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The average run length for continuous distribution process mean shift detection on a modified EWMA control chart

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Abstract

Herein, we provide an approximated average run length (ARL) solved by applying integral equations for detecting shifts in the process mean on a modified exponentially weighted moving average (EWMA) control chart when the observations are from continuous distributions such as gamma or Weibull. We compared numerical approximations of the ARL using four quadrature rules: the composite midpoint, trapezoidal, and Simpson's rules and the Gauss-Legendre rule. The shape and scale parameters of four continuous distributions of the observations: Gamma (2, 1), Gamma (3, 1), Weibull (2, 1), and Weibull (3, 1) were determined according to their skewness. The criterion for evaluating the performances of the four quadrature rules and control charts was the out-of-control ARL (ARL_1) and CPU Time. Our analysis reveals that the accuracies of the four quadrature rules to approximate the ARL on a modified EWMA control chart with observations from either gamma or Weibull distributions were similar. However, the Gauss-Legendre rule provided the simplest ARL calculation and achieved the highest accuracy for the given number of nodes. Meanwhile, the results reveal the superiority of the modified EWMA control chart over the standard one in terms of detecting a shift in the process mean. In addition, the efficacies of the control charts using the approximated ARL solutions were also demonstrated using a continuous distribution of real observations.

Keywords: Integral equations, ARL, Gamma distribution, Weibull distribution

1. Introduction

Control charts are extremely useful tools for statistical process control (SPC). Their main purpose is to enhance and ensure the quality of processes to satisfy and respond to customer requirements. To make the SPC process simpler, several control chart schemes have been proposed including the Shewhart [1], cumulative sum (CUSUM) [2], and exponentially weighted moving average (EWMA) control charts [3]. They have been used in many applications in several research fields and industrial processes [4-6]. Roberts [7] proposed the concept of the EWMA control chart, initially for normally distributed observations. Currently, the method is extensively used for the online monitoring of analytical processes [8], industrial production processes [9], public health surveillance [10], among others, even when the observations are sequential and do not necessarily follow a normal distribution. In terms of statistical performance, the EWMA control chart is a more powerful tool for detecting small-to-moderate shifts in the process mean than the Shewhart control chart. The common form of the two-sided EWMA control chart has been provided by several scholars, including Montgomery [11]. The EWMA statistic is the exponentially weighted average of the previous and current observations. The representative weight parameter (or smoothing parameter) λ ranges from 0 to 1, which works well for small values in the range of 0.05 and 0.25 when monitoring production processes. There have been several published studies on the effectiveness of the EWMA control chart, such as [12-13].

Several researchers have made an effort to develop a useful technique for the detection of shifts in the process mean at an earlier stage to avoid losses incurred in manufacturing processes. To this end, the modified EWMA control chart was initially proposed by Gan [14] for the monitoring of binomial counts. Afterward, three modified EWMA statistics to detect smaller shifts in the process mean than is possible with the Shewhart control chart were introduced by Gan [15]. Subsequently, the modified EWMA control chart for monitoring small shifts in the process mean when they are small and immediate was offered by Patel and Divecha [16] in 2011. Recently, to improve the rapid detecting ability of the EWMA statistic, Khan, et al [17] presented a modified EWMA statistic applied to independent normal observations. For the rapid detection of the small process shifts, a second parameter k was presented, which is different from the traditional EWMA statistic. Based on the observations, the modified EWMA statistic can detect the smaller and more abrupt mean shifts than the traditional EWMA statistic. The modified EWMA control chart for monitoring time-series observations was first suggested by Herdiani, et al [18] and later extended by [19-21]. The conclusions from all of these articles state that the modified EWMA control chart can detect an early shift in a process parameter better than the traditional EWMA control chart. Likewise, a comparison of standard and modified EWMA control charts for monitoring a process shift is offered in this article.

There are several continuous distributions for modeling lifetime data, such as exponential, gamma, and Weibull. The interest in this study is centered on gamma and Weibull distributions with two parameters since they are suitable for skewed data and can also be applied to model the time between events. Both distributions have been extensively applied in different fields. The gamma distribution is considered to be a good fit for life testing data. Many authors have reported the use of control chart charts for gamma-distributed observations; for instance, Sheu and Lin [22] designed a control chart when the data of interest follow a gamma distribution. Bhaumik and Gibbons [23] applied a two-parameter gamma distribution for environment monitoring and control studies. A random-shift model for measuring the average run length (ARL) for a gamma distribution was proposed by Zhang, et al [24]. Recently, a control chart for a specific variable following a gamma distribution by using a neutrosophic statistical interval method was provided by Aslam, et al [25]. There are several studies on control charts for Weibull distributions. Nelson [26] designed various control charts for the Weibull process. Bai and Choi [27] designed and range control charts to monitor the skewed data. Hawkins and Olwell [28] proposed the CUSUM control chart to monitor the scale parameter of a Weibull distribution. Chang and Bai [29] designed various control charts, including the EWMA control chart, using weighted standard deviations for skewed distributions.

Control charts can be compared by determining their ARLs in various situations. The average number of observations under the control limit for a process while it is in-control until a false signal for out-of-control occurs is denoted as ARL_0. The average number of observations under the control limit until a true out-of-control signal occurs is denoted as ARL_1. ARL_0 should be sufficiently large to keep the false alarm signals at an acceptable level, while conversely, ARL_1 should be kept as small as possible to capture a true out-of-control signal.

The ARL can be evaluated by applying many methods, such as Monte Carlo simulation, the Markov Chain approach, the Martingale approach, and integral equations. Fredholm integral equations of the second kind [30] are used in the integral equations, which depend on numerical quadrature rules to measure the integrals. The numerical integration (or quadrature) method is commonly applied as an approximation of the integral [31]. In this study, some of the basic quadrature rules, such as the composite midpoint, trapezoidal, and Simpson's rules and the Gauss-Legendre rule, are used in the derivation of the ARL for the modified EWMA control chart when the observations are from gamma or Weibull distributions.

The rest of this paper is organized as follows. In the next section, background on observations following continuous distributions of the modified EWMA control chart and characteristics of the ARL are introduced. Integral equations for approximating the ARL of the modified EWMA control chart derived by using the four previously mentioned quadrature rules are also presented and proved. In Section 3, the comparative performance of the four approximated ARLs on standard and modified EWMA control charts are covered in Section 4. In Section 5, applying the proposed approximated ARL using the Gauss-Legendre rule for monitoring the strength of single carbon fibers of gauge length 20 mm on the modified EWMA control chart is presented, while Section 6 offers conclusions on the study.

2. Continuous distributions of the observations on the modified EWMA control chart

The objective of this research is to estimate the ARL for detecting shifts in the process mean on a modified EWMA control chart when the observations are continuously distributed.

2.1 Gamma and weibull distributions

Assume that X as a two-parameter gamma or Weibull distributions with shape parameter $(\alpha > 0)$ and scale parameter $(\beta > 0)$.

Definition 2.1 For gamma-distributed random variable X denoted as $X \sim \text{Gamma}(\alpha, \beta)$, the corresponding probability density function in the shape-scale parametrization [32] is

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \text{for } x > 0,\ \alpha, \beta > 0. \tag{1}$$

Definition 2.2 For Weibull-distributed random variable X denoted as $X \sim \text{Weibull}(\alpha, \beta)$, the corresponding probability density function in the shape-scale parametrization [32] is

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^{\alpha}}, \text{ for } x > 0, \alpha, \beta > 0. \tag{2}$$

2.2 Control charts with continuously distributed observations

The standard and modified EWMA control charts and their characteristics are briefly discussed in this section.

2.2.1 The standard EWMA control chart

The concept of EWMA is to combine previous and current observations for detecting shifts in the process mean to enable more rapid detection than the Shewhart control chart, where the focus is only on the most current observations without considering the previous ones. This is achieved by applying weighting parameter λ, to determine the importance of each observation (X_t) and assign the highest weight to the most current observation. The weights on the observations depend on the time series, with the first observations having the lowest weight. The average for the t^{th} period, for $t = 1, 2, \ldots$, is defined as

$$Z_t = \lambda X_t + (1-\lambda) Z_{t-1} \ ; t = 1, 2, \ldots, \tag{3}$$

where $0 < \lambda \le 1$ and the starting value is $Z_0 = \mu_0$. The upper control limit (UCL) and lower control limit (LCL) of the EWMA chart are

$$\text{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)}$$

and

$$\text{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)}, \tag{4}$$

respectively, where μ_0 is the target mean, σ^2 is the process variance and L is a suitable control width limit.

2.2.2 The modified EWMA control chart

The properties of the modified EWMA control chart are user-friendliness and high effectiveness for detecting shifts in the process mean. Patel and Divecha [16] combined the useful properties of the Shewart and standard EWMA control charts for detecting small process changes from data from a single-order autocorrelated process. The modified EWMA control chart is defined as

$$Z_t = \lambda X_t + (1-\lambda) Z_{t-1} + (X_t - X_{t-1}); \ t = 1, 2, \ldots, \tag{5}$$

where Z_t is the average from the historical data, λ is a sequence of weighted parameters from 0 to 1, X_t are observations from a gamma (or Weibull) distribution, X_{t-1} are the values of the previous observations, Z_{t-1} is the target, $Z_0 = u$ and $X_0 = v$ are the initial values.

The control limits of the Modified EWMA chart are

$$\mathrm{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda} + \frac{2\lambda(1-\lambda)}{2-\lambda}}$$

and

$$\mathrm{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda} + \frac{2\lambda(1-\lambda)}{2-\lambda}}, \tag{6}$$

where μ_0 is the target mean, σ^2 is the process variance, and L a suitable control width limit.

The corresponding stopping time (τ_b) for the out-of-control process can be written as

$$\tau_b = \inf\{t > 0;\, Z_t > b\},\ u < b, \tag{7}$$

where b is a constant for the UCL.

2.3 Control chart characteristics

The ARL is a general characteristic of control charts. It is defined as the expectation of an alarm being sent to signify a possible change in a particular parameter' s distribution. In the in-control process, an acceptable ARL should be large enough to detect small changes in the parameter's distribution. In this paper, the following notation is applied:

$$\mathrm{ARL}_0 = E_\theta(\tau_b) = \mathrm{ARL}(u), \text{ where } \theta = \infty, \tag{8}$$

and $E_\infty(.)$ is the expectation depending on the target value that should be large enough and θ is the change point time. When the ARL process is out-of-control, it is called the average number of observations until the signal for a sequence with a constant expectation indicates the out-of-control state and is denoted by ARL_1. Herein, the following notation is applied:

$$\mathrm{ARL}_1 = E_\theta(\tau_b - \theta + 1 \mid \tau_b \geq \theta), \tag{9}$$

where $E_\theta(.)$ is the expectation under the assumption that a change point occurs when given $\theta = 1$.

If X_1 is the control limit, then

$$0 \leq (1-\lambda)u + (1+\lambda)X_1 - v \leq b.$$

The solution can be written in the form

$$\mathrm{ARL}(u) = 1 + \int_{\frac{-(1-\lambda)u+v}{1+\lambda}}^{\frac{b-(1-\lambda)u+v}{1+\lambda}} \mathrm{ARL}\big((1-\lambda)u + (1+\lambda)x - v\big) f(x)dx.$$

Let $k = (1-\lambda)u + (1+\lambda)x - v$ can be written as follows: (Crowder [12])

$$\mathrm{ARL}(u) = 1 + \frac{1}{\lambda+1}\int_0^b \mathrm{ARL}(k) f\left(\frac{k-(1-\lambda)u+v}{\lambda+1}\right)dk \tag{10}$$

The function of the ARL cannot be evaluated exactly in its closed form, and so approximate numerical integration is required to evaluate it. Thus, we applied the composite midpoint, trapezoidal, and Simpson's rules and the Gauss-Legendre rule to estimate the ARL.

3. The approximated ARL via numerical integration on a modified EWMA control chart

Numerical integration is a basic tool that can be applied to obtain an approximate answer for a definite integral by replacing it with the sum, while quadrature is equivalent to numerical integration in one dimension. Define $f(t)$ as a function on closed interval $[a,b]$ and the set of separated nodes $\{t_0, t_1, t_2, ..., t_n\}$. Thus, the approximation of the numerical integration can be defined as

$$\int_a^b f(t)\,dt \cong \sum_{i=0}^{n} w_i f(t_i), \tag{11}$$

where t_i are the quadrature nodes (or quadrature points) and w_i are the quadrature weights. The definite integrals can be evaluated by applying numerical integration methods. In this study, we applied the composite midpoint, trapezoidal, and Simpson's rules and the Gauss-Legendre rules [34].

3.1 The composite midpoint rules

The midpoint of each subinterval is defined by how likely it is to be close to the average point. The midpoint rule uses the midpoint in the sum. The i^{th} interval $[t_{i-1}, t_i]$, referred to as midpoint $\overline{t_i}$, is

$$\overline{t_i} = \frac{t_{i-1} + t_i}{2}.$$

Let $\Delta t_i = t_i - t_{i-1}$ be the length of each interval. Thus, we use the following formula to approximate the integral by using midpoints:

$$Mid_n = \sum_{i=1}^{n} f(\overline{t_i})\Delta t_i.$$

For even spacing, $\Delta t_i = h = (b-a)/n$, for which the formula is

$$Mid_n = h\sum_{i=1}^{n} f(\overline{t_i}) = h(\hat{x}_1 + \hat{x}_2 + \ldots, +\hat{x}_n),$$

where $\hat{x}_i = f(\overline{t_i})$. According to the midpoint rule and given $f(A_j) = f\left(\frac{a_j - (1-\lambda)u + v}{1+\lambda}\right)$, the one-sided ARL on a modified EWMA control chart in Equation (10) can be approximated by applying the following formula:

$$\mathrm{ARL}_{Mid}(u) \approx 1 + \frac{1}{1+\lambda}\sum_{j=1}^{n} w_j \,\mathrm{ARL}(a_j) f(A_j), \tag{12}$$

where $w_j = \frac{b}{n}$ and $a_j = \left(j - \frac{1}{2}\right); j = 1, 2, \ldots, n.$

3.2 The composite trapezoidal rule

The trapezoidal rule is obtained by integrating the first-order polynomial interpolation. It is written as:

$$Int = \int_a^b f(t)\,dt = \frac{b-a}{2}\left[f(b) + f(a)\right].$$

Closed intervals $[a,b]$ can be separated into n intervals with equal width h. These nodes are $a = t_0, t_1, t_2, \ldots, t_n = b$, where $t_i = t_0 + ih$, for all $i = 1, 2, \ldots, n.$ The value h is given as $(b-a)/n$.

The above $n-$interval case is relevant for

$$Int = \int_a^b f(t)\,dt = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(a+ih)\right]$$

$$= \frac{h}{2}\left[f(a) + f(b) + 2\left(f(t_1) + f(t_2) + \ldots + f(t_{n-1})\right)\right].$$

when $f(a) = f_0, f(a+h) = f_1$ and $f(a+ih) = f_i$ are replaced, the above case is relevant for

$$Int = \int_a^b f(t)\,dt = \frac{h}{2}\left[f_0 + f_i + 2\left(f_1 + f_2 + \ldots + f_{i-1}\right)\right].$$

According to the trapezoidal rule, given $f(A_j)=f\left(\frac{a_j-(1-\lambda)u+\nu}{1+\lambda}\right)$, the ARL in Equation (10) can be approximated as

$$\mathrm{ARL}_{Trapi}(u)\approx 1+\frac{1}{1+\lambda}\sum_{j=1}^{n+1}w_j\,\mathrm{ARL}(a_j)f(A_j), \tag{13}$$

where $a_j=jw_j$ and $w_j=\frac{b}{n}; j=1,2,...,n-1$, in other cases, $w_j=\frac{b}{2n}$.

3.3 The composite simpson's rule

Simpson's rule applies a first-order polynomial function and is used for equal data intervals with width h. For three points: $t_0=a,\ t_1=a+h$ and $t_2=b$. Simpson's rule is defined as

$$Int=\int_a^b f(t)dt=\frac{h}{3}\left[f(t_0)+f(t_2)+4f(t_1)\right].$$

Simpson's rule can accurately improve by separating closed intervals $[a,b]$ into sub-intervals as follows:

$$Int=\frac{h}{3}\left[f(a)+f(b)+2\sum_{\substack{i=2\\ i=even}}^{n-2}f(a+ih)+4\sum_{\substack{i=1\\ i=odd}}^{n-1}f(a+ih)\right].$$

By setting $f_i=f(a+ih)$ in the above relationship, the following formula is obtained:

$$Int=\frac{h}{3}\left[f_0+f_n+2(f_2+f_4+...+f_{n-2})+4(f_1+f_3+...+f_{n-1})\right].$$

According to Simpson's rule, the ARL in Equation (10) can be estimated by

$$\mathrm{ARL}_{Simp}(u)=1+\frac{1}{1+\lambda}\sum_{j=1}^{2n+1}w_j\,\mathrm{ARL}(a_j)f(A_j), \tag{14}$$

where $a_j=jw_j$ and $w_j=\frac{4}{3}\left(\frac{b}{2n}\right);\ j=1,3,...,2n-1,\ w_j=\frac{2}{3}\left(\frac{b}{2n}\right); j=2,4,...,2n-2,$ in other cases, $w_j=\frac{1}{3}\left(\frac{b}{2n}\right)$.

3.4 The gauss-legendre rule

The above three numerical methods use points to calculate function $f(t)$ on closed intervals $[a,b]$ that are equally spaced. One of the approximating integral methods is the Gaussian quadrature rule. Special values of weights and abscissas (referred to as evaluation or Gauss points) applied in quadrature rules are usually pre-computed and utilizable in most standard mathematical procedures. The two-point Gaussian quadrature rule for function $f(t)$ can be calculated between fixed limits a and b as follows:

$$Int=\int_a^b f(t)dt\approx k_1f(t_1)+k_2f(t_2),$$

where k_1,k_2,t_1 and t_2 are four unknown coefficients that must be defined by integrating exact cubic polynomial $f(t)=a_3t^3+a_2t^2+a_1t+a_0$. The results are more accurate when the number of Gaussian points is increased. Thus,

$$Int=\int_a^b f(t)dt\approx k_1f(t_1)+k_2f(t_2)+k_3f(t_3).$$

Hence, n nodes are applied to approximate $f(t)$ between fixed limits such that

$$Int=\int_a^b f(t)dt\approx k_1f(t_1)+k_2f(t_2)+...+k_nf(t_n).$$

Gaussian integration is based on the use of polynomials to approximate integrand $f(x)$ on closed interval $[-1, 1]$. The coefficients of this polynomial are unknown variables that can be determined by using any optimal method. The basic form of the Gaussian quadrature rule is focused on uniform weighting over the interval, while specific nodes of $f(x)$ are the roots of a particular class of Legendre polynomials over the interval.

Gaussian quadrature formulas can be evaluated by using abscissae and weights. In general, the integral on a general interval is used for variable t on closed intervals $[a,b]$ this interval is linearly mapped for t on closed interval $[-1,1]$ for x. A simple change of variable can be applied when the integral is not posted on closed interval $[-1,1]$. To revise any one of closed intervals $[a,b]$ as an integral on closed interval $[-1,1]$, let $t = mx + c$. Moreover, $x = -1$ when $t = a$ and $x = 1$ when $t = b$. Hence, $t = \frac{b-a}{2}x + \frac{b+a}{2}$. The following formula is arrived at after simplification:

$$Int = \int_a^b f(t)\,dt = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}dx$$

$$Int = \left(\frac{b-a}{2}\right)\sum_{i=1}^{n} k_i f\left(\frac{b+a}{2} + \frac{b-a}{2}x_i\right).$$

According to the Gauss-Legendre rule, the ARL in Equation (10) can be approximated by

$$\mathrm{ARL}_{Gaus}(u) = 1 + \frac{1}{1+\lambda}\sum_{j=1}^{n} w_j \,\mathrm{ARL}(a_j) f(A_j), \tag{15}$$

where $w_j = \frac{b}{n}$ and $a_j = \frac{b}{n}\left(j - \frac{1}{2}\right); j = 1, 2, \ldots, n.$

4. The performance of the approximated ARL and comparison of the standard and modified EWMA control charts

In this section, the numerical values of the approximated ARL and the CPU times to calculate them using the four methods on the modified EWMA control chart when the observations are from a continuous distribution are compared. The lowest ARL value with the shortest CPU time infers the best performance. Furthermore, these performance measures were used to compare their efficacies on both the standard and EWMA control charts. We determined the ARL_1 values of the process shift for ARL_0 = 370 or 500 as the in-control parameter and various process mean shift sizes (δ = 0.001, 0.005, 0.01, 0.03, 0.05, 0.1, 0.3, 0.5, and 1). For the gamma and Weibull distributions, the scale parameter (β) was varied from 2 to 3 and the shape parameter (α) was set as 1.

The results for ARL_0 = 370 and 500 are given in Tables 2 and 3, respectively. We used division point $m = 500$ nodes computed by using the Mathematica program. Accordingly, all values of b (the UCL of the modified EWMA control chart) for ARL_0 = 370 or 500 were calculated by using Equations (12)-(15).

Table 1 The values of the modified EWMA control limit (b) for continuous distributions.

Continuous Distribution	λ	ARL0	
		370	500
Gamma (2, 1)	0.05	23.92000	24.22650
	0.10	13.69800	13.88470
Gamma (3, 1)	0.05	22.99230	23.19750
	0.10	12.78770	12.90660
Weibull (2, 1)	0.05	24.22650	24.22650
	0.10	13.69735	13.88392
Weibull (3, 1)	0.05	22.99070	23.19650
	0.10	12.78720	12.90610

The results for b when $\mathrm{ARL}_0 = 370$ and 500 are reported in Table 1. It was found that as λ increased, b decreased on the modified EWMA control chart for both gamma and Weibull-distributed observations. When α was changed from 2 to 3, and β = 1, the findings indicate that α increased and b decreased for both distributions.

Table 2 The ARL values solved via integral equations on the modified EWMA control chart for continuously distributed observations at $ARL_0 = 370$.

Continuous Distribution	λ	Methods	δ								
			0.001	0.005	0.01	0.03	0.05	0.1	0.3	0.5	1
Gamma (2, 1)	0.05	Midpoint	363.232	335.761	305.727	220.115	168.732	105.135	46.141	31.893	19.121
			(2.28)	(2.18)	(2.03)	(2.22)	(2.20)	(2.18)	(2.25)	(2.21)	(2.25)
		Trapezoidal	362.720	335.305	305.331	219.876	168.577	105.067	46.127	31.886	19.119
			(2.03)	(2.06)	(2.04)	(2.03)	(2.05)	(2.25)	(2.31)	(2.11)	(2.19)
		Simpson's	363.042	335.591	305.58	220.026	168.674	105.11	46.136	31.890	19.120
			(8.15)	(8.67)	(9.04)	(13.55)	(10.25)	(9.42)	(9.71)	(9.69)	(10.16)
		Gaussian	362.979	335.533	305.527	219.993	168.653	105.101	46.1341	31.890	19.120
			(11.03)	(9.21)	(10.37)	(10.31)	(10.41)	(12.55)	(10.16)	(10.50)	(10.39)
	0.1	Midpoint	363.173	336.219	306.266	217.809	162.358	91.905	31.234	19.861	11.250
			(2.42)	(2.72)	(2.49)	(2.36)	(2.31)	(2.50)	(2.39)	(2.47)	(2.39)
		Trapezoidal	363.201	336.235	306.271	217.795	162.343	91.897	31.234	19.861	11.250
			(2.42)	(2.56)	(2.56)	(2.55)	(2.44)	(2.42)	(2.53)	(2.66)	(2.66)
		Simpson's	363.184	336.226	306.270	217.805	162.354	91.903	31.234	19.861	11.250
			(10.16)	(9.55)	(8.84)	(9.20)	(9.14)	(9.03)	(9.64)	(9.58)	(10.61)
		Gaussian	363.165	336.208	306.253	217.795	162.348	91.901	31.234	19.861	11.250
			(9.80)	(9.92)	(10.08)	(10.08)	(10.86)	(10.16)	(10.55)	(10.41)	(9.80)
Gamma (3, 1)	0.05	Midpoint	360.425	322.980	284.271	186.782	137.335	84.741	40.322	28.742	17.659
			(19.41)	(16.13)	(17.89)	(15.59)	(15.74)	(16.20)	(16.81)	(13.17)	(12.08)
		Trapezoidal	359.682	322.357	283.766	186.545	137.209	84.702	40.318	28.741	17.659
			(15.34)	(18.69)	(14.80)	(14.58)	(15.34)	(18.24)	(15.02)	(12.80)	(12.94)
		Simpson's	360.177	322.771	284.102	186.703	137.293	84.728	40.320	28.742	17.659
			(134.42)	(132.11)	(130.31)	(129.48)	(135.55)	(122.25)	(112.42)	(109.30)	(134.42)
		Gaussian	360.177	322.772	284.102	186.703	137.293	84.728	40.320	28.742	17.659
			(26.30)	(27.38)	(23.94)	(27.31)	(23.75)	(22.41)	(23.55)	(21.19)	(20.64)
	0.1	Midpoint	360.050	322.240	282.346	177.908	122.969	65.196	24.445	16.519	9.833
			(13.39)	(15.44)	(13.59)	(13.72)	(12.11)	(12.75)	(2.22)	(3.38)	(4.09)
		Trapezoidal	359.650	321.898	282.063	177.768	122.894	65.174	24.443	16.519	9.833
			(13.84)	(11.52)	(12.03)	(13.11)	(13.05)	(12.52)	(2.30)	(2.36)	(2.48)
		Simpson's	359.917	322.126	282.251	177.861	122.944	65.189	24.444	16.519	9.833
			(122.11)	(110.53)	(109.34)	(110.70)	(109.67)	(117.24)	(109.47)	(100.13)	(120.83)
		Gaussian	359.917	322.126	282.251	177.861	122.944	65.189	24.444	16.519	9.833
			(20.38)	(20.52)	(20.08)	(20.89)	(22.25)	(19.91)	(9.55)	(10.25)	(10.06)
Weibull (2, 1)	0.05	Midpoint	363.232	335.761	305.727	220.115	168.732	105.135	46.141	31.893	19.121
			(2.89)	(2.75)	(2.80)	(2.89)	(2.81)	(2.67)	(2.92)	(2.77)	(3.34)
		Trapezoidal	361.733	334.323	304.355	218.925	167.645	104.179	45.371	31.227	18.619
			(2.95)	(2.98)	(2.86)	(2.70)	(2.77)	(2.73)	(2.67)	(2.69)	(2.70)
		Simpson's	363.049	335.591	305.580	220.026	168.674	105.110	46.136	31.890	19.120
			(11.17)	(11.28)	(11.36)	(11.22)	(11.50)	(10.90)	(11.09)	(11.25)	(10.94)
		Gaussian	362.043	334.594	304.595	219.078	167.741	104.226	45.382	31.232	18.620
			(11.19)	(11.05)	(11.11)	(11.13)	(11.22)	(11.69)	(11.34)	(11.25)	(11.06)
	0.1	Midpoint	362.811	335.892	305.977	217.628	162.240	91.857	31.228	19.858	11.249
			(2.74)	(2.66)	(2.70)	(2.81)	(2.64)	(2.70)	(2.67)	(2.69)	(2.63)
		Trapezoidal	361.928	335.000	305.079	216.727	161.353	91.014	30.516	19.239	10.781
			(2.70)	(2.67)	(2.63)	(2.66)	(2.70)	(2.67)	(2.69)	(2.70)	(2.59)
		Simpson's	362.822	335.898	305.980	217.624	162.235	91.854	31.228	19.858	11.249
			(10.45)	(10.19)	(10.50)	(10.39)	(10.02)	(10.92)	(10.52)	(10.81)	(10.81)
		Gaussian	361.898	334.979	305.066	216.731	161.360	91.019	30.517	19.239	10.781
			(10.94)	(11.36)	(10.97)	(10.94)	(10.92)	(10.89)	(10.91)	(10.94)	(10.94)
Weibull (3, 1)	0.05	Midpoint	359.649	322.324	283.735	186.521	137.191	84.691	40.313	28.738	17.656
			(18.36)	(18.50)	(18.17)	(18.45)	(18.13)	(17.49)	(16.48)	(17.20)	(14.83)
		Trapezoidal	357.946	320.745	282.279	185.349	136.146	83.773	39.558	28.080	17.158
			(21.86)	(22.06)	(24.14)	(20.53)	(20.16)	(20.45)	(18.86)	(17.53)	(16.11)
		Simpson's	359.401	322.116	283.567	186.442	137.148	84.678	40.312	28.737	17.657
			(128.06)	(125.00)	(124.69)	(123.38)	(122.91)	(121.67)	(112.63)	(106.99)	(98.25)
		Gaussian	358.442	321.161	282.616	185.508	136.231	83.799	39.561	28.081	17.158
			(26.59)	(26.66)	(26.50)	(26.42)	(26.34)	(25.78)	(25.09)	(24.48)	(22.49)
	0.1	Midpoint	359.618	321.867	282.035	177.748	122.879	65.167	24.441	16.518	9.833
			(14.08)	(14.08)	(14.20)	(14.28)	(14.70)	(14.80)	(2.89)	(2.88)	(2.83)
		Trapezoidal	358.305	320.616	280.847	176.719	121.931	64.310	23.729	15.898	9.365
			(16.55)	(16.63)	(18.52)	(17.88)	(18.55)	(17.64)	(3.44)	(3.38)	(3.37)
		Simpson's	359.484	321.753	281.941	177.701	122.854	65.159	24.441	16.517	9.833
			(125.49)	(130.42)	(126.25)	(121.30)	(121.06)	(127.44)	(13.34)	(13.58)	(13.36)
		Gaussian	358.571	320.843	281.035	176.812	121.981	64.324	23.730	15.898	9.365
			(23.27)	(22.56)	(23.02)	(23.02)	(22.13)	(22.94)	(11.19)	(11.41)	(11.30)

The parentheses represent the CPU time in seconds.

Table 3 The ARL values solved via integral equation of the modified EWMA control chart for continuously distributed observations at $ARL_0 = 500$.

Continuous Distribution	λ	Methods	δ 0.001	0.005	0.01	0.03	0.05	0.1	0.3	0.5	1
Gamma (2, 1)	0.05	Midpoint	489.451	447.449	402.016	275.892	203.358	118.616	48.241	32.864	19.517
			(2.34)	(2.27)	(2.53)	(2.63)	(2.30)	(2.42)	(2.36)	(2.38)	(2.52)
		Trapezoidal	488.729	446.809	401.462	275.565	203.151	118.529	48.224	32.856	19.514
			(2.28)	(2.53)	(2.19)	(2.31)	(2.47)	(2.44)	(2.48)	(2.41)	(2.58)
		Simpson’s	489.182	447.212	401.811	275.771	203.282	118.585	48.235	32.861	19.516
			(9.39)	(9.09)	(8.98)	(9.14)	(9.69)	(8.78)	(9.56)	(9.97)	(9.30)
		Gaussian	489.142	447.162	401.756	275.724	203.25	118.572	48.233	32.860	19.516
			(10.05)	(10.34)	(10.02)	(10.30)	(9.80)	(10.92)	(10.58)	(10.50)	(10.27)
	0.1	Midpoint	490.027	450.381	406.626	279.634	202.241	107.770	33.181	20.617	11.513
			(2.44)	(2.27)	(2.33)	(2.16)	(2.28)	(2.63)	(2.75)	(2.77)	(2.37)
		Trapezoidal	489.446	449.883	406.214	279.427	202.128	107.738	33.180	20.617	11.513
			(2.59)	(2.05)	(2.44)	(2.45)	(2.44)	(2.72)	(2.47)	(2.39)	(2.47)
		Simpson’s	489.835	450.217	406.490	279.566	202.204	107.760	33.181	20.617	11.513
			(8.78)	(9.05)	(9.19)	(9.83)	(9.31)	(9.83)	(9.44)	(10.42)	(9.31)
		Gaussian	489.472	449.914	406.248	279.460	202.153	107.749	33.181	20.617	11.513
			(10.14)	(9.58)	(9.66)	(9.48)	(9.72)	(9.63)	(10.08)	(10.38)	(10.00)
Gamma (3, 1)	0.05	Midpoint	484.613	427.043	368.472	226.460	158.697	91.7628	41.4559	29.3061	17.9039
			(16.33)	(17.84)	(15.27)	(16.50)	(14.72)	(17.95)	(14.55)	(13.17)	(12.11)
		Trapezoidal	483.444	426.077	367.704	226.121	158.525	91.7146	41.4515	29.3048	17.9037
			(19.03)	(15.16)	(16.83)	(18.56)	(14.92)	(15.92)	(15.14)	(14.03)	(14.66)
		Simpson’s	484.222	426.72	368.215	226.347	158.639	91.7467	41.4544	29.3057	17.9038
			(132.02)	(131.36)	(128.92)	(132.48)	(132.95)	(132.20)	(124.17)	(113.11)	(103.05)
		Gaussian	484.224	426.722	368.216	226.347	158.64	91.7468	41.4544	29.3057	17.9038
			(23.56)	(24.36)	(28.77)	(27.11)	(24.75)	(24.69)	(23.64)	(24.80)	(19.42)
	0.1	Midpoint	485.289	429.56	371.356	222.737	147.693	72.9242	25.3159	16.8943	9.97975
			(14.08)	(12.39)	(12.36)	(12.61)	(15.17)	(15.33)	(11.53)	(2.48)	(2.58)
		Trapezoidal	484.684	429.048	370.938	222.539	147.59	72.8961	25.314	16.8937	9.97967
			(12.16)	(17.94)	(11.70)	(12.23)	(12.11)	(16.58)	(14.70)	(2.09)	(2.25)
		Simpson’s	485.087	429.389	371.217	222.671	147.658	72.9148	25.3153	16.8941	9.97972
			(110.64)	(116.47)	(107.75)	(114.11)	(112.64)	(118.36)	(118.28)	(8.33)	(8.91)
		Gaussian	485.087	429.389	371.217	222.671	147.658	72.9148	25.3153	16.8941	9.97972
			(22.22)	(25.98)	(21.69)	(21.70)	(22.42)	(20.19)	(19.69)	(12.33)	(9.67)
Weibull (2, 1)	0.05	Midpoint	489.451	447.449	402.016	275.892	203.358	118.616	48.241	32.864	19.517
			(2.672)	(2.77)	(2.67)	(2.69)	(2.70)	(2.84)	(2.78)	(2.72)	(2.73)
		Trapezoidal	487.933	446.001	400.642	274.716	202.293	117.681	47.480	32.202	19.016
			(2.97)	(2.81)	(2.88)	(2.95)	(2.98)	(2.84)	(2.83)	(2.97)	(2.84)
		Simpson’s	489.182	447.212	401.811	275.771	203.282	118.585	48.235	32.861	19.516
			(10.80)	(10.69)	(10.75)	(10.78)	(10.86)	(11.83)	(10.94)	(10.58)	(11.33)
		Gaussian	488.135	446.163	400.766	274.763	202.312	117.681	47.476	32.200	19.015
			(11.06)	(11.46)	(11.06)	(11.02)	(11.13)	(11.16)	(10.99)	(10.95)	(11.09)
	0.1	Midpoint	489.392	449.812	406.127	279.330	202.048	107.696	33.173	20.614	11.512
			(2.66)	(2.75)	(2.73)	(2.72)	(2.77)	(2.74)	(2.69)	(2.70)	(2.70)
		Trapezoidal	487.832	448.343	404.755	278.197	201.036	106.815	32.457	19.993	11.043
			(2.69)	(2.58)	(2.66)	(2.66)	(2.07)	(2.75)	(2.79)	(2.66)	(2.67)
		Simpson’s	489.202	449.649	405.993	279.263	202.011	107.685	33.172	20.614	11.512
			(10.45)	(10.41)	(10.17)	(10.19)	(10.38)	(10.39)	(10.56)	(10.25)	(10.45)
		Gaussian	487.918	448.429	404.838	278.264	201.085	106.838	32.460	19.994	11.043
			(10.88)	(10.94)	(10.92)	(10.92)	(11.36)	(11.17)	(11.28)	(11.28)	(10.95)
Weibull (3, 1)	0.05	Midpoint	483.868	426.423	367.975	226.232	158.577	91.726	41.450	29.303	17.903
			(17.59)	(18.17)	(18.42)	(18.47)	(17.50)	(16.78)	(16.80)	(15.47)	(14.34)
		Trapezoidal	481.739	424.501	366.254	224.959	157.488	90.798	40.695	28.646	17.404
			(21.48)	(22.24)	(21.69)	(21.44)	(21.63)	(21.61)	(21.31)	(19.77)	(17.75)
		Simpson’s	483.478	426.101	367.718	226.119	158.520	91.706	41.449	29.303	17.903
			(127.81)	(132.86)	(129.05)	(126.78)	(126.41)	(124.77)	(116.30)	(109.64)	(101.49)
		Gaussian	482.520	425.147	366.768	225.186	157.603	90.831	40.698	28.647	17.404
			(25.92)	(27.55)	(27.38)	(26.80)	(27.50)	(26.00)	(24.19)	(23.38)	(21.78)
	0.1	Midpoint	484.655	429.019	370.910	222.516	147.573	72.888	25.312	16.893	9.979
			(14.59)	(14.67)	(14.69)	(14.98)	(14.02)	(14.14)	(14.53)	(2.88)	(2.84)
		Trapezoidal	483.138	427.598	369.587	221.430	146.597	72.025	24.599	16.273	9.511
			(18.36)	(17.75)	(18.17)	(21.05)	(19.36)	(19.66)	(17.83)	(3.38)	(3.59)
		Simpson’s	484.454	428.848	370.771	222.450	147.539	72.879	25.312	16.893	9.979
			(106.91)	(106.77)	(114.39)	(109.45)	(107.66)	(106.69)	(107.02)	(10.78)	(11.06)
		Gaussian	483.540	427.938	369.865	221.561	146.666	72.044	24.601	16.273	9.511
			(22.81)	(23.77)	(23.02)	(22.89)	(22.80)	(23.37)	(21.70)	(11.20)	(11.53)

The parentheses represent the CPU time in seconds.

Tables 2 and 3 present results for the ARL values solved via integral equations using the four quadrature rules (the composite midpoint, trapezoidal, and Simpson’s rules and the Gauss-Legendre rule) obtained from the modified EWMA control chart. λ was fixed at either 0.05 or 0.10 to observe its effect on the UCL of the chart when the process was out-of-control ($\delta = 0.03 - 1$). The results signify that $\lambda = 0.10$ was better for detecting shifts in the process mean than $\lambda = 0.05$ for both distributions. Similarly, from the numerical results for both gamma and Weibull distributions, $\alpha = 3$ was better for detecting shifts in the process mean than $\alpha = 1$ for all shift sizes. Hence, the ARL values derived using the four quadrature rules obtained similar results for both gamma and Weibull-distributed observations. The CPU times for the four quadrature rules were similar in the following increasing order: Gauss-Legendre > composite Simpson' s > composite trapezoidal > composite midpoint. This ordering of CPU times agrees with the error estimates for the different quadrature rules given in Equation (15). Because the Gauss-Legendre rule achieved the highest accuracy for the given number of nodes, it was used in the rest of the computations.

4.1 Comparison of the standard and modified EWMA control charts

The performances of the ARL values with the Gauss-Legendre rule of the standard and modified EWMA control charts for continuously distributed observations were compared. The out-of-control ARL was measured for both charts at various values of process mean shift as a performance measure. The results of out-of-control ARL for $ARL_0 = 370$ and 500 are provided in Tables 4 and 5, respectively. They reveal that the modified EWMA control chart slightly outperformed the standard one by producing smaller out- of- control ARL increases with increasing process mean shift when $\lambda = 0.10$. However, when $\lambda = 0.05$, the performance of the modified EWMA control chart was better than the standard one when shift sizes in the process mean (δ) were less than or equal to 0.05 for Weibull(2, 1) and Weibull(3, 1) distributed observations.

Table 4 The ARL values derived with the Gauss-Legendre rule of the standard and modified EWMA control charts for continuously distributed observations at $ARL_0 = 370$.

Continuous Distribution	λ	b	Control Chart	δ 0.001	0.005	0.01	0.03	0.05	0.1	0.3	0.5	1
Gamma (2, 1)	0.05	2.50505	EWMA	364.620	344.208	321.029	248.764	199.463	129.193	53.229	35.143	20.162
		24.2265	Modified EWMA	362.979	335.533	305.527	219.993	168.653	105.101	46.1341	31.890	19.120
	0.10	2.8851149	EWMA	365.031	346.036	324.139	253.391	202.664	126.461	42.2333	24.9347	13.0357
		13.698	Modified EWMA	363.165	336.208	306.253	217.795	162.348	91.901	31.234	19.861	11.250
Gamma (3, 1)	0.05	3.60818	EWMA	363.573	351.196	312.550	232.412	181.242	113.768	47.670	32.143	18.802
		22.9923	Modified EWMA	360.177	322.772	284.102	186.703	137.293	84.728	40.320	28.742	17.659
	0.10	4.05761	EWMA	364.030	341.407	315.734	235.980	182.118	107.359	35.087	21.357	11.590
		12.7877	Modified EWMA	359.917	322.126	282.251	177.861	122.944	65.189	24.444	16.519	9.833
Weibull (2, 1)	0.05	1.0440182	EWMA	362.554	334.942	304.760	218.773	167.208	103.474	44.625	30.563	18.112
		23.92	Modified EWMA	362.043	334.594	304.595	219.078	167.741	104.226	45.382	31.232	18.620
	0.10	1.15445	EWMA	362.860	335.844	305.826	217.199	161.663	91.141	30.532	19.245	10.782
		13.69735	Modified EWMA	361.898	334.979	305.066	216.731	161.360	91.019	30.517	19.239	10.781
Weibull (3, 1)	0.05	0.99975	EWMA	359.403	321.819	282.974	185.189	135.635	83.006	38.801	27.413	16.652
		22.9907	Modified EWMA	358.442	321.161	282.616	185.508	135.231	83.799	39.561	28.081	17.158
	0.10	1.07167	EWMA	359.584	321.716	281.763	177.187	122.192	64.393	23.738	15.902	9.366
		12.7872	Modified EWMA	358.571	320.843	281.035	176.812	121.981	64.324	23.730	15.898	9.365

Table 5 The ARL values derived with the Gauss-Legendre rule of the standard and modified EWMA control charts for continuously distributed observations at $ARL_0 = 500$.

Continuous Distribution	λ	b	Control Chart	δ 0.001	0.005	0.01	0.03	0.05	0.1	0.3	0.5	1
Gamma (2, 1)	0.05	2.55384	EWMA	491.903	461.037	426.228	319.617	181.724	152.330	57.165	36.851	20.805
		24.2265	Modified EWMA	489.142	447.162	401.756	275.724	203.250	118.572	48.233	32.860	19.516
	0.10	2.9473575	EWMA	492.737	465.037	433.236	331.588	259.968	155.211	46.787	26.614	13.548
		13.8847	Modified EWMA	489.472	449.914	406.248	279.460	202.153	107.749	33.181	20.617	11.513
Gamma (3, 1)	0.05	3.66665	EWMA	490.224	453.663	412.231	295.278	222.519	131.170	50.419	33.373	19.287
		23.1975	Modified EWMA	484.224	426.722	368.216	226.347	158.640	91.747	41.454	29.306	17.904
	0.10	4.1306	EWMA	492.202	459.110	421.742	307.142	231.361	129.456	38.070	22.467	11.952
		12.9066	Modified EWMA	485.087	429.389	371.217	222.671	147.658	72.915	25.315	16.894	9.980
Weibull (2, 1)	0.05	1.05853	EWMA	488.630	446.485	400.906	274.430	201.743	116.907	46.714	31.529	18.505
		2265.24	Modified EWMA	488.135	446.163	400.766	274.763	202.312	117.681	47.476	32.200	19.015
	0.10	1.17139	EWMA	489.397	449.732	405.957	278.903	201.475	106.981	32.476	20.000	11.045
		88392.13	Modified EWMA	487.918	448.429	404.838	278.264	201.085	106.838	32.460	19.994	11.043
Weibull (3, 1)	0.05	1.009505	EWMA	483.826	426.068	367.314	224.911	157.011	90.027	39.934	27.977	16.897
		23.1965	Modified EWMA	482.520	425.147	366.768	225.186	157.603	90.831	40.698	28.647	17.404
	0.10	1.08245	EWMA	484.618	428.858	370.624	221.936	146.869	72.105	24.607	16.276	9.512
		12.9061	Modified EWMA	483.540	427.938	369.865	221.561	146.666	72.044	24.601	16.273	9.511

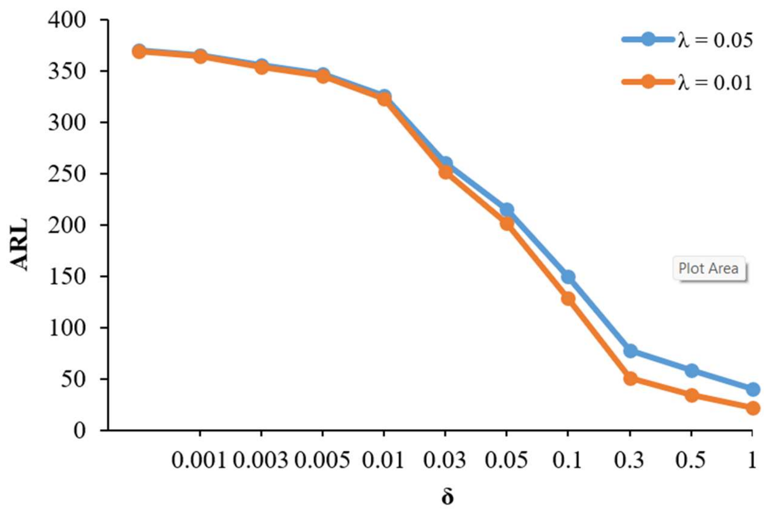

Figure 1 The ARL values derived with the Gauss-Legendre rule of the modified EWMA control chart for Weibull-distributed observations.

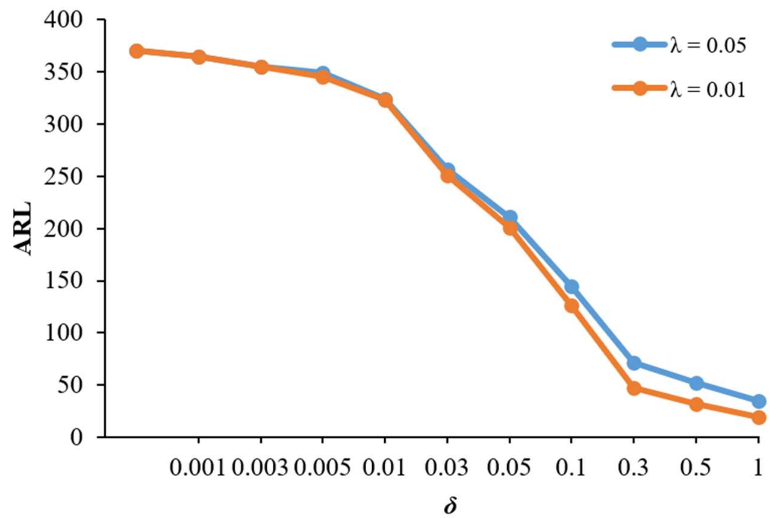

Figure 2 The ARL values derived with the Gauss-Legendre rule of the standard EWMA control chart for Weibull-distributed observations.

5. Application of the approximated ARL using the gauss-legendre rule with real data

The approximated ARL was solved by applying integral equations for detecting shifts in the process mean of a modified EWMA control chart for gamma and Weibull distributions. The Gauss-Legendre method is unsuitable when speed is desired and computer storage is limited. Moreover, Romberg integration should be considered when accuracy is paramount, and all other factors can be disregarded. When the expression to be integrated can be expressed exactly as a polynomial of degree n, then Gauss-Legendre integration should be used. On the other hand, when the expression cannot be expressed exactly as an nth degree polynomial, then it can be expressed as function $g(x)(1-x)^{\alpha}(1+x)^{\beta}$, where $\alpha, \beta > -1$, and the Gauss-Jacobi method should be used.

Table 6 The ARL values of the standard and modified EWMA control charts for strength data for single carbon fibers of gauge length 20 at $ARL_0 = 370$.

δ	Weilbull(5.049422, 3.314562)			
	$\lambda = 0.05$		$\lambda = 0.10$	
	Modified EWMA	EWMA	Modified EWMA	EWMA
0.001	364.647	365.030	363.574	364.893
0.003	355.274	355.319	353.782	354.933
0.005	346.288	349.039	344.355	345.344
0.010	325.397	324.473	322.282	322.897
0.030	259.944	257.021	251.647	251.113
0.050	214.953	210.787	201.773	200.482
0.100	150.128	144.456	128.478	126.201
0.300	77.470	70.949	50.454	47.517
0.500	58.393	52.034	34.272	31.388
1.000	40.209	34.429	21.835	19.213

6. Conclusion

The approximated average run length (ARL) solved by applying integral equations for detecting shifts in the process mean of a modified exponentially weighted moving average (EWMA) control chart for gamma and weibull distribution. The speed is desired and computer storage is limited then the Gauss-Legendre methods would be eliminated. The accuracy and disregarded all other factors, then Romberg integration would be considered. The expression to be integrated can be expressed exactly as a polynomial of degree n, then Gauss-Legendre integration should be used. On the other hand, if the expression cannot be expressed exactly as a n^{th} degree polynomial, or as then the expression can be expressed as a function $g(x)(1-x)^{\alpha}(1+x)^{\beta}$ where $\alpha, \beta > -1,$ and the Gauss-Jacobi should be used.

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8. References

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