\\0\end{pmatrix}=6^n\begin{pmatrix}2&3\\1&2\end{pmatrix}^n\begin{pmatrix}3\\1\end{pmatrix}.$$

And by (2.2) we obtain

$$(x_1,y_1)=(u_1,v_1)=(3,1)$$

$$(x_3,y_3)=\left(\frac{u_3}{6^1},\frac{v_3}{6^1}\right)=(9,5)$$

$$(x_5,y_5)=\left(\frac{u_5}{6^2},\frac{v_5}{6^2}\right)=(33,19)$$

$$\vdots$$

$$\begin{pmatrix}x_{2n+1}\\y_{2n+1}\end{pmatrix}=\frac{1}{6^n}\begin{pmatrix}u_{2n+1}\\v_{2n+1}\end{pmatrix}=\begin{pmatrix}2&3\\1&2\end{pmatrix}^n\begin{pmatrix}3\\1\end{pmatrix}.$$

Hence

$$u_1^2-3v_1^2=6^1=6$$

$$u_3^2-3v_3^2=6^3=216$$

$$u_5^2-3v_5^2=6^5=7776$$

$$\vdots$$

$$u_{2n+1}^2-3v_{2n+1}^2=6^{2n+1}.$$

And

$$x_{2n+1}^2-3y_{2n+1}^2=6$$

for $n\ge 0$.

Lemma 2.2 Let (x_1,y_1) be the fundamental solution of the Pell equation $x^2-Dy^2=-N$ and let

$$\begin{pmatrix}u_{2n+1}\\v_{2n+1}\end{pmatrix}=\begin{pmatrix}x_1&Dy_1\\y_1&x_1\end{pmatrix}^{2n+1}\begin{pmatrix}1\\0\end{pmatrix}$$

for $n\ge 0$. Then the integer solutions of the Pell equation $x^2-Dy^2=-N$ are (x_{2n+1},y_{2n+1}) where

$$(x_{2n+1},y_{2n+1})=\left(\frac{u_{2n+1}}{N^n},\frac{v_{2n+1}}{N^n}\right)$$

for u_{2n+1} and v_{2n+1} are multiples of N^n.

Proof. This lemma can be proved as in the same way that lemma 2.1 was proved.

Example 2.2 Consider the Pell equation $x^2-6y^2=-2$. The fundamental solution is $(x_1,y_1)=(2,1)$. By (2.1) we have

$$\begin{pmatrix}u_1\\v_1\end{pmatrix}=\begin{pmatrix}2&6\\1&2\end{pmatrix}^1\begin{pmatrix}1\\0\end{pmatrix}=2^0\begin{pmatrix}5&12\\2&5\end{pmatrix}^0\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix}u_3\\v_3\end{pmatrix}=\begin{pmatrix}2&6\\1&2\end{pmatrix}^3\begin{pmatrix}1\\0\end{pmatrix}=2^1\begin{pmatrix}5&12\\2&5\end{pmatrix}^1\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}44\\18\end{pmatrix}$$

$$\begin{pmatrix}u_5\\v_5\end{pmatrix}=\begin{pmatrix}2&6\\1&2\end{pmatrix}^5\begin{pmatrix}1\\0\end{pmatrix}=2^2\begin{pmatrix}5&12\\2&5\end{pmatrix}^2\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}872\\356\end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix}u_{2n+1}\\v_{2n+1}\end{pmatrix}=\begin{pmatrix}2&6\\1&2\end{pmatrix}^{2n+1}\begin{pmatrix}1\\0\end{pmatrix}=2^n\begin{pmatrix}5&12\\2&5\end{pmatrix}^n\begin{pmatrix}2\\1\end{pmatrix}.$$

And by (2.2) we obtain

$$(x_1,y_1)=(u_1,v_1)=(2,1)$$

$$(x_3,y_3)=\left(\frac{u_3}{2^1},\frac{v_3}{2^1}\right)=(22,9)$$

$$(x_5,y_5)=\left(\frac{u_5}{2^2},\frac{v_5}{2^2}\right)=(218,89)$$

$$\vdots$$

$$\begin{pmatrix}x_{2n+1}\\y_{2n+1}\end{pmatrix}=\frac{1}{2^n}\begin{pmatrix}u_{2n+1}\\v_{2n+1}\end{pmatrix}=\begin{pmatrix}5&12\\2&5\end{pmatrix}^n\begin{pmatrix}2\\1\end{pmatrix}.$$

Hence

$$u_1^2-6v_1^2=-2^1=-2$$

$$u_3^2-6v_3^2=-2^3=-8$$

$$u_5^2-6v_5^2=-2^5=-32$$

$$\vdots$$

$$u_{2n+1}^2-6v_{2n+1}^2=-2^{2n+1}.$$

And

$$x_{2n+1}^2-6y_{2n+1}^2=-2$$

for $n\ge 0$.

3. Main results

In this section, we will find the solutions of Pell equation $x^2-Dy^2=\pm N$ where $D>1$ is a non-square integer and N is a positive integer.

Theorem 3.1 Let (x_1,y_1) be the fundamental solution of the Pell equation $x^2-Dy^2=N$ then

$$x_{2n+1}=\frac{\left(x_1^2+Dy_1^2\right)x_{2n-1}+2Dx_1y_1y_{2n-1}}{N}$$

$$y_{2n+1}=\frac{2x_1y_1x_{2n-1}+\left(x_1^2+Dy_1^2\right)y_{2n-1}}{N}$$

and

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = -2x_1y_1$$

for $n \geq 1$, where (2.1) and (2.2) hold.

Proof. By (2.1) we have

$$u_{2n+1} = \left(x_1^2 + Dy_1^2\right)u_{2n-1} + 2Dx_1y_1v_{2n-1}$$

$$v_{2n+1} = 2x_1y_1u_{2n-1} + \left(x_1^2 + Dy_1^2\right)v_{2n-1}$$

and by (2.2) we have

$$u_{2n+1} = N^n x_{2n+1}$$

$$v_{2n+1} = N^n y_{2n+1}.$$

We get

$$N^n x_{2n+1} = u_{2n+1}$$

$$= \left(x_1^2 + Dy_1^2\right)u_{2n-1} + 2Dx_1y_1v_{2n-1}$$

$$= \left(x_1^2 + Dy_1^2\right)N^{n-1}x_{2n-1} + 2Dx_1y_1N^{n-1}y_{2n-1}$$

$$x_{2n+1} = \frac{\left(x_1^2 + Dy_1^2\right)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{N}$$

and

$$N^n y_{2n+1} = v_{2n+1}$$

$$= 2x_1y_1u_{2n-1} + \left(x_1^2 + Dy_1^2\right)v_{2n-1}$$

$$= 2Dx_1y_1N^{n-1}x_{2n-1} + \left(x_1^2 + Dy_1^2\right)N^{n-1}y_{2n-1}$$

$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right)y_{2n-1}}{N}.$$

Hence,

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = x_{2n+1}y_{2n-1} - x_{2n-1}y_{2n+1}$$

$$= \left(\frac{\left(x_1^2 + Dy_1^2\right)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{N}\right)y_{2n-1}$$

$$-x_{2n-1}\left(\frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right)y_{2n-1}}{N}\right)$$

$$= \frac{2Dx_1y_1y_{2n-1}^2 - 2x_1y_1x_{2n-1}^2}{N}$$

$$= -\frac{2x_1y_1\left(x_{2n-1}^2 - Dy_{2n-1}^2\right)}{N}$$

$$= -\frac{2x_1y_1(N)}{N}$$

$$= -2x_1y_1.$$

This complete the proof.

Theorem 3.2 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = -N$ then

$$x_{2n+1} = \frac{\left(x_1^2 + Dy_1^2\right)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{N}$$

$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right)y_{2n-1}}{N}$$

and

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = 2x_1y_1$$

for $n \geq 1$, where (2.1) and (2.2) hold.

Proof. This theorem can be proved as in the same way that theorem 3.1 was proved.

Theorem 3.3 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = N$ then (x_{2n+1}, y_{2n+1}) satisfy the following recurrence relations

$$x_{2n+1} = \left(\frac{4}{N}x_1^2 + s\right)x_{2n-1}$$

$$+\left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)x_{2n-3} + (s+2)x_{2n-5}$$

$$y_{2n+1} = \left(\frac{4}{N}x_1^2 + s\right)y_{2n-1}$$

$$+\left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)y_{2n-3} + (s+2)y_{2n-5} \tag{3.4}$$

for $n \geq 3$ and s is an integer where (2.1) and (2.2) hold,

$$x_3 = \frac{x_1}{N}\left(4x_1^2 - 3N\right)$$

$$y_3 = \frac{y_1}{N}\left(4x_1^2 - N\right)$$

$$x_5 = \frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)$$

$$y_5 = \frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right).$$

Proof. We prove theorem by mathematical induction on n. By theorem 3.1, we have

$$x_3 = \frac{\left(x_1^2 + Dy_1^2\right)x_1 + 2Dx_1y_1^2}{N}$$

$$= \frac{x_1\left(x_1^2 + 3Dy_1^2\right)}{N}$$

$$= \frac{x_1}{N}\left(4x_1^2 - 3N\right)$$

$$y_3 = \frac{2x_1^2 y_1 + \left(x_1^2 + Dy_1^2\right) y_1}{N}$$

$$= \frac{y_1\left(3x_1^2 + Dy_1^2\right)}{N}$$

$$= \frac{y_1}{N}\left(4x_1^2 - N\right)$$

$$x_5 = \frac{\left(x_1^2 + Dy_1^2\right) x_3 + 2Dx_1 y_1 y_3}{N}$$

$$= \frac{1}{N}\left[\left(x_1^2 + Dy_1^2\right)\left(\frac{x_1}{N}\left(4x_1^2 - 3N\right)\right)\right.$$
$$\left. + 2Dx_1 y_1\left(\frac{y_1}{N}\left(4x_1^2 - N\right)\right)\right]$$

$$= \frac{x_1}{N^2}\left[\left(2x_1^2 - N\right)\left(4x_1^2 - 3N\right)\right.$$
$$\left. + 2\left(x_1^2 - N\right)\left(4x_1^2 - N\right)\right]$$

$$= \frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)$$

$$y_5 = \frac{2x_1 y_1 x_3 + \left(x_1^2 + Dy_1^2\right) y_3}{N}$$

$$= \frac{1}{N}\left[2x_1 y_1\left(\frac{x_1}{N}\left(4x_1^2 - 3N\right)\right)\right.$$
$$\left. + \left(x_1^2 + Dy_1^2\right)\left(\frac{y_1}{N}\left(4x_1^2 - N\right)\right)\right]$$

$$= \frac{y_1}{N^2}\left[2x_1^2\left(4x_1^2 - 3N\right)\right.$$
$$\left. + \left(2x_1^2 - N\right)\left(4x_1^2 - N\right)\right]$$

$$= \frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right).$$

And

$$x_7 = \frac{\left(x_1^2 + Dy_1^2\right) x_5 + 2Dx_1 y_1 y_5}{N}$$

$$= \frac{1}{N}\left[\left(x_1^2 + Dy_1^2\right)\left(\frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)\right)\right.$$
$$\left. + 2Dx_1 y_1\left(\frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right)\right]$$

$$= \frac{x_1}{N^3}\left[\left(2x_1^2 - N\right)\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)\right.$$
$$\left. + 2\left(x_1^2 - N\right)\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right]$$

$$= \frac{x_1}{N^3}\left(64x_1^6 - 112Nx_1^4 + 56N^2 x_1^2 - 7N^3\right)$$

$$y_7 = \frac{2x_1 y_1 x_5 + \left(x_1^2 + Dy_1^2\right) y_5}{N}$$

$$= \frac{1}{N}\left[2x_1 y_1\left(\frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)\right)\right.$$
$$\left. + \left(x_1^2 + Dy_1^2\right)\left(\frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right)\right]$$

$$= \frac{y_1}{N^3}\left[2x_1^2\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)\right.$$
$$\left. + \left(2x_1^2 - N\right)\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right]$$

$$= \frac{y_1}{N^3}\left(64x_1^6 - 80Nx_1^4 + 24N^2 x_1^2 - N^3\right).$$

For $n = 3$, we have

$$\left(\frac{4}{N}x_1^2 + s\right) x_5 + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right) x_3$$
$$+ (s+2) x_1$$

$$= \left(\frac{4}{N}x_1^2 + s\right)\left(\frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)\right)$$
$$+ \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)\left(\frac{x_1}{N}\left(4x_1^2 - 3N\right)\right)$$
$$+ (s+2) x_1$$

$$= x_1\left(\frac{64}{N^3}x_1^6 - \frac{112}{N^2}x_1^4 + \frac{56}{N}x_1^2 - 7\right)$$

$$= \frac{x_1}{N^3}\left(64x_1^6 - 112Nx_1^4 + 56N^2 x_1^2 - 7N^3\right)$$

$$= x_7$$

$$\left(\frac{4}{N}x_1^2 + s\right) y_5 + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right) y_3$$
$$+ (s+2) y_1$$

$$= \left(\frac{4}{N}x_1^2 + s\right)\left(\frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right)$$
$$+ \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)\left(\frac{y_1}{N}\left(4x_1^2 - N\right)\right)$$
$$+ (s+2) y_1$$

$$= y_1\left(\frac{64}{N^3}x_1^6 - \frac{80}{N^2}x_1^4 + \frac{24}{N}x_1^2 - 1\right)$$

$$= \frac{y_1}{N^3}\left(64x_1^6 - 80Nx_1^4 + 24N^2x_1^2 - N^3\right)$$
$$= y_7 .$$

Therefore (3.4) is true for $n = 3$. With the assumption that (3.4) is satisfied for $n \geq 3$.
To prove that (3.4) is true for $n+1$.

$$\left(\frac{4}{N}x_1^2 + s\right)x_{2n+1} + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)x_{2n-1} + (s+2)x_{2n-3}$$
$$= \left(\frac{4}{N}x_1^2 + s\right)\left(\frac{\left(x_1^2 + Dy_1^2\right)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{N}\right)$$
$$+\left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)\left(\frac{\left(x_1^2 + Dy_1^2\right)x_{2n-3} + 2Dx_1y_1y_{2n-3}}{N}\right)$$
$$+(s+2)\left(\frac{\left(x_1^2 + Dy_1^2\right)x_{2n-5} + 2Dx_1y_1y_{2n-5}}{N}\right)$$
$$= \frac{1}{N}\Bigg[\left(x_1^2 + Dy_1^2\right)\Bigg(\left(\frac{4}{N}x_1^2 + s\right)x_{2n-1} + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)x_{2n-3} + (s+2)x_{2n-5}\Bigg)$$
$$+2Dx_1y_1\Bigg(\left(\frac{4}{N}x_1^2 + s\right)y_{2n-1} + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)y_{2n-3} + (s+2)y_{2n-5}\Bigg)\Bigg]$$
$$= \frac{\left(x_1^2 + Dy_1^2\right)x_{2n+1} + \left(2Dx_1y_1\right)y_{2n+1}}{N}$$
$$= x_{2n+3}$$

$$\left(\frac{4}{N}x_1^2 + s\right)y_{2n+1} + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)y_{2n-1} + (s+2)y_{2n-3}$$
$$= \left(\frac{4}{N}x_1^2 + s\right)\left(\frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right)y_{2n-1}}{N}\right)$$
$$+\left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)\left(\frac{2x_1y_1x_{2n-3} + \left(x_1^2 + Dy_1^2\right)y_{2n-3}}{N}\right)$$
$$+(s+2)\left(\frac{2x_1y_1x_{2n-5} + \left(x_1^2 + Dy_1^2\right)y_{2n-5}}{N}\right)$$
$$= \frac{1}{N}\Bigg[2x_1y_1\Bigg(\left(\frac{4}{N}x_1^2 + s\right)x_{2n-1} + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)x_{2n-3} + (s+2)x_{2n-5}\Bigg)$$
$$+\left(x_1^2 + Dy_1^2\right)\Bigg(\left(\frac{4}{N}x_1^2 + s\right)y_{2n-1} + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)y_{2n-3} + (s+2)y_{2n-5}\Bigg)\Bigg]$$
$$= \frac{\left(2x_1y_1\right)x_{2n+1} + \left(x_1^2 + Dy_1^2\right)y_{2n+1}}{N}$$
$$= y_{2n+3} .$$

This complete the proof.

Example 3.1 From example 2.1, by theorem 3.3 with $s = -2$ we have

$$x_1 = 3, y_1 = 1$$
$$x_3 = \frac{x_1}{N}\left(4x_1^2 - 3N\right) = \frac{3}{6}\left(4(3)^2 - 3(6)\right) = 9$$
$$y_3 = \frac{y_1}{N}\left(4x_1^2 - N\right) = \frac{1}{6}\left(4(3)^2 - 6\right) = 5$$
$$x_5 = \frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right) = \frac{3}{6^2}\left(16(3)^4 - 20(6)(3)^2 + 5(6)^2\right) = 33$$
$$y_5 = \frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right) = \frac{1}{6^2}\left(16(3)^4 - 12(6)(3)^2 + (6)^2\right) = 19$$
$$x_7 = \left(\frac{4}{N}x_1^2 - 2\right)x_5 - x_3 = \left(\frac{4}{6}(3)^2 - 2\right)(33) - 9 = 123$$
$$y_7 = \left(\frac{4}{N}x_1^2 - 2\right)y_5 - y_3$$

$$= \left(\frac{4}{6}(3)^2 - 2\right)(19) - 5$$
$$= 71$$
$$x_9 = \left(\frac{4}{N}x_1^2 - 2\right)x_7 - x_5$$
$$= \left(\frac{4}{6}(3)^2 - 2\right)(123) - 33$$
$$= 459$$
$$y_9 = \left(\frac{4}{N}x_1^2 - 2\right)y_7 - y_5$$
$$= \left(\frac{4}{6}(3)^2 - 2\right)(71) - 19$$
$$= 265$$

Hence
$$x_{2n+1} = 4x_{2n-1} - x_{2n-3}$$
$$y_{2n+1} = 4y_{2n-1} - y_{2n-3}$$
for $n \geq 3$.

Theorem 3.4 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = -N$ then (x_{2n+1}, y_{2n+1}) satisfy the following recurrence relations

$$x_{2n+1} = \left(\frac{4}{N}x_1^2 + s\right)x_{2n-1} + \left(-\frac{4}{N}(s-2)x_1^2 - 2s + 3\right)x_{2n-3} + (s-2)x_{2n-5}$$

$$y_{2n+1} = \left(\frac{4}{N}x_1^2 + s\right)y_{2n-1} + \left(-\frac{4}{N}(s-2)x_1^2 - 2s + 3\right)y_{2n-3} + (s-2)y_{2n-5}$$

for $n \geq 3$ and s is an integer where (2.1) and (2.2) hold,

$$x_3 = \frac{x_1}{N}\left(4x_1^2 + 3N\right)$$
$$y_3 = \frac{y_1}{N}\left(4x_1^2 + N\right)$$
$$x_5 = \frac{x_1}{N^2}\left(16x_1^4 + 20Nx_1^2 + 5N^2\right)$$
$$y_5 = \frac{y_1}{N^2}\left(16x_1^4 + 12Nx_1^2 + N^2\right).$$

Proof. This theorem can be proved as in the same way that theorem 3.3 was proved.

Example 3.2 From example 2.2, by theorem 3.4 with $s = 2$ we have

$$x_1 = 2, y_1 = 1$$
$$x_3 = \frac{x_1}{N}\left(4x_1^2 + 3N\right)$$
$$= \frac{2}{2}\left(4(2)^2 + 3(2)\right)$$
$$= 22$$
$$y_3 = \frac{y_1}{N}\left(4x_1^2 + N\right)$$
$$= \frac{1}{2}\left(4(2)^2 + 2\right)$$
$$= 9$$
$$x_5 = \frac{x_1}{N^2}\left(16x_1^4 + 20Nx_1^2 + 5N^2\right)$$
$$= \frac{2}{2^2}\left(16(2)^4 + 20(2)(2)^2 + 5(2)^2\right)$$
$$= 218$$
$$y_5 = \frac{y_1}{N^2}\left(16x_1^4 + 12Nx_1^2 + N^2\right)$$
$$= \frac{1}{2^2}\left(16(2)^4 + 12(2)(2)^2 + (2)^2\right)$$
$$= 89$$
$$x_7 = \left(\frac{4}{N}x_1^2 + 2\right)x_5 - x_3$$
$$= \left(\frac{4}{2}(2)^2 + 2\right)(218) - 22$$
$$= 2158$$
$$y_7 = \left(\frac{4}{N}x_1^2 + 2\right)y_5 - y_3$$
$$= \left(\frac{4}{2}(2)^2 + 2\right)(89) - 9$$
$$= 881$$
$$x_9 = \left(\frac{4}{N}x_1^2 + 2\right)x_5 - x_3$$
$$= \left(\frac{4}{2}(2)^2 + 2\right)(2158) - 218$$
$$= 21362$$
$$y_9 = \left(\frac{4}{N}x_1^2 + 2\right)y_5 - y_3$$
$$= \left(\frac{4}{2}(2)^2 + 2\right)(881) - 89$$
$$= 8721$$

Hence

$$x_{2n+1} = 10x_{2n-1} - x_{2n-3}$$
$$y_{2n+1} = 10y_{2n-1} - y_{2n-3}$$

for $n \geq 3$.

4. Conclusions

In this paper, we considered the Pell equation $x^2 - Dy^2 = \pm N$ where $D > 1$ is a non-square integer and N is a positive integer, and obtained some formulas its integer solutions.

Acknowledgements

The authors would like to thank the editors and referees for their comments and suggestions on the manuscript. This work was supported by the Surindra Rajabhat University.

Declaration of conflicting interests

The authors declared that they have no conflicts of interest in the research, authorship, and this article's publication.

References

1. Chanduol A. The Pell Equation $x^2 - Dy^2 = \pm k^2$. Adv Pure Math. 2011; 1: 16-22.
2. Kaplan P, Williams, KS. Pell's Equation $x^2 - my^2 = -1, -4$ and Continued Fractions. J Number Theory. 1986; 23: 169-182.
3. Matthews K. The Diophantine Equation $x^2 - Dy^2 = N, D > 0$. Expo Math. 2000; 18: 323-331.
4. Ramya D, Seethalakshmi V, Durai AD. The Pell Equation $x^2 - Dy^2 = 390625$. Int J Comput Organ Trend. 2015; 19(1) :57-59.
5. Shabani AS. The Proof of Two Conjectures Related to Pell's Equation $x^2 - Dy^2 = \pm 4$. Int J Comput Math Sci. 2008; 2(1) :24-27.
6. Tekcan A. Pell Equation $x^2 - Dy^2 = 2$, II. Bull Irish Math Soc. 2004; 54: 73-89.
7. Tekcan A. The Pell Equation $x^2 - Dy^2 = \pm 4$. Appl Math Sci. 2007; 1(8): 363-369.

