CLAIMS MODELING WITH AN ALTERNATIVE GAMMA-EXPONENTIATED WEIBULL DISTRIBUTION AND RUIN PROBABILITY APPROXIMATION

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ABSTRACT

Title of Dissertation Claims Modeling with an Alternative Gamma-Exponentiated Weibull Distribution and Ruin Probability Approximation

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In this dissertation, two studies that are beneficial for actuaries and the insurance business are proposed. In the first study, an exponentiated Weibull distribution using gamma-generated distribution is modified to obtain an alternative gamma-exponentiated Weibull (AGEW) distribution; its sub-models include both gamma and Weibull distributions, both of which are popular in claims modeling by insurance companies. Its basic structural properties such as distribution function, density function, and moments were investigated. Moreover, the maximum likelihood method to estimate the AGEW distribution's parameters was utilized, then the distribution was applied to a real-life dataset to show its superiority over gamma and Weibull distributions by comparing fitness between them.

In the second study, a new approximation method to obtain the ruin probability referring to the risk of insolvency of an insurance company is proposed by modifying the Pollaczek-Khinchin approximation. The proposed approximation is simpler and requires fewer assumptions than other methods mentioned in the literature. The results from a simulation study show that, in some cases, the proposed method gave better approximated ruin probability values in terms of the overall deviation from the exact values. Insurance companies are interested in calculating the

initial capital using ruin probability, and so with this in mind, the proposed method was applied to estimate the minimum initial capital that needs to be reserved to ensure that the ruin probability does not exceed an acceptable quantity. To illustrate the performance of the approximation, the ruin probability and the minimum initial capital modeled by the AGEW distribution were estimated with a real-life dataset.

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CHAPTER 1

INTRODUCTION

1.1 Background

Since the foundation of modern risk theory was suggested by Filip Lundberg in 1903, actuaries have attracted much attention in the insurance business. Actuarial science became a formal discipline involving mathematical and statistical methods to analyze insurance claims data. In this study, we concern about the important tasks of the actuary such as the claims modeling and, the approximation of ruin probability and minimum initial capital.

The claims modeling is an important procedure that leads to the pricing of premium and risk analysis for an insurance company. Sasithorn Anantasopon, Pairote Sattayatham and Tosaporn Talangtam (2015) suggested that the distribution of claim amounts should be modeled as a mixed distribution of non-negative continuous random variables and proposed the infinite mixture distribution that can be applied to non-life insurance claims data. Thus, attempts have been made to modify the exponentiated Weibull (EW) distribution, which is a non-negative continuous and flexible distribution. Several authors have constructed alternative distributions using the EW distribution as a baseline. Mahmoudi and Sepahdar (2013) introduced an exponentiated Weibull Poisson distribution, Pinho, Cordeiro and Nobre (2012) use a gamma-generated distribution to develop a gamma-exponentiated Weibull distribution, and a beta-exponentiated Weibull distribution was proposed by Percontini, Blas and Cordeiro (2013). The first purpose of this study is to construct the new mixed distribution using EW as a baseline to achieve a better fit with real-life claims data than classic distributions usually used to model the claim amounts distribution.

The ruin probability refers to the risk that the monetary surplus of an insurance company becomes less than zero, which leads to insolvency. Recent studies have presented many methods to obtain the ruin probability. Cizek, Härdle and Weron (2005) concluded that simple analytic results for the ruin probability using the classical model exist when the claim amounts distribution is exponential or close to it, but for other claim amounts distributions, they are not easy to obtain. Several authors have proposed the approximation method for ruin probability. De Vylder (1978) suggested the approximation of ruin probability based on the idea of replacing the surplus process with an exponential claim amounts distribution such that the first three moments coincide; in this case, the approximation gives the exact result. However, for other claim amounts distributions, this approximation has the condition that the first three moments of the claim amounts distribution must exist. Using Monte Carlo simulation, Asmussen and Binswanger (1997) proposed computer approximations that are independent of the first three moments and can be chosen as the reference method for calculating the ruin probability in infinite time. For some claim amounts distributions, simulation of the algorithm for computing the approximate ruin probability by this method is complicated, and so in this study, an algorithm simpler than computing the approximate ruin probability is proposed.

Generally, an insurance company must reserve the initial capital for managing the ruin probability to an acceptable level, i.e. the insurance company is unlikely to become insolvent. Thus, the proposed approximate ruin probability was also applied to estimate the minimum of initial capital that an insurance company should hold in reserve. To demonstrate the proposed approximation, it was applied to a real-life sample of motor insurance claims data from one company.

1.2 Objectives of the Study

The objectives of this study are as follows:

1) To introduce a new claim amounts distribution as a combination of several non-negative continuous distributions.

2) To propose an approximation method to obtain the probability that the cash surplus of an insurance company is less than zero for a given initial capital, and to find minimum initial capital for an insurance company that makes it unlikely to face insolvency.

3) To demonstrate the proposed approximation using real-life data modeled by the proposed new mixed distribution.

1.3 Scope of the Study

The individual claim amounts, the ruin probability, and the initial capital of insurance company are considered under the following conditions:

1) The effect of interest and reinsurance are not considered.

2) The claim amounts that response difference policy are independent.

3) The study is within the framework of the classic continuous-time surplus model.

4) The insurance company has to reserve the initial capital for managing the ruin probability in infinite time not greater than the given quantity.

1.4 Usefulness of the Study

The benefits of this study are as follows:

1) The new mixed distribution will achieve a better fit with claims data than some classic distributions.

2) The proposed approximation method can help insurance companies to estimate the ruin probability that they face, and that the reserve of initial capital for managing the ruin probability is not greater than the given quantity.

CHAPTER 2

LITERATURE REVIEW

In this chapter, a review of the literature on the following topics is presented. Detail of claims modeling and distributions that are often used for modeling claim amounts distributions are discussed in Section 2.1. The mixed distributions that are applied in this study to model the claim amounts distribution are introduced in Section 2.2. The probability of ruin (the risk of insolvency) of the insurance company and its approximation are discussed in Sections 2.3 and 2.4. Finally, the minimum initial capital that an insurance company must reserve for managing the ruin probability that is not greater than an acceptable quantity is introduced in Section 2.5.

2.1 Claims Modeling and Claim Amounts Distribution

In actuarial science, the modeling of claims is important work that leads to claims estimation, premium pricing, and risk analysis. Furthermore, the modeling of claims is separated into claim frequency and claim severity. In this study, the claim frequency is assume to be a Poisson process with intensity $\lambda > 0$, and so the focus here is on claim severity in reference to the amount of an insurance claim. Generally, this monetary loss is not less than zero, so it should be modeled with non-negative continuous distributions, and some well-known ones often used for modeling claim amounts distributions are discussed in this section.

2.1.1 The Exponential Distribution

The exponential distribution is a simple distribution usually used in risk theory studies. For an exponential claim amounts distribution with rate β, the probability density function (pdf) and the cumulative distribution function (cdf) of a claim amounts variable $X \sim Expo(\beta)$ are given by

$$f(x) = \beta e^{-\beta x},$$

and

$$F(x) = 1 - e^{-\beta x},$$

respectively, where $x, \beta > 0$. The expected value and the variance of $X \sim Expo(\beta)$ are derived as follows

$$E(X) = 1/\beta \ ,$$

and

$$Var(X) = 1/\beta^2,$$

respectively.

2.1.2 The Gamma Distribution

The gamma distribution has often been used in claims modeling for automobile insurance. For the gamma claim amounts distribution with shape α and rate β or $Gamma(\alpha, \beta)$, the pdf of the claim amounts variable is given by

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x, \alpha, \beta > 0,$$

where $\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt$ is the gamma function. The cdf that correspond to the above pdf is derived as

$$F(x) = \frac{1}{\Gamma(\alpha)} \gamma(\alpha, \beta x), \quad x, \alpha, \beta > 0,$$

where $\gamma(a, x) = \int_0^{x} t^{a-1} e^{-t} dt$ is the lower incomplete gamma function. The expected value and the variance of $X \sim Gamma(\alpha, \beta)$ are derived as

$$E(X) = \alpha / \beta \ ,$$

and

$$Var\left(X\right)=\alpha/\beta^{2},$$

respectively.

2.1.3 The Weibull Distribution

The Weibull distribution is usually used in engineering problems such as survival analysis, reliability analysis, and failure analysis, but in actuality, this distribution has been used for claims modeling of reinsurance claims. The pdf and the cdf of the claim amounts variable with the Weibull distribution with two parameters α and β or $Weibull\left(\alpha,\beta\right)$ are as follows

$$f\left(x\right)=\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}},$$

and

$$F\left(x\right)=1-e^{-(\beta x)^{\alpha}},$$

where $x,\alpha,\beta>0$, respectively. The expected value and the variance of $X\sim Weibull\left(\alpha,\beta\right)$ are derived as

$$E\left(X\right)=\Gamma\left(1+1/\alpha\right)/\beta,$$

and

$$Var\left(X\right)=\left[\Gamma\left(1+2/\alpha\right)-\left(\Gamma\left(1+1/\alpha\right)\right)^{2}\right]/\beta^{2},$$

respectively.

2.2 The Mixed Distribution

It is reasonable to model a claim amounts distribution with a mixed distribution that includes the well-known non-negative continuous distributions in the previous section as sub-models. Thus, some mixed distributions are discussed in this section as:

2.2.1 The Exponentiated Weibull Distribution

Mudholkar and Srivastava (1993) proposed an exponentiated Weibull (EW) distribution that is an extension of the Weibull family; it is obtained by adding a second shape parameter to the Weibull distribution. If W is a random variable of an EW distribution, then the pdf and the cdf of $W \sim EW(\alpha,\beta,\tau)$ are given by

$$g(w;\alpha,\beta,\tau)=\alpha\beta\tau^{\beta}w^{\beta-1}v(1-v)^{\alpha-1},\ w>0,\alpha,\beta,\tau>0, \tag{2.1}$$

where $v=e^{-(\tau w)^{\beta}}$, and

$$G(w)=(1-v)^{\alpha}, \tag{2.2}$$

respectively. Choudhury (2005) derived the k^{th} moment of $W \sim EW(\alpha,\beta,\tau)$ when α is a positive integer as

$$E(W^k)=\Gamma\left(\frac{k}{\beta}+1\right)\frac{\alpha}{\tau^k}\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\frac{(-1)^j}{(j+1)^{\frac{k}{\beta}+1}}. \tag{2.3}$$

Therefore, we can derived the expected value and the variance of W when α is a positive integer as

$$E(W)=\Gamma\left(\frac{1}{\beta}+1\right)\frac{\alpha}{\tau}\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\frac{(-1)^j}{(j+1)^{\frac{1}{\beta}+1}},$$

and

$$Var(W)=\Gamma\left(\frac{2}{\beta}+1\right)\frac{\alpha}{\tau^2}\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\frac{(-1)^j}{(j+1)^{\frac{2}{\beta}+1}}-\left[\Gamma\left(\frac{1}{\beta}+1\right)\frac{\alpha}{\tau}\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\frac{(-1)^j}{(j+1)^{\frac{1}{\beta}+1}}\right]^2,$$

respectively.

2.2.2 The Gamma-Generated Distribution

Zografos and Balakrishnan (2009) proposed a technique to develop new flexible probability distributions that extend well-known distributions by inserting a well-known cdf $G(x)$ into a new cdf as follows:

$$F(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log \bar{G}(x)} t^{\delta-1} e^{-t} dt, \; x > 0, \; \delta > 0, \tag{2.4}$$

where $\bar{G}(x) = 1 - G(x)$, and the correspond pdf is defined as

$$f(x) = \frac{1}{\Gamma(\delta)} \left[-\log \bar{G}(x)\right]^{\delta-1} g(x). \tag{2.5}$$

In the same way, Ristić and Balakrishnan (2011) proposed an alternative gamma-generated distribution with the cdf and pdf as

$$H(x) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log G(x)} t^{\delta-1} e^{-t} dt, \; x > 0, \; \delta > 0 \tag{2.6}$$

and

$$h(x) = \frac{1}{\Gamma(\delta)} \left[-\log G(x)\right]^{\delta-1} g(x), \tag{2.7}$$

respectively.

2.2.3 The Gamma-Exponentiated Weibull Distribution

Pinho et al. (2012) proposed a gamma-exponentiated Weibull (GEW) distribution by inserting (2.2) into the cdf of the gamma-generated distribution in (2.6). The correspond pdf as (2.7) is derived as

$$h(x; \delta, \alpha, \beta, \tau) = \frac{\beta \theta^{\beta} \alpha^{\delta}}{\Gamma(\delta)} x^{\beta-1} v (1-v)^{\alpha-1} \left[-\log(1-v)\right]^{\delta-1},$$

where $v = e^{-(\tau x)^{\beta}}$.

In their study, the gamma-exponentiated Weibull distribution included, as special cases several models such as the exponential, Weibull and exponentiated Weibull distributions.

The work in this study is based on the gamma-generated distribution proposed by Zografos and Balakrishnan (2009). In an alternative to the gamma-exponentiated Weibull distribution, a gamma distribution is included in the sub-models.

The application of the proposed distribution is considered with the important tasks of the actuary in mind: claims modeling, the approximation of ruin probability, and the estimation of the minimum initial capital. In the insurance business, the modeling of claims is important work leading to claims estimation, insurance premium pricing, and risk analysis. There are two kinds of claims modeling: claims frequency and claims severity. In this study, it is assumed that the claims frequency in the proposed distribution is a Poisson process with intensity $\lambda > 0$, and the focus is on the modeling of claims severity, which refers to the monetary loss of an insurance claim by the insurer. The proposed distribution includes both gamma and Weibull distributions in its sub-models, and it should be useful for modeling the claim amounts distribution for an insurance company.

2.3 Ruin Probability

In this section, the probability of ruin or the probability that the cash surplus of an insurance company becomes less than zero for a given initial capital is discussed. The concern in this study is with the probability of ruin in a classical compound Poisson continuous time surplus process. The surplus process at time t is defined as

$$U(t) = u + ct - S(t), \qquad (2.8)$$

where u is the initial capital; c is the rate of premium income per unit of time; and the aggregate claims process $S(t) = X_1 + X_2 + ... + X_{N(t)}$, where $N(t)$ is the number of claims at time t. The number of claims process $\{N(t);t\geq 0\}$ is assumed to be a Poisson process with intensity $\lambda>0$. The sequence of claim sizes $\{X_i;i=1,2,\ldots,N(t)\}$ is assumed to be a sequence of positive independent and identically distributed (i.i.d.) random variables with distribution function F_X and a finite mean $E[X_i]=\mu_1$, and are independent of $N(t)$. The premium rate c is calculated using the expected value premium principle, i.e.

$$c=(1+\theta)\lambda\mu_1, \tag{2.9}$$

where $\theta>0$ is the relative security loading. The risk of insolvency of any insurance company happens when its monetary surplus falls to less than zero, and with a given initial capital u or the probability of ruin over infinite time, is defined as

$$\psi(u)=Pr\Big(U(t)<0,\ \ for\ some\ t>0\Big|U(0)=u\Big). \tag{2.10}$$

Bowers, Gerber, Hickmann, Jones and Nesbitt (1997) showed that the ruin probability when the surplus process is based upon a compound Poisson aggregate claims process with the claim amounts distribution being $Expo(\beta)$ is in the form

$$\psi(u)=\frac{1}{1+\theta}\exp\left(\frac{-\theta\beta u}{1+\theta}\right), \tag{2.11}$$

for all initial capital $u\geq 0$. The derivative of (2.11) is a basic example of the risk theory study. For the claim amounts distributed as $Gamma(2,\beta)$, the ruin probability in Yuanjian, Xucheng and Zhang (2003) is derived as follows:

$$\psi(u)=-\left[\frac{\nu_2(\nu_1+\beta)^2}{(\nu_1-\nu_2)\beta^2}e^{\nu_1 u}+\frac{\nu_1(\nu_2+\beta)^2}{(\nu_2-\nu_1)\beta^2}e^{\nu_2 u}\right], \tag{2.12}$$

where $\nu_1=\left(\lambda-2c\beta+\sqrt{\lambda^2+4c\beta\lambda}\right)/(2c)$ and $\nu_2=\left(\lambda-2c\beta-\sqrt{\lambda^2+4c\beta\lambda}\right)/(2c)$.

However, for other claim amounts distributions, the ruin probability is not easy to obtain, thus approximations of the ruin probability are of interest. Some approximation methods are discussed in the next section.

2.4 Approximation of the Ruin Probability

There are many approximation methods to obtain the ruin probability, but only three methods are discussed in this section: the De Vylder approximation in De Vylder (1978), the Bowers approximation in Bowers, Gerber, Hickmann, Jones and Nesbitt (1997), and the Pollaczek-Khinchin approximation in Asmussen and Binswanger (1997).

2.4.1 The De Vylder Approximation

This approximation is based on the idea of replacing the surplus process $U(t)$ in (2.8) with a surplus process

$$\tilde{U}(t) = u + \tilde{c}t - \tilde{S}(t),$$

where the aggregate claims process $\tilde{S}(t) = \tilde{X}_1 + \tilde{X}_2 + ... + \tilde{X}_{\tilde{N}(t)}$, in which the number of claims process $\{\tilde{N}(t); t \geq 0\}$ is assumed to be a Poisson process with intensity $\tilde{\lambda} > 0$. The sequence of claim sizes $\{\tilde{X}_i; i = 1, 2, \ldots, \tilde{N}(t)\}$ is assumed to be a sequence of i.i.d. random variables with distribution $Expo(\tilde{\beta})$ and a finite mean $E[\tilde{X}_i] = \tilde{\beta}^{-1}$, and are independent of $\tilde{N}(t)$. The premium rate $\tilde{c}$ is define as $\tilde{c} = (1+\tilde{\theta})\tilde{\lambda}\tilde{\beta}^{-1}$, so that the ruin probability with $\tilde{U}(t)$ is defined as in (2.11). The De Vylder Approximation is derived by setting $\tilde{\theta}$, $\tilde{\beta}$, and $\tilde{\lambda}$ in $\tilde{U}(t)$ in such a way that

$$E\left[\tilde{U}(t)^k\right] = E\left[U(t)^k\right], \text{ for } k = 1, 2, 3 \text{ and } t \geq 0.$$

De Vylder (1978) defined the characteristic function $U(t)$ as follows:

$$\varphi_{U(t)}(s) = E\left[e^{isU(t)}\right] = \exp\left(uis + \mu_1\lambda\theta tis - \frac{1}{2}\mu_2\lambda ts^2 + \frac{1}{6}\mu_3\lambda tis^3 \cdots\right),$$

where $\mu_j = E\left[X_i^j\right]$, for $j = 1,2,3$. A similar expansion is valid for the characteristic function of $\tilde{U}(t)$. By the property $E\left[U(t)^k\right] = -i^k\left[\frac{\partial^k}{\partial s^k}\varphi_{U(t)}(s)\right]_{s=0}$ and the condition that $E\left[\tilde{U}(t)^k\right] = E\left[U(t)^k\right]$, for $k = 1,2,3$ and $t \geq 0$, the following equation can be derived:

$$\mu_1\lambda\theta = \tilde{\beta}^{-1}\tilde{\lambda}\tilde{\theta}, \mu_2\lambda = 2\tilde{\beta}^{-2}\tilde{\lambda}, \mu_3\lambda = 6\tilde{\beta}^{-3}\tilde{\lambda}.$$

From above result it is expressed

$$\tilde{\theta} = \frac{2\mu_1\mu_3\theta}{3\mu_2^2}, \ \tilde{\beta} = \frac{3\mu_2}{\mu_3}, \ \tilde{\lambda} = \frac{9\mu_2^3\lambda}{2\mu_3^2}.$$

Subsequently, the De Vylder approximation for the ruin probability with $\tilde{U}(t)$ is define as

$$\psi_{DV}(u) = \frac{1}{1+\tilde{\theta}}\exp\left(\frac{-\tilde{\theta}\tilde{\beta}u}{1+\tilde{\theta}}\right), \tag{2.13}$$

where $\tilde{\theta} = (2\mu_1\mu_3\theta)/(3\mu_2^2)$, $\tilde{\beta} = 3\mu_2/\mu_3$, and $\mu_j = E\left[X_i^j\right]$, for $j = 1,2,3$. If the claim amounts variables $X_i \sim Expo(\beta)$, then $\tilde{\theta} = \theta$, $\tilde{\beta} = \beta$, and $\psi_{DV}(u)$ is equal to $\psi(u)$. It should be noted here that the De Vylder approximation requires the existence of the first three moments of the claim amounts distribution.

2.4.2 The Bowers Approximation

The well-known Lundberg upper bound of ruin probability is defined from the right hand side of the inequality below:

$$\psi(u) \leq e^{-Ru}, \tag{2.14}$$

for any initial capital $u \geq 0$, and the adjustment coefficient R is defined as the smallest positive root of

$$M_{S(t)-ct}(r) = E\left[e^{r(S(t)-ct)}\right] = e^{-rct} M_{S(t)}(r) = 1. \tag{2.15}$$

The ruin probability $\psi(u)$ is a non-increasing function in u, so that its lower bound, defined on p. 415 in Bowers, Gerber, Hickmann, Jones and Nesbitt (1997), is

$$\psi(0) = \frac{1}{1+\theta}. \tag{2.16}$$

The Bowers approximation uses the fact that $(1+\theta)^{-1} \leq \psi(u) \leq e^{-Ru}$, giving the approximated ruin probability as

$$\psi_B(u) = \frac{1}{1+\theta} e^{-Ku}. \tag{2.17}$$

To obtain a reasonable constant K, the process of aggregate claims over premiums received $\{S(t)-ct\}$ or $\left\{\sum_{i=1}^{N(t)} X_i - ct\right\}$ is considered. The number of claims process $N(t)$ is a Poisson process and the sequence of claim amounts $\{X_i; i \geq 1\}$ consists of non-negative i.i.d. random variables, let $\{T_i; i \geq 1\}$ be the sequence of the timing of the claims corresponding to claim amounts $\{X_i; i \geq 1\}$. Thus, the process $\{S(t)-ct\}$ increments by height X_i for $t = T_i$; $i = 1, 2, 3, \ldots$ and decreases by slope c otherwise. The process $\{S(t)-ct\}$ contains the values of insurance company loss at time t. If the value of the process $\{S(t)-ct\}$ is very large, then the company is insolvent and the process $\{S(t)-ct\}$ has ended. Thus, under the assumption that the insolvency or ruin can occur, the process $\{S(t)-ct\}$ is bound. Let M be the number of claims where the process $\{S(t)-ct\}$ becomes a maximum at time T_M and let Y_1 be the value of the process $\{S(t)-ct\}$ that reaches above zero for the first time. Next, let Y_2 be the value of excess that reaches above the value of Y_1 for the first time. Variables $Y_3, Y_4, \ldots$ are sequentially defined in the same way. The number of iterations of the process $\{S(t)-ct\}$ carried out in this sequence N is called the number of new record highs and the values Y_k, for $k = 1, 2, \cdots, N$ are referred to as new record highs. The last new record high Y_N occurs at time T_M so that $\{S(t)-ct\}$ is a maximum for all time intervals $t \geq 0$. Generally, for each claim $X_1, X_2, \ldots, X_M$ that makes the process $\{S(t)-ct\}$ increment, the new record highs $Y_1, Y_2, \ldots, Y_N$ may or may not occur, so $N \leq M$. Hence, the maximum of the process $\{S(t)-ct\}$ or the maximal aggregate loss L is illustrated as

$$L = \max_{t \geq 0}\{S(t)-ct\} = Y_1 + Y_2 + \ldots + Y_N. \quad (2.18)$$

Figure 2.1 shows a graph of L for $M = 5$ and $N = 3$.

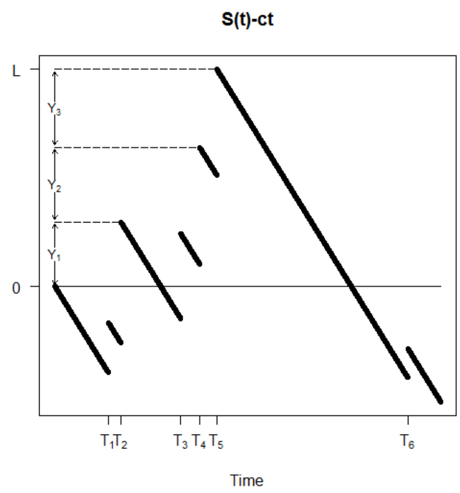

Figure 2.1 Maximal aggregate loss for the number of new record highs $N = 3$.

Since a stationary and independent increment of process $S(t)$ is assumed, $\{Y_k\}$ is a sequence of i.i.d. variables with the density

$$f_Y(y) = \bar{F}_X(y) / \mu_1, \tag{2.19}$$

where $\bar{F}_X(y) = 1 - F_X(y)$ and μ_1 is the expected value of the claim amounts. The number of new record highs N is geometric distributed with parameter $1-\psi(0)$, and its probability mass function is

$$\Pr(N = n) = [1-\psi(0)][\psi(0)]^n = \theta\left(\frac{1}{1+\theta}\right)^{n+1}, \tag{2.20}$$

where $n = 0,1,2,...$ The ruin probability with an infinite horizon time in (2.10) can be represented in the form of a distribution function of L derived as follows:

$$\psi(u) = Pr\left(\max_{t\geq 0}\{S(t) - ct\} > u\right) = \Pr(L > u) = 1 - F_L(u), \tag{2.21}$$

where F_L is a distribution function of the maximal aggregate loss L. Using the property of the expected values, it can be written

$$E[L] = \int_0^\infty [1 - F_L(u)]du = \int_0^\infty \psi(u)du.$$

The maximal aggregate loss L can be expressed in terms of a geometric process with the expected value

$$E[L] = E[Y_k]E[N],$$
$$= E[Y_k]\frac{1}{\theta}.$$

From the density function in (2.19), the moment generating function of the new record highs $Y_1, Y_2, \ldots, Y_N$ can be denoted as

$$M_Y(t) = \int_0^\infty e^{ty} f_Y(y)\,dy,$$
$$= \frac{1}{\mu_1}\int_0^\infty e^{ty}\left[1 - F_X(y)\right]dy,$$
$$= \frac{1}{\mu_1}\left\{\frac{e^{ty}}{t}\left[1 - F_X(y)\right]\Big|_0^\infty + \frac{1}{t}\int_0^\infty e^{ty} f_x(y)\,dy\right\},$$
$$= \frac{1}{\mu_1 t}\left[M_x(t) - 1\right].$$

By substituting $M_X(t) = 1 + \mu_1 t + \mu_2 \frac{t^2}{2} + \mu_3 \frac{t^3}{6} + \cdots$ in the above equation, it can be expressed as

$$M_Y(t) = 1 + \frac{\mu_2}{\mu_1}\frac{t}{2} + \frac{\mu_3}{\mu_1}\frac{t^2}{6} + \frac{\mu_4}{\mu_1}\frac{t^3}{24} + \cdots.$$

From the above moment generating function, the expected value of the new record highs can be calculated as $E[Y_k] = \frac{\mu_2}{2\mu_1}$, and so $E[L] = \frac{\mu_2}{2\theta\mu_1}$. The constant K in (2.17) is chosen such that the approximated value conforms to

$$E[L] = \int_0^\infty \left[1 - F_L(u)\right]du = \int_0^\infty \psi(u)\,du = \frac{\mu_2}{2\theta\mu_1}. \tag{2.22}$$

Thus,

$$\frac{\mu_2}{2\theta\mu_1} = \int_0^\infty \psi_B(u)\,du = \int_0^\infty \frac{1}{1+\theta} e^{-Ku}\,du\ ,$$

so a reasonable K is

$$K = \frac{2\theta\mu_1}{(1+\theta)\mu_2}, \tag{2.23}$$

and the Bowers approximation in (2.17) becomes

$$\psi_B(u) = \frac{1}{1+\theta} e^{\frac{-2\theta\mu_1}{(1+\theta)\mu_2}u}$$

(Bowers, Gerber, Hickmann, Jones and Nesbitt, 1997: 418-423). One advantage of the Bowers approximation over the De Vylder approximation is that it requires only the first two moments of the claim amounts distribution.

2.4.3 The Pollaczek-Khinchin Approximation

This algorithm only requires the first moment of the claim amounts distribution, and the method is based on a Monte Carlo simulation using (2.21). To obtain $1-F_L(u)$ in (2.21), the density f_Y is first defined as in (2.19) and the number of new record highs N is generated with a density as in (2.20). Next, a sequence of new record highs $\{Y_1, Y_2, \ldots, Y_N\}$ with f_Y is generated. Let $L = Y_1 + Y_2 + \ldots + Y_N$ and define indicator Z as

$$Z = \begin{cases} 0 & ; L \le u, \\ 1 & ; L > u, \end{cases} \tag{2.24}$$

where $E[Z] = \psi(u)$. Repeat this process n times, so we have $Z_1, Z_2, \ldots, Z_n$ and $\bar{Z} = \sum_{i=1}^{n} Z_i / n$ converges to $\psi(u)$ as n becomes large. The algorithm for computing the approximation of the ruin probability can be presented as follows:

1) Assume F_X is known, then obtain the density f_Y from $f_Y(y) = [1 - F_X(y)] / \mu_1$.

2) Set the number of iterations n to be some large number, e.g. 10,000 or 50,000. Generate $N_i, i = 1, 2, \ldots, n$ from $Geometric(q)$, where $q = \theta / (1+\theta)$, and set it to be the number of new record highs.

3) Generate Y_j^i, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, N_i$, from the density f_Y in step 1 and obtain $L_i = Y_1^i + \ldots + Y_{N_i}^i$.

4) For each i, if $L_i > u$, then $Z_i = 1$, otherwise $Z_i = 0$.

5) Calculate $\bar{Z} = \sum_{i=1}^{n} Z_i / n$.

6) Increase the number of iterations n by 5,000 or 10,000 and repeat steps 1 to 5 until $\bar{Z}$ remains constant.

The above algorithm describes the steps for the Pollaczek-Khinchin approximation denoted by $\psi_{PK}(u) = \bar{Z}$. One difficulty with this method is at the step of simulating $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ with density f_Y. For example, when the claim amounts distribution is $Gamma(\eta, \beta)$ with shape parameter η as an integer, the density f_Y can be derived as the density of a mixture of η gamma distributions with equal weights $1/\eta$, scale parameter β, and shape parameters $\{1, 2, \ldots, \eta\}$. However, when shape parameter η is non-integer, to simulate the amount of each new record high $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ with f_Y is complicated.

In this study, a simple algorithm to approximate $\psi(u)$ based on the amount of each new record high $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ is proposed (see Chapter 4).

2.5 The Minimum Initial Capital

The minimum initial capital is defined in Pairote Sattayatham, Kiat Sangaroon and Watcharin Klongdee (2013) as follows.

Definition 1 Let $\{U(t), t \ge 0\}$ be a surplus process driven by the compound Poisson claims process $\{S(t), t \ge 0\}$ and $c > 0$ be the premium rate. For any $\alpha \in (0,1)$, let $u \ge 0$ be the initial capital. If $\psi(u) \le \alpha$, then u is referred to as an acceptable initial capital level corresponding to $(\alpha, c, \{S(t), t \ge 0\})$. In particular, if

$$u_\alpha^* = \min_{u \ge 0} \{u : \psi(u) \le \alpha\}$$

exists, u_α^* is called the minimum initial capital corresponding to $(\alpha, c, \{S(t), t \ge 0\})$.

In this study, an approximation method for computing the minimum initial capital u_{α}^{*} is proposed (see Chapter 4).

CHAPTER 3

THE ALTERNATIVE GAMMA-EXPONENTIATED WEIBULL DISTRIBUTION

In this chapter, the new distribution, namely the alternative gamma-exponentiated Weibull (AGEW) distribution, is obtained by mixing the exponentiated Weibull and gamma distributions. Moreover, its basic structural properties such as distribution function, density function, moments, sub-models, and parameter estimation with the maximum likelihood estimator (MLE) method are presented.

3.1 The Distribution Function and the Probability Density Function

Theorem 3.1 Let X be a random variable of the AGEW distribution with parameters δ, α, β, and τ. The cdf of X is defined by

$$F(x) = \frac{\gamma\left\{\delta, -\log\left[1-(1-v)^{\alpha}\right]\right\}}{\Gamma(\delta)}, \tag{3.1}$$

where $v = e^{-(\tau x)^{\beta}}$, $x > 0$, parameters $\delta, \alpha, \beta, \tau > 0$, and $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function.

Proof

We can obtain the cdf of X by inserting $G(x)$ from equation (2.2) into equation (2.5). Hence, the cdf of the AGEW distribution can be written as

$$F(x)=\frac{1}{\Gamma(\delta)}\int_{0}^{-\log\left[1-(1-v)^{\alpha}\right]} t^{\delta-1}e^{-t}dt,$$

$$=\frac{\gamma\left\{\delta,-\log\left[1-(1-v)^{\alpha}\right]\right\}}{\Gamma(\delta)}.$$

Theorem 3.2 Let X be a random variable of the AGEW distribution with parameters δ,α,β, and τ. The pdf of X is given by

$$f(x)=\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}v(1-v)^{\alpha-1}\left[-\log\left(1-(1-v)^{\alpha}\right)\right]^{\delta-1}, \qquad (3.2)$$

where $v=e^{-(\tau x)^{\beta}}$, $x>0$, and parameters $\delta,\alpha,\beta,\tau>0$.

Proof

We can obtain the pdf of X by inserting $g(x)$ from equation (2.1) into equation (2.6), and so, the pdf of the AGEW distribution becomes

$$f(x)=\frac{1}{\Gamma(\delta)}\left[-\log\left(1-(1-v)^{\alpha}\right)\right]^{\delta-1}\alpha\beta\tau^{\beta}x^{\beta-1}v(1-v)^{\alpha-1},$$

$$=\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}v(1-v)^{\alpha-1}\left[-\log\left(1-(1-v)^{\alpha}\right)\right]^{\delta-1}.$$

Since $x>0$ and parameters $\delta,\alpha,\beta,\tau>0$, then $v=e^{-(\tau x)^{\beta}}$ is between 0 and 1,

$$0<(1-v)^{\alpha-1}<1,$$

$$-\log\left(1-(1-v)^{\alpha}\right)>0.$$

Thus, $f(x)>0$. If we let $z=-\log\left(1-(1-v)^{\alpha}\right)$, then we can write

$$\int_{0}^{\infty}f(x)dx=\int_{0}^{\infty}\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}v(1-v)^{\alpha-1}\left[-\log\left(1-(1-v)^{\alpha}\right)\right]^{\delta-1}dx$$

$$=\int_{0}^{\infty}\frac{1}{\Gamma(\delta)}z^{\delta-1}e^{-z}dz,$$

$$=1.$$

By the above properties, equation (3.2) is the pdf of the AGEW distribution.

Theorem 3.3 Let $a_s = (s+2)^{-1}$, $b_{0,m} = a_0^m$, and $b_{s,m} = \frac{1}{sa_0}\sum_{k=1}^{m}\left[m(k+1)-s\right]a_k b_{s-k,m}$ for positive integers s and m. Let X be a random variable of the AGEW distribution with positive parameters δ, α, β, and τ. When δ is a positive integer, the pdf of X can be defined as the linear combination of EW density functions as

$$f(x) = \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty} c_{\delta,s,m}\, g\left(x; \alpha(\delta+s+m), \beta, \tau\right), \tag{3.3}$$

where $c_{\delta,s,m} = \binom{\delta-1}{m}\frac{b_{s,m}}{(\delta+s+m)\Gamma(\delta)}$ and $g(\bullet)$ is the pdf of EW, as defined in equation (2.1).

Proof

Let $w = (1-v)^{\alpha}$, then the pdf in equation (3.2) can be rewritten as

$$f(x) = \frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)} x^{\beta-1} v(1-v)^{\alpha-1}\left[-\log(1-w)\right]^{\delta-1}.$$

By using the power series $-\log(1-w) = \sum_{i=0}^{\infty}\frac{w^{i+1}}{i+1}$, we can obtain

$$\begin{aligned} f(x) &= \frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)} x^{\beta-1} v(1-v)^{\alpha-1}\left[\sum_{i=0}^{\infty}\frac{w^{i+1}}{i+1}\right]^{\delta-1}, \\ &= \frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)} x^{\beta-1} v(1-v)^{\alpha-1}\left[w + \sum_{i=1}^{\infty}\frac{w^{i+1}}{i+1}\right]^{\delta-1}, \\ &= \frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)} x^{\beta-1} v(1-v)^{\alpha-1}\left[w + \sum_{s=0}^{\infty}\frac{w^{s+2}}{s+2}\right]^{\delta-1}. \end{aligned}$$

By applying the binomial theorem, when δ is a positive integer, we can write

$$f(x)=\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}v(1-v)^{\alpha-1}\left[\sum_{m=0}^{\delta-1}\binom{\delta-1}{m}w^{\delta-1-m}\left(\sum_{s=0}^{\infty}\frac{w^{s+2}}{s+2}\right)^{m}\right],$$

$$=\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}v(1-v)^{\alpha-1}\left[\sum_{m=0}^{\delta-1}\binom{\delta-1}{m}w^{\delta-1+m}\left(\sum_{s=0}^{\infty}\frac{w^{s}}{s+2}\right)^{m}\right],$$

$$=\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}\frac{v}{(1-v)}w^{\delta}\left[\sum_{m=0}^{\delta-1}\binom{\delta-1}{m}w^{m}\left(\sum_{s=0}^{\infty}\frac{w^{s}}{s+2}\right)^{m}\right].$$

Let $a_s=(s+2)^{-1}$, and consider the result on a power series raised to a positive integer (Gradshteyn and Ryzhik, 2000: 17)

$$\left(\sum_{s=0}^{\infty}a_s y^s\right)^{m}=\sum_{s=0}^{\infty}b_{s,m}y^s ,$$

where $b_{0,m}=a_0^m$ and $b_{s,m}=\frac{1}{sa_0}\sum_{k=1}^{m}\left[m(k+1)-s\right]a_k b_{s-k,m}$. We can express the pdf of the AGEW distribution in terms of a linear combination of the EW distribution as

$$f(x)=\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}\frac{v}{(1-v)}w^{\delta}\left[\sum_{m=0}^{\delta-1}\binom{\delta-1}{m}w^{m}\sum_{s=0}^{\infty}b_{s,m}w^{s}\right],$$

$$=\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}\frac{v}{(1-v)}w^{\delta}\left[\sum_{m=0}^{\delta-1}\binom{\delta-1}{m}w^{m}\sum_{s=0}^{\infty}b_{s,m}w^{s}\right],$$

$$=\sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty}\binom{\delta-1}{m}\frac{\alpha\beta\tau^{\beta}}{\Gamma(\delta)}x^{\beta-1}\frac{v}{(1-v)}b_{s,m}w^{\delta+s+m}, \qquad ,$$

$$=\sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty}\binom{\delta-1}{m}\frac{b_{s,m}}{(\delta+s+m)\Gamma(\delta)}\alpha(\delta+s+m)\beta\tau^{\beta}x^{\beta-1}v(1-v)^{\alpha(\delta+s+m)-1},$$

$$=\sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty}c_{\delta,s,m}g\left(x;\alpha(\delta+s+m),\beta,\tau\right),$$

where $c_{\delta,s,m}=\binom{\delta-1}{m}\frac{b_{s,m}}{(\delta+s+m)\Gamma(\delta)}$ and $g(\bullet)$ is the pdf of EW, as defined in equation (2.1).

To show the various of shapes of this distribution, some specified parameters of the AGEW distribution and their density functions are provided in Figure 3.1: (a) fixed parameters $\alpha=4,\beta=0.5,\tau=1.5$ and varied parameter δ; and (b) fixed parameter $\delta = 4$ and varied parameters α, β and τ. Hence, we can see that the AGEW distribution is suitable for fitting to various shapes of data.

(a) (b)

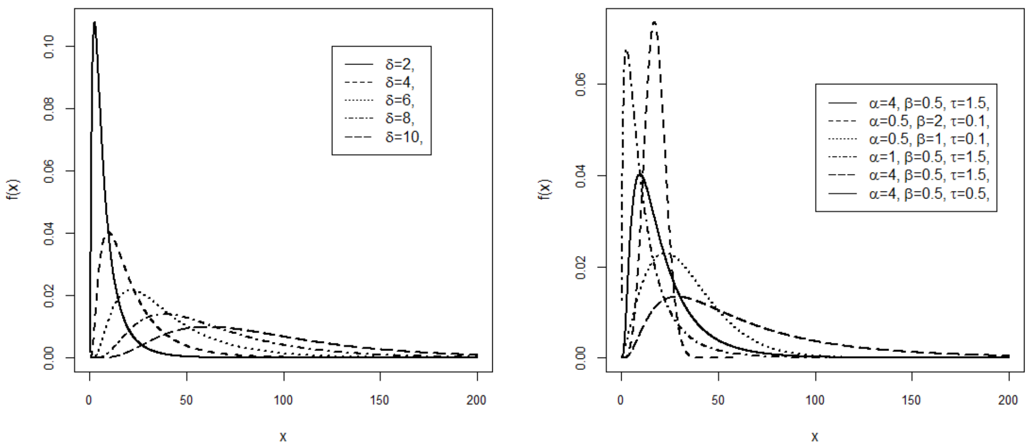

Figure 3.1 The density function of the AGEW distribution.

3.2 Moments for the AGEW distribution

Theorem 3.4 Let $a_s = (s+2)^{-1}$, $b_{0,m} = a_0^m$ and $b_{s,m} = \frac{1}{sa_0}\sum_{k=1}^{m}\left[m(k+1)-s\right]a_k b_{s-k,m}$ for positive integers s and m. Let X be a random variable of the AGEW distribution with positive parameters δ, α, β and τ. When δ and α are positive integers, the moment of X can be written as

$$\mu_k = E\left(X^k\right) = \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty} c_{\delta,s,m}\mu_k^*\left(\alpha(\delta+s+m),\beta,\tau\right), \tag{3.4}$$

where $c_{\delta,s,m} = \frac{\binom{\delta-1}{m}b_{s,m}}{(\delta+s+m)\Gamma(\delta)}$ and $\mu_k^*(\alpha,\beta,\tau) = \Gamma\left(\frac{k}{\beta}+1\right)\frac{\alpha}{\tau^k}\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\frac{(-1)^j}{(j+1)^{\frac{k}{\beta}+1}}$.

Proof

Let $Y \sim EW\left(\alpha^*, \beta^*, \tau^*\right)$ be a random variable with a density function as in equation (2.1). We let $\mu_k^*\left(\alpha^*, \beta^*, \tau^*\right)$ be the k^{th} moment of Y that corresponds to $\left(\alpha^*, \beta^*, \tau^*\right)$. When δ is a positive integer, we can apply the definition of the moment to the linear combination of the EW density function in Theorem 3.3 such that

$$\begin{aligned}
\mu_k = E\left(X^k\right) &= \int_0^\infty x^k f\left(x\right)dx, \\
&= \int_0^\infty x^k \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty} c_{\delta,s,m}\, g\left(x;\alpha\left(\delta+s+m\right),\beta,\tau\right)dx, \\
&= \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty} c_{\delta,s,m}\int_0^\infty x^k g\left(x;\alpha\left(\delta+s+m\right),\beta,\tau\right)dx, \\
&= \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty} c_{\delta,s,m}\,\mu_k^*\left(\alpha\left(\delta+s+m\right),\beta,\tau\right).
\end{aligned},$$

Choudhury (2005) derived $\mu_k^*\left(\alpha^*, \beta^*, \tau^*\right)$ when α^* is a positive integer as

$$\mu_k^*\left(\alpha^*, \beta^*, \tau^*\right) = E\left(Y^k\right) = \Gamma\left(\frac{k}{\beta^*}+1\right)\frac{\alpha^*}{\tau^{*k}}\sum_{j=0}^{\alpha^*-1}\binom{\alpha^*-1}{j}\frac{(-1)^j}{\left(j+1\right)^{\frac{k}{\beta^*}+1}}.$$

Therefore, we can obtain the moment of AGEW distribution as

$$\begin{aligned}
\mu_k = E\left(X^k\right) &= \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty} c_{\delta,s,m}\,\mu_k^*\left(\alpha\left(\delta+s+m\right),\beta,\tau\right), \\
&= \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty} c_{\delta,s,m}\,\Gamma\left(\frac{k}{\beta}+1\right)\frac{\alpha\left(\delta+s+m\right)}{\tau^k}\sum_{j=0}^{\alpha(\delta+s+m)-1}\binom{\alpha\left(\delta+s+m\right)-1}{j}\frac{(-1)^j}{\left(j+1\right)^{\frac{k}{\beta}+1}},
\end{aligned}$$

where δ and α are positive integers. The proof of Theorem 3.4 is complete.

From Theorem 3.4, we can derive the expected value and the variance of the AGEW distribution as

$$E\left(X\right) = \mu_1 = \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty}\sum_{j=0}^{\alpha(\delta+s+m)-1} c_{\delta,s,m}\,\Gamma\left(\frac{\beta+1}{\beta}\right)\binom{\alpha\left(\delta+s+m\right)-1}{j}\frac{\alpha\left(\delta+s+m\right)(-1)^j}{\tau\left(j+1\right)^{\frac{\beta+1}{\beta}}}$$

and

$$Var(X) = \mu_2 - (\mu_1)^2,$$

$$= \sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty}\sum_{j=0}^{\alpha(\delta+s+m)-1} c_{\delta,s,m}\Gamma\left(\frac{\beta+2}{\beta}\right)\binom{\alpha(\delta+s+m)-1}{j}\frac{\alpha(\delta+s+m)(-1)^j}{\tau(j+1)^{\frac{\beta+2}{\beta}}}$$

$$-\left[\sum_{m=0}^{\delta-1}\sum_{s=0}^{\infty}\sum_{j=0}^{\alpha(\delta+s+m)-1} c_{\delta,s,m}\Gamma\left(\frac{\beta+1}{\beta}\right)\binom{\alpha(\delta+s+m)-1}{j}\frac{\alpha(\delta+s+m)(-1)^j}{\tau(j+1)^{\frac{\beta+1}{\beta}}}\right]^2,$$

where δ and α are positive integers.

3.3 The Sub-models

If we let parameter δ in equation (3.1) be 1, then the cdf of the AGEW distribution defined as in equation (2.2) can be written as

$$F(x) = \frac{\gamma\left\{1, -\log\left[1-(1-v)^{\alpha}\right]\right\}}{\Gamma(1)},$$

$$= \int_0^{-\log\left[1-(1-v)^{\alpha}\right]} e^{-t}dt,$$

$$= \left[1-(1-v)^{\alpha}\right],$$

$$= \left(1-e^{-(\tau x)^{\beta}}\right)^{\alpha}.$$

In the same fashion, we can obtain some of the sub-models of the AGEW distribution, as shown in Table 3.1.

Table 3.1 The sub-models table for the AGEW distribution.

Distribution	parameters				$F(x)$
	δ	α	β	τ	
1 Exponentiated Weibull (EW)	1	α	β	τ	$F(x) = \left(1-e^{-(\tau x)^{\beta}}\right)^{\alpha}$
2 Weibull	1	1	β	τ	$F(x) = 1-e^{-(\tau x)^{\beta}}$
3 Gamma	δ	1	1	τ	$F(x) = \frac{1}{\Gamma(\delta)}\gamma(\delta, \tau x)$

The AGEW distribution includes Weibull and gamma distributions in its sub-models, both of which are popular in claims modeling. Hence, it is of interest to use this model to fit claim amounts data from insurance companies.

3.4 Parameter Estimation

In this subsection, we presume that the data with sample size n are drawn from an AGEW distributed population, and that $\Theta = \left(\delta, \alpha, \beta, \tau\right)^T$ is the parameter vector of the distribution. Subsequently, the likelihood function of AGEW distribution is given by

$$L\left(\underline{x}\right) = \prod_{i=1}^{n} \frac{\alpha\beta\tau^{\beta}}{\Gamma\left(\delta\right)} x_i^{\beta-1} v_i \left(1-v_i\right)^{\alpha-1} \left[-\log\left(1-\left(1-v_i\right)^{\alpha}\right)\right]^{\delta-1},$$

where $v_i = e^{-\left(\tau x_i\right)^{\beta}}$. Following this, the log likelihood function can be written as

$$l\left(\underline{x}\right) = \sum_{i=1}^{n} \Big(\log\left(\alpha\right) + \log\left(\beta\right) + \beta\log\left(\tau\right) - \log\Gamma\left(\delta\right) + \left(\beta-1\right)\log x_i - \left(\tau x_i\right)^{\beta} + \left(\alpha-1\right)\log\left(1-v_i\right)$$
$$+\left(\delta-1\right)\log\left[-\log\left(1-\left(1-v_i\right)^{\alpha}\right)\right]\Big).$$

Therefore, by using the MLE method, the elements of score vector

$U = \left(\frac{\partial l}{\partial\delta}, \frac{\partial l}{\partial\alpha}, \frac{\partial l}{\partial\beta}, \frac{\partial l}{\partial\tau}\right)^T$ are set to zero, where

$$\frac{\partial l}{\partial\delta} = \sum_{i=1}^{n} \left(-\Psi\left(\delta\right) + \log\left[-\log\left(1-\left(1-v_i\right)^{\alpha}\right)\right]\right),$$

$$\frac{\partial l}{\partial\alpha} = \sum_{i=1}^{n} \left(\frac{1}{\alpha} + \log\left(1-v_i\right) - \frac{\left(\delta-1\right)\left(1-v_i\right)^{\alpha}\log\left(1-v_i\right)}{\left[1-\left(1-v_i\right)^{\alpha}\right]\log\left[1-\left(1-v_i\right)^{\alpha}\right]}\right),$$

$$\frac{\partial l}{\partial \beta}=\sum_{i=1}^{n}\left(\frac{1}{\beta}+\left[1-\left(\tau x_i\right)^{\beta}\right]\log\left(\tau x_i\right)+\frac{(\alpha-1)\left(\tau x_i\right)^{\beta}u_i\log\left(\tau x_i\right)}{\left(1-v_i\right)}\right.$$

$$\left.-\frac{(\delta-1)\alpha\left(\tau x_i\right)^{\beta}v_i\left(1-v_i\right)^{\alpha-1}\log\left(\tau x_i\right)}{\left[1-\left(1-v_i\right)^{\alpha}\right]\log\left[1-\left(1-v_i\right)^{\alpha}\right]}\right),$$

$$\frac{\partial l}{\partial \tau}=\sum_{i=1}^{n}\left(\frac{\beta}{\tau}-\beta\tau^{\beta-1}x_i^{\beta}+\frac{(\alpha-1)\beta\tau^{\beta-1}x_i^{\beta}v_i}{\left(1-v_i\right)}-\frac{(\delta-1)\alpha\beta\tau^{\beta-1}x_i^{\beta}v_i\left(1-v_i\right)^{\alpha-1}}{\left[1-\left(1-v_i\right)^{\alpha}\right]\log\left[1-\left(1-v_i\right)^{\alpha}\right]}\right),$$

where $\Psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=\frac{\frac{\delta}{\delta x}\Gamma(x)}{\Gamma(x)}$ is a digamma function. The maximum likelihood estimator $\hat{\Theta}=\left(\hat{\delta},\hat{\alpha},\hat{\beta},\hat{\tau}\right)^{T}$ is the solution to the above score equations which they are not in closed form, but they can be calculated using different Barzilai-Borwein (BB) steplengths from the BB package included in the R statistical software, as discussed by Varadhan and Gilbert (2009).

3.5 Application of the Proposed Distribution

To measure the superiority of the AGEW distribution over its sub-models, the AGEW model and its sub-models were fitted to 363 real-life claim datasets of motor insurance collected from five car dealers. The fitness of the AGEW model was compared with its sub-models: gamma, Weibull, and EW using a graphical approach. The comparison of the estimated pdfs with real-life data as a histogram is presented in Figure 3.2. However, we can see that it is not easy to detect any differences in these graphs, so the discrepancy distribution of each estimated model was compared with real-life data using the Kolmogorov-Smirnov (K-S) method and also by considering the mean squared errors (MSEs) of the distributions as

$$MSE=\frac{\sum_{i=1}^{n}\left[F\left(x_i\right)-\hat{F}\left(x_i\right)\right]^{2}}{n},$$

with n sample size data, and where $x_1, x_2, \ldots, x_n$ are observed, $F(x)$ is the value of the theoretical cdf, and the empirical cdf $\hat{F}(x)$ is the fraction of observations less than or equal to x. The maximum likelihood estimates of the parameters, K-S statistics, and corresponding p-values for the fitted models and the MSEs are shown in Table 3.2. The K-S test showed that only the EW and AGEW distributions were accepted as the model at the 0.05 significance level. The MSE verified that the AGEW distribution with the lowest MSE was superior to the gamma and Weibull sub-distributions. Both AGEW and EW gave similar MSE values, but the AGEW distribution was chosen since it had a slightly lower MSE than EW when modeling the claim amounts distribution.

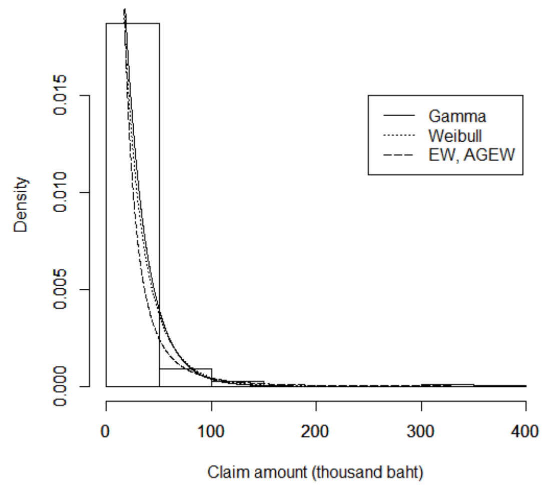

Figure 3.2 The pdfs of the distributions with real-life claim amounts data.

Table 3.2 Maximum likelihood estimates, K-S statistics with corresponding p-values, and MSEs for the claim amounts data.

Fitted Distribution	δ	α	β	τ	K-S	p-value	MSE
Gamma	0.9169	-	-	0.0467	0.1184	0.0001	0.004378
Weibull	-	-	0.8721	0.0557	0.0927	0.0039	0.003167
EW	-	22.6383	0.3014	6.2811	0.0300	0.8998	0.000175
AGEW	0.6242	33.4948	0.2966	5.5117	0.0293	0.9140	0.000170

CHAPTER 4

THE PROPOSED APPROXIMATION

In this chapter, the Pollaczek-Khinchin approximation is modified in Subsection 2.4.3 and a new algorithm is presented without using density f_Y for approximating the ruin probability. From the idea of the proposed approximation of the ruin probability, a new algorithm to compute the approximation of the minimum initial capital u_α^* is also proposed and discussed in Section 2.5.

4.1 The Proposed Modified Ruin Probability Approximation

From Subsection 2.4.3, the Pollaczek-Khinchin approximation is based on the idea of generating the number of new record highs N from $Geometric(q)$ where $q = \theta / (1+\theta)$ and $\theta > 0$ is the relative security loading. Next, the sequence of new record highs $\{Y_1, Y_2, \ldots, Y_N\}$ is generated using density f_y as in (2.19), and the maximal aggregate loss $L = Y_1 + Y_2 + \ldots + Y_N$ is computed. The approximation of the ruin probability $\psi(u) = \Pr(L < u)$, where u is the initial capital, is computed by repeating this process. However, simulating the amount of each new record highs $\{Y_1, Y_2, \ldots, Y_N\}$ with f_y is complicated. Thus, a new simpler algorithm to approximate $\psi(u)$ without using density f_y is presented.

We represent the timing of claims in the form of $T_n = W_1 + ... + W_n$, $n = 1, 2, 3, \ldots$, and W_n is the time difference between consecutive claims T_n and T_{n-1}. For the Poisson claims number process with intensity $\lambda > 0$, the sequence of time difference between consecutive claims $\{W_n = T_n - T_{n-1}, n = 1, 2, 3, \ldots\}$ is a sequence of i.i.d. random variables with $Expo(\lambda)$ using the Poisson distribution property.

To simulate the amount of the first new record high Y_1 in (2.17), we generate the time difference between consecutive claims $\{W_1, W_2, \ldots\}$ and the claim amount random variables $\{X_1, X_2, \ldots\}$. We compute the timing of claims $T_j = W_1 + \ldots + W_j$ and the value of process $\{S(t) - ct\}$, for $t = T_1, T_2, \ldots$, until the process $\{S(t) - ct\}$ reaches above the zero level for the first time.

However, it is possible that the process $\{S(t) - ct\}$, which computes forms $\{W_1, W_2, \ldots\}$ and $\{X_1, X_2, \ldots\}$, does not reach above the zero level at any time t. To mitigate this, we set a large positive integer D to be the limit on the number of simulated claims; the constant D also refers to the number of elements to truncate. Let T_D be the timing of the claim that corresponds to D. When the process $\{S(t) - ct; 0 < t \le T_D\}$ reaches above the zero level at least once, then let $Y_{1,D}$ be the value of the process $\{S(t) - ct; 0 < t \le T_D\}$ that reaches above zero for the first time.

Theorem 4.1 If the number of truncated elements D is large, then $Y_{1,D}$ converges in distribution to the first order of new record highs Y_1.

Proof

Let $F_{Y_{1,D}}$ and F_{Y_1} be distribution functions of $Y_{1,D}$ and Y_1, respectively, then

$$F_{Y_{1,D}}(y) = \Pr(Y_{1,D} < y) = \Pr\left(0 < S(t) - ct < y \middle| S(t) - ct > 0, \textit{ for some } 0 < t \le T_D\right),$$

for some $y > 0$. When the number of truncated elements $D \to \infty$, it is also obvious that the time at which the D^{th} claim occurred $T_D \to \infty$. Thus,

$$\begin{aligned}\lim_{D\to\infty} F_{Y_{1,D}}\left(y\right) &= \lim_{D\to\infty} \Pr\left(0 < S\left(t\right) - ct < y \middle| S\left(t\right) - ct > 0,\ for\ some\ \ 0 < t \le T_D\right),\\ &= \Pr\left(0 < S\left(t\right) - ct < y \middle| S\left(t\right) - ct > 0,\ for\ some\ 0 < t < \infty\right),\\ &= \Pr\left(0 < S\left(t\right) - ct < y \middle| S\left(t\right) - ct > 0,\ for\ some\ t > 0\right),\\ &= F_{Y_1}\left(y\right).\end{aligned}$$

From Theorem 4.1, we can approximate Y_1 by $Y_{1,D}$. The sequence of new record highs $\{Y_1, Y_2, \ldots, Y_N\}$ is an i.i.d. sequence of random variables, and so $Y_{i,D}$ converges in distribution to Y_i, for $i = 2,3,...,N$, when D is large. To simulate the amount of each new record high $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ for the i^{th} loop, generate the time difference between consecutive claims $\{W_1, W_2, \ldots, W_D\}$ from i.i.d. $Exp(\lambda)$ and the claim amount random variables $\{X_1, X_2, \ldots, X_D\}$ with F_X. We compute the timings of the claims $\{T_1, T_2, \ldots, T_D\}$ from $T_j = W_1 + ... + W_j$ and let $Y_{1,D}^i$ be the value of the process $\{S(t) - ct; 0 \le t \le T_D\}$ that reaches above zero for the first time. If the process $\{S(t) - ct\}$ does not reach above zero for all $0 < t \le T_D$, then repeatedly generate the timings of the claims and the claim amount random variables until $Y_{1,D}^i$ occurs. We approximate Y_1^i by $Y_{1,D}^i$ and repeat this process until the values $\{Y_{2,D}^i, Y_{3,D}^i, \ldots, Y_{N_i,D}^i\}$ that approximate $\{Y_2^i, Y_3^i, \ldots, Y_{N_i}^i\}$ are obtained. In a real-life situation where the claim amounts distribution F_X is unknown, F_X can be approximated based on the real-life claim amounts data. The proposed algorithm to obtain an approximation of the ruin probability is as follows:

1) Approximate F_X based on real-life data.

2) Set the number of iterations n and number of truncated elements D to be some large number, e.g. 10,000 or 50,000. Generate $N_i, i = 1, 2, \ldots, n$ from $Geometric(q)$, where $q = \theta / (1 + \theta)$, and set them to be the number of new record highs.

3) Generate sequence $\{W_1, W_2, \cdots, W_D\}$ from i.i.d. $Exp(\lambda)$ and $\{X_1, X_2, \ldots, X_D\}$ with F_X. Let $T_j = W_1 + ... + W_j$ and $S_j = X_1 + ... + X_j$ be the timings of the claims and the values of the claims process, respectively. Compute the value of the process $\{S(t) - ct; 0 \le t \le T_D\}$ using $V_j = S_j - cT_j$, for $j = 1, 2, \ldots, D$.

4) If $V_j > 0$ for some $j = 1, 2, \ldots, D$, then let $Y_{1,D}^i$ be the first V_j above zero, else repeat step 3.

5) Obtain the amount of $\{Y_{2,D}^i, Y_{3,D}^i, ..., Y_{N_i,D}^i\}$ by repeating steps 2 to 4 and let $L_{i,D} = Y_{1,D}^i + ... + Y_{N_i,D}^i$.

6) For each i, if $L_{i,D} > u$, then $Z_i = 1$, otherwise $Z_i = 0$.

7) Calculate $\bar{Z} = \sum_{i=1}^{n} Z_i / n$.

8) Increase n and D of 5,000 or 10,000, and repeat steps 1 to 5 until $\bar{Z}$ remains constant.

In this study, the approximation $\psi_M(u) = \bar{Z}$ is proposed to approximate $E[Z_i] = Pr(L_{i,D} > u)$. From Theorem 4.1 and the fact that $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ is a sequence of i.i.d. random variables, $\{Y_{1,D}^i, Y_{2,D}^i, \ldots, Y_{N_i,D}^i\}$ converges in distribution to $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ when D is large, so that $Pr(L_{i,D} > u) = Pr(Y_{1,D}^i + ... + Y_{N_i,D}^i > u)$ converges to $Pr(Y_1^i + ... + Y_{N_i}^i > u) = \psi(u)$. Thus, $\psi_M(u)$ converges to $\psi(u)$ when n and D are large.

4.2 The Proposed Minimum Initial Capital Approximation

In this subsection, the algorithm to compute the minimum initial capital $u_\alpha^* = \min_{u \ge 0}\{u : \psi(u) \le \alpha\}$ for any acceptable levels $0 < \alpha < 1$ is proposed. From (2.21), the ruin probability is monotone non-increasing with initial capital u and

$\psi\left(u_\alpha^*\right)=\Pr\left(L>u_\alpha^*\right)=\alpha$, where L is the maximal aggregate loss. For discussing the proposed minimum initial capital approximation, we state Lemma 4.1 and Theorem 4.2 are stated as follows.

Lemma 4.1 Let non-ruin probability $\phi(u)=1-\psi(u)$, then the transformation through function $g(z)$ is defined for any probability $0<z<1$ by $g(z)=\phi^{-1}(z)$. We obtain the first derivative of g as

$$g'(z)=\frac{d}{dz}\phi^{-1}(z)=\frac{1}{\phi'\left(\phi^{-1}(z)\right)}.$$

Proof

Let $y=\phi^{-1}(z)$ iff $\phi(y)=z$, then $\phi'(y)dy=dz$ and $\frac{dy}{dz}=\frac{1}{\phi'(y)}=\frac{1}{\phi'\left(\phi^{-1}(z)\right)}$.

Theorem 4.2 Let $\{L_1,L_2,\ldots\}$ be a sequence of maximal aggregate loss with each $L_j, j=1,2,\ldots,m$ being independent random variables and distributed according to non-ruin probability $\phi(u)=1-\psi(u)$. Let $L_{[m(1-\alpha)]}$ be the $m(1-\alpha)^{th}$ order statistic based on $\{L_1,L_2,\ldots,L_m\}$ and $\alpha\geq\theta/(1+\theta)$, where $\theta>0$ is the relative security loading. Therefore,

$$\sqrt{m}\left(L_{[m(1-\alpha)]}-u_\alpha^*\right)\xrightarrow{d}W \sim N\left(0,\frac{\alpha(1-\alpha)}{\left[\phi'\left(u_\alpha^*\right)\right]^2}\right).$$

Proof

Suppose the sequence of maximal aggregate loss $L_1,L_2,\ldots,L_m$ consists of i.i.d. continuous random variables from a distribution with non-ruin probability ϕ. Let $\bar{Z}_m(u)$ be a random variable defined for positive initial capital u by

$$\bar{Z}_m(u) = \frac{1}{m}\sum_{i=1}^{m} Z_i(u),$$

where

$$Z_i(u) = \begin{cases} 1 & ; L_i \le u, \\ 0 & ; L_i > u. \end{cases},$$

Subsequently, $Z_i(u)$ has the expectation $E\left[Z_i(u)\right] = \Pr(L_i \le u) = \phi(u)$ and the variance $\sigma^2(u) = \phi(u)\left[1-\phi(u)\right]$, then by the central limit theorem,

$$\sqrt{m}\left(\bar{Z}_m(u) - \phi(u)\right) \xrightarrow{d} W \sim N\left(0, \phi(u)\left[1-\phi(u)\right]\right).$$

By Lemma 4.1, using the Delta method,

$$\sqrt{m}\left[\phi^{-1}\left(\bar{Z}_m(u)\right) - \phi^{-1}\left(\phi(u)\right)\right] \xrightarrow{d} W \sim N\left(0, \frac{\phi(u)\left[1-\phi(u)\right]}{\phi'\left(\phi^{-1}\left(\phi(u)\right)\right)}\right),$$

and, by replacing u with the minimum initial capital u_α^*, we obtain

$$\sqrt{m}\left[\phi^{-1}\left(\bar{Z}_m\left(u_\alpha^*\right)\right) - u_\alpha^*\right] \xrightarrow{d} W \sim N\left(0, \frac{\alpha(1-\alpha)}{\phi'\left(u_\alpha^*\right)}\right).$$

Now $\phi^{-1}\left(\bar{Z}_m(u)\right)$ is a random variable that lies between the order $\left[100(1-\alpha)-1\right]^{st}$ and $100(1-\alpha)^{th}$ sample quantile that can be written using order statistic notation as $L_{[m(1-\alpha)]}$. In fact,

$$\left|L_{[m(1-\alpha)]} - \phi^{-1}\left(\bar{Z}_m(u)\right)\right| \xrightarrow{a.s.} 0.$$

Hence, it follows that

$$\sqrt{m}\left(L_{[m(1-\alpha)]} - u_\alpha^*\right) \xrightarrow{d} W \sim N\left(0, \frac{\alpha(1-\alpha)}{\left[\phi'\left(u_\alpha^*\right)\right]^2}\right),$$

and the proof of Theorem 4.2 is complete.

Our proposed approximation of the minimum initial capital u_α^* is based on the idea of generating the number of new record highs N from $Geometric(q)$, where $q=\theta/(1+\theta)$. Next, the sequence $\{Y_{1,D},Y_{2,D},\ldots,Y_{N,D}\}$ is generated with the algorithm as in Subsection 4.1 when the number of truncated element D is large and the sequence $\{Y_{1,D},Y_{2,D},\ldots,Y_{N,D}\}$ converges to the sequence of new record highs $\{Y_1,Y_2,\ldots,Y_N\}$. We compute the maximal aggregate loss $L=Y_{1,D}+Y_{2,D}+\ldots+Y_{N,D}$, and by repeating this process m times, we obtain the sequence of maximal aggregate loss $\{L_1,L_2,\ldots,L_m\}$. From Theorem 4.2, the $m(1-\alpha)^{th}$ order statistic based on $\{L_1,L_2,\ldots,L_m\}$ defined by $L_{\lceil m(1-\alpha)\rceil}$ converges to u_α^* when m is large. For accuracy, we repeat the above process n times to obtain $\{L_{\lceil m(1-\alpha)\rceil,1},L_{\lceil m(1-\alpha)\rceil,2},\ldots,L_{\lceil m(1-\alpha)\rceil,n}\}$. Thus, the estimator $\bar{u}_\alpha=\sum_{i=1}^{n}L_{\lceil m(1-\alpha)\rceil,i}/n$ converges to u_α^* when n and m are large. The algorithm is as follows:

1) Approximate F_X based on real-life data.

2) Set the numbers of iterations n,m, and the number of truncated elements D to be some large numbers e.g. 5,000 or 10,000. Generate $N_i, i=1,2,\ldots,n$ from $Geometric(q)$, where $q=\theta/(1+\theta)$, and set them to be the number of new record highs.

3) Generate sequence $\{W_1,W_2,\cdots,W_D\}$ from i.i.d. $Exp(\lambda)$ and $\{X_1,X_2,\ldots,X_D\}$ with F_X. Let $T_j=W_1+\ldots+W_j$ and $S_j=X_1+\ldots+X_j$ be the timing of claims and values of the claims process, respectively. Compute the value of the process $\{S(t)-ct; 0\le t\le T_D\}$ using $V_j=S_j-cT_j$, for $j=1,2,\ldots,D$.

4) If $V_j>0$ for some $j=1,2,\ldots,D$, then let $Y_{1,D}^i$ be the first V_j above zero, else repeat step 3.

5) Obtain the amount of $\{Y_{2,D}^{i}, Y_{3,D}^{i}, ..., Y_{N_i,D}^{i}\}$ by repeating steps 2 to 4 and let $L_{i,D} = Y_{1,D}^{i} + ... + Y_{N_i,D}^{i}$.

6) Repeat steps 2 to 6 m times.

7) Let u_{α}^{j} be the $\left[m(1-\alpha)\right]^{th}$ smallest observation in $L_{N_1,D}, L_{N_2,D}, ..., L_{N_m,D}$.

8) Repeat steps 2 to 7 n times.

9) Estimate u_{α}^{*} by $\overline{u}_{\alpha} = \sum_{j=1}^{n} u_{\alpha}^{j} / n$.

10) Increase n, m, and D by 500 or 1,000, and repeat steps 2 to 9 until $\overline{u}_{\alpha}$ remains constant.

From previous section, $\left\{Y_{1,D}^{i}, Y_{2,D}^{i}, \ldots, Y_{N_i,D}^{i}\right\}$ in step 5 converges in distribution to the sequence of new record highs $\left\{Y_{1}^{i}, Y_{2}^{i}, \ldots, Y_{N_i}^{i}\right\}$ for the i^{th} loop and when D is large, so that their quantiles coincide. Moreover, Theorem 4.2 shows that u_{α}^{j} converges in distribution to u_{α}^{*} when m is large. Thus, the proposed approximated $\overline{u}_{\alpha} = \sum_{j=1}^{n} u_{\alpha}^{j} / n$ converges to u_{α}^{*} as n, m, and D become large.

CHAPTER 5

SIMULATION STUDY AND APPLICATION WITH REAL DATA

In this chapter, a report on the performance of the algorithm proposed in the previous chapter tested using a simulation study is presented. The proposed approximation was also applied to real-life data from an insurance company, a record of which is included after the simulation study.

5.1 Simulation Study for the Proposed Ruin Probability Approximation

To measure the performance of the proposed algorithm in Section 4.1, a numerical evaluation was used. From Section 2.3, the surplus process is based on the Poisson claims number process with intensity $\lambda > 0$, the initial capital $u \geq 0$, and the security loading $\theta > 0$ can be used to compute the exact ruin probability in the form

$$\psi(u) = \frac{1}{1+\theta} \exp\left(\frac{-\theta \beta u}{1+\theta} \right)$$

when the claim amounts distribution is $Expo(\beta)$. Moreover, it can be derived as

$$\psi(u) = -\left[\frac{\nu_2 (\nu_1 + \beta)^2}{(\nu_1 - \nu_2)\beta^2} e^{\nu_1 u} + \frac{\nu_1 (\nu_2 + \beta)^2}{(\nu_2 - \nu_1)\beta^2} e^{\nu_2 u} \right],$$

where $\nu_1 = \left(\lambda - 2c\beta + \sqrt{\lambda^2 + 4c\beta\lambda}\right)/(2c)$, $\nu_2 = \left(\lambda - 2c\beta - \sqrt{\lambda^2 + 4c\beta\lambda}\right)/(2c)$, and the premium rate $c = (1+\theta)2\lambda/\beta$ when the claim amounts distribution is $Gamma(2,\beta)$. Thus, the claim amounts distributions $Expo(\beta)$ and $Gamma(2,\beta)$, whose exact ruin probabilities are obtained when $\beta = 1,2$, were considered. Besides the proposed approximation $\psi_M(u)$ in Section 4.1, three other approximations were computed: the De Vylder approximation $\psi_{DV}(u)$, the Bowers approximation $\psi_{B}(u)$, and the Pollaczek-Khinchin approximation $\psi_{PK}(u)$. Approximated values from the proposed method were compared with the previously considered methods using their maximum absolute errors defined as $\max\left|\psi(u)-\hat{\psi}(u)\right|$ for exact ruin probability $\psi(u)$ and all considered approximations $\hat{\psi}(u)$. Moreover, all approximated values were also compared with the Lundberg upper bound of ruin probability in (2.14) as

$$\psi(u) \le e^{-Ru},$$

for any initial capital $u \ge 0$, and where the adjustment coefficient R is defined as in (2.15), whereby all approximated values should be lower than the Lundberg upper bound.

The intensity of the number of claims process was set to $\lambda = 1$. The exact ruin probability $\psi(u)$ is zero when both the security loading θ and the initial capital u are large, so only the results with $\theta = 0.1, 0.3, 0.5$ and $u = 0, 5, 10, \ldots, 30$ were considered. In the simulation, the Pollaczek-Khinchin approximated $\psi_{PK}(u)$ was computed with the number of iterations set at $n = 500{,}000$, and the proposed approximation $\psi_{M}(u)$ was computed with the number of iterations set at $n = 500{,}000$ and the number of truncated elements set at $D = 100$. The exact and all approximated ruin probability are shown in Tables 5.1-5.2.

Table 5.1 The exact ruin probability $\psi(u)$, the Lundberg upper bound e^{-Ru}, the Pollaczek-Khinchin approximated ruin probability $\psi_{PK}(u)$, and the proposed approximated ruin probability $\psi_M(u)$ for claim amounts distribution $Expo(\beta)$.

β	θ	R	u	$\psi(u)$	e^{-Ru}	$\psi_{PK}(u)$	$\psi_M(u)$
1	0.1	0.0909	0	0.9091	1.0000	0.9092	0.9092
			5	0.5770	0.6347	0.5774	0.5782
			10	0.3663	0.4029	0.3670	0.3659
			15	0.2325	0.2557	0.2326	0.2333
			20	0.1476	0.1623	0.1485	0.1472
			25	0.0937	0.1030	0.0936	0.0939
			30	0.0595	0.0654	0.0597	0.0594
	0.3	0.2308	0	0.7692	1.0000	0.7691	0.7697
			5	0.2426	0.3154	0.2424	0.2423
			10	0.0765	0.0995	0.0764	0.0771
			15	0.0241	0.0314	0.0243	0.0239
			20	0.0076	0.0099	0.0075	0.0078
			25	0.0024	0.0031	0.0024	0.0024
			30	0.0000	0.0000	0.0000	0.0000
	0.5	0.3333	0	0.6667	1.0000	0.6656	0.6677
			5	0.1259	0.1889	0.1262	0.1250
			10	0.0238	0.0357	0.0239	0.0240
			15	0.0045	0.0067	0.0045	0.0045
			20	0.0008	0.0013	0.0008	0.0009
			25	0.0002	0.0002	0.0002	0.0002
			30	0.0000	0.0000	0.0000	0.0000

Table 5.1 (Continued)

β	θ	R	u	$\psi(u)$	e^{-Ru}	$\psi_{PK}(u)$	$\psi_M(u)$
2	0.1	0.1818	0	0.9091	1.0000	0.9092	0.9095
			5	0.3663	0.4029	0.3672	0.3653
			10	0.1476	0.1623	0.1482	0.1478
			15	0.0595	0.0654	0.0592	0.0594
			20	0.0240	0.0263	0.0238	0.0239
			25	0.0097	0.0106	0.0097	0.0095
			30	0.0039	0.0043	0.0040	0.0040
	0.3	0.4615	0	0.7692	1.0000	0.7691	0.7696
			5	0.0765	0.0995	0.0760	0.0764
			10	0.0076	0.0099	0.0076	0.0076
			15	0.0008	0.0010	0.0008	0.0008
			20	0.0001	0.0001	0.0001	0.0001
			25	0.0000	0.0000	0.0000	0.0000
			30	0.0000	0.0000	0.0000	0.0000
	0.5	0.6667	0	0.6667	1.0000	0.6671	0.6665
			5	0.0238	0.0357	0.0236	0.0237
			10	0.0008	0.0013	0.0009	0.0008
			15	0.0000	0.0000	0.0000	0.0000
			20	0.0000	0.0000	0.0000	0.0000
			25	0.0000	0.0000	0.0000	0.0000
			30	0.0000	0.0000	0.0000	0.0000
Maximum Absolute Error						0.0011	0.0013

Note: For the exponential, claim amounts distribution, $\psi_{DV}(u)$ and $\psi_B(u)$ are equal to the exact ruin probability $\psi(u)$. Hence, the value of $\psi_{DV}(u)$ and $\psi_B(u)$ are not shown in Tables 5.1.

The results shown in Table 5.1 show that $\psi_{PK}(u)$ and $\psi_M(u)$ were close to the exact ruin probability $\psi(u)$ and none of them were higher than e^{-Ru}, which means that both of them gave reasonable values.

Table 5.2 The exact ruin probability $\psi(u)$, the Lundberg upper bound e^{-Ru}, the De Vylder approximated ruin probability $\psi_{DV}(u)$, the Bowers approximated ruin probability $\psi_B(u)$, the Pollaczek-Khinchin approximated ruin probability $\psi_{PK}(u)$, and the proposed approximated ruin probability $\psi_M(u)$ for claim amounts distribution $Gamma(2,\beta)$.

β	θ	R	u	$\psi(u)$	e^{-Ru}	$\psi_{DV}(u)$	$\psi_B(u)$	$\psi_{PK}(u)$	$\psi_M(u)$
1	0.1	0.0613	0	0.9091	1.0000	0.9184	0.9091	0.9090	0.9091
			5	0.6767	0.7360	0.6762	0.6714	0.6237	0.6773
			10	0.4982	0.5417	0.4979	0.4959	0.4391	0.4991
			15	0.3668	0.3987	0.3666	0.3663	0.3224	0.3679
			20	0.2700	0.2935	0.2699	0.2705	0.2420	0.2703
			25	0.1988	0.2160	0.1987	0.1998	0.1849	0.1995
			30	0.1463	0.1590	0.1463	0.1476	0.1433	0.1467
	0.3	0.1584	0	0.7692	1.0000	0.7895	0.7692	0.7702	0.7692
			5	0.3600	0.4529	0.3585	0.3564	0.3204	0.3600
			10	0.1631	0.2052	0.1628	0.1652	0.1511	0.1626
			15	0.0739	0.0929	0.0739	0.0765	0.0793	0.0736
			20	0.0335	0.0421	0.0336	0.0355	0.0448*	0.0332
			25	0.0152	0.0191	0.0152	0.0164	0.0258*	0.0154
			30	0.0069	0.0086	0.0069	0.0076	0.0159*	0.0069

Table 5.2 (Continued)

β	θ	R	u	$\psi(u)$	e^{-Ru}	$\psi_{DV}(u)$	$\psi_B(u)$	$\psi_{PK}(u)$	$\psi_M(u)$
	0.5	0.2324	0	0.6667	1.0000	0.6923	0.6667	0.6680	0.6667
			5	0.2199	0.3129	0.2184	0.2195	0.1971	0.2191
			10	0.0688	0.0979	0.0689	0.0722	0.0710	0.0692
			15	0.0215	0.0306	0.0217	0.0238	0.0301	0.0213
			20	0.0067	0.0096	0.0069	0.0078	0.0142*	0.0067
			25	0.0021	0.0030	0.0022	0.0026	0.0070*	0.0020
			30	0.0007	0.0009	0.0007	0.0008	0.0038*	0.0006
2	0.1	0.1225	0	0.9091	1.0000	0.9184	0.9091	0.9077	0.9091
			5	0.4982	0.5420	0.4979	0.4959	0.4408	0.4986
			10	0.2700	0.2938	0.2699	0.2705	0.2408	0.2710
			15	0.1463	0.1592	0.1463	0.1476	0.1424	0.1472
			20	0.0793	0.0863	0.0793	0.0805	0.0889*	0.0800
			25	0.0430	0.0468	0.0430	0.0439	0.0574*	0.0432
			30	0.0233	0.0253	0.0233	0.0240	0.0381*	0.0236
	0.3	0.3168	0	0.7692	1.0000	0.7895	0.7692	0.7687	0.7684
			5	0.1631	0.2052	0.1628	0.1652	0.1506	0.1638
			10	0.0335	0.0421	0.0336	0.0355	0.0444	0.0334
			15	0.0069	0.0086	0.0069	0.0076	0.0161*	0.0066
			20	0.0014	0.0018	0.0014	0.0016	0.0064*	0.0014
			25	0.0003	0.0004	0.0003	0.0004	0.0028*	0.0003
			30	0.0001	0.0001	0.0001	0.0001	0.0014*	0.0001
	0.5	0.4648	0	0.6667	1.0000	0.6923	0.6667	0.6662	0.6663
			5	0.0688	0.0979	0.0689	0.0722	0.0713	0.0689
			10	0.0067	0.0096	0.0069	0.0078	0.0143*	0.0068
			15	0.0007	0.0009	0.0007	0.0008	0.0037*	0.0006
			20	0.0001	0.0001	0.0001	0.0001	0.0012*	0.0001
			25	0.0000	0.0000	0.0000	0.0000	0.0004*	0.0000
			30	0.0000	0.0000	0.0000	0.0000	0.0001*	0.0000
	Maximum Absolute Error					0.0256	0.0053	0.0591	0.0011

Note: *the approximation exceeds the Lundberg upper bound

The maximum absolute error results in Table 5.2 show that when the claim amounts distribution was $Gamma(2,\beta)$, $\psi_M(u)$ performed the best in terms of overall deviation from the exact value with the maximum absolute error 0.0011. Moreover, all of the values of $\psi_M(u)$ were less than e^{-Ru}, whereas most values of $\psi_{PK}(u)$ were higher than e^{-Ru} when the security loading θ was large. The value of $\psi_{DV}(u)$ produced a high maximum absolute error because there were more derivations between $\psi(0)$ and $\psi_{DV}(0)$. Overall, both $\psi_M(u)$ and $\psi_{DV}(u)$ gave similar results. However, in the case where the first three moments of claim amounts distribution did not exist, $\psi_{DV}(u)$ was undefined.

5.2 Simulation Study for the Proposed Minimum Initial Capital Approximation

To evaluate the performance for the approximation of the minimum initial capital, $\bar{u}_\alpha$ proposed in Section 4.2, a numerical study was carried out. The claim amounts distributions $Expo(1)$, $Expo(2)$, $Gamma(2,1)$, and $Gamma(2,2)$, where the minimum initial capital u_α^* was obtained by setting (2.11) and (2.12) equal to the acceptable level α, were used. The Lundberg upper bound (2.14) and the fact that the ruin probability is a non-increasing function in u means that the upper bound of the minimum initial capital can derived as follows:

$$\begin{aligned} \psi\left(u_\alpha^*\right) = \alpha \quad & \leq e^{-Ru_\alpha^*}, \\ u_\alpha^* \quad & \leq \frac{\ln \alpha}{R}, \end{aligned} \tag{5.1}$$

where the adjustment coefficient R is defined as in (2.15). If the simulation results show that the proposed estimate was greater than the upper bound for any cases, then the proposed estimator was considered not reasonable. The intensity of the number of claims processes was set as $\lambda = 1$. The exact minimum initial capital u_α^* is close to zero when both of the security loading θ and the acceptable level α are large, so only the results with $\theta = 0.1, 0.3, 0.5$ and $\alpha = 0.05, 0.1, 0.2$ were considered. The approximation $\bar{u}_\alpha$ in Section 4.2 was computed after the numbers of iterations $m = 5,000, n = 1,000$ and the number of truncated elements $D = 100$ were completed. The exact minimum initial capital u_α^***,** the proposed estimator $\bar{u}_\alpha$, and the upper bound of the minimum initial capital $\ln \alpha / R$ are shown in Table 5.3.

Table 5.3 The minimum initial capital u_α^*, the proposed estimator $\bar{u}_\alpha$, and the upper bound of the minimum initial capital $\ln \alpha / R$ with the claim amounts distributions $Expo(1)$, $Expo(2)$, $Gamma(2,1)$, and $Gamma(2,2)$.

θ	$F(x)$	α	u_α^*	$\bar{u}_\alpha$	$\ln \alpha / R$
0.1	$Expo(1)$	0.05	31.9046	31.5599	32.9531
	$R = 0.0909$	0.10	24.2800	24.6094	25.3284
		0.20	16.6554	16.0938	17.7038
	$Expo(2)$	0.05	15.9523	15.9683	16.4765
	$R = 0.1818$	0.10	12.1400	12.3779	12.6642
		0.20	8.3277	8.4085	8.8519
	$Gamma(2,1)$	0.05	47.5332	47.7969	48.9090
	$R = 0.0613$	0.10	36.2167	36.5589	37.5926
		0.20	24.9003	24.3273	26.2761
	$Gamma(2,2)$	0.05	23.7666	23.5672	24.4545
	$R = 0.1225$	0.10	18.1084	18.0899	18.7963
		0.20	12.4501	12.7812	13.1380
0.3	$Expo(1)$	0.05	11.8446	11.8493	12.9815
	$R = 0.2308$	0.10	8.8410	8.8602	9.9779
		0.20	5.8373	5.8541	6.9742
	$Expo(2)$	0.05	5.9223	5.8299	6.4908
	$R = 0.4615$	0.10	4.4205	4.3561	4.9889
		0.20	2.9187	2.9367	3.4871

Table 5.3 (Continued)

θ	$F(x)$	α	u_α^*	$\bar{u}_\alpha$
$Gamma(2,1)$	0.05	17.4632	17.5061	18.9140
$R = 0.1584$	0.10	13.0869	13.0808	14.5377
	0.20	8.7106	8.9683	10.1614
$Gamma(2,2)$	0.05	8.7316	8.6122	9.4570
$R = 0.3168$	0.10	6.5435	6.6101	7.2689
	0.20	4.3553	4.4709	5.0807
$Expo(1)$	0.05	7.7708	7.9780	8.9881
$R = 0.3333$	0.10	5.6914	5.5932	6.9084
	0.20	3.6119	3.5740	4.8288
$Expo(2)$	0.05	3.8854	3.9890	4.4934
$R = 0.6667$	0.10	2.8457	2.7966	3.4537
	0.20	1.8060	1.7870	2.4140
$Gamma(2,1)$	0.05	11.3745	11.5783	12.8904
$R = 0.2324$	0.10	8.3920	8.1980	9.9079
	0.20	5.4092	5.2827	6.9253
$Gamma(2,2)$	0.05	5.6872	5.7891	6.4452
$R = 0.4648$	0.10	4.1960	4.0990	4.9539
	0.20	2.7046	2.6413	3.4626

From Table 5.3, we can see that the proposed estimator $\bar{u}_\alpha$ was close to the exact minimum initial capital u_α^* for all exponential and gamma claim amounts distributions considered. Moreover, all of the values $\bar{u}_\alpha$ were within the upper bound of the minimum initial capital. Thus, the proposed approximation $\bar{u}_\alpha$ is reasonable for computing the minimum initial capital.

5.3 Application of the Proposed Approximation to Real-life Data

In this section, the proposed approximation was applied to claims data from a motor insurance company. From a survey in 2013, the insurance company under consideration had an average of 13.1275 claims per day. As mentioned previously, it is presumed that the number of claims follow a Poisson distribution with estimated parameter $\hat{\lambda}=13.1275$. From Section 3.5, the claim amounts distribution was fitted as AGEW with estimated parameters $\hat{\delta}=0.6242$, $\hat{\alpha}=33.4948$, $\hat{\beta}=0.2966$, and $\hat{\tau}=5.5117$.

5.3.1 The Ruin Probability Approximation

Under the AGEW claim amounts distribution, the first three moments of the claim amounts distribution are complicated to define, and the De Vylder approximation and the Bowers approximation cannot be applied to this claim amounts data. However, it was possible to use the proposed ruin probability approximation method since it does not require the second and third moments of the claim amounts distribution. For premium rate c, as in equation (2.9), which depends on the expected value μ_1 of the claim amounts distribution, the sample mean of the claim amounts data was use to estimate it. From equations (2.8) to (2.10), the ruin probability is monotone non-increasing with security loading θ and initial capital u. Thus, the security loading $\theta = 0.1, 0.3, 0.8$ and the initial capital $u = 0, 10, 20, 30$ were set so that they gave an overview of the exact ruin probability. For an approximation based on simulation, the number of iterations were set at $n = 500,000$ and the number of truncated elements at $D = 100$. The approximated ruin probability for this company is shown in Table 5.4.

It can be seen that the approximated ruin probabilities in Table 5.4 decrease when the security loading θ and the initial capital u increase, which is concordant with the theoretical ruin probability. Therefore, the approximated ruin probability for the claim amounts distribution under the AGEW model is reasonable.

Table 5.4 The approximate ruin probability with real-life data.

θ	u (1,000 Baht)	$\psi_M(u)$
0.1	0	0.9088
	10	0.8665
	20	0.8317
	30	0.8025
0.3	0	0.7697
	10	0.6820
	20	0.6162
	30	0.5654
0.8	0	0.5561
	10	0.4419
	20	0.3662
	30	0.3107

5.3.2 The Minimum Initial Capital Approximation

As well as setting the security loading $\theta = 0.1, 0.3, 0.8$ and the acceptable level $\alpha = 0.0, 0.10,$ and 0.30 for the minimum initial capital approximation based on simulation, we set the number of iterations to $n = 1,000$, $m = 5,000$ and the number of truncated elements to $D = 100$. The approximated minimum initial capital for this company is shown in Table 5.5, from which we can see that when the security loading θ and the acceptable level α are small, the minimum initial capital is large. These results are reasonable because if the premium and the acceptable risk are low, then the insurance company must reserve more initial capital to decrease the ruin probability.

Table 5.5 The approximate minimum initial capital with real-life data

θ	α	$\bar{u}_{\alpha}$ (1,000 Baht)
0.1	0.05	1,118.3910
	0.10	847.6619
	0.30	390.6305
0.3	0.05	451.4544
	0.10	307.8866
	0.30	117.9063
0.8	0.05	201.2936
	0.10	123.1191
	0.30	32.8554

CHAPTER 6

CONCLUSIONS

6.1 Conclusions

We proposed a new distribution, the AGEW distribution, for modeling the claim amounts distribution of insurance companies. It was obtained by mixing gamma and EW distributions, and contains sub-models that are well-known: gamma and Weibull, and its basic mathematical properties such as distribution function, density function, and moments were determined. Moreover, the MLE method was applied to estimate its parameters and the fitness of the AGEW distribution was tested with a real-life dataset from an insurance company. We compared the AGEW distribution with its sub-models: gamma, Weibull, and EW, and the K-S statistic and MSE results in Table 3.1 show that the AGEW distribution provided a better fit than the gamma and Weibull distributions. Both AGEW and EW gave similar results, but the AGEW distribution was chosen since it had a slightly lower MSE than EW when modeling the claim amounts distribution.

When the claim amounts distribution is exponential or closely related to it, the ruin probability over infinite time with a classical continuous time surplus process exists. However, for other claim amounts distributions, the approximate ruin probability is used. In this study, a new simple approximate ruin probability for use with any claim amounts distribution was proposed. The numerical studies showed that almost all of the approximated ruin probabilities by the proposed approximation were reasonable and close to the exact ruin probability. In some situations, the proposed method gave better approximated values than other previously reported approximation methods.

By application of the proposed ruin probability approximation, a method to find the approximate minimum initial capital for any claim amounts distribution was proposed. The numerical study showed that the proposed approximation was close to the exact minimum initial capital. Therefore, the proposed approximation is reasonable and useful for reserving the initial capital for managing the ruin probability of companies over infinite time since it was no larger than the given quantity. The results from Tables 5.6 and 5.7 also support these findings and show that the approximate ruin probabilities and minimum initial capital under for the claim amounts distributions with the AGEW model is more preferable.

In summary, for applying the proposed procedure to real-life situations, the recommended procedure is as follows:

1) Choose candidate distributions where the shape of the density function is fitted using a histogram of the real-life data.

2) Estimate the parameters of each candidate distribution (the MLE method was used in this study).

3) Test the candidate distributions with a goodness-of-fit test (such as the K-S test). If none of them are accepted, then other candidate distributions should be chosen and tested.

4) If a number of candidate distributions are accepted, then they should be compared with a criterion such mean squared error (MSE) to determine which one has the best fit.

5) Approximate the ruin probability and the minimum initial capital by using the proposed algorithm.

6.2 Recommendations for Future Research

A possible extension of this study could be to improve the AGEW distribution by replacing the EW distribution with the exponentiated Weibull Poisson distribution described by Percontini, Blas and Cordeiro (2013). Furthermore, it could be interesting to extend the proposed approximation for the ruin probability and the minimum initial capital to situations where the surplus process is controlled by reinsurance or investment in a financial market.

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APPENDIX

Appendix

R SYNTAX

R Syntax for Calculate the Approximate Ruin Probability

```
ap_ruin2_gamma=function (n,u,zeta,alpha,beta,lamda)
{
        #This program aproximate ruin probability based on classic continuous
        #surplus process which gamma(alpha,beta) claim amount
        #u is initial surplus
        #zeta is security loading
        #n is number of simulation

        q=zeta/(1+zeta)
        N=rgeom(n,q)
        L=rep(0,n)
        Z=rep(0,n)
        c=(1+zeta)*lamda*alpha/beta

        for(i in 1:n){
        m=0
        pre_Lmmc=0
        while(m<N[i]){
                S=0
                T=0
                out=0
                limit=0
                        while((out<=0)&(limit<=100)){
                                S=S+rgamma(1,alpha,beta)
                                T=T+rexp(1,lamda)
                                limit=limit+1
                                out=S-c*T
                        }
                pre_Lmmc=c(pre_Lmmc,out)
                Lmmc=pre_Lmmc[pre_Lmmc>0]
                m=length(Lmmc)
        }

                if(N[i]>0)(L[i]=sum(Lmmc))
                if(L[i]>u)(Z[i]=1)
        }
```

```
    ruin=mean(Z)
    return(ruin)
}

ap_ruin2_gamma(n=5000,u=0,zeta=0.1,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=5,zeta=0.1,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=10,zeta=0.1,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=15,zeta=0.1,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.1,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.1,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.1,alpha=1,beta=1,lamda=1)

ap_ruin2_gamma(n=500000,u=0,zeta=0.1,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=5,zeta=0.1,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=10,zeta=0.1,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=15,zeta=0.1,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.1,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.1,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.1,alpha=1,beta=2,lamda=1)

ap_ruin2_gamma(n=500000,u=0,zeta=0.1,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=5,zeta=0.1,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=10,zeta=0.1,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=15,zeta=0.1,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.1,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.1,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.1,alpha=2,beta=1,lamda=1)

ap_ruin2_gamma(n=500000,u=0,zeta=0.1,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=5,zeta=0.1,alpha=2,beta=2,lamda=1)
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ap_ruin2_gamma(n=500000,u=20,zeta=0.1,alpha=2,beta=2,lamda=1)
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ap_ruin2_gamma(n=500000,u=30,zeta=0.3,alpha=1,beta=1,lamda=1)

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ap_ruin2_gamma(n=500000,u=15,zeta=0.3,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.3,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.3,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.3,alpha=2,beta=1,lamda=1)

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ap_ruin2_gamma(n=500000,u=10,zeta=0.3,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=15,zeta=0.3,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.3,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.3,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.3,alpha=2,beta=2,lamda=1)

ap_ruin2_gamma(n=500000,u=0,zeta=0.5,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=5,zeta=0.5,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=10,zeta=0.5,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=15,zeta=0.5,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.5,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.5,alpha=1,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.5,alpha=1,beta=1,lamda=1)

ap_ruin2_gamma(n=500000,u=0,zeta=0.5,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=5,zeta=0.5,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=10,zeta=0.5,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=15,zeta=0.5,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.5,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.5,alpha=1,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.5,alpha=1,beta=2,lamda=1)
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ap_ruin2_gamma(n=500000,u=0,zeta=0.5,alpha=2,beta=1,lamda=1)
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ap_ruin2_gamma(n=500000,u=10,zeta=0.5,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=15,zeta=0.5,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.5,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.5,alpha=2,beta=1,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.5,alpha=2,beta=1,lamda=1)

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ap_ruin2_gamma(n=500000,u=5,zeta=0.5,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=10,zeta=0.5,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=15,zeta=0.5,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=20,zeta=0.5,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=25,zeta=0.5,alpha=2,beta=2,lamda=1)
ap_ruin2_gamma(n=500000,u=30,zeta=0.5,alpha=2,beta=2,lamda=1)

rgew=function (n,delta=1,alpha=1,beta=1,zeta=1)
{
#Return random sample size n with Gamma Exponentiated Weibull distribution
        x=rgamma(n,delta)
        v=(log(1/(1-(1-exp(-x))^(1/alpha)))^(1/beta))/zeta
        return(v)
}


delta=0.6242355
alpha=33.4948214
beta=0.2966394
theta=5.5116698
lamda=13.1275
mean= 19.62752

ap_ruin2_gew=function (n,u,zeta,delta,alpha,beta,theta,mean,lamda)
{
        #This program aproximate ruin probability based on classic continuous
        #surplus process which AGEW(delta,alpha,beta,theta) claim amount
        #u is initial surplus
        #zeta is security loading
        #n is number of simulation

        q=zeta/(1+zeta)
        N=rgeom(n,q)
        L=rep(0,n)
        Z=rep(0,n)
```

```
        c=(1+zeta)*lamda*mean

        for(i in 1:n){
        m=0
        pre_Lmmc=0
        while(m<N[i]){
                S=0
                T=0
                out=0
                limit=0
                        while((out<=0)&(limit<=100)){
                                S=S+rgew(1, delta,alpha,beta,theta)
                                T=T+rexp(1,lamda)
                                limit=limit+1
                                out=S-c*T
                        }
                pre_Lmmc=c(pre_Lmmc,out)
                Lmmc=pre_Lmmc[pre_Lmmc>0]
                m=length(Lmmc)
        }

                if(N[i]>0)(L[i]=sum(Lmmc))
                if(L[i]>u)(Z[i]=1)
        }

        ruin=mean(Z)
        return(ruin)
}

ap_ruin2_gew(n=500000,u=0,zeta=0.1,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=10,zeta=0.1,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=20,zeta=0.1,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=30,zeta=0.1,delta,alpha,beta,theta,mean=mean,lamda=lamda)

ap_ruin2_gew(n=500000,u=0,zeta=0.3,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=10,zeta=0.3,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=20,zeta=0.3,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=30,zeta=0.3,delta,alpha,beta,theta,mean=mean,lamda=lamda)


ap_ruin2_gew(n=500000,u=0,zeta=0.8,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=10,zeta=0.8,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=20,zeta=0.8,delta,alpha,beta,theta,mean=mean,lamda=lamda)
ap_ruin2_gew(n=500000,u=30,zeta=0.8,delta,alpha,beta,theta,mean=mean,lamda=lamda)
```

R Syntax for Calculate the Approximate Initial Capital

```
map_mic_gamma=function (m,n,a,zeta,lamda,eta,beta)
{
	#Writed by Pawat Paksarnuwat 08/07/57
	#This program aproximate mic based on classic continuous
	#surplus process which gamma(eta,beta) claim amount
	#u is initial surplus
	#zeta is security loading
	#m and n is number of simulation

	u=rep(0,n)
	q=zeta/(1+zeta)
	N=rgeom(m,q)
	L=rep(0,m)
	Z=rep(0,m)
	c=(1+zeta)*lamda*eta/beta

	for(j in 1:n){
	for(i in 1:m){
	k=0
	pre_Lmmc=0
	while(k<N[i]){
		S=0
		T=0
		out=0
		limit=0
			while((out<=0)&(limit<=100)){
				S=S+rgamma(1,eta,beta)
				T=T+rexp(1,lamda)
				limit=limit+1
				out=S-c*T
			}
		pre_Lmmc=c(pre_Lmmc,out)
		Lmmc=pre_Lmmc[pre_Lmmc>0]
		k=length(Lmmc)
	}
	if(N[i]>0)(L[i]=sum(Lmmc))
	}
	u[j]=quantile(L,(1-a))
	}

	mic=mean(u)
```

```
        return(mic)
}

map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.1,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.1,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.1,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.1,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.1,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.1,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.1,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.1,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.1,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.1,lamda=1,eta=2,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.1,lamda=1,eta=2,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.1,lamda=1,eta=2,beta=2)

map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.3,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.3,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.3,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.3,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.3,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.3,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.3,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.3,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.3,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.3,lamda=1,eta=2,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.3,lamda=1,eta=2,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.3,lamda=1,eta=2,beta=2)

map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.5,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.5,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.5,lamda=1,eta=1,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.5,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.5,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.5,lamda=1,eta=1,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.5,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.5,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.5,lamda=1,eta=2,beta=1)
map_mic_gamma(m=5000,n=1000,a=0.05,zeta=0.5,lamda=1,eta=2,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.1,zeta=0.5,lamda=1,eta=2,beta=2)
map_mic_gamma(m=5000,n=1000,a=0.2,zeta=0.5,lamda=1,eta=2,beta=2)

map_mic_gew=function (m,n,a,zeta,lamda,delta,alpha,beta,theta,mean)
```

```
{
	#This program aproximate mic based on classic continuous
	#surplus process which AGEW(delta,alpha,beta,theta) claim amount
	#u is initial surplus
	#zeta is security loading
	#m and n is number of simulation

	u=rep(0,n)
	q=zeta/(1+zeta)
	N=rgeom(m,q)
	L=rep(0,m)
	Z=rep(0,m)
	c=(1+zeta)*lamda*mean

	for(j in 1:n){
	for(i in 1:m){
	k=0
	pre_Lmmc=0
	while(k<N[i]){
		S=0
		T=0
		out=0
		limit=0
			while((out<=0)&(limit<=100)){
				S=S+ rgew(1, delta,alpha,beta,theta)
				T=T+rexp(1,lamda)
				limit=limit+1
				out=S-c*T
			}
		pre_Lmmc=c(pre_Lmmc,out)
		Lmmc=pre_Lmmc[pre_Lmmc>0]
		k=length(Lmmc)
	}
	if(N[i]>0)(L[i]=sum(Lmmc))
	}
	u[j]=quantile(L,(1-a))
	}

	mic=mean(u)
	return(mic)
}
map_mic_gew(m=5000,n=1000,a=0.05,zeta=0.1,lamda=lamda,delta,alpha,beta,theta,mean=mean)
map_mic_gew(m=5000,n=1000,a=0.1,zeta=0.1,lamda=lamda,delta,alpha,beta,theta,mean=mean)
map_mic_gew(m=5000,n=1000,a=0.3,zeta=0.1,lamda=lamda,delta,alpha,beta,theta,mean=mean)
```

```
map_mic_gew(m=5000,n=1000,a=0.05,zeta=0.3,lamda=lamda,delta,alpha,beta,theta,mean=mean)
map_mic_gew(m=5000,n=1000,a=0.1,zeta=0.3,lamda=lamda,delta,alpha,beta,theta,mean=mean)
map_mic_gew(m=5000,n=1000,a=0.3,zeta=0.3,lamda=lamda,delta,alpha,beta,theta,mean=mean)

map_mic_gew(m=5000,n=1000,a=0.05,zeta=0.8,lamda=lamda,delta,alpha,beta,theta,mean=mean)
map_mic_gew(m=5000,n=1000,a=0.1,zeta=0.8,lamda=lamda,delta,alpha,beta,theta,mean=mean)
map_mic_gew(m=5000,n=1000,a=0.3,zeta=0.8,lamda=lamda,delta,alpha,beta,theta,mean=mean)
```

BIOGRAPHY

NAME	Mr. Pawat Paksaranuwat
ACADEMIC BACKGROUND	A Bachelor's Degree with a major in Applied Statistics from King Mongkut's Institute of Technology, North Bangkok, Bangkok, Thailand in 2007 Master's Degree in Applied Statistics from King Mongkut's University of Technology, North Bangkok, Bangkok, Thailand in 2009