TESTS FOR GAMMA DISTRIBUTION BASED ON ITS INDEPENDENCE PROPERTY

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A Dissertation Submitted in Partial
Fulfillment of the Requirements for the Degree of
Doctor of Philosophy (Statistics)
School of Applied Statistics
National Institute of Development Administration
2015

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December 2015

ABSTRACT

Title of Dissertation Tests for Gamma Distribution Based on Its Independence Property

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Degree Doctor of Philosophy (Statistics)

Year 2015

There are two test statistics proposed in this study in order to test whether data come from a gamma distribution. Both of the proposed test statistics are developed from a modified Kendall coefficient based on the independence property of a gamma distribution. The first one is asymptotically distributed as standard normal and the limit distribution function of the second one was improved using an Edgeworth expansion and the Jackknife method. They are invariant to scale parameters and perform substantially better than existing tests in terms of Type I error rate and test power, especially in cases with samples of moderate size.

ACKNOWLEDGEMENTS

I am greatly indebted to my advisor, Associate Professor Dr. Pachitjanut Siripanich, for her invaluable assistance. Without her guidance and encouragement throughout the PhD study process, I could not have made it this far.

I would like to thank my thesis committee members: Professor Dr. Prachoom Suwattee, Associate Professor Dr. Jirawan Jitthavech and Professor Dr. Samruam Chongcharoen. Not only their very useful comments but also attending their courses helped me gain the knowledge necessary for conducting this research.

I feel delighted in extending my gratitude towards my friends at the School of Applied Statistics, National Institute of Development Administration (NIDA), Thailand-even though their names are not mentioned here; their assistance was unforgettable.

Lastly, I am also grateful to the Ministry of Science and Technology, Thailand, for their financial support.

Bandhita Plubin

May 2016

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CHAPTER 1

INTRODUCTION

1.1 Statement of the Problem

In data analysis, it is generally known that statistical methods are applied with careful consideration in order to reach reliable conclusions. When testing for significance, parametric tests are often chosen because, in many situations, they can provide a higher power than nonparametric ones. However, one of the assumptions of parametric tests is the distribution of the random variable involved. If the distribution is ignored, or not determined, before conducting the analysis, high error rates may occur and lead to the wrong conclusion.

A gamma distribution, a general type of statistical distribution, is one of the most well-known shape-scale distributions for nonnegative variables. This distribution is commonly found in a wide variety of fields; for example, marketing, insurance, investment, hydrology, medicine, sociology, and demography, since data from these sources usually contain variables with nonnegative values. Furthermore, it is commonly used in models for analyzing life-time data (Raja and Mir, 2011: 393-400), which refers to, for instance, the human lifespan, the lifespan of a mechanism before it fails, or the survival time of a patient after diagnosis of death. A number of studies have used data that have come from a gamma population, such as ecology studies (Ellis, Mahlooji, Lascano and Matis, 2005: 1602-1615), the quantity of daily rainfall (Husak, Michaelsen and Funk, 2007: 935-944; Sharma and Singh, 2010: 40-49), the ascension curve of the hydrograph model in hydrological analysis (Volkova, Longobardi and Krasnogorskaya, 2014: 88-95), inventory control and queuing problems (Krever, Wunderink, Dekker and Schorr, 2005: 342-358), and reliability (Amari, 2012: 1-6). A gamma distribution with nonnegative values is usually skewed, as has been found in various fields of study. Nevertheless, there is a need to determine whether data come from a gamma distribution or not.

There are many different ways to test for whether a distribution is the same as the hypothesized distribution and they are categorized into two major approaches: one is based on the distance between the empirical distribution function (EDF) and the hypothesized distribution, and the other is based on certain characteristics of gamma distributions.

In the first approach, three popular tests: the Cramér-von Mises (CM), the Kolmogorov-Smirnov (KS), and the Anderson-Darling (AD) tests (Stephens, 1974: 730-737) can be used to test whether the sample has been drawn from a population with a specified distribution function $F(x)$. In this study, the specified distribution function $F(x)$ is a gamma distribution. All of these tests require an underlying continuous distribution with known parameters, so the parameters need to be estimated first before the distribution of the tests can be obtained. Consequently, the accuracy of the results depends on the accuracy of the parameter estimation. Thus, many studies have focused on the estimation step by modifying the three tests. Examples of modified tests can be found in Woodruff, Viviano, Moore and Dunne (1984: 241-245); Shawky and Bakoban (2009: 1-17).

The second approach is a process based on considering the properties of certain characteristics of the distribution being tested, which is gamma in our case, and there have been a number of studies in this area, e.g., Wilding and Mudholkar, (2008: 3813-3821); Villaseñor and González-Estrada (2015: 281-286); Baringhaus and Gaigall (2015: 193-208). In recent years, there has been increasing interest in testing for gamma distribution by means of this approach. For instance, by employing the gamma characteristic of Hwang and Hu (1999: 749-753), Wilding and Mudholkar (2008: 3813-3821) proposed a test based on modifying Pearson's correlation coefficient between the sample mean and the sample CV of n Jackknife sub-samples. In addition, the Baringhaus and Gaigall test (2015: 193-208) was achieved by applying a consistent nonparametric independence test to the gamma distribution property proposed by Lukacs (1955: 1-5). However, there are many other characteristics of a gamma distribution that have been used in this approach, such as the test proposed by Villaseñor and González-Estrada (2015: 281-286); Henze, Meintanis and Ebner (2012: 1543-1556).

It is noticeable that, if the true distribution is similar to a gamma distribution, that is, the distances between these distributions are close together, then using a test based on the first approach is hardly able to distinguish them. In other words, the null hypothesis is more likely to be accepted. The proposed tests in this study were chosen to be developed under the second approach. The focus of this study is on ascertaining goodness-of-fit tests for a gamma distribution based on its characteristics. The hypothesis for a gamma distribution can be written as:

H_0 : A random sample is drawn from a gamma distribution
H_1 : A random sample is not drawn from a gamma distribution (1.1)

1.2 Objectives of the Study

The objectives of the study are as follows:

1) To propose tests using the independence characteristic approach for a gamma distribution

2) To investigate some underlying properties of the proposed tests

3) To compare the proposed tests with some previously reported ones

1.3 Scope of the Study

1) The proposed tests were developed based on the characteristic of a gamma distribution, proposed by Lee and Lim (2009: 1-5), that $\sum_{k=1}^{m} X_k$ and $X_i X_j \left(\sum_{k=1}^{m} X_k\right)^{-2}$ are independent.

2) The properties of the proposed tests to be investigated are:

(1) The asymptotic distribution of the proposed test statistics under the null hypothesis

(2) Empirical Type I error rates under nominal conditions

(3) Empirical power of the proposed tests under various alternative distributions with various parameters

3) The competitive tests used for comparison were the Cramér-von Mises (CM), the Kolmogorov-Smirnov (KS), the Anderson-Darling (AD) tests, the tests proposed by Wilding and Mudholkar (2008: 3813-3821); Villaseñor and González-Estrada (2015: 281-286), denoted by WM and VGE, respectively.

1.4 Advantages of the Study

The outcomes of this study are conducive to a possible alternative goodness-of-fit test that can be used to determine the optimal initially implemented gamma model for the analysis of data. The advantages of the study are as follows:

1) Testing is not required to estimate the parameters.

2) It can be used as a guide to create tests for other distributions.

CHAPTER 2

LITERATURE REVIEW

2.1 The Gamma Distribution

A gamma distribution, as proposed by Karl Pearson during the late 19th century (quoted in Shakil, Kibria and Singh, 2010: 259-278), is created from the sum of independent exponentially distributed random variables. It is a continuous distribution function with two positive parameters, the shape and scale parameter denoted by λ and δ, respectively. This distribution is widely used in many applications involving life-time data: in biology, engineering, monthly rainfall data in meteorology, and insurance claims and loan defaults in business. These data are nonnegative and their distributions are usually skewed. Frequently, a gamma distribution is one that fits well with data from these sources.

A random variable X that is gamma-distributed with shape parameter λ and scale parameter δ is denoted by

$$X \sim \Gamma(\lambda,\delta) \equiv \text{Gamma}(\lambda,\delta).$$

The probability density function of random variable X can be written in the form

$$f(x;\lambda,\delta) = \frac{\delta^{\lambda}}{\Gamma(\lambda)} x^{\lambda-1} e^{-x\delta}, \text{ for } x \geq 0,\ \lambda, \delta > 0, \tag{2.1}$$

where $\Gamma(c)$ is defined by using an integral formula as $\Gamma(c) = \int_0^{\infty} \mathrm{t}^{c-1}\mathrm{e}^{-t}\,\mathrm{dt}$, for $c > 0$, $\Gamma(c+1) = c\Gamma(c)$, and $\Gamma(c+1) = c\,!$ for integer c.

2.1.1 Basic Characteristics of a Gamma Distribution

The basic characteristics of a gamma(λ,δ) distribution are as follows:

Mean $= \lambda / \delta$,

Mode $= (\lambda - 1)/\delta$ if $\lambda \geq 1$,

Variance $= \lambda / \delta^2$,

Coefficient of variation $= \sqrt{\lambda}$,

Skewness $= 2/\sqrt{\lambda}$, and

Excess kurtosis $= 6/\lambda$.

Proof of the above gamma distribution properties can be found in many statistics text books, such as in "Statistical Distributions" (Forbes, Evans, Hastings and Peacock, 2011: 109-113). There are many types of gamma-related distribution and some examples of those are exhibited in Table 2.1 below.

Table 2.1 Examples of Gamma-Related Distribution

Gamma Distribution	Shape Parameter	Scale Parameter	Related Distribution
Gamma$(1, c)$	1	c	Exponential: exp(c)
Gamma$\left(\frac{n}{2}, 2\right)$	$\frac{n}{2}$	2	Chi-Square: $\chi^2_{(n)}$
Gamma(n, δ)	n	δ	Erlang: Erl(n, δ)

Note: c and δ are positive real numbers and n is a positive integer.

2.1.2 The Independence Property of a Gamma Distribution

Some independence properties of gamma distribution are reviewed here because they play an important role in this study. Suppose $X_1, X_2, ..., X_m$ are positive i.i.d. random variables with a common absolutely continuous distribution function $F(x)$ and $E(X^2) < \infty$, we find that all of the following pairs of statistics are independent if and only if $X_1, X_2, ..., X_m$ are distributed as gamma. Note that $\bar{X}$ and S are respectively the sample mean and standard deviation.

1) Lukacs (1955: 319-324)

$X_i + X_j$ and $X_i \big/ \left(X_i + X_j\right)$, for $i, j = 1, 2, ..., m$ and $i \neq j$.

2) Hwang and Hu (1999: 749-753)

$\bar{X}$ and $CV = S / \bar{X}$, for $m \geq 3$.

3) Hwang and Hu (2000: 427-437)

(1) $\bar{X}$ and $\sum_{i=1}^{m} a_i X_{(i)} \Big/ \bar{X}$,

where $a_1 \leq a_2 \leq ... \leq a_m$ are not all equal, $\sum_{i=1}^{m} a_i = 0$, and

$X_{(1)} \leq X_{(2)} \leq \cdots$ $_{(\dots)}$.

(2) $\bar{X}$ and $\dfrac{1}{m(m-1)\bar{X}} \sum_{i,j=1}^{m} \left| X_i - X_j \right|$.

(3) $\bar{X}$ and $\left(X_{(m)} - X_{(1)}\right) \big/ \bar{X}$,

where $X_{(m)} - X_{(1)}$ is the sample range.

4) Lee and Lim (2007: 411 – 418)

$\sum_{k=1}^{m} X_k$ and $\dfrac{X_1 + X_2 + \ldots + X_{m^*}}{X_1 + X_2 + \ldots + X_m}$, for $1 \leq m^* < m$.

5) Lee and Lim (2009: 1-5)

$\sum_{k=1}^{m} X_k$ and $(X_i X_j)\left(\sum_{k=1}^{m} X_k \right)^{-2}$, for $1 \leq i < j \leq m$.

2.2 Kendall's Tau, a Measure of Association

The measure of association between two random variables can be obtained by both parametric and nonparametric methods. The two most commonly used nonparametric measures for association are Kendall's tau (1938: 81-89); Spearman's rho (1904: 72-101), while the most commonly used parametric measure for association is Pearson's correlation (1895: 240-242). In this study we emphasized on Kendall's correlation coefficient, denoted by τ, developed by Maurice Kendall in 1938. Suppose a random sample of n^* pairs, (X_1, Y_1), (X_2, Y_2), (X_3, Y_3), …,(X_{n^*}, Y_{n^*}), comes from a bivariate population. Then the Kendall's tau (τ) can be defined as (Kendall, 1938: 81-89)

$$\tau = p_c - p_d ,$$

where $p_c = P\left[\left(X_j - Y_i\right)\left(X_j - Y_i\right) > 0\right]$ and $p_d = P\left[\left(X_j - Y_i\right)\left(X_j - Y_i\right) < 0\right]$.

The Kendall's τ can be estimated by K', called Kendall's coefficient, and is defined as

$$K' = \frac{2}{n^*(n^*-1)} \sum_{1 \le i <}^{n^*-1} \sum_{j \le n^*}^{n^*} A'_{ij} \quad , \tag{2.2}$$

where $A'_{ij} = sgn\left(X_j - X_i\right) sgn\left(Y_j - Y_i\right)$, and the values of A'_{ij} are as follows:

$$A'_{ij} = \begin{cases} 1 & \text{if these pairs are concordant,} \\ -1 & \text{if these pairs are discordant,} \\ 0 & \text{if these pairs are neither concordant nor discordant} \end{cases}$$

Note that Kendall also proved that K' is an unbiased estimate of τ. If two random variables X and Y are independent, then τ is zero. Although it is not true in general that $\tau = 0$ implies independence, it can be shown by using general limit theorems as

$n^* \to \infty$, so the inverse is obtained. Hence, Kendall's coefficient is one of the nonparametric statistics for testing whether random variables X and Y are independent or not (Gibbons and Chakraborti, 2011: 385-429). That is, for $n^* \to \infty$, hypothesis (1.1) is equivalent to hypothesis (2.3).

H_0 : Random variables X and Y are independent
H_1 : Random variables X and Y are not independent. (2.3)

Kendall's τ is equal to zero when two variables are independent. Two assumptions must be held throughout this study:

(A1) X and Y have continuous marginal probability distributions
and (A2) X and Y are independent.

The expectation and variance of the K' statistic under the null hypothesis, as given by Kendall, are

$$\mathrm{E}\left(K' \mid H_0\right) = 0 \text{ and } \mathrm{V}(K' \mid H_0) = \frac{2(2n^*+5)}{9n^*(n^*-1)}, \text{ respectively.}$$

Consequently, the standardized form of the K' statistic is

$$Z = \frac{K'}{\sqrt{\dfrac{2(2n^*+5)}{9n^*(n^*-1)}}}, \tag{2.4}$$

which is suitable for use as a statistic for testing that hypothesis (2.3) is equivalent to hypothesis (1.1), as previously mentioned. In addition, the asymptotic distribution of Z is standard normal.

2.3 Tests for a Gamma Distribution

Tests for a gamma distribution are used to determine whether sample data are drawn from a population having a gamma distribution and have been developed along two approaches. The first is Based on the EDF where tests have been constructed from various measures to examine the distance between the EDF and the gamma distribution function with known shape parameter α_1 and scale parameter α_2 in (2.1). For the second approach, many tests of gamma distribution function have been developed from its characteristics, such as the independence property attribute.

2.3.1 Tests Based on the EDF

The EDF is an important estimator function in statistics. It is constructed from the observations and used to estimate the unknown probability distribution function. Suppose $X_1, X_2, ..., X_m$ are i.i.d. random variables with distribution function $F(x)$ and $X_{(1)}, X_{(2)}, ..., X_{(m)}$ are order statistics according to these variables, then we can easily show that the maximum likelihood estimator of the distribution function $F\left(x\right)$ (quoted in Feller, 1948:177) is

$$F_m\left(x\right) = \begin{cases} 0, & x \le X_{(1)}, \\ \dfrac{k}{m}, & X_{(k)} < x \le X_{(k+1)} ,1 \le k \le m-1, \\ 1, & x > X_{(m)} , \end{cases}$$

where $F_m(x)$ is a step function that increases by $\frac{1}{m}$ at each point. That is, the EDF of $X_1, X_2, \ldots, X_m$ is obtained by multiplying the probability $\frac{1}{m}$ at each value of x_k.

The hypotheses for goodness-of-fit tests are:

$$\mathbf{H}_0\text{: } F(x) = F_0(x) \text{ against } \mathbf{H}_1\text{: } F(x) \neq F_0(x) \text{ ,} \tag{2.5}$$

where $F(x)$ denotes the cumulative distribution function (CDF) of the true population distribution that generated the sample, and $F_0(x)$ denotes the theoretical CDF of the distribution being tested.

In addition, from the Cantelli-Glivenko Lemma, Glivenko (1933, quoted in Darling, 1957: 824) asserted that "If $F_m(x)$ is the EDF for a random sample of size m taken from a population with distribution function $F(x)$ then the EDF converges, with probability 1 and uniformly in x, to the distribution function $F(x)$". Therefore, in (2.5), the distance between the EDF, $F_m(x)$, and the hypothetical CDF, $F_0(x)$, is the initial idea in developing a goodness-of-fit test Based on the EDF.

There are a number of methods employed in the test for a gamma distribution constructed from various distance measures and, in each method, the criterion for measuring the distance is defined in a different way. However, testing whether EDF is close to the hypothetical CDF is generally based on the criterion of minimum distance. The three well-known traditional tests in this class are the Cramér–von Mises (CM), Kolmogorov–Smirnov (KS), and Anderson–Darling (AD) tests; these are said to be valid when there are no unknown parameters in the hypothesized distribution.

The tests previously mentioned and the theorems involved are described below.

2.3.1.1 The Cramér-von Mises Test

The Cramér-von Mises (CM) test was first developed by Cramér in 1928 (Darling, 1957: 825-827) Based on the squared distance ω^2 between the EDF and the hypothetical CDF, namely $(F_m(x) - F(x))^2$ as

$$\omega^2 = \int_{-\infty}^{+\infty} (F_m(x) - F(x))^2 dW(x), \tag{2.6}$$

where $W(x)$ is a proper non-decreasing weight function. The hypothesis in (2.5) is rejected if ω^2 is too large. In 1931, Von Mises proposed some properties of the statistic ω^2. Later, in 1936, Smirnov (1936, quoted in Birnbaum, 1953: 1-2) proposed the CM statistic modified from the statistic ω^2 in (2.6) as follows:

$$\mathrm{CM} = m \int_{-\infty}^{+\infty} (F_m(x) - F(x))^2 \psi[F(x)]\, dF(x) \,, \tag{2.7}$$

where $\Psi(t)$ is a weight function. The CM statistic may also be written as

$$\mathrm{CM} = \frac{1}{12m} + \sum_{i=1}^{m} \left\{ F\left(x_i, \theta\right) - \frac{2i-1}{2m} \right\}^2 .$$

2.3.1.2 The Kolmogorov-Smirnov Test

The Kolmogorov-Smirnov (KS) test was first developed Based on the Cantelli-Glivenko Lemma in 1933 (Kolmogorov, 1933 quoted in Birnbaum, 1953: 3-4). The test statistic, KS, is the largest distance from all points of observations, which can be written as

$$\mathrm{KS} = \sup_{-\infty \le x \le \infty} \left| F_m\left(x\right) - F\left(x\right) \right| ,$$

where $F_m\left(x\right)$ is the EDF, m is the sample size, and $F\left(x\right)$ is the theoretical distribution function. A large value of KS indicates that the difference between the EDF and the hypothetical CDF at any point x is large, or it can be said that the EDF does not correspond to the hypothetical CDF. Consequently, if KS is too large, the hypothesized distribution would be rejected. Moreover, Smirnov (1948: 279-281) proposed a table for estimating the goodness-of-fit of the EDF. The KS statistic may also be written as

$$\text{KS} = \max(D_m^+, D_m^-),$$

where m is the sample size, $D_m^+ = \max\limits_{1\le i\le m}\left\{\frac{i}{m} - F\left(x_i,\theta\right)\right\}$, $D_m^- = \max\limits_{1\le i\le m}\left\{F\left(x_i,\theta\right) - \frac{i-1}{m}\right\}$ and $x_1, x_2, ..., x_m$ are sample values in increasing order.

2.3.1.3 The Anderson–Darling Test

One weakness of the Kolmogorov-Smirnov and Cramér-von Mises tests is the difference between the EDF and the hypothetical CDF tends towards zero when $x \to -\infty$ or $x \to +\infty$. Another test proposed by Anderson and Darling (1952: 193-212, 1954: 765-769) is a test weighing the tail of the distribution more than the center. The weight function $\Psi(t)$ used in (2.7) is $\frac{1}{F\left(x\right)\left[1-F\left(x\right)\right]}$. The weight tends to ∞ as x tends towards either $-\infty$ or $+\infty$ and is smaller around the median of the distribution. The Anderson–Darling test statistic is in the form

$$\text{AD} = m\int_{-\infty}^{+\infty} \frac{\left[F_m(x) - F(x)\right]^2}{F(x)\left[1-F(x)\right]}\, dF(x).$$

The AD statistic may also be written as

$$\text{AD} = -m - \frac{1}{m}\sum_{i=1}^{m}\left(2i-1\right)\left[\ln F\left(x_i,\theta\right) + \ln\left(1 - F\left(x_{m-i+1},\theta\right)\right)\right].$$

2.3.2 Tests Based on the Independence Property

Tests for a gamma distribution based on a characteristic are equivalent to independence tests between statistics based on a specified characteristic of a gamma distribution. The idea behind this type of testing is very useful because it is not necessary to know the parameters of the unknown distribution, and so estimated values of unknown parameters are not needed. If a pair of statistics follows a gamma characteristic of independent, then it implies that a random sample came from a gamma population.

The independence property between the sample mean and coefficient of variation (CV) has been applied to construct test WM for determining goodness-of-fit test of a gamma distribution (Wilding and Mudholkar, 2008: 3813-3821). Suppose that $X_1, X_2, \ldots, X_m$ are positive i.i.d. random variables with a common absolutely continuous distribution function $F(x)$ and $E(X^2) < \infty$. For a random sample of size m, where $m \geq 3$, create m new random samples, each of which is obtained by removing the i^{th} observation from the original sample data. The sample mean and CV are then computed for each new random sample. Suppose that $\bar{x}_{-i}$ and c_{-i}, $i = 1, 2, 3, \ldots, m$ are the i^{th} sample mean and the i^{th} sample CV, respectively, of the new sample data from which the i^{th} observation has been removed. Thus, there are m pairs of sample mean and CV, $(\bar{x}_{-i}, c_{-i}), i = 1, 2, \ldots, m$. For testing whether the sample mean and CV are independent, the test statistic WM is given by

$$\text{WM} = \frac{Z(G) - \mu_{m,\alpha}}{\sqrt{\sigma^2_{m,\alpha}}} \sim N(0,1) \quad ,$$

where $$Z(G) = \frac{1}{2}\log\left[\frac{1 - r(G)}{1 + r(G)}\right], \quad r(G) = \frac{\sum_{i=1}^{m}(\bar{x}_{-i} - \bar{x})(c_{-i} - \tilde{c}}{\sqrt{\sum_{i=1}^{m}(\bar{x}_{-i} - \bar{x})^2 \sum_{i=1}^{m}(c_{-i} - \tilde{c}}} ,$$

$$\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i , \quad \tilde{c} \quad \frac{1}{m}\sum_{i=1}^{m} \;{}_{i} , \qquad \mu_{m,\alpha} = -\frac{5}{m^{0.75}\alpha^{0.63}} + \frac{7}{m\alpha^{0.84}} ,$$

$$\sigma^2_{m,\alpha} = \frac{1}{m}\left(3 + \frac{10}{\alpha}\right) - \frac{1}{m^{1.25}}\left(\frac{1.5}{\alpha^{0.1}} + \frac{25}{\alpha^{1.2}}\right) + \frac{1}{m^{1.5}}\left(\frac{0.1}{\alpha^{0.07}} + \frac{21}{\alpha^{1.3}}\right),$$

and $N(0,1)$ is a standard normal distribution.

2.3.3 Tests Based on the Ratio of Two Variance Estimators

The VGE test developed by Villaseñor and González-Estrada (2015: 281-286) is based on the ratio of two variance estimators of the random variable X that is gamma-distributed with shape parameter ϑ and scale parameter γ. The probability density function of random variable X can be written in the form

$$f(x;\vartheta,\gamma) = \frac{1}{\Gamma(\vartheta)\gamma^{\vartheta}} x^{\vartheta-1} e^{-x/\gamma} \text{, for } x \geq 0 \text{ , } \vartheta, \gamma > 0 .$$

Suppose that $X_1, X_2, \ldots, X_m$ is a positive random sample of size m with a common absolutely continuous distribution function and $E(X^2) < \infty$. The sample mean and the unbiased sample variance are $\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i$ and $S^2 = \frac{1}{m-1}\sum_{i=1}^{m}(X_i - \bar{X})^2$ respectively. Let $Z_i = \log(X_i)$, $i = 1, 2, 3, \ldots, m$. Compute $\bar{Z} = \frac{1}{m}\sum_{i=1}^{m} Z_i$, the sample mean of Z, and $S_{XZ} = \frac{1}{m}\sum_{i=1}^{m}(X_i - \bar{X})(Z_i - \bar{Z})$, the sample covariance of X and Z. For testing whether the population distribution is gamma with shape parameter ϑ, Villaseñor and González-Estrada (2015: 281-286) proposed a test based on the statistic VGE as

$$\text{VGE} = \sqrt{\frac{m\hat{\vartheta}}{2}}(V-1) \sim N(0,1),$$

where $\hat{\vartheta} = \frac{\bar{X}}{S_{XZ}}$, $V = \frac{S^2}{\hat{\sigma}^2}$ and $\hat{\sigma}^2 = \bar{X}S_{XZ}$. For instance, the shape parameter and variance of the gamma distribution are estimated, respectively, by $\hat{\vartheta}$ and $\hat{\sigma}^2$.

In this study, goodness-of-fit tests for a gamma distribution along the lines of the second approach were proposed. The independence property proposed by Lee and Lim (2009: 1-5) was used as the basis for constructing the test, and Kendall's Tau was modified to elucidate the independence of the selected property (Kendall, 1938: 81-89). The efficiency of the proposed tests were compared with previously reported goodness-of-fit tests based on the distance between the empirical distribution function (EDF) and the hypothesized distribution, and tests Based on certain characteristics of gamma distributions. The proposed tests were compared with the KS, CM, and AD tests from the first approach and the WM and VGE tests from the second approach.

CHAPTER 3

THE PROPOSED TESTS

As aforementioned, testing for a gamma distribution can be classified into two approaches: the empirical distribution function (EDF) and a characteristic of a gamma distribution. In this study, the second approach was applied by using an independence property associated to a gamma distribution. One of the main reasons for not using the EDF approach is that, often, it cannot differentiate between a near-gamma distribution, such as Weibull, and a gamma distribution and, consequently, this makes it difficult to reject the null hypothesis for a gamma distribution.

3.1 Conceptual Framework

Let $X_1, X_2, \ldots, X_m$ be positive random variables with a common absolute continuous distribution function and $E\left(X^2\right)$ exists. Lee and Lim (2009: 1-5) showed that $\frac{(X_i X_j)}{\left(\sum_{k=1}^{m} X_k\right)^2}$ and $\sum_{k=1}^{m} X_k$, for all $1 \le i < j \le m$ and $m \ge 1$, are independent if and only if the population is gamma distributed. (3.1)

When $m = 2$, the statistics in (3.1) become $U = \frac{X_1 X_2}{\left(X_1 + X_2\right)^2}$ and $V = X_1 + X_2$, respectively. In order to test whether the bivariate random variables $\left(U, V\right)$ are independent, one of the most well-known nonparametric tests is the Kendall test (Kendall, 1970). Recall that testing whether U and V are independent by using Kendall's K statistic (or Z_K test statistic) is equivalent to testing hypothesis (1.1) for a large sample size.

3.2 Kendall Tests

3.2.1 Kendall's Coefficient

To apply Kendall's coefficient for testing whether a random sample comes from a gamma distribution based on its independence characteristic, see (3.1), it is necessary to construct n bivariate random statistics. Suppose that $X_1, X_2, \ldots, X_m$ is a random sample of size m and $m = cn$, then the original sample can be partitioned into n subsamples, each of size c. Therefore, the desired bivariate random statistics, $\big((U_1,V_1),(U_2,V_2),\ldots\ \ldots)\big)$, are obtained, where

$$U_j = \prod_{k=1}^{c} X_{j_k} \Big/ \left(\sum_{k=1}^{c} X_{j_k} \right)^c \text{ and } V_j = \sum_{k=1}^{c} X_{j_k},\ j = 1,2,\ldots,n. \tag{3.2}$$

Here, a sample of even number (i.e. $m = 2n$) is assumed and, hence, $c = 2$ is selected so that the number of subsamples n is maximized. Therefore, the newly constructed sample $\big((U_1,V_1),(U_2,V_2),\ldots\ \ldots)\big)$ has as large a sample size n as possible. Therefore, (U_j, V_j) in equation (3.2) becomes

$$U_j = \frac{X_{j_1} X_{j_2}}{\left(X_{j_1} + X_{j_2} \right)^2} \text{ and } V_j = X_{j_1} + X_{j_2}, \tag{3.3}$$

where $j = 1,2,\ldots,n$; $j_k = 1, 2, \ldots, m$; $k = 1,2$; and $j_1 \neq j_2$. The modified Kendall's coefficient in (2.2) is obtained as

$$K = \frac{2}{n(n-1)} \sum_{1 \le i <}^{n} \sum_{j \le n}^{n} A_{ij}, \tag{3.4}$$

where $A_{ij} = \operatorname{sgn}\left(U_j - U_i\right) \operatorname{sgn}\left(V_j - V_i\right)$, for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,n$.

The following section concerns some properties of the K statistic.

3.2.2 Some Properties of Kendall's Coefficient

1) Mean and Variance

Theorem 3.1

Let $\big((U_1,V_1),(U_2,V_2),\ldots$ [illegible] $)\big)$ be a bivariate random sample of size n. Under the null hypothesis $H_0:\ \tau = 0$, the expectation and variance of Kendall's coefficient statistic K in (3.4) are $\mathrm{E}(K \mid H_0) = 0$ and $\mathrm{V}(K \mid H_0) = \dfrac{2(2n+5)}{9n(n-1)}$, respectively.

Proof. See Kendall (1970: 69-71).

2) Distribution of Kendall's Coefficient

The exact distribution of Kendall's coefficient in (3.4) is obtained from the function of n and all possible value of $\sum\limits_{1\le i<}^{n} \sum\limits_{j\le n}^{n} A_{ij}$ in the aggregate of $n!$. Let S be $\sum\limits_{1\le i<}^{n} \sum\limits_{j\le n}^{n} A_{ij}$. For n bivariate random variables, the number of all possible different values of K and S is $\dfrac{n(n-1)}{2}+1$. Let $u(n,S_q)$ be the number of values of S_q in the aggregate of $n!$ and $f(K_q)$ be the probability distribution of K_q proposed by Kendall (1970: 67-69). We can compute that

$$f(K_q) = \frac{u(n,S_q)}{n!},\ q=1,\ 2,\ldots,\frac{n(n-1)}{2}+1, \tag{3.5}$$

where $u(n+1,S_q) = u(n,S_q-n)+u(n,S_q-n+2)+\ldots+u(n,S_q+n-2)+u(n,S_q+n)$;

$S_q = -\dfrac{n(n-1)}{2}+2q-2$, for $q=1,\ 2,\ldots,\dfrac{n(n-1)}{2}+1$; and $n \ge 2$.

For $n=2$, there are two values of S, either -1 or 1, and $u(n=2,\ S_1=-1)=1$ and $u(n=2,\ S_1=1)=1$. In the case of $n<10$, the probability distributions of K in (3.5) are shown in Tables 3.1-3.8.

Table 3.1 The Probability Distribution of K for $n = 2$

S_q	$u(n,\ S_q)$	$K_q = \frac{2S_q}{n(n-1)}$	$f(K_q)=\frac{u(n,\ S_q)}{n!}$
-1	1	-1	0.5
1	1	1	0.5
	Total		1.0

Table 3.2 The Probability Distribution of K for $n = 3$

S_q	$u(n,\ S_q)$	$K_q = \frac{2S_q}{n(n-1)}$	$f(K_q)=\frac{u(n,\ S_q)}{n!}$
-3	1	-0.50000	0.16667
-1	2	-0.16667	0.33333
1	2	0.16667	0.33333
3	1	0.50000	0.16667
	Total		1.00000

Table 3.3 The Probability Distribution of K for $n = 4$

S_q	$u(n,\ S_q)$	$K_q = \frac{2S_q}{n(n-1)}$	$f(K_q)=\frac{u(n,\ S_q)}{n!}$
-6	1	-1.00000	0.04167
-4	3	-0.66667	0.12500
-2	5	-0.33333	0.20833
0	6	0.00000	0.25000
2	5	0.33333	0.20833
4	3	0.66667	0.12500
6	1	1.00000	0.04167
	Total		1.00000

Table 3.4 The Probability Distribution of K for $n = 5$

S_q	$u(n, S_q)$	$K_q = \frac{2S_q}{n(n-1)}$	$f(K_q) = \frac{u(n, S_q)}{n!}$
-10	1	-1.00000	0.00833
-8	4	-0.80000	0.03333
-6	9	-0.60000	0.07500
-4	15	-0.40000	0.12500
-2	20	-0.20000	0.16667
0	22	0.00000	0.18333
2	20	0.20000	0.16667
4	15	0.40000	0.12500
6	9	0.60000	0.07500
8	4	0.80000	0.03333
10	1	1.00000	0.00833
	Total		1.00000

Table 3.5 The Probability Distribution of K for n = 6

S_q	$u(n, S_q)$	$K_q = \frac{2S_q}{n(n-1)}$	$f(K_q)=\frac{u(n, S_q)}{n!}$
-15	1	-1.00000	0.00139
-13	5	-0.86667	0.00694
-11	14	-0.73333	0.01944
-9	29	-0.60000	0.04028
-7	49	-0.46667	0.06806
-5	71	-0.33333	0.09861
-3	90	-0.20000	0.12500
-1	101	-0.06667	0.14028
1	101	0.06667	0.14028
3	90	0.20000	0.12500
5	71	0.33333	0.09861
7	49	0.46667	0.06806
9	29	0.60000	0.04028
11	14	0.73333	0.01944
13	5	0.86667	0.00694
15	1	1.00000	0.00139
	Total		1.00000

Table 3.6 The Probability Distribution of K for $n = 7$

S_q	$u(n, S_q)$	$K_q = \frac{2S_q}{n(n-1)}$	$f(K_q) = \frac{u(n, S_q)}{n!}$
-21	1	-1.00000	0.00020
-19	6	-0.90476	0.00119
-17	20	-0.80952	0.00397
-15	49	-0.71429	0.00972
-13	98	-0.61905	0.01944
-11	169	-0.52381	0.03353
-9	259	-0.42857	0.05139
-7	359	-0.33333	0.07123
-5	455	-0.23810	0.09028
-3	531	-0.14286	0.10536
-1	573	-0.04762	0.11369
1	573	0.04762	0.11369
3	531	0.14286	0.10536
5	455	0.23810	0.09028
7	359	0.33333	0.07123
9	259	0.42857	0.05139
11	169	0.52381	0.03353
13	98	0.61905	0.01944
15	49	0.71429	0.00972
17	20	0.80952	0.00397
19	6	0.90476	0.00119
21	1	1.00000	0.00020
	Total		1.00000

Table 3.7 The Probability Distribution of K for n = 8

S_q	$u(n, S_q)$	$K_q = \dfrac{2S_q}{n(n-1)}$	$f(K_q) = \dfrac{u(n, S_q)}{n!}$
-28	1	-1.00000	0.00002
-26	7	-0.92857	0.00017
-24	27	-0.85714	0.00067
-22	76	-0.78571	0.00188
-20	174	-0.71429	0.00432
-18	343	-0.64286	0.00851
-16	602	-0.57143	0.01493
-14	961	-0.50000	0.02383
-12	1415	-0.42857	0.03509
-10	1940	-0.35714	0.04812
-8	2493	-0.28571	0.06183
-6	3017	-0.21429	0.07483
-4	3450	-0.14286	0.08557
-2	3736	-0.07143	0.09266
0	3836	0.00000	0.09514
2	3736	0.07143	0.09266
4	3450	0.14286	0.08557
6	3017	0.21429	0.07483
8	2493	0.28571	0.06183
10	1940	0.35714	0.04812
12	1415	0.42857	0.03509
14	961	0.50000	0.02383
16	602	0.57143	0.01493
18	343	0.64286	0.00851
20	174	0.71429	0.00432
22	76	0.78571	0.00188
24	27	0.85714	0.00067
26	7	0.92857	0.00017
28	1	1.00000	0.00002
		Total	1.00000

Table 3.8 The Probability Distribution of K for $n = 9$

S_q	$u(n, S_q)$	$K_q = \frac{2S_q}{n(n-1)}$	$f(K_q) = \frac{u(n, S_q)}{n!}$
-36	1	-1.00000	0.00000
-34	8	-0.94444	0.00002
-32	35	-0.88889	0.00010
-30	111	-0.83333	0.00031
-28	285	-0.77778	0.00079
-26	628	-0.72222	0.00173
-24	1230	-0.66667	0.00339
-22	2191	-0.61111	0.00604
-20	3606	-0.55556	0.00994
-18	5545	-0.50000	0.01528
-16	8031	-0.44444	0.02213
-14	11021	-0.38889	0.03037
-12	14395	-0.33333	0.03967
-10	17957	-0.27778	0.04948
-8	21450	-0.22222	0.05911
-6	24584	-0.16667	0.06775
-4	27073	-0.11111	0.07461
-2	28675	-0.05556	0.07902
0	29228	0.00000	0.08054
2	28675	0.05556	0.07902
4	27073	0.11111	0.07461
6	24584	0.16667	0.06775
8	21450	0.22222	0.05911
10	17957	0.27778	0.04948
12	14395	0.33333	0.03967
14	11021	0.38889	0.03037
16	8031	0.44444	0.02213
18	5545	0.50000	0.01528
20	3606	0.55556	0.00994
22	2191	0.61111	0.00604
24	1230	0.66667	0.00339
26	628	0.72222	0.00173
28	285	0.77778	0.00079

Table 3.8 (Continued)

S_q	$u(n, S_q)$	$K_q = \frac{2S_q}{n(n-1)}$	$f(K_q) = \frac{u(n, S_q)}{n!}$
30	111	0.83333	0.00031
32	35	0.88889	0.00010
34	8	0.94444	0.00002
36	1	1.00000	0.00000
	Total		1.00000

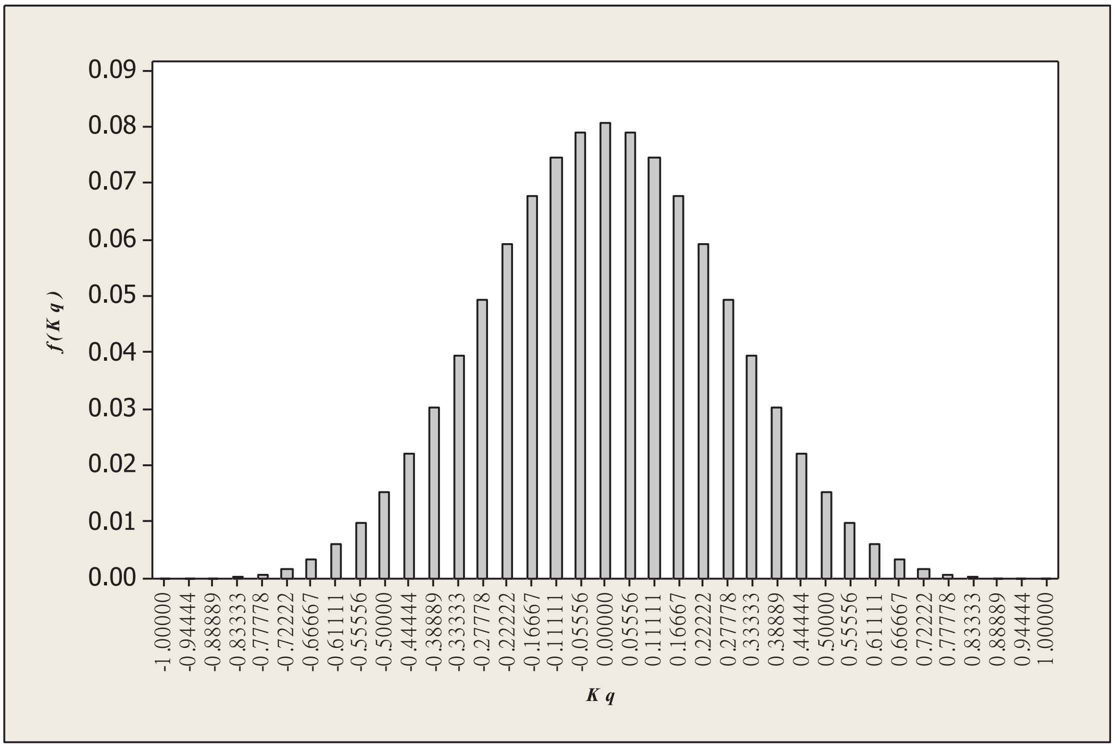

Figure 3.1 The Probability Distribution of K for $n = 9$

3.3 The Proposed Tests

3.3.1 The Proposed Test Statistic

In order to test hypothesis $H_0:\ \tau = 0$ using Kendall's coefficient, a sample size of at least 5 or 6 for, respectively, 0.05 and 0.01 significance levels should be used because these sample sizes can be applied to determine the critical regions corresponding to the specified significance levels. However, Kendall's Tau is not appropriate for a large sample size because the calculation is complicated (albeit using a computer).

For n large, the distribution of $\sum_{1 \le i <}^{n} \sum_{j \le n}^{n} A_{ij}$ in (3.4) tends towards standard normal under the null hypothesis (Kendall, 1970: 69-71). Therefore, the sample Kendall's coefficient K in (3.4) is approximately normally distributed and so, in this study, the test statistic for testing the hypothesis in (1.1) is defined as

$$Z_K = \frac{K}{\left(\frac{2(2n+5)}{9n(n-1)}\right)^{1/2}}. \tag{3.6}$$

3.3.2 Asymptotic Distribution of the Proposed Test Statistic

1) The Normal Approximation

Theorem 3.2

Let $X_1, X_2, ..., X_m$ be a positive continuous random sample of size $m = 2n$ drawn from a gamma population (H_0) and K is defined as in (3.4), then the test statistic defined as (3.6) has an asymptotic distribution approaching standard normal as $n \to \infty$.

Proof. From theorem 3.1, it is easy to show that $\mathrm{E}(K\,|\,\mathrm{H}_0)=0$ and $\mathrm{V}(K\,|\,\mathrm{H}_0)=\dfrac{2(2n+5)}{9n(n-1)}$, and so the expectation and variance of Kendall's coefficient statistic Z_K under the null hypothesis are given by

$$\mathrm{E}(\mathrm{Z}_K\,|\,\mathrm{H}_0)=\mathrm{E}\left(\frac{K}{\left(\dfrac{2(2n+5)}{9n(n-1)}\right)^{1/2}}\Big|\mathrm{H}_0\right)=\frac{1}{\left(\dfrac{2(2n+5)}{9n(n-1)}\right)^{1/2}}\mathrm{E}\left(K\big|\mathrm{H}_0\right)=0\text{, and}$$

$$\mathrm{V}(\mathrm{Z}_K\,|\,\mathrm{H}_0)=\mathrm{V}\left(\frac{K}{\left(\dfrac{2(2n+5)}{9n(n-1)}\right)^{1/2}}\Big|\mathrm{H}_0\right)=\frac{9n(n-1)}{2(2n+5)}\mathrm{V}\left(K\big|\mathrm{H}_0\right)=1\text{, respectively.}$$

Thus, by standardization of the K statistic under H_0, Z_K follows a standard normal distribution. The proof is complete.

2) The Edgeworth Approximation

When applying the proposed test Z_K in (3.6), which is asymptotically normal, to test a sample that is not sufficiently large, a high error in testing may occur. Since the distribution of the Z_K test statistic may not be close to a normal distribution, some researchers have proposed the use of the third and fourth moments to improve on attaining a normal distribution (Wallace, 1958: 635-654; Field and Ronchetti, 1990: 1-140; Bentkus, Götze and van Zwet, 1997: 851-896), and an Edgeworth expansion is one of these methods; it is used to approximate a distribution, especially in the case of a small sample size, since it can be approximately close to the exact distribution (Ghosh and Jammalamadaka, 1998: 245-261). Therefore, the probability distribution of the proposed test statistic Z_K can be revised by applying the Edgeworth expansion method to produce test statistic T_K as follows:

$$\mathrm{T}_K = \frac{K}{\left(\frac{2(2n+5)}{9n(n-1)}\right)^{1/2}} . \tag{3.7}$$

The asymptotic distribution of T_K can be obtained through an Edgeworth expansion so that the density and the limit distribution function can be written as

$$\hat{f}_{\mathrm{T}_K}(\mathrm{t}_K) = \phi(\mathrm{t}_K)+\phi(\mathrm{t}_K)\left\{\frac{\mu_3}{6\sqrt{n}}\left(\mathrm{t}_K^3-3\mathrm{t}_K\right)+\frac{1}{n}\left[\frac{(\mu_4-3)}{24}\left(\mathrm{t}_K^4-6\mathrm{t}_K^2+3\right) + \frac{\mu_3^2}{72}\left(\mathrm{t}_K^6-15\,\mathrm{t}_K^4+15\mathrm{t}_K^2-15\right)\right]\right\} \tag{3.8}$$

$$\hat{\mathrm{F}}_{\mathrm{T}_K}(\mathrm{t}_K) = \Phi(\mathrm{t}_K)-\phi(\mathrm{t}_K)\left\{\frac{\mu_3}{6\sqrt{n}}\left(\mathrm{t}_K^2-1\right)+\frac{1}{n}\left[\frac{(\mu_4-3)}{24}\left(\mathrm{t}_K^3-3\mathrm{t}_K\right) + \frac{\mu_3^2}{72}\left(\mathrm{t}_K^5-10\,\mathrm{t}_K^3+15\mathrm{t}_K\right)\right]\right\}, \tag{3.9}$$

where Φ and ϕ are, respectively, the CDF and probability density function of a standard normal distribution, and μ_p is the p^{th} central moment of the test statistic T_K. For instance, under H_0 in (2.3), $\mu_3 = 0$ because of the symmetrical distribution of T_K (Kendall, 1970: 72-73). Therefore, equations (3.8) and (3.9) can be rewritten as

$$\hat{f}_{\mathrm{T}_K}(\mathrm{t}_K) = \phi(\mathrm{t}_K)+\phi(\mathrm{t}_K)\frac{(\mu_4-3)}{24n}\left(\mathrm{t}_K^4-6\mathrm{t}_K^2+3\right), \tag{3.10}$$

$$\hat{\mathrm{F}}_{\mathrm{T}_K}(\mathrm{t}_K) = \Phi(\mathrm{t}_K)-\phi(\mathrm{t}_K)\left\{\frac{(\mu_4-3)}{24n}\left(\mathrm{t}_K^3-3\mathrm{t}_K\right)\right\}. \tag{3.11}$$

Theorem 3.3

Let $X_1, X_2, ..., X_m$ be a positive continuous random sample of size $m = 2n$ drawn from a gamma population, T_K be defined as in (3.7), and $\hat{\mu}_4$ be the estimator of the fourth central moment of the test statistic T_K, then the Edgeworth density (3.10) and distribution function (3.11) of T_K as $n \to \infty$ can be written as

$$\hat{f}^*_{\mathrm{T}_K}(\mathrm{t}_K) = \phi(\mathrm{t}_K) + \phi(\mathrm{t}_K)\frac{(\hat{\mu}_4 - 3)}{24n}\left(\mathrm{t}_K^4 - 6\mathrm{t}_K^2 + 3\right), \tag{3.12}$$

$$\hat{\mathrm{F}}^*_{\mathrm{T}_K}(\mathrm{t}_K) = \Phi(\mathrm{t}_K) - \phi(\mathrm{t}_K)\left\{\frac{(\hat{\mu}_4 - 3)}{24n}\left(\mathrm{t}_K^3 - 3\mathrm{t}_K\right)\right\}. \tag{3.13}$$

Next, an estimator of μ_4, the 4th central moment of the test statistic T_K needs to be found. The jackknife technique was first developed by Quenouille (1949: 355-375) to estimate the bias of an estimator. In 1998, John W. Tukey applied a technique to estimate variance and named this technique jackknife sampling (Tukey, 1998, quoted in Abdi and Williams, 2010: 656-661). It is very useful for estimating the variance and bias of an estimator.

For this study, the jackknife estimator of the 4th central moment of T_K is obtained and μ_4 is estimated by $\hat{\mu}_4$ using the jackknife method. The h^{th}, $h = 1, 2, 3, \ldots, n$ pseudo sample is constructed by deleting the h^{th} sample pair value (U_h, V_h). The statistic K_{-h} is the h^{th} pseudo sample Kendall's coefficient computed by deleting the h^{th} sample pair value and $\bar{\mathrm{K}}^*_{-}$ is the mean of the jackknife Kendall's rank coefficient, as shown below:

$$K_{-h} = \frac{1}{\binom{n-1}{2}} \sum_{\substack{1 \le i < \\ i \ne h}}^{n-1} \sum_{\substack{j \le n \\ , j \ne h}}^{n} \mathrm{A_{ij}} \;, \tag{3.14}$$

$$\bar{K}_{-}^{*} = \frac{\sum_{h=1}^{n} K_{-h}}{n} \quad , \tag{3.15}$$

$$\hat{\mu}_4 = \left(\frac{2(2n+5)}{9n(n-1)} \right)^2 \sum_{h=1}^{n} \frac{\left(K_{-h} - \bar{K}_{-}^{*} \right)^4}{n} . \tag{3.16}$$

Theorem 3.4

Let $X_1, X_2, ..., X_m$ be a positive continuous random sample of size $m = 2n$ drawn from a gamma population, K be defined as in (3.4) and K_{-h} as in (3.14), and $h = 1, 2, 3, \ldots, n$ be the h^{th} pseudo sample K statistic computed by deleting the h^{th} pseudo sample out of (U_j, V_j); j = 1, 2 , …, n in (3.3). Consequently, $E(\bar{K}_{-}^{*}) = \tau$ and $\hat{\mu}_4$ is the jackknife estimator of μ_4, where $\bar{K}_{-}^{*}$ and $\hat{\mu}_4$ are defined in (3.15) and (3.16), respectively.

Proof. Let $\bar{K}_{-}^{*}$ be the mean of the jackknife Kendall's coefficient

$$\bar{K}_{-}^{*} = \frac{1}{n} \sum_{h=1}^{n} K_{-h}$$

$$= \frac{1}{n} \sum_{h=1}^{n} \left(\frac{\sum_{\substack{1 \le i < \\ i \ne h}}^{n-1} \sum_{\substack{j \le n \\ , j \ne h}}^{n} A_{ij}}{\binom{n-1}{2}} \right)$$

$$= \frac{1}{n} \sum_{h=1}^{n} \left(\frac{\sum_{\substack{1 \le i < \\ i \ne h}}^{n-1} \sum_{\substack{j \le n \\ , j \ne h}}^{n} A_{ij}}{\binom{n-1}{2}} \right)$$

$$= \frac{2(n-2)}{n(n-1)(n-2)} \sum_{1 \le i <}^{n-1} \sum_{j \le n}^{n} A_{ij}$$

$$= \frac{2}{n(n-1)} \sum_{1 \le i <}^{n-1} \sum_{j \le n}^{n} A_{ij}$$

$$= K,$$

$$E(\bar{\mathrm{K}}_{-}^{*}) = E(K) = \tau .$$

Thus, $\bar{\mathrm{K}}_{-}^{*}$ is an unbiased estimator of τ using the jackknife method.

The jackknife estimator $\hat{\mu}_4$ is a good estimator of μ_4 to approximate the probability distribution function T_K, as shown in Appendix B.

3.4 Some Properties of the Proposed Tests

3.4.1 The Invariance Property of the Proposed Tests

Lemma 3.1.

A_{ij} is invariant under a group of scalar transformations $X \to cX$, where c is positive real and

$$A_{ij} = sgn\left(U_j - U_i\right) sgn\left(V_j - V_i\right),\ i \ne j,\ i, j = 1, 2, ..., n,$$

$$U_j = \frac{X_{2j-1} X_{2j}}{(X_{2j-1} + X_{2j})^2},\ V_j = X_{2j-1} + X_{2j},\ j = 1, 2, ..., n.$$

Proof. To show that A_{ij} is invariant under a group of scalar transformations $X \to cX$, where c is positive real, and define $Y_i = cX_i$, where c is positive real, then

$$U_j^y = \frac{Y_{2j-1} Y_{2j}}{(Y_{2j-1} + Y_{2j})^2}$$

$$= \frac{cX_{2j-1}cX_{2j}}{(cX_{2j-1}+cX_{2j})^2} = U_j,$$

$$V_j^y = Y_{2j-1} + Y_{2j}$$

$$= cX_{2j-1} + cX_{2j} = cV_j, \text{ and}$$

$$A_{ij}^y = sgn\left(U_j^y - U_i^y\right) sgn\left(V_j^y - V_i^y\right) \quad , i \neq j,\ i, j = 1, 2, \ldots, n.$$

$$= sgn\left(U_j - U_i\right) sgn\left(cV_j - cV_i\right)$$

$$= sgn\left(U_j - U_i\right) sgn\left(c\left(V_j - V_i\right)\right)$$

$$= sgn\left(U_j - U_i\right) sgn\left(V_j - V_i\right)$$

$$= A_{ij}.$$

Theorem 3.5

Let $X_1, X_2, \ldots, X_m$ be random variables having a continuous density function $f(x)$. Hence, K, Z_K, and T_K are invariant under a group of scalar transformations $X \to cX$, where c is positive real and K, Z_K and T_K are defined as in (3.4), (3.6) and (3.7), respectively.

Proof. K, Z_K, and T_K are functions of A_{ij}, which, from Lemma 3.1, is a function of scale invariance transformation, and so they provide the same property.

3.4.2 The Size of the Proposed Tests

The size of a test, often called the significance level, is the probability of meeting a Type I error, which occurs if the null hypothesis is rejected when true, and is denoted by α.

3.4.2.1 The Asymptotic Size of the Test for Z_K

The asymptotic size of the test for Z_K can be approximated by $\hat{\alpha}(\mathrm{Z}_K)$.

$$
\begin{aligned}
\hat{\alpha}(\mathrm{Z}_K) &= \mathrm{P}\left(\text{Reject } \mathrm{H}_0 \,|\, \mathrm{H}_0 \text{ is true}\right) \\
&= \mathrm{P}\left(|\mathrm{Z}_K| > \mathrm{Z}_{K_{\frac{\alpha}{2}}} \,|\, F_0(x) \right) \\
&= \mathrm{P}\left(\frac{\mathrm{K}}{\sqrt{V(K)}} > \mathrm{Z}_{K_{\frac{\alpha}{2}}} \right) + \mathrm{P}\left(\frac{\mathrm{K}}{\sqrt{V(K)}} < -\mathrm{Z}_{K_{\frac{\alpha}{2}}} \right) \\
&= 1 - \Phi(\mathrm{Z}_{K_{\frac{\alpha}{2}}}) + \Phi(-\mathrm{Z}_{K_{\frac{\alpha}{2}}}) .
\end{aligned}
$$

3.4.2.2 The Asymptotic Size of the Test for T_K

The asymptotic size of the test for T_K can be approximated by $\hat{\alpha}(\mathrm{T}_K)$.

$$
\begin{aligned}
\hat{\alpha}(\mathrm{T}_K) &= \mathrm{P}\left(\text{Reject } \mathrm{H}_0 \,|\, \mathrm{H}_0 \text{ is true}\right) \\
&= \mathrm{P}\left(|\mathrm{T}_K| > \mathrm{T}_{K_{\frac{\alpha}{2}}} \,|\, F_0(x) \right) \\
&= \mathrm{P}\left(\frac{\mathrm{K}}{\sqrt{V(K)}} > \mathrm{T}_{K_{\frac{\alpha}{2}}} \right) + \mathrm{P}\left(\frac{\mathrm{K}}{\sqrt{V(K)}} < -\mathrm{T}_{K_{\frac{\alpha}{2}}} \right) \\
&= 1 - \left(\Phi(\mathrm{t}_K) - \phi(\mathrm{t}_K) \left\{ \frac{(\hat{\mu}_4 - 3)}{24n} \left(\mathrm{t}_K^3 - 3\,\mathrm{t}_K \right) \right\} \right) \\
&\quad + \left(\Phi(-\mathrm{t}_K) - \phi(-\mathrm{t}_K) \left\{ \frac{(\hat{\mu}_4 - 3)}{24n} \left(-\mathrm{t}_K^3 + 3\,\mathrm{t}_K \right) \right\} \right) .
\end{aligned}
$$

3.4.3 The Power of the Proposed Test

The power of a statistical test is the probability that it will correctly lead to the rejection of a false null hypothesis. The probability distribution of K is asymptotically normal for sample pairs from any bivariate population. Since E(K) = τ for any distribution, and using the consistent estimator $\hat{\mu}_4$ for μ_4, the asymptotic distribution of Z_K remains standard normal and the asymptotic distribution of T_K is $\hat{\mathrm{F}}^*_{\mathrm{T}_K}(\mathrm{t}_K)$.

3.4.3.1 The Power of the Test for Z_K

The power of the test for Z_K can be approximated by $1-\hat{\beta}(Z_K)$.

$$1-\hat{\beta}(Z_K) \;\; = \mathrm{P}\left(\text{Reject } \mathrm{H}_0 | \mathrm{H}_1 \text{ is true}\right)$$

$$1-\hat{\beta}(Z_K) = \mathrm{P}\left(|Z_K| > Z_{K_{\frac{\alpha}{2}}} | F_0'(x) \right)$$

$$= \mathrm{P}\left(\frac{K-\tau}{\sqrt{V(K|H_1)}} > \frac{Z_{K_{\frac{\alpha}{2}}}\sqrt{V(K|H_0)}-\tau}{\sqrt{V(K|H_1)}} \right)$$

$$+\,\mathrm{P}\left(\frac{K-\tau}{\sqrt{V(K|H_1)}} < \frac{-Z_{K_{\frac{\alpha}{2}}}\sqrt{V(K|H_0)}-\tau}{\sqrt{V(K|H_1)}} \right)$$

$$= 1 \; - \Phi\left(\frac{Z_{K_{\frac{\alpha}{2}}}\sqrt{V(K|H_0)}-\tau}{\sqrt{V(K|H_1)}} \right)$$

$$+\;\Phi\left(\frac{-Z_{K_{\frac{\alpha}{2}}}\sqrt{V(K|H_0)}-\tau}{\sqrt{V(K|H_1)}} \right).$$

3.4.3.2 The Power of the Test for T_K

The power of the test for T_K can be approximated by $1-\hat{\beta}(\mathrm{T}_K)$.

$$1-\hat{\beta}(\mathrm{T}_K) = \mathrm{P}\left(\text{Reject } \mathrm{H}_0 | \mathrm{H}_1 \text{ is true}\right)$$

$$= \mathrm{P}\left(|\mathrm{T}_K| > \mathrm{T}_{K_{\frac{\alpha}{2}}} | F_0'(x) \right)$$

$$1-\hat{\beta}(\mathrm{T}_K) = \mathrm{P}\left(\frac{K-\tau}{\sqrt{V(K|H_1)}} > \frac{\mathrm{T}_{K_{\frac{\alpha}{2}}}\sqrt{V(K|H_0)}-\tau}{\sqrt{V(K|H_1)}} \right)$$

$$+\mathrm{P}\left(\frac{K-\tau}{\sqrt{V(K|H_1)}} < \frac{-\mathrm{T}_{K_{\frac{\alpha}{2}}}\sqrt{V(K|H_0)}-\tau}{\sqrt{V(K|H_1)}} \right)$$

$$= 1 - \hat{\mathrm{F}}^*_{\mathrm{T}_K}\left(\frac{\mathrm{T}_{K_{\frac{\alpha}{2}}}\sqrt{V(K|H_0)}-\tau}{\sqrt{V(K|H_1)}} \right)$$

$$+ \hat{\mathrm{F}}^*_{\mathrm{T}_K}\left(\frac{-\mathrm{T}_{K_{\frac{\alpha}{2}}}\sqrt{V(K|H_0)}-\tau}{\sqrt{V(K|H_1)}} \right)$$

CHAPTER 4

SIMULATION STUDY

Tests for a gamma distribution are nonparametric and can be classified into two approaches: one is based on the empirical distribution function (EDF) and the other is based on certain characteristics of a gamma distribution. The first approach measures the distance between the EDF and the gamma distribution and the second utilizes the independence property of a gamma distribution. In this study, the two proposed tests are based on the second approach of using the independence property shown in equation (3.1). These are called Z_K and T_K, and are modifications of Kendall's correlation coefficient. The difference between these test statistics is their distribution: Z_K is asymptotically distributed as standard normal whereas the distribution of T_K is obtained using an Edgeworth expansion and the jackknife method.

4.1 Simulation Entities

The performances of the two proposed tests Z_K and T_K were compared to five selected previously reported tests, namely Kolmogorov-Smirnov (KS), Cramér–von Mises (CM), Anderson–Darling (AD), Wilding and Mudholkar (WM), and Villaseñor and González-Estrada (VGE), using a Monte Carlo simulation study with the entities shown in Table 4.1

Table 4.1 The Entities in the Simulation Study

Entity	Under Null Hypothesis	Under Alternative Hypothesis
Distribution	Gamma (λ, 1)	Weibull (λ, 1) Fréchet (λ, 1) Lognormal (50, θ)
λ	0.1, 0.2, 0.3, 0.4, 0.5, 0.7, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10	0.5, 1, 2, 3, 5, 7, 9, 10
θ	-	0.5, 1, 2, 3, 5, 7, 9, 10
Sample size (m)	20, 30, 40, 50, 70, 100	20, 30, 40, 50, 70, 100
Replication	10,000	10,000
Significance level (α)	0.01, 0.05	0.01, 0.05

Remark that since the four competitive tests, Z_{K}, T_{K}, WM, and VGE, possess invariance under a scale transformation, the simulations were conducted under the condition of only varying the shape parameter. That is, the shape parameter for the gamma, Weibull, and Fréchet distributions was fixed at one and, in a similar fashion, the location parameter of the lognormal distribution was fixed at 50.

4.2 Empirical Type I Error Rate and Power of the Test

The efficacy of the tests were considered based on empirical Type I error rates and powers of the tests, which are defined as follows:

Empirical Type I error rate of test J is

$$\hat{\alpha}_J = \frac{\text{number of rejections}}{\text{number of replications}}. \quad (4.1)$$

Empirical power of test J is

$$\hat{\beta}_J = \frac{\text{number of rejections}}{\text{number of replications}}, \quad (4.2)$$

where J = 1, 2, 3, 4, 5, 6, and 7 for KS, AD, CM, WM, VGE, $\mathrm{Z_K}$, and T_K, respectively. To confirm whether the empirical Type I error rates of the tests fell within the given 0.05 nominal significance level, the well-known two-proportion test was applied to all cases at the nominal significance level. However, the empirical powers were only evaluated for the tests which were close to the nominal significance level.

4.3 Simulation Procedure

Denote that

K_{-h}, h=1,2,…,n, is CDF of $\mathrm{T_K}$ under an Edgeworth expansion,

$\hat{\mathrm{F}}_{\mathrm{T}_K}(t)$ is CDF of $\mathrm{T_K}$ under an Edgeworth expansion, and

$\hat{\mu}_4$ is an estimate of the 4^{th} central moment for the test statistic T_K.

The procedure in Figure 4.1 shows the simulation process in order to obtain the empirical Type I error rates for all seven tests corresponding to hypothesis (1.1). Under the alternative hypothesis, steps in finding the empirical power are similar except that the population distribution in the first step is varied according to the particular distribution (see Table 4.1) and, instead of $\hat{\alpha}_J$, $\hat{\beta}_J$'s are computed for all J's.

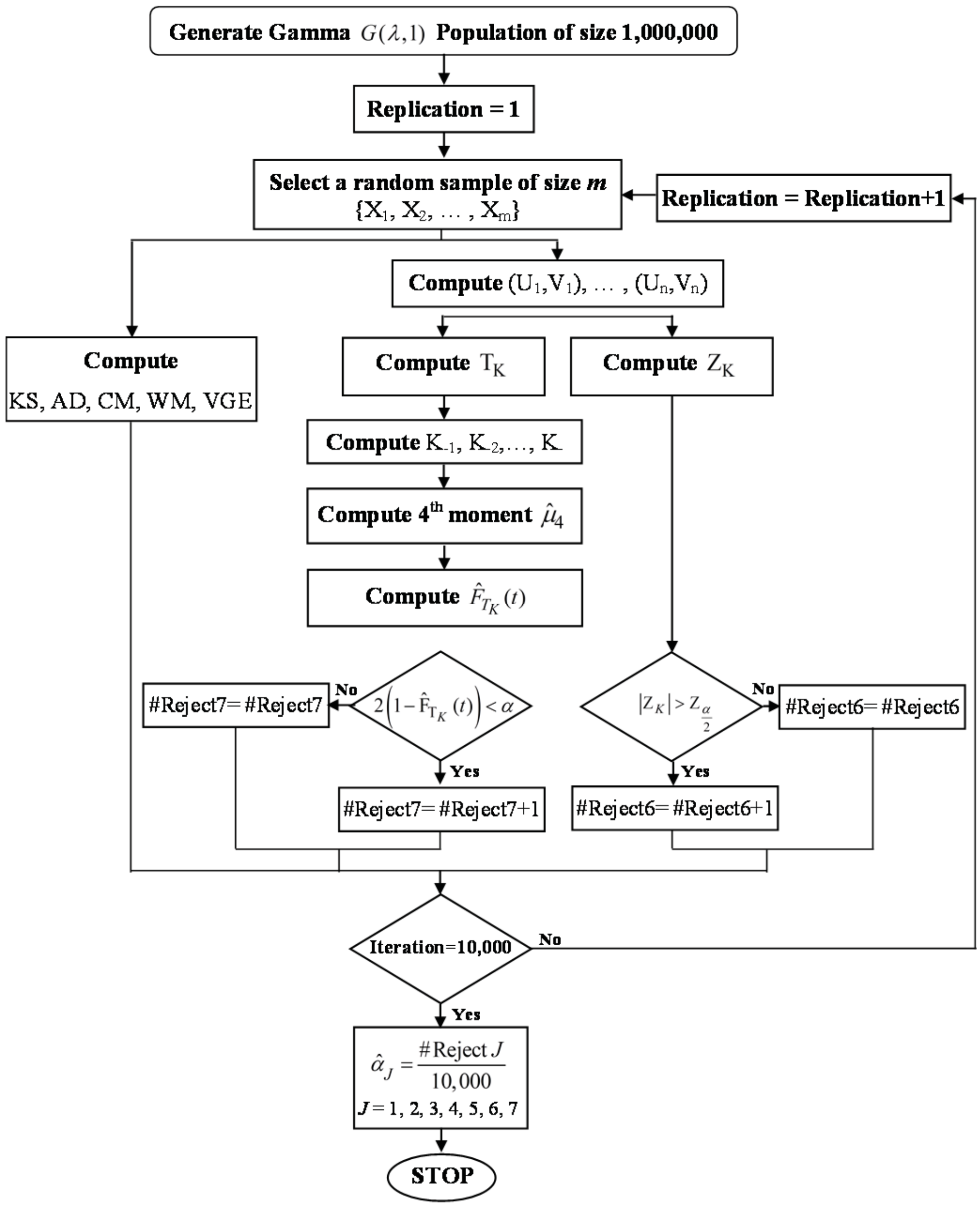

Figure 4.1 Simulation Procedure for the Z_K and T_K Tests

4.4 Results

4.4.1 Empirical Type I Error Rates

The empirical Type I error rates of all selected tests are shown in Tables A.1-A.2 in Appendix A and Figures 4.2- 4.7. It was found that the empirical Type I error rates of the tests KS, AD, and CM, as can be seen in Tables A.1 - A.2 in Appendix A, were all far lower than both nominal significance levels, α = 0.01 and 0.05, and tended towards zero in all situations.

The empirical Type I error rates of the VGE test for most shape parameters were substantially lower than the nominal significance 0.01, especially when the sample size was less than 60, the empirical Type I error rates were lower than the nominal significance values 0.01 for all shape of parameters, as can be seen in the bottom left graph in Figure 4.2. For significance level 0.05, the empirical Type I error rates of the VGE test for all shape parameters were lower than the nominal significance 0.05 for all sample sizes, as can be seen in the bottom left graph in Figure 4.3. However, the empirical Type I error rates tended to converge to the given nominal significance values when the sample size increased.

Conversely, the Type I error rates of the WM test were substantially higher than the nominal significance values 0.01 and 0.05 for all shape parameters. Additionally, when the shape parameter and the sample size increase, the empirical Type I error rate of WM are closer to the nominal significance levels, as can be seen in the bottom right graphs in Figures 4.2 and 4.3. The shape parameter had a significant impact on the WM test.

The empirical Type I error rates of the Z_K test were found to be lower than the nominal level values 0.01 and 0.05 when the sample size was less than 50 and 40, respectively, as can be seen in the top left graph of Figures 4.2 and 4.3, respectively. It tended to converge to the given nominal significance values for all shapes considered as the sample size increased.

Empirical Type I error rates for the proposed test statistic T_K attained the nominal significance level values 0.01 for all shape parameters, as can be seen in the top right graph in Figure 4.2. For the nominal significance level 0.05, results were lower than it when the sample size was less than 40, as can be seen in the top right graph in Figure 4.3.

Findings from the two-proportion test indicated that the Type I error rates of the five previously reported tests (KS, AD, CM, WM, and VGE) were significantly different from the nominal significance level for all cases, even when the sample size was less than 100, as can be seen in Tables A.1-A.2 in Appendix A and Figures 4.4-4.7. Consequently, their empirical powers were not investigated. The empirical Type I error rates of the proposed test statistics Z_K and T_K were close to the nominal significance level for all shape parameters considered and, by observation, tended towards the nominal significance level when the sample size increased. To be more precise, T_K performed better than Z_K for all shape parameters at the 0.01 significance level, as can be seen in Figures 4.4-4.5.

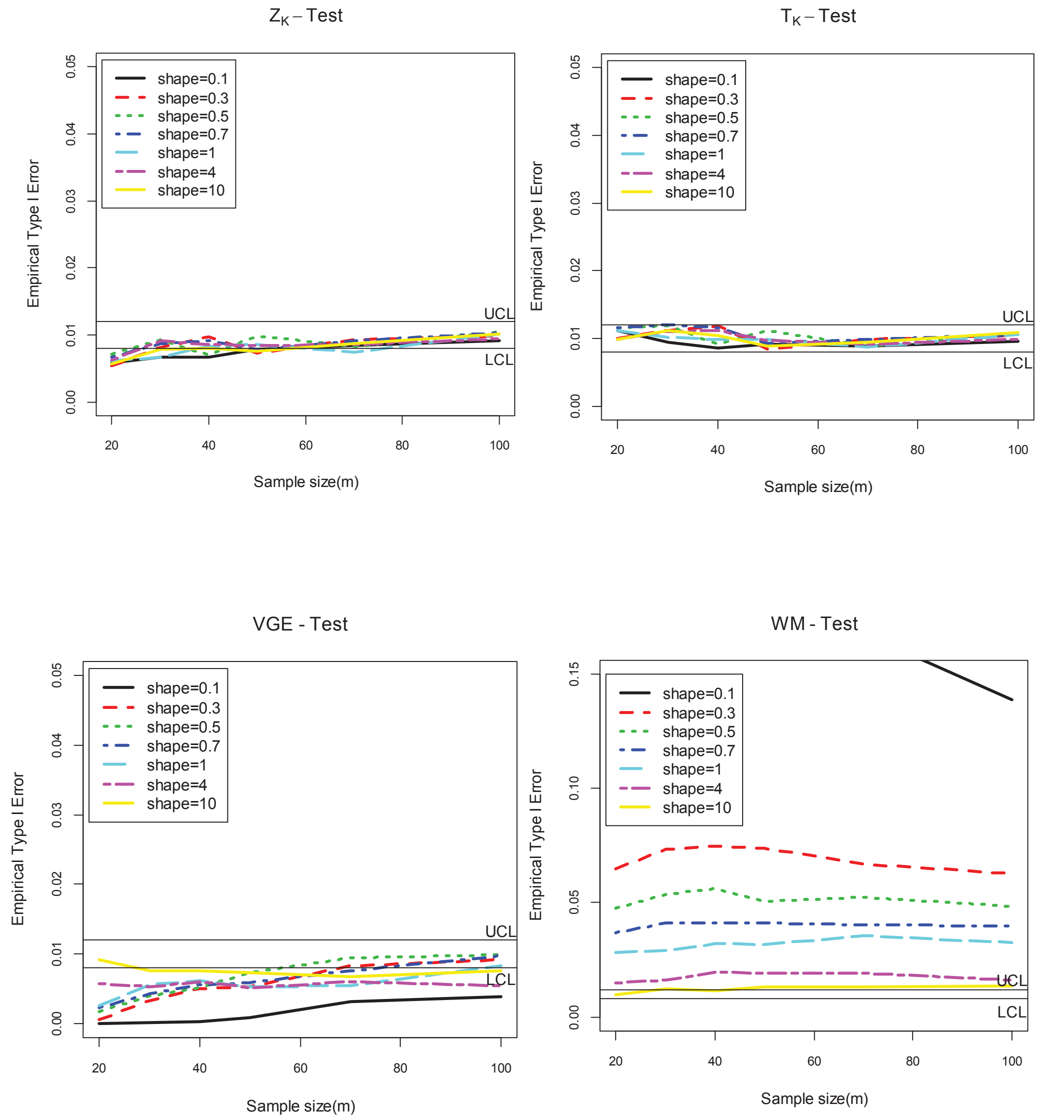

Figure 4.2 Empirical Type I Error Rates at the 0.01 Significance Level

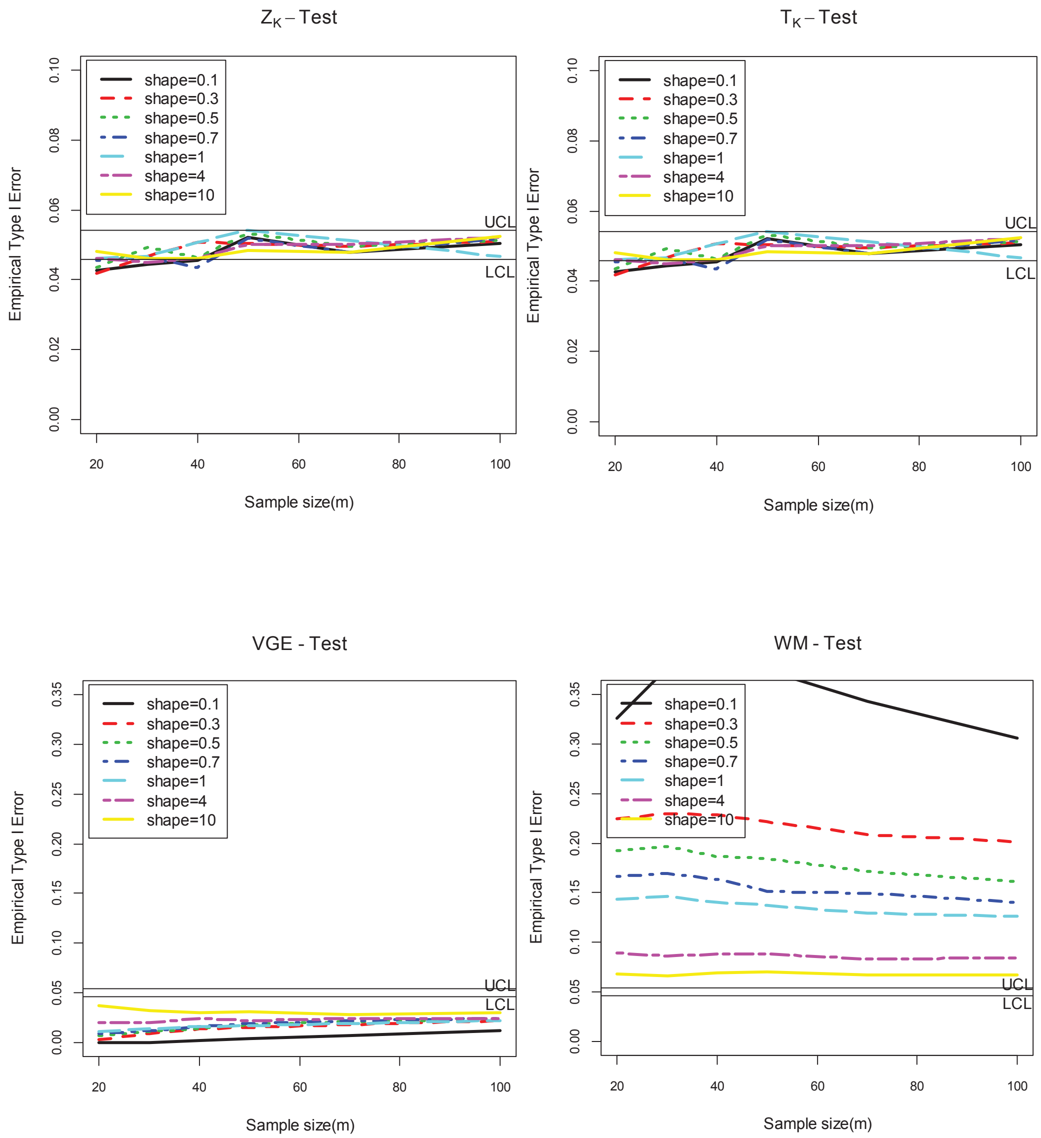

Figure 4.3 Empirical Type I Error Rates at the 0.05 Significance Level

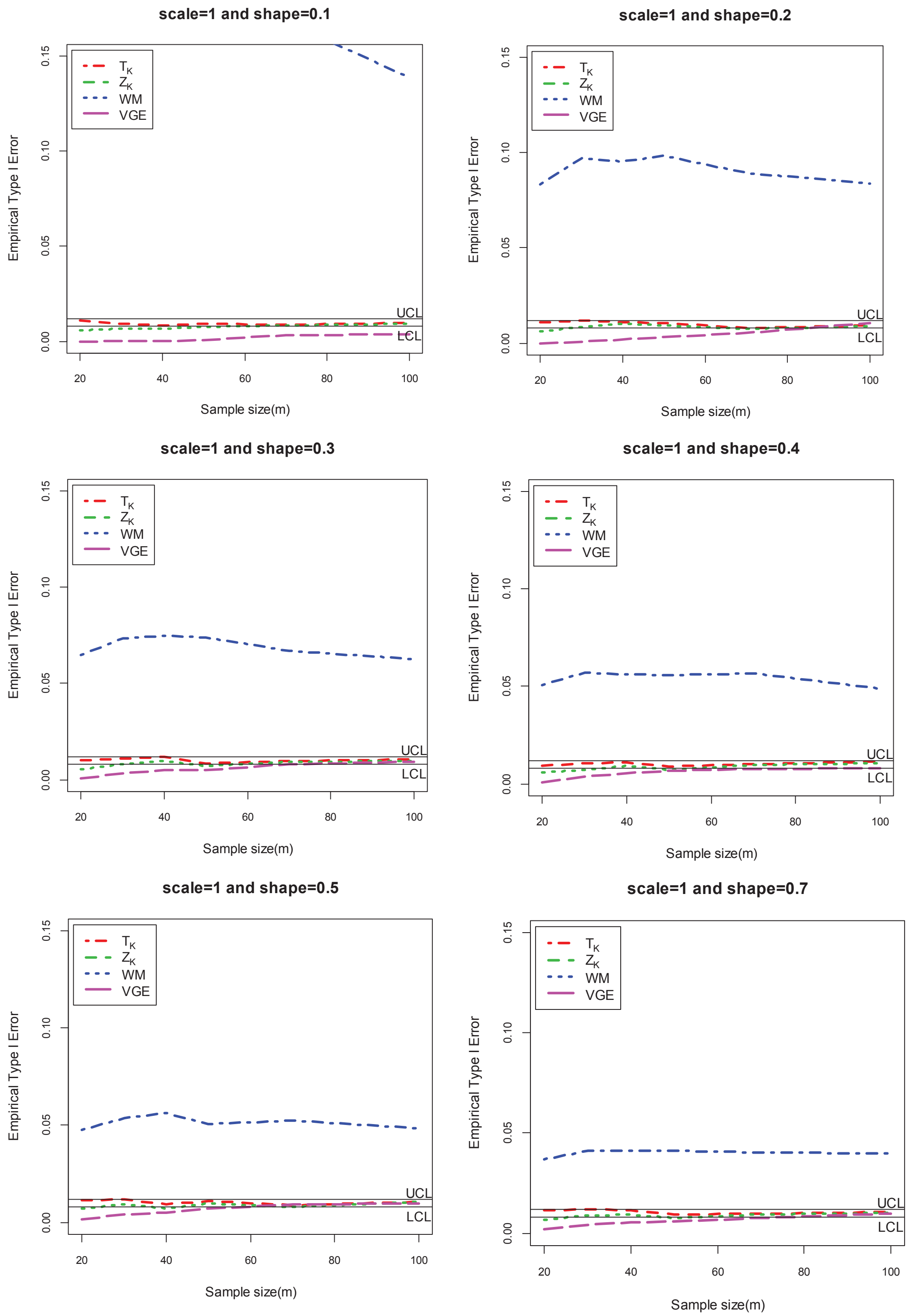

Figure 4.4 Empirical Type I Error Rates at $\alpha = 0.01$ When Scale = 1 and Shape = 0.1, 0.2, 0.3, 0.4, 0.5, and 0.7

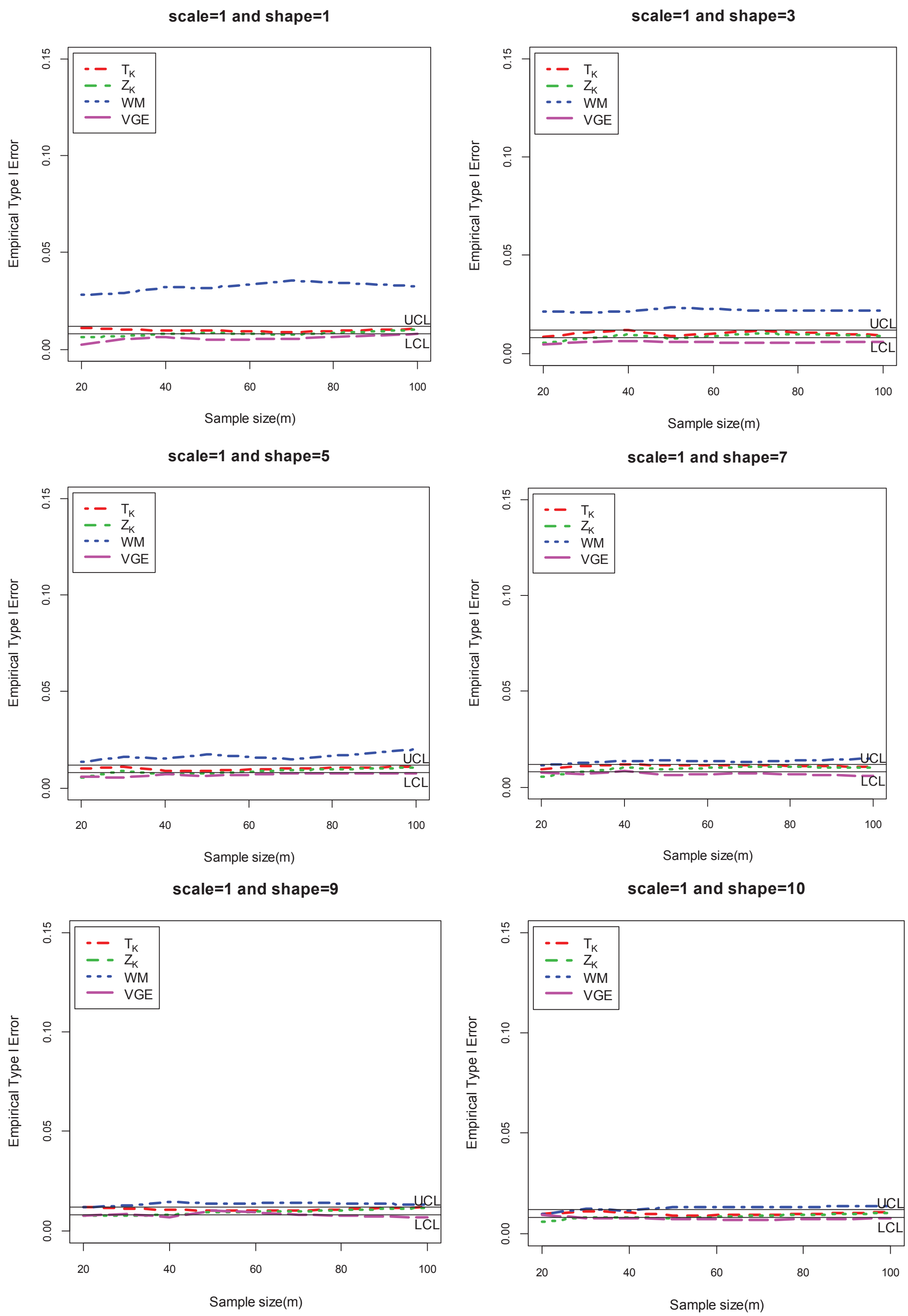

Figure 4.5 Empirical Type I Error Rates at α = 0.01 When Scale = 1 and Shape = 1, 3, 5, 7, 9, and 10

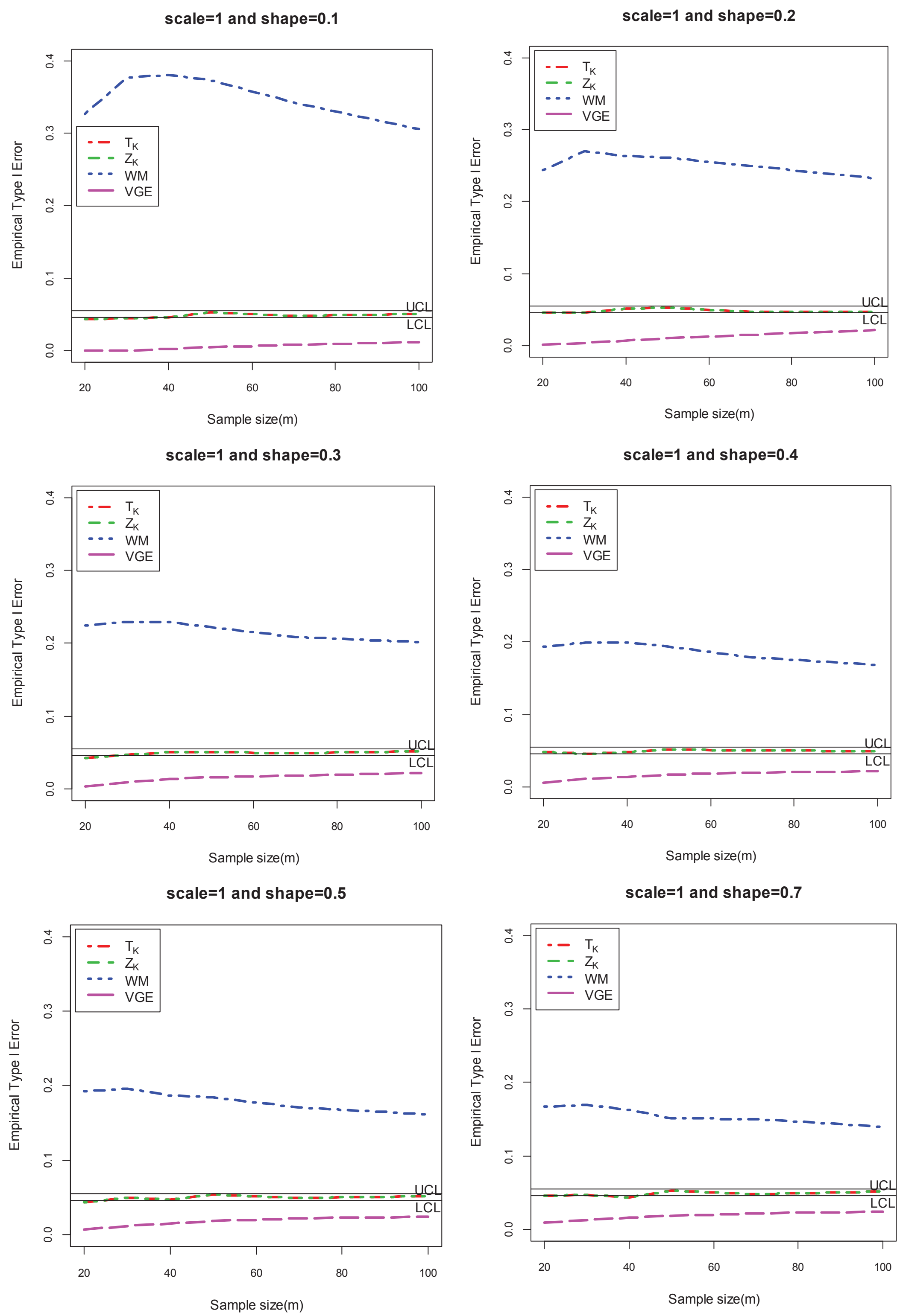

Figure 4.6 Empirical Type I Error Rates at α = 0.05 When Scale = 1 and Shape = 0.1, 0.2, 0.3, 0.4, 0.5, and 0.7

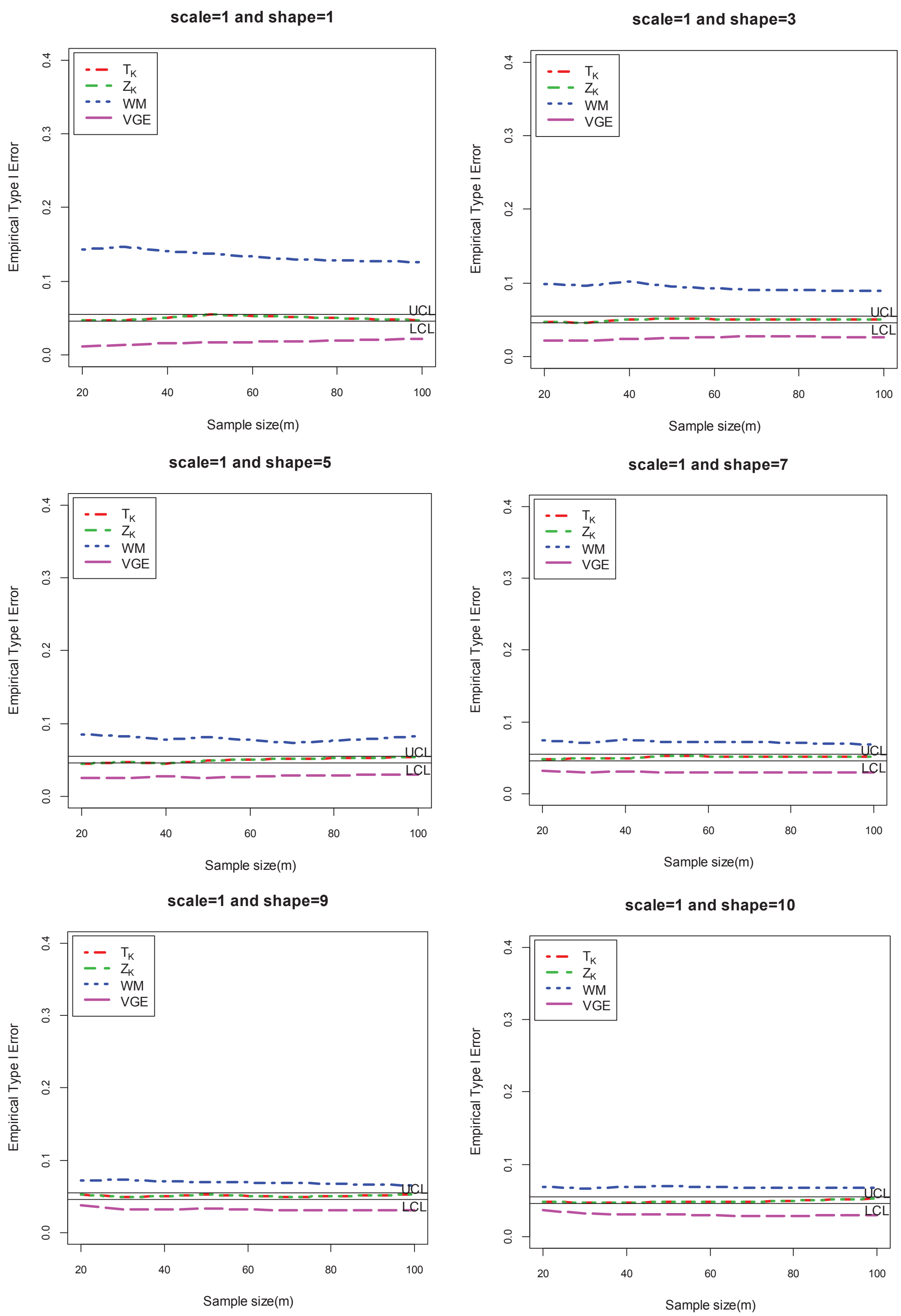

Figure 4.7 Empirical Type I Error Rates at α = 0.05 When Scale = 1 and Shape = 1, 3, 5, 7, 9, and 10

4.4.2 Empirical Powers of the Tests

Naturally, the empirical power of the Z_K and T_K tests increased as the sample size increased for all three alternative distributions: lognormal, Weibull, and Fréchet. The empirical powers of the tests for a Weibull distribution, as can be seen in Figures 4.8 - 4.9, were very low when the shape parameter equaled one for all sample sizes. The reason is that the magnitude of the empirical power depends on setting the distribution to the alternative hypothesis. A Weibull distribution with both shape and scale parameters set to one is a gamma distribution, which results in the null hypothesis with, as expected, the empirical power being very low and tending towards the nominal significance level when the sample size is large. When the sample size was less than 50, the empirical powers for all shape parameters were found to be quite low. The empirical powers of the proposed tests Z_K and T_K tended towards one when the sample size increased, and rapidly inclined towards one when the shape parameter was large.

Under a lognormal distribution, the empirical powers of the proposed tests increased when the sample size and the scale parameter increased, and rapidly tended towards one as the scale parameter increased, similar to the Weibull distribution, as can be seen in Figures 4.10-4.11, whereas, under a Fréchet distribution, the empirical powers of the proposed tests varied inversely with the shape parameter of this distribution, i.e. the lower the shape parameter, the higher the empirical power. It was quite high and close to one when the sample size was larger than 100, as can be seen in Figures 4.12-4.13.

Using the T_K test at the 0.01 significance level for a sample size of less than 50 generated higher empirical powers than the Z_K test, but this relationship was not obvious at the 0.05 significance level, as can be seen in Tables A.3 - A.4 in Appendix A.

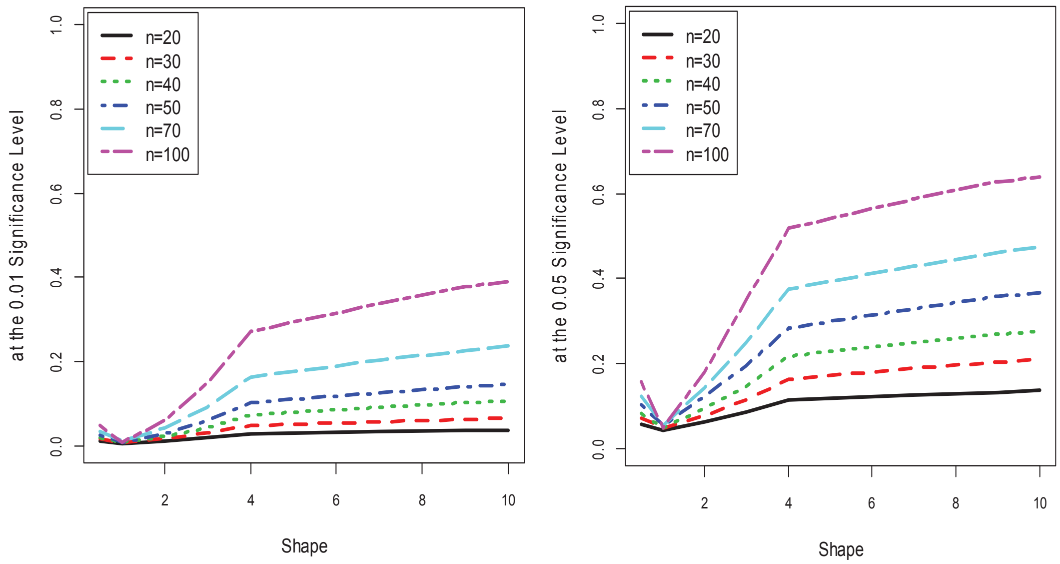

Figure 4.8 Empirical Power of the Z_K Test under a Weibull Distribution

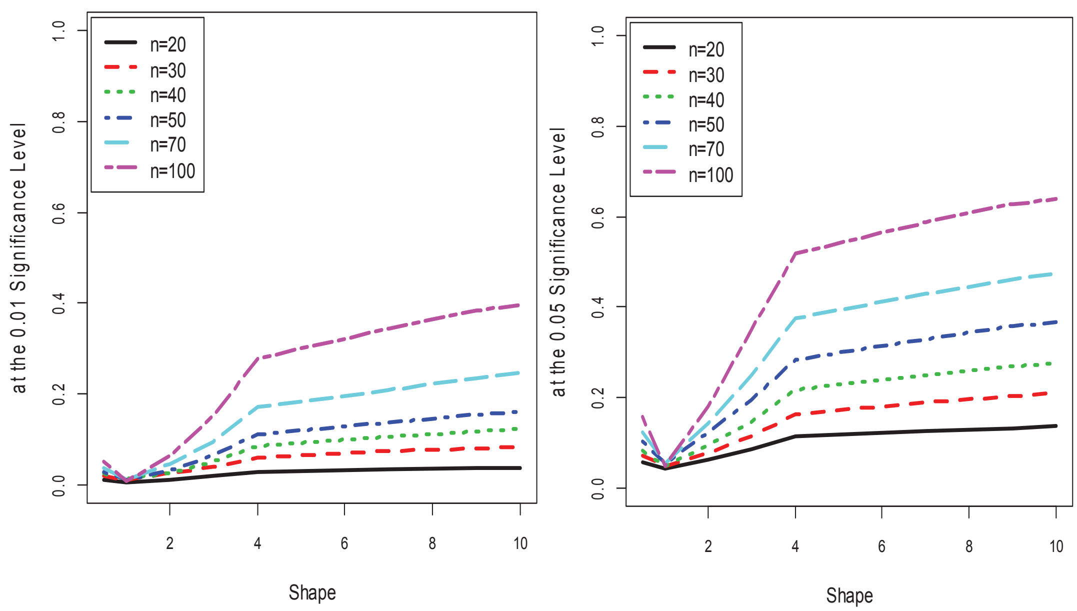

Figure 4.9 Empirical Power of the T_K Test under a Weibull Distribution

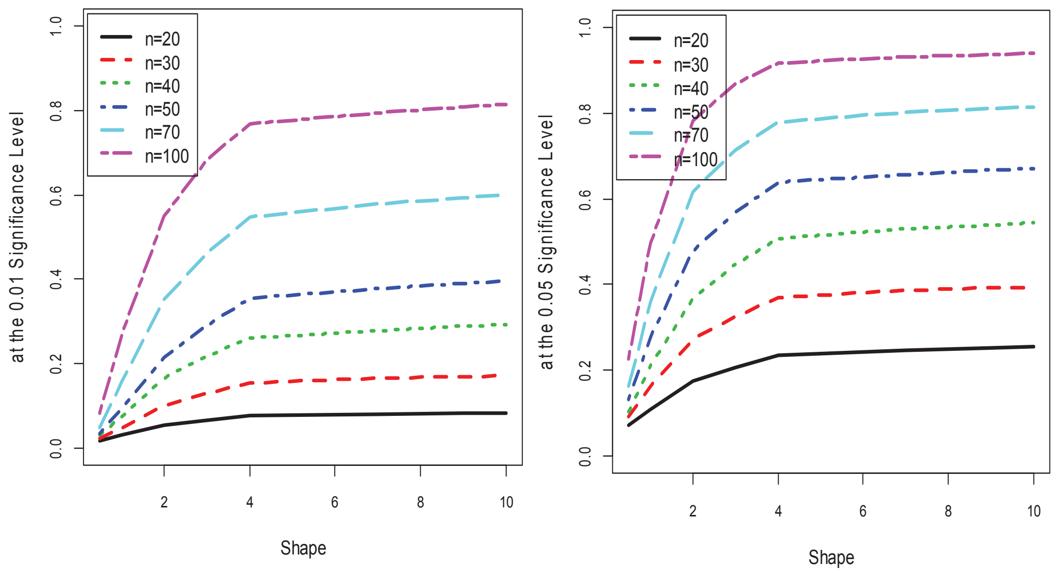

Figure 4.10 Empirical Power of the Z_K Test under a Lognormal Distribution

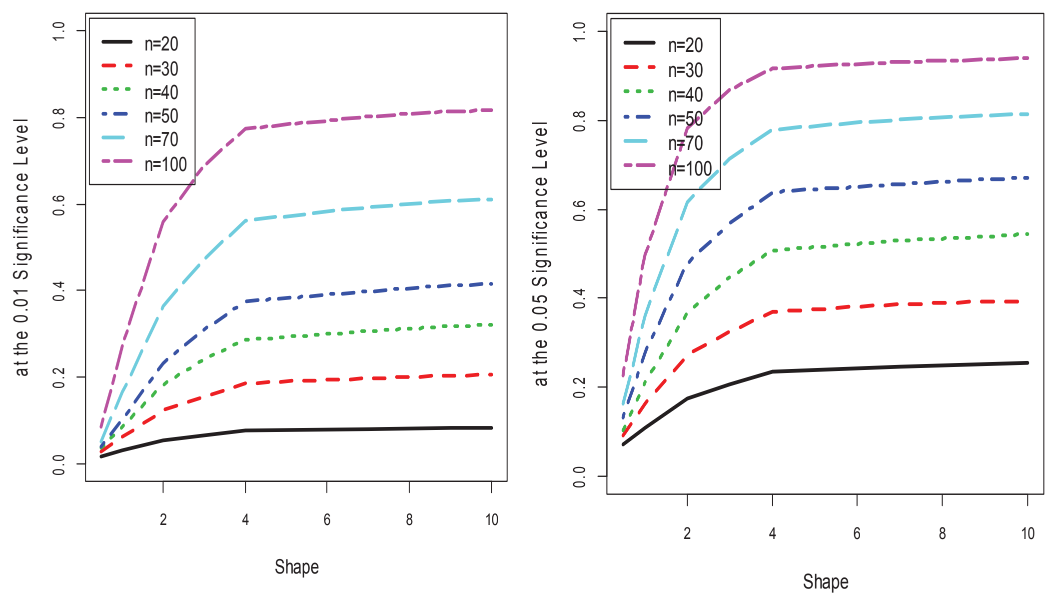

Figure 4.11 Empirical Power of the T_K Test under a Lognormal Distribution

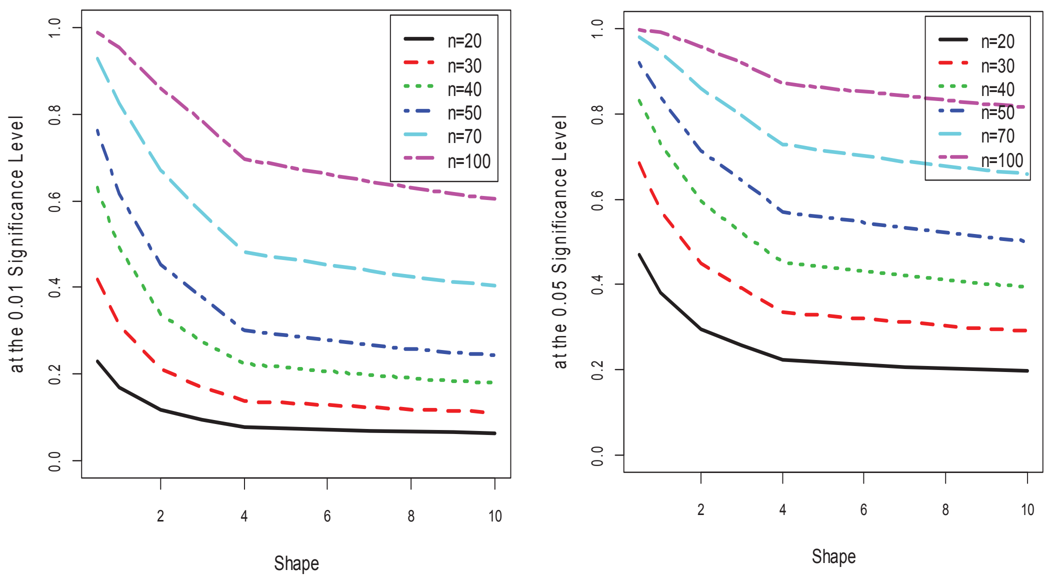

Figure 4.12 Empirical Power of the Z_K Test under a Fréchet Distribution

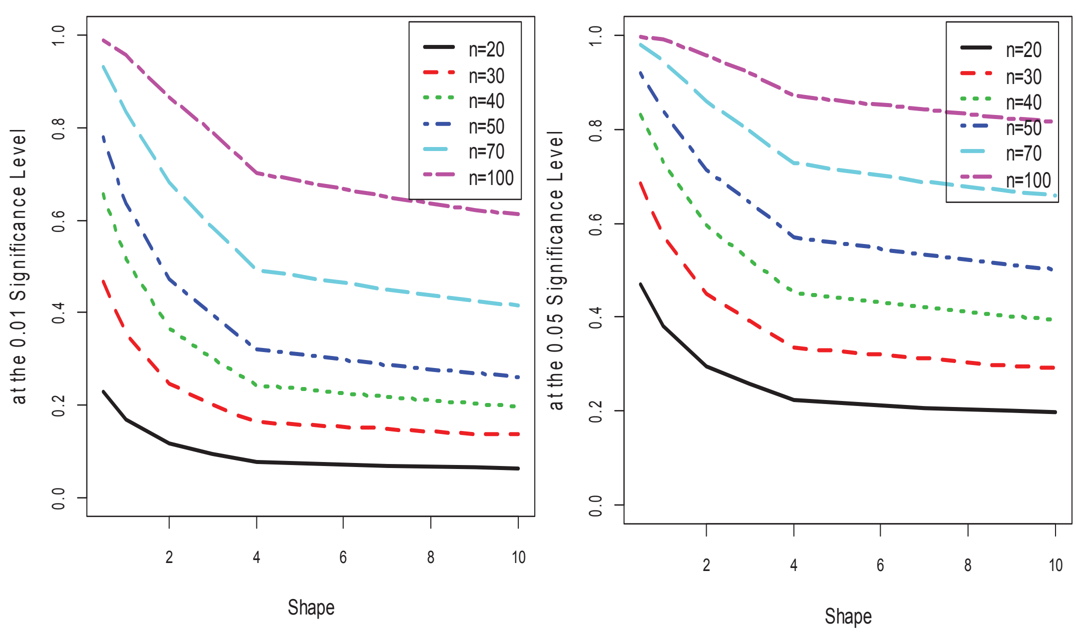

Figure 4.13 Empirical Power of the T_K Test under a Fréchet Distribution

4.5 Illustration of the Proposed Tests on a Real-life Dataset

4.5.1 Honey Bee Transit Time Data

The spread of Africanized honey bees (AHBs) has become problematic since the cross breeding between them and European honey bees (EHBs) first occurred in Brazil in 1956 and has become prominent in this century. They have had a major impact on humans and pets because they are very aggressive. Data from the East-West trap line of the AHB front in Northern Guatemala and in the Atlantic and Pacific coastal areas of Mexico were made available in 1989. These traps consisted of bait hives, each hive being a 25 liter cardboard box baited with a chemical attractant. To determine whether an AHB swarm had been captured, the hives were checked at least once a month (Matis, Rubink and Makela, 1992: 436-440).

The data on the distances (unit: 100 km) and time intervals (unit: months) between 45 consecutive first capture dates along these trap lines were collected. The speeds (distance/time) and transit times were calculated in order to make predictions of future movements. The transit time data are shown in Table 4.2 below:

Table 4.2 Honey Bee Transit Time Data

Transit Time Data (months / 100 km)								
5.3	1.8	4.2	5.7	3.8	0.8	1.4	3.5	17.5
4.6	0.8	6.3	2.9	0.6	1.9	2.0	6.7	5.5
2.5	2.2	6.7	5.7	10.0	3.3	3.5	20.0	1.6
8.3	4.8	20.0	3.6	8.2	1.3	4.0	5.0	1.7
2.0	2.9	19.2	1.1	1.4	1.5	3.2	8.6	2.2

Note that it has been assumed that these data were drawn from a gamma population (Matis, Rubink and Makela, 1992: 436-440). In this study, the proposed test was applied not only to confirm the results above, i.e. that the population is gamma, but also to demonstrate that the test does not depend on selecting pairs of $(X_{i_1}, X_{j_1}),(X_{i_2}, X_{j_2}),(X_{i_3}, X_{j_3}),...,(X_{i_n}, X_{j_n})$. Note that the test was carried out at the 0.05 significance level.

4.5.2 Results of Honey Bee Transit Time Data

Obviously, the distribution of transit time data was not normal, as can be seen in Figure 4.14, and so, according to (Matis, Rubink and Makela, 1992: 436-440), the use of a gamma distribution model is better suited. Because of a limitation of the statistic, which requires an even-numbered sample size, a modification in the testing method had to be introduced whereby 44 units were randomly selected for analysis. In order to confirm the validity of the test results, 45 possible outcomes were presented (i represents the unit that was omitted from analysis, i = 1, 2, 3, …, 45) The test results showing all possible outcomes yielding a p-value > 0.05 are presented in Table A.5 in Appendix A, which shows that all 45 cases of different pair matching resulted in the same conclusion, that the observations were drawn from a population with a gamma distribution, i.e., the null hypothesis was not rejected though the values of Z_K were different.

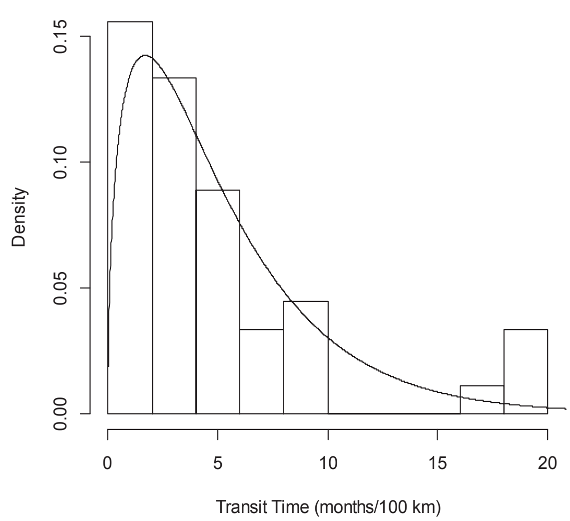

Figure 4.14 Histogram of Honey Bee Transit Time Data with the Fitted Gamma Model

CHAPTER 5

CONCLUSIONS AND DISCUSSION

In this study, test statistics for examining the distribution of data to determine whether they come from a gamma distribution, i.e. the null hypothesis is that the random variable is distributed as gamma, are presented. The proposed asymptotic tests for a gamma distribution are based on a key characteristic, the independence property, using a nonparametric approach.

5.1 Conclusions

The proposed tests Z_K and T_K were constructed based on a key characteristic of a gamma distribution, the independence property. The development of the proposed test Z_K is a modification of Kendall's coefficient, and T_K was developed from Kendall's tau. An Edgeworth expansion and the jackknife method were used to estimate the probability distribution of the T_K statistic. Both Z_K and T_K had the same assumption that the random variables $X_1, X_2, \ldots, X_m$ are i.i.d. of size m drawn from a population having a continuous density function f*(x)* on $\mathbb{R}^+$, the second moment about the origin exists, and $E(X^2) < \infty$. The sample size *m* is an even number equal to or exceeding 8, i.e. the sample size satisfies $m = 2n$, $n = 4, 5, 6, \ldots$.

Lee and Lim (2009: 1-5) proved that $\dfrac{(X_i X_j)}{\left(\sum_{k=1}^{m} X_k\right)^2}$, $1 \le i < j \le m$ and $\sum_{k=1}^{m} X_k$,

$m \ge 1$ are independent if and only if the population is gamma distributed under the condition that $X_1, X_2, \ldots, X_m$ are positive random variables with a common absolute continuous distribution function and $E\left(X^2\right)$ exists. In the case where $m=2$, the proof of Lee and Lim (2009: 1-5) is the reverse proof of the independence property of a gamma distribution, as mentioned previously. Therefore, if it can be determined that data are concordant with the characteristics of a gamma distribution, it can be said that these data follow a gamma distribution. Here, two asymptotic tests are presented: $\mathrm{Z}_K = \dfrac{K}{\left(\dfrac{2(2n+5)}{9n(n-1)}\right)^{1/2}}$ with a standard normal distribution, and T_K using an Edgeworth expansion to improve the limit distribution function of Z_K by taking the 3rd and 4th central moments to approximate a limit distribution function. The 3rd central moment equals zero because of the symmetry of T_K whereas the 4th central moment was estimated by the jackknife method. The limit distribution function of T_K is

$$\hat{\mathrm{F}}^*_{\mathrm{T}_K}(\mathrm{t}_K) = \Phi(\mathrm{t}_K) - \phi(\mathrm{t}_K)\left\{\frac{(\hat{\mu}_4 - 3)}{24n}\left(\mathrm{t}_K^3 - 3\mathrm{t}_K\right)\right\}.$$

The proposed test statistics have the favorable property of invariance under a scalar transformation. Importantly, they performed well even though they are not dependent on the shape and scale parameters of the gamma distribution. One of the main advantages of the new tests is that it is not necessary to estimate the parameter of the gamma distribution, which ensures that the proposed tests do not depend on this when testing for it. In addition, obtaining the proposed test values is much simpler than for other reported methods for testing such data. Because the sample size affected the performance of the tests, both under the null and alternative hypotheses, in practice, it is recommended that the sample size of the data to be tested by Z_K and T_K should be as large as possible, or at least 60 and 40, respectively.

5.2 Discussion

The simulation results showed that the existing KS, AD, and CM tests are more conservative than the proposed tests; they make it more difficult to reject the null hypothesis, which is similar to the findings from previous studies that tested whether a random sample came from a normal population when the parameters were unknown (Linnet, 1988: 180-186).

The proposed test statistics are not only easy to calculate but also one of their noteworthy characteristics is that it is not necessary to estimate the parameter when constructing the test. Consequently, the Type I error rate does not depend on the shape and scale parameters. For most of the tests reviewed (Wilding and Mudholkar, 2008: 3813-3821; Villaseñor and González-Estrada, 2015: 281-286; Baringhaus and Gaigall, 2015: 193-208), their test statistics depend on either the shape or scale parameter, or both. Therefore, the method of parameter estimation can cause considerable problems and have a significant effect on the test statistic. Moreover, there are some issues when selecting the new samples $(X_{i_1}, X_{j_1}),(X_{i_2}, X_{j_2}),$ $(X_{i_3}, X_{j_3}),...,(X_{i_n}, X_{j_n})$ in order to construct a set of ordered pairs $((U_1, V_1),$ $(U_2, V_2),(U_3, V_3),...,(U_n, V_n))$. Randomly selected (X_i, X_j) are suggested from which many values of Z_K can be obtained. However, this is not disadvantageous for the Z_K test since the results of testing with honey dee transit time data (as seen in Table A.5 in Appendix A) were found to be the same.

In addition, the proposed tests could be adapted to use other independent properties of a gamma distribution. When using the independence property between the sample mean and CV (Hwang and Hu 1999: 749-753) to construct a set of ordered pairs (U_j, V_j), we get the same Z_K and T_K, as shown in Appendix C.

BIBLIOGRAPHY

Abdi, Hervé and Williams, Lynne J. 2010. Jackknife. In **Encyclopedia of Research Design**. Salkind, Neil J., ed. Thousand Oaks, CA: Sage. Pp. 656-661.

Amari, Suprasad V. 2012. Reliability of k-out-of-n Standby Systems with Gamma Distributions. **Reliability and Maintainability Symposium**. 7 (January): 1-6.

Anderson, Theodore W. and Darling, Donald A. 1952. Asymptotic Theory of Certain "Goodness-of-Fit" Criteria Based on Stochastic Processes. **The Annals of Mathematical Statistics**. 23 (June): 193-212.

Anderson, Theodore W. and Darling, Donald A. 1954. A Test of Goodness-of-Fit. **Journal of the American Statistical Association**. 49 (December): 765-769.

Baringhaus, Ludwig and Gaigall, Daniel. 2015. On an Independence Test Approach to the Goodness-of-Fit Problem. **Journal of Multivariate Analysis**. 140 (September): 193-208.

Bentkus, Vidmantas; Götze, F. and van Zwet, W. R. 1997. An Edgeworth Expansion for Symmetric Statistics. **The Annals of Statistics**. 25 (April): 851-896.

Birnbaum, Zygmunt W. 1953. Distribution-Free Tests of Fit for Continuous Distribution Functions. **The Annals of Mathematical Statistics**. 24 (March): 1-8.

Darling, Donald A. 1957. The Kolmogorov-Smirnov, Cramer-von Mises Tests. **The Annals of Mathematical Statistics**. 28 (December): 823-838.

Ellis, W. C.; Mahlooji, M.; Lascano, C. E. and Matis, James H. 2005. Effects of Size of Ingestively Masticated Fragments of Plant Tissues on Kinetics of Digestion of NDF. **Journal of Animal Science**. 83 (July): 1602-1615.

Feller William. 1948. On the Kolmogorov-Smirnov Limit Theorems for Empirical Distributions. **The Annals of Mathematical Statistics**. 19 (June): 177-189.

Forbes, Catherine; Evans, Merran; Hastings, Nicholas and Peacock, Brian. 2011 **Statistical Distributions**. 4th ed. Hoboken, NJ: Wiley.

Field, Christopher and Ronchetti, Elvezio. 1990. Small Sample Asymptotics. **Institute of Mathematical Statistics Lecture Notes-Monograph Series**. 13 (July): 1-140.

Ghosh, Kaushik and Jammalamadaka, S. Rao. 1998. Small Sample Approximations for Spacing Statistics. **Journal of Statistical Planning and Inference**. 69 (June): 245-261.

Gibbons, Jean D. and Chakraborti, Subhabrata. 2011. **Nonparametric Statistical Inference**. 5th ed. New York: Chapman and Hall.

Henze, Norbert; Meintanis, Simos G. and Ebner, Bruno. 2012. Goodness-of-Fit Tests for the Gamma Distribution Based on the Empirical Laplace Transform. **Communications in Statistics-Theory and Methods**. 41(9): 1543-1556.

Husak, Gregory J.; Michaelsen, Joel and Funk, Chris. 2007. Use of the Gamma Distribution to Represent Monthly Rainfall in Africa for Drought Monitoring Applications. **International Journal of Climatology**. 27 (June): 935-944.

Hwang, Tea Y. and Hu, Chin Y. 1999. On a Characterization of the Gamma Distribution: The Independence of the Sample Mean and the Sample Coefficient of Variation. **Annals of the Institute of Statistical Mathematics**. 51 (December): 749-753.

Hwang, Tea Y. and Hu, Chin Y. 2000. On some Characterizations of Population Distributions. **Taiwanese Journal of Mathematics**. 4 (September): 427-437.

Kendall, Maurice G. 1938. A New Measure of Rank Correlation. **Biometrika** 30 (1-2): 81-89.

Kendall, Maurice G. 1970. **Rank Correlation Methods**. 4th ed. London: Griffin.

Krever, Maarten; Wunderink, Sophia; Dekker, Rommert and Schorr, Benno. 2005. Inventory Control Based on Advanced Probability Theory, an Application. **European Journal of Operational Research**. 162 (April): 342-358.

Lee, Min Y. and Lim, Eun H. 2007. Characterizations of Gamma Distribution. **Journal of the Chungcheong Mathematical Society**. 20 (December): 411-418.

Lee, Min Y. and Lim, Eun H. 2009. A Characterization of Gamma Distribution by Independent Property. **Journal of the Chungcheong Mathematical Society**. 22 (March): 1-5.

Linnet K. 1988. Testing Normality of Transformed Data. **Applied Statistics**. 37 (3): 180-186.

Lukacs, Eugene. 1955. A Characterization of the Gamma Distribution. **The Annals of Mathematical Statistics**. 26 (June): 319-324.

Matis, James H.; Rubink, W. L. and Makela, M. 1992. Use of the Gamma Distribution for Predicting Arrival Times of Invading Insect Populations. **Environmental Entomology**. 2 (3): 436-440.

Pearson, Karl. 1895. Notes on Regression and Inheritance in the Case of Two Parents. **Proceedings of the Royal Society of London.** 58 (347-352): 240-242.

Quenouille, Maurice H. 1949. Problems in Plane Sampling. **The Annals of Mathematical Statistics**. 20 (September): 355-375.

Raja, T. A. and Mir, A. H. 2011. On Extension of Some Exponentiated Distributions with Application. **International Journal of Contemporary Mathematical Sciences**. 6 (8): 393-400.

Shakil, Mohammad; Kibria, Golam B. M. and Singh, Jai N. 2010. A New Family of Distributions Based on Generalized Pearson Differential Equation with Some Applications. **Austrian Journal of Statistics**. 39 (3): 259-278.

Sharma, Mohita A. and Singh, Jai B. 2010. Use of Probability Distribution in Rainfall Analysis. **New York Science Journal**. 3 (September): 40-49.

Shawky, A. I. and Bakoban R. A. 2009. Modified Goodness-of-Fit Tests for Exponentiated Gamma Distribution with Unknown Shape Parameter. **Interstat**. 3 (July): 1-17.

Smirnov, Nikolai. 1948. Table for Estimating the Goodness of Fit of Empirical Distributions. **The Annals of Mathematical Statistics**. 19 (June): 279-281.

Spearman, Charles. 1904. The Proof and Measurement of Association between Two Things. **American Journal of Psychology**. 15 (January): 72-101.

Stephens, Michael A. 1974. EDF Statistics for Goodness-of-Fit and Some Comparisons. **Journal of the American Statistical Association**. 69 (September): 730-737.

Villaseñor, José A. and González-Estrada, E. 2015. A Variance Ratio Test of Fit for Gamma Distributions. **Statistics and Probability Letters**. 96 (January): 281-286.

Volkova, Tcverans; Longobardi, Antonia and Krasnogorskaya, Nataliya. 2014. Gamma Distribution Function as a Tool for Monthly Precipitation Generation in the Bashkortostan Republic, Russian Federation, In **Latest Trends in Energy, Environment and Development**. Breton, Julia G.; Quartieri, Joseph; Guida, Michele; Guida, Domenico and Guarnaccia, Claudio, eds. Salerno, Italy: WSEAS Press. Pp. 88-95.

Wallace, David L. 1958. Asymptotic Approximations to Distributions. **The Annals of Mathematical Statistics**. 29 (September): 635-654.

Wilding, Gregory E. and Mudholkar, Govind S. 2008. A Gamma Goodness-of-Fit Test Based on Characteristic Independence of the Mean and the Coefficient of Variation. **Journal of Statistical Planning and Inference**. 138 (December): 3813-3821.

Woodruff, BrianW.; Viviano, PhilipJ.; Moore, Albert H. and Dunne, Edward J. 1984. Modified Goodness-of-Fit Tests for Gamma Distributions with Unknown Location and Scale Parameters. **IEEE Transactions on Reliability**. 33 (August): 241-245.

APPENDICES

APPENDIX A

TABLES A.1-A.4

APPENDIX A

TABLES A.1-A.4

Table A.1 Empirical Type I Error Rates under the Null Hypothesis at Nominal Significance Level 0.01

Shape Parameter	Sample Size (m)	Comparative Test					Proposed Test	
		KS	AD	CM	WM	VGE	Z_K	T_K
0.1	20	0.0002*	0.0000*	0.0000*	0.1747*	0.0000*	0.0059	0.0112
	30	0.0001*	0.0000*	0.0000*	0.2030*	0.0001*	0.0067	0.0094
	40	0.0001*	0.0000*	0.0000*	0.2064*	0.0002*	0.0067	0.0086
	50	0.0001*	0.0000*	0.0000*	0.1969*	0.0008*	0.0078	0.0092
	70	0.0005*	0.0000*	0.0000*	0.1678*	0.0031*	0.0084	0.0089
	100	0.0001*	0.0000*	0.0000*	0.1390*	0.0038*	0.0092	0.0096
0.2	20	0.0000*	0.0000*	0.0000*	0.0834*	0.0000*	0.0063*	0.0112
	30	0.0000*	0.0000*	0.0000*	0.0972*	0.0006*	0.0084	0.0123
	40	0.0000*	0.0000*	0.0000*	0.0953*	0.0022*	0.0102	0.0112
	50	0.0001*	0.0000*	0.0000*	0.0984*	0.0035*	0.0091	0.0106
	70	0.0000*	0.0000*	0.0000*	0.0892*	0.0056*	0.0077	0.0081
	100	0.0001*	0.0000*	0.0000*	0.0837*	0.0106	0.0087	0.0093
0.3	20	0.0000*	0.0000*	0.0000*	0.0649*	0.0005*	0.0054*	0.0100
	30	0.0001*	0.0000*	0.0000*	0.0736*	0.0033*	0.0081	0.0111
	40	0.0001*	0.0000*	0.0000*	0.0748*	0.0050*	0.0097	0.0119
	50	0.0001*	0.0000*	0.0000*	0.0737*	0.0052*	0.0073*	0.0084
	70	0.0002*	0.0000*	0.0000*	0.0670*	0.0082	0.0093	0.0099
	100	0.0001*	0.0000*	0.0000*	0.0628*	0.0092	0.0098	0.0104
0.4	20	0.0000*	0.0000*	0.0000*	0.0506*	0.0009*	0.0057*	0.0094
	30	0.0000*	0.0000*	0.0000*	0.0573*	0.0036*	0.0073*	0.0104
	40	0.0000*	0.0000*	0.0000*	0.0563*	0.0054*	0.0091	0.0111
	50	0.0000*	0.0000*	0.0000*	0.0558*	0.0067*	0.0073*	0.0090
	70	0.0001*	0.0000*	0.0000*	0.0565*	0.0075*	0.0097	0.0103
	100	0.0001*	0.0000*	0.0000*	0.0491*	0.0080*	0.0105	0.0114

Table A.1 (Continued)

Shape Parameter	Sample Size (m)	Comparative Test KS	AD	CM	WM	VGE	Proposed Test Z_K	T_K
0.5	20	0.0000*	0.0000*	0.0000*	0.0475*	0.0016*	0.0071*	0.0116
	30	0.0000*	0.0000*	0.0000*	0.0535*	0.0040*	0.0093	0.0119
	40	0.0001*	0.0000*	0.0000*	0.0563*	0.0051*	0.0070*	0.0091
	50	0.0000*	0.0000*	0.0000*	0.0505*	0.0073*	0.0098	0.0112
	70	0.0000*	0.0000*	0.0000*	0.0522*	0.0094	0.0082	0.0089
	100	0.0000*	0.0000*	0.0000*	0.0484*	0.0099	0.0104	0.0107
0.7	20	0.0000*	0.0000*	0.0000*	0.0369*	0.0022*	0.0067*	0.0116
	30	0.0000*	0.0000*	0.0000*	0.0410*	0.0042*	0.0087	0.0120
	40	0.0000*	0.0000*	0.0000*	0.0410*	0.0055*	0.0092	0.0115
	50	0.0001*	0.0000*	0.0000*	0.0412*	0.0059*	0.0078*	0.0093
	70	0.0000*	0.0000*	0.0000*	0.0401*	0.0075*	0.0091	0.0098
	100	0.0000*	0.0000*	0.0000*	0.0397*	0.0097	0.0103	0.0106
1	20	0.0000*	0.0000*	0.0000*	0.0284*	0.0025*	0.0061*	0.0112
	30	0.0000*	0.0000*	0.0000*	0.0292*	0.0055*	0.0067*	0.0101
	40	0.0000*	0.0000*	0.0000*	0.0321*	0.0061*	0.0082	0.0099
	50	0.0000*	0.0000*	0.0000*	0.0316*	0.0052*	0.0086	0.0099
	70	0.0000*	0.0000*	0.0000*	0.0354*	0.0054*	0.0074*	0.0087
	100	0.0000*	0.0000*	0.0000*	0.0327*	0.0082	0.0102	0.0105
2	20	0.0000*	0.0000*	0.0000*	0.0220*	0.0055*	0.0057*	0.0095
	30	0.0000*	0.0000*	0.0000*	0.0226*	0.0062*	0.0082	0.0120
	40	0.0000*	0.0000*	0.0000*	0.0222*	0.0072*	0.0102	0.0119
	50	0.0000*	0.0000*	0.0000*	0.0240*	0.0069*	0.0089	0.0097
	70	0.0000*	0.0000*	0.0000*	0.0238*	0.0077*	0.0102	0.0111
	100	0.0000*	0.0000*	0.0000*	0.0214*	0.0064*	0.0096	0.0102
3	20	0.0002*	0.0000*	0.0000*	0.0214*	0.0047*	0.0055*	0.0083
	30	0.0002*	0.0000*	0.0000*	0.0208*	0.0057*	0.0078*	0.0107
	40	0.0001*	0.0000*	0.0000*	0.0214*	0.0065*	0.0098	0.0119
	50	0.0001*	0.0000*	0.0000*	0.0235*	0.0060*	0.0078*	0.0089
	70	0.0005*	0.0000*	0.0000*	0.0217*	0.0055*	0.0103	0.0114
	100	0.0003*	0.0000*	0.0000*	0.0219*	0.0059*	0.0090	0.0091
4	20	0.0001*	0.0000*	0.0000*	0.0148*	0.0057*	0.0063*	0.0099
	30	0.0005*	0.0000*	0.0000*	0.0163*	0.0053*	0.0091	0.0110
	40	0.0005*	0.0000*	0.0000*	0.0195*	0.0060*	0.0086	0.0112
	50	0.0003*	0.0000*	0.0000*	0.0190*	0.0051*	0.0084	0.0097
	70	0.0001*	0.0000*	0.0000*	0.0193*	0.0060*	0.0085	0.0092
	100	0.0000*	0.0000*	0.0000*	0.0160*	0.0054*	0.0094	0.0098

Table A.1 (Continued)

Shape Parameter	Sample Size (m)	Comparative Test KS	AD	CM	WM	VGE	Proposed Test Z_K	T_K
5	20	0.0001*	0.0001*	0.0001*	0.0135*	0.0059*	0.0054*	0.0100
	30	0.0001*	0.0000*	0.0000*	0.0160*	0.0056*	0.0088	0.0110
	40	0.0000*	0.0000*	0.0000*	0.0155*	0.0071*	0.0070*	0.0088
	50	0.0002*	0.0000*	0.0000*	0.0173*	0.0062*	0.0075*	0.0087
	70	0.0003*	0.0000*	0.0000*	0.0151*	0.0075*	0.0094	0.0102
	100	0.0001*	0.0000*	0.0000*	0.0200*	0.0078*	0.0107	0.0112
6	20	0.0001*	0.0000*	0.0000*	0.0134*	0.0067*	0.0047*	0.0095
	30	0.0003*	0.0000*	0.0000*	0.0139*	0.0068*	0.0064*	0.0090
	40	0.0000*	0.0000*	0.0000*	0.0152*	0.0066*	0.0076*	0.0100
	50	0.0001*	0.0000*	0.0000*	0.0154*	0.0066*	0.0091	0.0100
	70	0.0001*	0.0000*	0.0000*	0.0166*	0.0077*	0.0095	0.0100
	100	0.0003*	0.0000*	0.0000*	0.0195*	0.0072*	0.0090	0.0096
7	20	0.0001*	0.0000*	0.0000*	0.0116	0.0076*	0.0054*	0.0093
	30	0.0002*	0.0000*	0.0000*	0.0126*	0.0067*	0.0079*	0.0109
	40	0.0000*	0.0000*	0.0000*	0.0138*	0.0084*	0.0102	0.0120
	50	0.0002*	0.0000*	0.0000*	0.0140*	0.0063*	0.0095	0.0114
	70	0.0002*	0.0001*	0.0001*	0.0133*	0.0071*	0.0108	0.0115
	100	0.0003*	0.0000*	0.0000*	0.0151*	0.0060*	0.0101	0.0108
8	20	0.0001*	0.0000*	0.0000*	0.0117	0.0068*	0.0043*	0.0081
	30	0.0002*	0.0000*	0.0000*	0.0152*	0.0077*	0.0086	0.0117
	40	0.0001*	0.0000*	0.0000*	0.0128*	0.0077*	0.0081	0.0097
	50	0.0000*	0.0000*	0.0000*	0.0127*	0.0080*	0.0078*	0.0095
	70	0.0004*	0.0000*	0.0000*	0.0142*	0.0063*	0.0103	0.0108
	100	0.0001*	0.0000*	0.0000*	0.0148*	0.0084	0.0098	0.0102
9	20	0.0002*	0.0000*	0.0000*	0.0119	0.0076*	0.0076*	0.0117
	30	0.0003*	0.0000*	0.0000*	0.0126*	0.0084*	0.0077*	0.0110
	40	0.0003*	0.0000*	0.0000*	0.0144*	0.0068*	0.0082	0.0106
	50	0.0001*	0.0000*	0.0000*	0.0137*	0.0101*	0.0091	0.0102
	70	0.0001*	0.0000*	0.0000*	0.0141*	0.0081	0.0098	0.0103
	100	0.0001*	0.0000*	0.0000*	0.0131*	0.0068*	0.0113	0.0117
10	20	0.0005*	0.0000*	0.0000*	0.0098	0.0092*	0.0057*	0.0098
	30	0.0002*	0.0000*	0.0000*	0.0125*	0.0075*	0.0079*	0.0111
	40	0.0002*	0.0000*	0.0000*	0.0114	0.0075*	0.0080	0.0104
	50	0.0000*	0.0000*	0.0000*	0.0134*	0.0073*	0.0076*	0.0089
	70	0.0000*	0.0000*	0.0000*	0.0133*	0.0067*	0.0087	0.0094
	100	0.0001*	0.0000*	0.0000*	0.0135*	0.0075*	0.0101	0.0108

Note: * means significantly different from H_0:α=0.01 at significance level 0.05

Table A.2 Empirical Type I Error Rates under the Null Hypothesis at Nominal Significance Level 0.05

Shape Parameter	Sample Size (m)	Comparative Test					Proposed Test	
		KS	AD	CM	WM	VGE	Z_K	T_K
0.1	20	0.0026*	0.0004*	0.0006*	0.3265*	0.0000*	0.0427*	0.0427*
	30	0.0035*	0.0004*	0.0005*	0.3769*	0.0001*	0.0443*	0.0443*
	40	0.0028*	0.0002*	0.0003*	0.3807*	0.0019*	0.0455*	0.0455*
	50	0.0026*	0.0003*	0.0004*	0.3740*	0.0038*	0.0520	0.0520
	70	0.0033*	0.0006*	0.0008*	0.3430*	0.0071*	0.0477	0.0477
	100	0.0019*	0.0004*	0.0006*	0.3059*	0.0114*	0.0504	0.0504
0.2	20	0.0011*	0.0001*	0.0003*	0.2437*	0.0006*	0.0457	0.0457
	30	0.0011*	0.0000*	0.0001*	0.2702*	0.0036*	0.0453*	0.0453*
	40	0.0017*	0.0000*	0.0004*	0.2632*	0.0070*	0.0514	0.0514
	50	0.0020*	0.0002*	0.0005*	0.2608*	0.0094*	0.0524	0.0524
	70	0.0010*	0.0002*	0.0002*	0.2496*	0.0147*	0.0462	0.0462
	100	0.0012*	0.0001*	0.0002*	0.2328*	0.0215*	0.0471	0.0471
0.3	20	0.0008*	0.0001*	0.0001*	0.2243*	0.0031*	0.0417*	0.0417*
	30	0.0014*	0.0002*	0.0002*	0.2294*	0.0088*	0.0468	0.0468
	40	0.0017*	0.0001*	0.0002*	0.2286*	0.0132*	0.0506	0.0506
	50	0.0013*	0.0002*	0.0001*	0.2219*	0.0152*	0.0504	0.0504
	70	0.0018*	0.0002*	0.0002*	0.2090*	0.0181*	0.0495	0.0495
	100	0.0013*	0.0002*	0.0002*	0.2012*	0.0219*	0.0509	0.0509
0.4	20	0.0007*	0.0000*	0.0000*	0.1938*	0.0051*	0.0479	0.0479
	30	0.0010*	0.0000*	0.0000*	0.1993*	0.0113*	0.0456*	0.0456*
	40	0.0009*	0.0000*	0.0000*	0.1989*	0.0134*	0.0483	0.0483
	50	0.0008*	0.0000*	0.0000*	0.1933*	0.0171*	0.0511	0.0511
	70	0.0005*	0.0000*	0.0000*	0.1783*	0.0193*	0.0498	0.0498
	100	0.0005*	0.0000*	0.0001*	0.1682*	0.0217*	0.0487	0.0487
0.5	20	0.0005*	0.0000*	0.0001*	0.1924*	0.0069*	0.0435*	0.0435*
	30	0.0005*	0.0000*	0.0000*	0.1963*	0.0111*	0.0493	0.0493
	40	0.0008*	0.0000*	0.0001*	0.1869*	0.0141*	0.0464	0.0464
	50	0.0007*	0.0000*	0.0000*	0.1845*	0.0180*	0.0532	0.0532
	70	0.0008*	0.0001*	0.0001*	0.171*	0.0212*	0.0495	0.0495
	100	0.0004*	0.0001*	0.0002*	0.1613*	0.0238*	0.0514	0.0514
0.7	20	0.0003*	0.0000*	0.0000*	0.1666*	0.0085*	0.0454*	0.0454*
	30	0.0008*	0.0000*	0.0001*	0.1689*	0.0120*	0.0463	0.0463
	40	0.0005*	0.0000*	0.0001*	0.1630*	0.0154*	0.0435*	0.0435*
	50	0.0005*	0.0000*	0.0001*	0.1508*	0.0183*	0.0519	0.0519
	70	0.0005*	0.0000*	0.0000*	0.1495*	0.0213*	0.0477	0.0477
	100	0.0004*	0.0000*	0.0000*	0.1399*	0.0233*	0.0517	0.0517

Table A.2 (Continued)

Shape Parameter	Sample Size (m)	KS	AD	CM	WM	VGE	Z_K	T_K
		Comparative Test					Proposed Test	
1	20	0.0002*	0.0000*	0.0000*	0.1431*	0.0106*	0.0461	0.0461
	30	0.0001*	0.0000*	0.0000*	0.1460*	0.0138*	0.0467	0.0467
	40	0.0003*	0.0000*	0.0000*	0.1402*	0.0159*	0.0506	0.0506
	50	0.0002*	0.0000*	0.0000*	0.1368*	0.0164*	0.0542	0.0542
	70	0.0003*	0.0000*	0.0000*	0.1292*	0.0183*	0.0513	0.0513
	100	0.0005*	0.0000*	0.0000*	0.1259*	0.0213*	0.0468	0.0468
2	20	0.0001*	0.0000*	0.0000*	0.1156*	0.0175*	0.0475	0.0475
	30	0.0001*	0.0000*	0.0000*	0.1097*	0.0214*	0.0471	0.0471
	40	0.0001*	0.0000*	0.0000*	0.1045*	0.0213*	0.0441*	0.0441*
	50	0.0001*	0.0000*	0.0000*	0.1046*	0.0220*	0.0508	0.0508
	70	0.0003*	0.0000*	0.0000*	0.0981*	0.0242*	0.0487	0.0487
	100	0.0003*	0.0000*	0.0000*	0.0944*	0.0236*	0.0498	0.0498
3	20	0.0002*	0.0000*	0.0000*	0.0987*	0.0214*	0.0465	0.0465
	30	0.0002*	0.0000*	0.0000*	0.0958*	0.0217*	0.0454*	0.0454*
	40	0.0001*	0.0000*	0.0000*	0.1016*	0.0241*	0.0497	0.0497
	50	0.0001*	0.0000*	0.0000*	0.0951*	0.0250*	0.0515	0.0515
	70	0.0005*	0.0000*	0.0000*	0.0902*	0.0266*	0.0503	0.0503
	100	0.0003*	0.0000*	0.0000*	0.089*	0.0261*	0.0506	0.0506
4	20	0.0001*	0.0000*	0.0000*	0.0888*	0.0200*	0.0460	0.0460
	30	0.0005*	0.0000*	0.0000*	0.0861*	0.0199*	0.0450*	0.0450*
	40	0.0005*	0.0000*	0.0000*	0.0880*	0.0235*	0.0460	0.0460
	50	0.0003*	0.0000*	0.0000*	0.0876*	0.0216*	0.0500	0.0500
	70	0.0001*	0.0000*	0.0000*	0.0830*	0.0235*	0.0502	0.0502
	100	0.0000*	0.0000*	0.0000*	0.0839*	0.0235*	0.0521	0.0521
5	20	0.0001*	0.0001*	0.0001*	0.0840*	0.0253*	0.0439*	0.0439*
	30	0.0001*	0.0000*	0.0000*	0.0820*	0.0254*	0.0464	0.0464
	40	0.0000*	0.0000*	0.0000*	0.078*	0.0268*	0.0446*	0.0446*
	50	0.0002*	0.0000*	0.0000*	0.0811*	0.0254*	0.0492	0.0492
	70	0.0003*	0.0000*	0.0000*	0.0734*	0.0287*	0.0515	0.0515
	100	0.0001*	0.0000*	0.0000*	0.0822*	0.0289*	0.0533	0.0533
6	20	0.0001*	0.0000*	0.0000*	0.0820*	0.0254*	0.0456*	0.0456*
	30	0.0003*	0.0000*	0.0000*	0.0774*	0.0275*	0.0474	0.0474
	40	0.0000*	0.0000*	0.0000*	0.0756*	0.0270*	0.0461	0.0461
	50	0.0001*	0.0000*	0.0000*	0.0786*	0.0269*	0.0535	0.0535
	70	0.0001*	0.0000*	0.0000*	0.0753*	0.0290*	0.0499	0.0499
	100	0.0003*	0.0000*	0.0000*	0.0747*	0.0283*	0.0514	0.0514

Table A.2 (Continued)

Shape Parameter	Sample Size (m)	Comparative Test					Proposed Test	
		KS	AD	CM	WM	VGE	Z_K	T_K
7	20	0.0001*	0.0000*	0.0000*	0.0742*	0.0315*	0.0477	0.0477
	30	0.0002*	0.0000*	0.0000*	0.0710*	0.0289*	0.0486	0.0486
	40	0.0000*	0.0000*	0.0000*	0.0755*	0.0308*	0.0492	0.0492
	50	0.0002*	0.0000*	0.0000*	0.0720*	0.0289*	0.0525	0.0525
	70	0.0002*	0.0001*	0.0001*	0.0714*	0.0291*	0.0516	0.0516
	100	0.0003*	0.0000*	0.0000*	0.0689*	0.0297*	0.0513	0.0513
8	20	0.0001*	0.0000*	0.0000*	0.0700*	0.0346*	0.0471	0.0471
	30	0.0002*	0.0000*	0.0000*	0.0694*	0.0297*	0.0445*	0.0445*
	40	0.0001*	0.0000*	0.0000*	0.0695*	0.0318*	0.0474	0.0474
	50	0.0000*	0.0000*	0.0000*	0.0678*	0.0285*	0.0506	0.0506
	70	0.0004*	0.0000*	0.0000*	0.0663*	0.0289*	0.0503	0.0503
	100	0.0001*	0.0000*	0.0000*	0.0678*	0.0294*	0.0536	0.0536
9	20	0.0002*	0.0000*	0.0000*	0.0717*	0.0370*	0.0520	0.0520
	30	0.0003*	0.0000*	0.0000*	0.0732*	0.0318*	0.0490	0.0490
	40	0.0003*	0.0000*	0.0000*	0.0708*	0.0315*	0.0496	0.0496
	50	0.0001*	0.0000*	0.0000*	0.0701*	0.0332*	0.0521	0.0521
	70	0.0001*	0.0000*	0.0000*	0.0689*	0.0311*	0.0494	0.0494
	100	0.0001*	0.0000*	0.0000*	0.0651*	0.0311*	0.0529	0.0529
10	20	0.0005*	0.0000*	0.0000*	0.0681*	0.0366*	0.0481	0.0481
	30	0.0002*	0.0000*	0.0000*	0.0656*	0.0316*	0.0461	0.0461
	40	0.0002*	0.0000*	0.0000*	0.0690*	0.0301*	0.0462	0.0462
	50	0.0000*	0.0000*	0.0000*	0.0698*	0.0307*	0.0483	0.0483
	70	0.0000*	0.0000*	0.0000*	0.0668*	0.0281*	0.0478	0.0478
	100	0.0001*	0.0000*	0.0000*	0.0673*	0.0296*	0.0524	0.0524

Note: * means significantly different from $H_0:\alpha=0.05$ at significance level 0.05

Table A.3 Empirical Powers at Nominal Significance Level 0.01

Shape Parameter	Sample Size (m)	Fréchet		Weibull	
		Z_K	T_K	Z_K	T_K
0.5	20	0.2284	0.2284	0.0111	0.0111
	30	0.4183**	0.4653**	0.0148*	0.0188*
	40	0.6299**	0.6558**	0.0201	0.0237
	50	0.7637*	0.7788*	0.0244	0.0278
	70	0.9273	0.9313	0.0343	0.0367
	100	0.9897	0.9899	0.0488	0.0494
1	20	0.1695	0.1695	0.0061	0.0061
	30	0.3106**	0.3552**	0.0089	0.0115
	40	0.4923**	0.5169**	0.0090*	0.0126*
	50	0.6157**	0.6363**	0.0087	0.0105
	70	0.8256	0.8339	0.0099	0.0103
	100	0.9552	0.9570	0.0085	0.0087
2	20	0.1171	0.1171	0.0114	0.011
	30	0.2106**	0.2447**	0.0166**	0.0235**
	40	0.3372**	0.3654**	0.0208*	0.0258*
	50	0.4521**	0.4715**	0.0276	0.0313
	70	0.6702	0.6816	0.0414	0.0445
	100	0.8602	0.8646	0.0596	0.0612
3	20	0.0950	0.0950	0.0181	0.0181
	30	0.1686**	0.2003**	0.0298**	0.0391**
	40	0.2753**	0.3008**	0.0427**	0.0513**
	50	0.3772*	0.3939*	0.0584	0.0641
	70	0.5741	0.5854	0.0907	0.0951
	100	0.7851	0.7915	0.1487	0.1518
5	20	0.0763	0.0763	0.0281	0.0281
	30	0.1370**	0.1626**	0.0469**	0.0598**
	40	0.2217**	0.2433**	0.0740**	0.0849**
	50	0.3012**	0.3207**	0.1031*	0.1121*
	70	0.4813	0.4935	0.1617	0.1698
	100	0.6961	0.7019	0.2716	0.2776
7	20	0.0687	0.0687	0.0328	0.0328
	30	0.1228**	0.1473**	0.0554**	0.0726**
	40	0.1961**	0.2171**	0.0913**	0.1048**
	50	0.2658**	0.2854**	0.1263*	0.1381*
	70	0.4374	0.4486	0.2013	0.2095
	100	0.6457	0.6517	0.3374	0.3434

Scale Parameter	Sample Size (m)	Lognormal	
		Z_K	T_K
0.5	20	0.0166	0.0166
	30	0.0206**	0.0273**
	40	0.0305*	0.0364*
	50	0.0328	0.0378
	70	0.0487	0.0514
	100	0.0811	0.0842
1	20	0.0303	0.0303
	30	0.0482**	0.0609**
	40	0.0736**	0.0839**
	50	0.0947	0.1019
	70	0.1576	0.1657
	100	0.2692	0.2746
2	20	0.0521	0.0521
	30	0.0993**	0.1211**
	40	0.1643**	0.1835**
	50	0.2141**	0.2307**
	70	0.3519	0.3647
	100	0.5500	0.5577
3	20	0.0655	0.0655
	30	0.1274**	0.1543**
	40	0.2164**	0.2399**
	50	0.2880**	0.3079**
	70	0.4600	0.4735
	100	0.6819	0.6874
5	20	0.0762	0.0762
	30	0.1525**	0.1844**
	40	0.2601**	0.2870**
	50	0.3545**	0.3749**
	70	0.5473*	0.5612*
	100	0.7677	0.7726
7	20	0.0798	0.0798
	30	0.1640**	0.1969**
	40	0.2784**	0.3061**
	50	0.3773**	0.3989**
	70	0.5795	0.5918
	100	0.7952	0.8013

Table A.3 (Continued)

Shape Parameter	Sample Size (m)	Fréchet Z_K	Fréchet T_K	Weibull Z_K	Weibull T_K
9	20	0.0648	0.0648	0.0359	0.0359
	30	0.1150**	0.1381**	0.0618**	0.0791**
	40	0.1827**	0.2020**	0.1024**	0.1168**
	50	0.2496**	0.2673**	0.1406**	0.1543**
	70	0.4119	0.4234	0.2263	0.2352
	100	0.6159	0.6227	0.3766	0.3826
10	20	0.0631	0.0631	0.0370	0.0370
	30	0.1122**	0.1360**	0.0641**	0.0822**
	40	0.1789**	0.1972**	0.1061**	0.1211**
	50	0.2440**	0.2602**	0.1460**	0.1606**
	70	0.4026	0.4143	0.2368	0.2451
	100	0.6055	0.6118	0.3894	0.3950

Scale Parameter	Sample Size (m)	Lognormal Z_K	Lognormal T_K
9	20	0.0817	0.0817
	30	0.1688**	0.2032**
	40	0.2876**	0.3164**
	50	0.3906**	0.4119**
	70	0.5937	0.6059
	100	0.8090	0.8144
10	20	0.0822	0.0822
	30	0.1710**	0.2057**
	40	0.2904**	0.3196**
	50	0.3936**	0.4152**
	70	0.5989	0.6097
	100	0.8132	0.8179

Note: * means significantly different from H_0: $\beta_6 = \beta_7$ at significance level 0.05

** means significantly different from H_0: $\beta_6 = \beta_7$ at significance level 0.01

Table A.4 Empirical Powers at Nominal Significance Level 0.05

Shape Parameter	Sample Size (m)	Fréchet Z_K	Fréchet T_K	Weibull Z_K	Weibull T_K
0.5	20	0.4707	0.4707	0.0571	0.0571
	30	0.6857	0.6857	0.0713	0.0713
	40	0.8310	0.8310	0.0834	0.0834
	50	0.9197	0.9197	0.1021	0.1021
	70	0.9805	0.9805	0.1221	0.1221
	100	0.9984	0.9984	0.1564	0.1564
1	20	0.3815	0.3815	0.0430	0.0430
	30	0.5740	0.5740	0.0472	0.0472
	40	0.7317	0.7317	0.0470	0.0470
	50	0.8411	0.8411	0.0536	0.0536
	70	0.9456	0.9456	0.0527	0.0527
	100	0.9905	0.9905	0.0482	0.0482
2	20	0.2952	0.2952	0.0614	0.0614
	30	0.4494	0.4494	0.0772	0.0772
	40	0.5971	0.5971	0.0928	0.0928
	50	0.7137	0.7137	0.1210	0.1210
	70	0.8586	0.8586	0.1430	0.1430
	100	0.9570	0.9570	0.1806	0.1806
3	20	0.2561	0.2561	0.0839	0.0839
	30	0.3909	0.3909	0.1146	0.1146
	40	0.5209	0.5209	0.1464	0.1464
	50	0.6442	0.6442	0.1951	0.1951
	70	0.7969	0.7969	0.2478	0.2478
	100	0.9191	0.9191	0.3488	0.3488
5	20	0.2220	0.2220	0.1132	0.1132
	30	0.3356	0.3356	0.1630	0.1630
	40	0.4521	0.4521	0.2161	0.2161
	50	0.5707	0.5707	0.2822	0.2822
	70	0.7287	0.7287	0.3736	0.3736
	100	0.8718	0.8718	0.5183	0.5183
7	20	0.2070	0.2070	0.1254	0.1254
	30	0.3104	0.3104	0.1874	0.1874
	40	0.4201	0.4201	0.2487	0.2487
	50	0.5315	0.5315	0.3302	0.3302
	70	0.6886	0.6886	0.4285	0.4285
	100	0.8431	0.8431	0.5865	0.5865

Scale Parameter	Sample Size (m)	Lognormal Z_K	Lognormal T_K
0.5	20	0.0712	0.0712
	30	0.0898	0.0898
	40	0.1036	0.1036
	50	0.1303	0.1303
	70	0.1627	0.1627
	100	0.2248	0.2248
1	20	0.1094	0.1094
	30	0.1630	0.1630
	40	0.2113	0.2113
	50	0.2768	0.2768
	70	0.3609	0.3609
	100	0.4988	0.4988
2	20	0.1739	0.1739
	30	0.2727	0.2727
	40	0.3672	0.3672
	50	0.4779	0.4779
	70	0.6171	0.6171
	100	0.783	0.783
3	20	0.2043	0.2043
	30	0.3243	0.3243
	40	0.4464	0.4464
	50	0.5676	0.5676
	70	0.7149	0.7149
	100	0.8692	0.8692
5	20	0.2338	0.2338
	30	0.3687	0.3687
	40	0.5075	0.5075
	50	0.6384	0.6384
	70	0.779	0.779
	100	0.9177	0.9177
7	20	0.2469	0.2469
	30	0.385	0.385
	40	0.5301	0.5301
	50	0.6573	0.6573
	70	0.8019	0.8019
	100	0.9319	0.9319

Table A.4 (Continued)

Shape Parameter	Sample Size (m)	Fréchet Z_K	Fréchet T_K	Weibull Z_K	Weibull T_K	Scale Parameter	Sample Size (m)	Lognormal Z_K	Lognormal T_K
9	20	0.1993	0.1993	0.1322	0.1322	9	20	0.2515	0.2515
	30	0.2948	0.2948	0.2030	0.2030		30	0.3911	0.3911
	40	0.4019	0.4019	0.2680	0.2680		40	0.5396	0.5396
	50	0.5086	0.5086	0.3584	0.3584		50	0.6678	0.6678
	70	0.6665	0.6665	0.4622	0.4622		70	0.812	0.812
	100	0.8236	0.8236	0.6260	0.6260		100	0.9374	0.9374
10	20	0.1966	0.1966	0.1355	0.1355	10	20	0.2535	0.2535
	30	0.2903	0.2903	0.2078	0.2078		30	0.3934	0.3934
	40	0.3936	0.3936	0.2749	0.2749		40	0.5434	0.5434
	50	0.5002	0.5002	0.3666	0.3666		50	0.6706	0.6706
	70	0.6579	0.6579	0.4739	0.4739		70	0.8145	0.8145
	100	0.8166	0.8166	0.6383	0.6383		100	0.9389	0.9389

Note: * means significantly different from H_0: $\beta_6 = \beta_7$ at significance level 0.05

** means significantly different from H_0: $\beta_6 = \beta_7$ at significance level 0.01

Table A.5 Test Statistic and P-Value for Honey Bee Transit Time Data under the Null Hypothesis.

Sample i	z_K	p-value	Sample i	z_K	P-value	Sample i	z_K	p-value
1	-1.216	0.224	16	0.646	0.518	31	-1.254	0.210
2	-0.912	0.362	17	0.228	0.820	32	-1.292	0.197
3	-1.216	0.224	18	0.380	0.704	33	-1.368	0.171
4	-1.595	0.111	19	0.912	0.362	34	-1.557	0.119
5	-1.292	0.197	20	0.912	0.362	35	-1.785	0.074
6	-1.557	0.119	21	0.760	0.447	36	-1.557	0.119
7	-0.722	0.470	22	0.684	0.494	37	-1.481	0.138
8	-0.798	0.425	23	0.076	0.939	38	-1.785	0.074
9	-0.114	0.909	24	-0.266	0.790	39	-1.406	0.160
10	0.152	0.879	25	-0.190	0.849	40	-1.937	0.053
11	0.722	0.470	26	-1.481	0.138	41	-1.633	0.102
12	1.026	0.305	27	-1.064	0.287	42	-1.671	0.095
13	1.026	0.305	28	-1.140	0.254	43	-1.785	0.074
14	0.570	0.569	29	-1.444	0.149	44	-2.013	0.044
15	0.646	0.518	30	-0.798	0.425	45	-1.102	0.271

APPENDIX B

COMPARISON OF THE ACCURACY OF DETERMINING THE ASYMPTOTIC DISTRIBUTION BY NORMAL AND EDGEWORTH APPROXIMATION

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COMPARISON OF THE ACCURACY OF DETERMINING THE ASYMPTOTIC DISTRIBUTION BY NORMAL AND EDGEWORTH APPROXIMATION

The accuracy of an appropriate estimator for the fourth central moment of T_K through of an Edgeworth approximation is considered here. The fourth central moment of T_K, estimated by using the jackknife method, is used in an Edgeworth approximation to improve the proposed test statistic Z_K to be asymptotic normal when the sample size is insufficiently large. Therefore, the efficiency of Z_K and T_K to approximate an exact distribution K are considered through a comparison of relative error (RE): $RE = \dfrac{\left|F(x) - \hat{F}(x)\right|}{\min\left(\left(1 - F(x)\right), F(x)\right)}$, and absolute error (AE): $AE = \left|F(x) - \hat{F}(x)\right|$; *RE* and *AE* are used to measure the discrepancy between the exact distribution of *K* and the approximated distributions of Z_K and T_K.

The accuracy of approximating a distribution for the proposed tests was measured by mean relative error (MRE) and mean absolute error (MAE) in a simulation study. The discrepancy between the exact distribution and the distributions of Z_K and T_K were compared by their MRE and MAE with the number of iterations set at 10,000.

$$MRE_{\mathrm{Z}_K} = \sum_{i=1}^{10,000} \frac{\left|F_i(K) - F_i(\mathrm{z}_K)\right|}{\min\left(\left(1 - F_i(K)\right), F_i(K)\right)} \Big/ 10,000,$$

$$MRE_{\mathrm{T}_K} = \sum_{i=1}^{10,000} \frac{\left|F_i(K) - \hat{F}_i^*(\mathrm{t}_K)\right|}{\min\left(\left(1 - F_i(K)\right), F_i(K)\right)} \Big/ 10,000,$$

$$MAE_{\mathrm{T}_K} = \frac{\sum_{i=1}^{10,000} \left|F_i(K) - \hat{F}_i^*(\mathrm{t}_K)\right|}{10,000},$$

$$MAE_{\mathrm{Z}_K} = \frac{\sum_{i=1}^{10,000} \left|F_i(K) - F_i(\mathrm{z}_K)\right|}{10,000} .$$

Table B.1 The MRE of Z_K under the Null Hypothesis

Shape Parameter	Sample Size							
	10	**20**	**30**	**40**	**50**	**70**	**80**	**90**
0.5	0.24724	0.12137	0.07298	0.04840	0.03573	0.02299	0.01898	0.01614
1	0.24727	0.12110	0.07230	0.04827	0.03613	0.02289	0.01896	0.01632
2	0.24656	0.12234	0.07348	0.04804	0.03647	0.02301	0.01913	0.01641
3	0.24683	0.12246	0.07185	0.04909	0.03653	0.02299	0.01914	0.01628
5	0.24867	0.12114	0.07218	0.04816	0.03587	0.02265	0.01897	0.01626
7	0.24871	0.12158	0.07188	0.04856	0.03574	0.02304	0.01904	0.01638
9	0.24702	0.12231	0.07193	0.04780	0.03580	0.02309	0.01900	0.01617
10	0.24625	0.12236	0.07153	0.04868	0.03577	0.02279	0.01894	0.01638

Table B.2 The MRE of T_K under the Null Hypothesis

Shape Parameter	Sample Size							
	10	20	30	40	50	70	80	90
0.5	0.23597	0.11150	0.06507	0.04194	0.03054	0.01904	0.01545	0.01289
1	0.23579	0.11152	0.06478	0.04195	0.03128	0.01890	0.01544	0.01298
2	0.23410	0.11170	0.06604	0.04188	0.03126	0.01897	0.01536	0.01302
3	0.23600	0.11255	0.06428	0.04281	0.03099	0.01892	0.01553	0.01291
5	0.23770	0.11079	0.06432	0.04154	0.03051	0.01877	0.01539	0.01293
7	0.23621	0.11139	0.06391	0.04241	0.03076	0.01903	0.01533	0.01303
9	0.23501	0.11258	0.06438	0.04171	0.03046	0.01893	0.01547	0.01291
10	0.23428	0.11257	0.06411	0.04250	0.03060	0.01896	0.01533	0.01299

Table B.3 The MAE of Z_K under the Null Hypothesis

Shape Parameter	Sample Size							
	10	20	30	40	50	70	80	90
0.5	0.05899	0.02840	0.01598	0.01031	0.00758	0.00483	0.00396	0.00337
1	0.06011	0.02837	0.01590	0.01031	0.00757	0.00483	0.00396	0.00338
2	0.05937	0.02843	0.01594	0.01027	0.00753	0.00483	0.00398	0.00340
3	0.05941	0.02835	0.01592	0.01022	0.00753	0.00484	0.00396	0.00338
5	0.05963	0.02840	0.01602	0.01034	0.00758	0.00487	0.00398	0.00336
7	0.05948	0.02838	0.01596	0.01027	0.00755	0.00483	0.00400	0.00338
9	0.05962	0.02842	0.01595	0.01029	0.00763	0.00485	0.00397	0.00337
10	0.05925	0.02833	0.01590	0.01028	0.00765	0.00484	0.00397	0.00337

Table B.4 The MAE of T_K under the Null Hypothesis

Shape Parameter	Sample Size							
	10	20	30	40	50	70	80	90
0.5	0.05320	0.02450	0.01335	0.00832	0.00599	0.00369	0.00296	0.00249
1	0.05418	0.02451	0.01327	0.00831	0.00598	0.00369	0.00297	0.00249
2	0.05353	0.02451	0.01332	0.00829	0.00595	0.00369	0.00297	0.00250
3	0.05354	0.02446	0.01331	0.00823	0.00593	0.00370	0.00296	0.00249
5	0.05375	0.02450	0.01336	0.00835	0.00598	0.00373	0.00298	0.00248
7	0.05366	0.02447	0.01331	0.00830	0.00597	0.00369	0.00299	0.00248
9	0.05376	0.02451	0.01331	0.00832	0.00603	0.00370	0.00296	0.00249
10	0.05340	0.02445	0.01328	0.00829	0.00605	0.00371	0.00297	0.00248

Table B.5 The Difference between the MRE of Z_K and T_K under the Null Hypothesis:

$$Diff_{MRE} = MRE_{Z_K} - MRE_{T_K}$$

Shape Parameter	Sample Size							
	10	20	30	40	50	70	80	90
0.5	0.01127	0.00987	0.00791	0.00646	0.00519	0.00396	0.00353	0.00325
1	0.01148	0.00958	0.00753	0.00632	0.00485	0.00399	0.00352	0.00334
2	0.01247	0.01065	0.00744	0.00617	0.00521	0.00404	0.00377	0.00339
3	0.01083	0.00991	0.00757	0.00628	0.00554	0.00407	0.00361	0.00337
5	0.01097	0.01036	0.00786	0.00662	0.00536	0.00387	0.00358	0.00333
7	0.01249	0.01019	0.00797	0.00615	0.00498	0.00401	0.00371	0.00335
9	0.01200	0.00973	0.00755	0.00609	0.00534	0.00416	0.00353	0.00327
10	0.01196	0.00979	0.00742	0.00617	0.00517	0.00384	0.00361	0.00339

Table B.6 The Difference between the MAE of Z_K and T_K under the Null Hypothesis: $Diff_{MAE} = MAE_{Z_K} - MAE_{T_K}$

Shape Parameter	Sample Size							
	10	**20**	**30**	**40**	**50**	**70**	**80**	**90**
0.5	0.00579	0.00390	0.00263	0.00199	0.00159	0.00114	0.00100	0.00088
1	0.00593	0.00385	0.00263	0.00200	0.00159	0.00114	0.00100	0.00089
2	0.00584	0.00392	0.00262	0.00198	0.00158	0.00114	0.00100	0.00090
3	0.00587	0.00389	0.00262	0.00199	0.00160	0.00115	0.00100	0.00089
5	0.00588	0.00390	0.00266	0.00199	0.00160	0.00114	0.00100	0.00089
7	0.00582	0.00391	0.00264	0.00197	0.00159	0.00114	0.00101	0.00090
9	0.00585	0.00391	0.00264	0.00197	0.00160	0.00115	0.00100	0.00088
10	0.00585	0.00388	0.00262	0.00199	0.00160	0.00113	0.00100	0.00089

From the simulation results in Tables B.1 to B.6, it can be seen that, in the case of a small sample size, the distribution function of T_K was closer to the exact distribution K than Z_K, and so we can say that $\hat{\mu}_4$ is a good estimator. As a result, the Edgeworth approximation was shown to improve the asymptotic normal distribution of Z_K.

APPENDIX C

FOR TESTING THE NULL HYPOTHESIS

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H_0 : A random variable is distributed as a gamma distribution
against H_1 : A random variable is not distributed as a gamma distribution,
the proposed tests and modifications of the independence characteristic of a gamma distribution, such as those presented by Lee and Lim (2009) or Hwang and Hu (1999), are based on the same conclusion.

Lee and Lim theorem (2009)

If $X_1, X_2, \ldots, X_m$ is positively i.i.d. with a common absolutely continuous distribution function $F(x)$ and $E(X^2) < \infty$, then $\sum_{k=1}^{m} X_k$ and $(X_i X_j)\left(\sum_{k=1}^{m} X_k\right)^{-2}$, for $1 \le i < j \le m$, $m \ge 1$ if and only if $X_1, X_2, \ldots, X_m$ are distributed as gamma.

Hwang and Hu theorem (1999)

If $X_1, X_2, \ldots, X_m$ is positively i.i.d. with a common absolutely continuous distribution function $F(x)$ and $E(X^2) < \infty$, then the independence of mean $\bar{X}$ and coefficient of variation CV is assured if and only if $X_1, X_2, \ldots, X_m$ are distributed as gamma.

When the sample size of each random samples equals 2, (*m=2*) for the j^{th} sample, $j = 1, 2, \ldots, n$, $X_{j_1}, X_{j_2} > 0$,

let $\quad V_j = X_{j1} + X_{j2}$,

$$W_j = \frac{X_{j1} + X_{j2}}{2},$$

$$V_j = X_{j1} + X_{j2} = 2W_j,$$

$$\therefore\ V_j \propto W_j \quad . \tag{C.1}$$

Let $U_j = \dfrac{X_{j1}X_{j2}}{\left(X_{j1} + X_{j2}\right)^2}$, and

$$D_j = \frac{SD_j}{\bar{X}_j} = \sqrt{X_{j1}^2 + X_{j2}^2 - \frac{\left(X_{j1} + X_{j2}\right)^2}{2}}\left(\frac{2}{X_{j1} + X_{j2}}\right) = \frac{\left|X_{j1} - X_{j2}\right|\sqrt{2}}{X_{j1} + X_{j2}}$$

$$\begin{aligned} D_j^2 &= \frac{2\left(X_{j1}^2 - 2X_{j1}X_{j2} + X_{j2}^2\right)}{\left(X_{j1} + X_{j2}\right)^2} \\ &= \frac{2\left(X_{j1}^2 + X_{j2}^2\right)}{\left(X_{j1} + X_{j2}\right)^2} - \frac{4X_{j1}X_{j2}}{\left(X_{j1} + X_{j2}\right)^2} \\ &= \frac{2\left(X_{j1}^2 + X_{j2}^2\right)}{\left(X_{j1} + X_{j2}\right)^2} - 4U_j \end{aligned}$$

$$\therefore\ D_j^2 \propto \frac{1}{U_j} \rightarrow D_j \propto \frac{1}{U_j} \quad . \tag{C.2}$$

Let $K = \dfrac{2}{n(n-1)} \sum_{1 \le i <}^{n} \sum_{j \le n}^{n} A_{ij}$, (C.3)

where $A_{ij} = \operatorname{sgn}\left(U_j - U_i\right)\operatorname{sgn}\left(V_j - V_i\right)$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Let $K_b = \dfrac{2}{n(n-1)} \sum_{1 \le i <}^{n} \sum_{j \le n}^{n} B_{ij}$, (C.4)

where $B_{ij} = \operatorname{sgn}\left(D_j - D_i\right)\operatorname{sgn}\left(W_j - W_i\right)$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

From (C.1) and (C.2), $K = -K_b$.

Test statistics Z_K and T_K are functions of K modified by the gamma independence characteristic (Lee and Lim, 2009). By using K_b instead of K in the test statistics Z_K and T_K, K_b is modified by the independence characteristic of a gamma distribution (Hwang and Hu, 1999), and so $Z_K = -Z_{K_b}$ and $T_K = -T_{K_b}$. However, the p-value for testing the null hypothesis are equal because the distribution of Z_K and T_K are symmetrical about zero.

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Panarat Plubin and Puttipong Bookkamana. 2003. The Application of Hierarchical Logistic Regression Technique in Attitude of Self-Donated Cadavera Survey. In **Integrating Technology in The Mathematical Sciences, 14-15 April 2003.** Penang, Malaysia.

Kunan Bangphoomi; Panarat Plubin and Puttipong Bookkamana. Udder Health Status in Dairy Farm. In **Thailand. Regional Conference on Ecological and Environmental Modeling (ECOMOD 2004), 15-16 September 2004**. Kamphangsean, Nakhon-Pathom.

Narongchai Sathavornvichit; Puttipong Bookkamana and Panarat Plubin. 2006. Central Composite Design in Optimization of the Factors of Automatic Flux Cored Arc Welding for Steel ST37. In **The 2th IMT-GT 2006 Regional Conference on Mathematics, Statistics and Applications, 13-15 June 2006**. Penang, Malaysia.

Panarat Plubin and Noparat Techapunratanakul. 2006. Ordinal Regression Analysis in Factors Related to Sensorial Hearing Loss of the Employees in Industrial Factory. In **The 2th IMT-GT 2006 Regional Conference on Mathematics, Statistics and Applications, 13-15 June 2006**. Lampang, Thailand.

Kamol Sanittham; Panarat Plubin and Puttipong Bookkamana. 2008. Correspondence Analysis in Back Pain Symptom of the Employees in the Industries. In **International Conference on Business and Information, 7-9 July 2008**. Lampang, Thailand.

Puttipong Bookkamana; Kamol Sanittham and Panarat Plubin. The Study of Employees' Health Care in Para Wood-Product Factories. In **International Conference on Business and Information, 7-9 July 2008**. Lampang, Thailand.