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Original Article

The Fermat-type equation with signature (2, 2, n) and Bunyakovsky conjecture

Sawian Jaidee*, and Korakot Saosoong

Department of Mathematics, Faculty of Science,
Khon Kaen University, Mueang, Khon Kaen, 40002 Thailand

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Abstract

We first discuss the Fermat-type equation with signature $(2, m, n)$, which is the Diophantine equation in the shape $x^2 + y^m = z^n$, where x, y and z are unknown integers, and m, n are fixed positive integers greater than 1. This kind of equations has been particularly focused on our work here in the case $m = 2, n = 5$ and $y = p$ is a fixed rational prime. Then the first result describing the condition of such a prime p in order to illustrate that this certain equation has an integer solution (x, y) when $p \equiv 1 (\mathrm{mod}\ 4)$ and $\gcd(x, p) = 1$, and the second result stating that the equation has no integer solution (x, y) when $p \equiv 3 (\mathrm{mod}\ 4)$ are provided. Lastly, we will indicate that the results of Be´rczes and Pink about solving the equation $x^2 + p^{2k} = z^n$ in 2008 have been generalized in the particular cases $(n, k) = (3,1)$ and $(5,1)$, and additionally present that the first result and also its analogous result in the particular case $n = 3$ can be linked to the Bunyakovsky conjecture.

Keywords: fermat- type equations, Gaussian integers, unique factorization, Bunyakovsky conjecture

1. Introduction

We call the Diophantine equation in the form of

$$x^l + y^m = z^n \tag{1.1}$$

the generalized Fermat equation or simply says the Fermat - type equation, where x, y and z are unknown integers, and the exponents l, m and n are fixed positive integers greater than 1. Our attention is always to consider the integer solutions (x, y, z) of (1.1) as primitive solutions since the others are not interesting and will be fairly set them in the sense of a trivial fashion. The triple (l, m, n) is said to be the signature of (1.1). For instance, the equation with signatures (n, n, n) is called the family of the *Fermat's Last Theorem*, which has been eventually proven by Andrew Wile in the early 1990s, and he was awarded the Abel prize in 2016 (Mckenzie, 2016; Wikipedia contributors, 2021). This is the reason why we call the equation (1.1) the generalized Fermat equation. More detail about these may be seen in (Wiles,1995; Faltings,1995; John Wiles,1996). In fact, the quantity of $\sigma(l, m, n) = \frac{1}{l} + \frac{1}{m} + \frac{1}{n}$ allows us to understand about the behaviour of such primitive solutions. Following the papers appearing in (Beukers,1998 ;Bennett, Mihăilescu & Siksek, 2016; Nils,1999), we shall give more precise statements as follows: (i) if $\sigma(l, m, n) < 1$, then (1.1) has at most finitely many integer solutions, for instance, the signatures $(n, n, n), n \geq 4$ have no non-trivial solution, (ii) if $\sigma(l, m, n) = 1$, then all possible signatures of (1.1) can happen only

$$(l, m, n) = (2, 6, 3), (2, 4, 4), (4, 4, 2), (3, 3, 3) \text{ or } (2, 3, 6),$$

arising from a corresponding elliptic curve of rank 0 over $\mathbb{Q}$, and the last equation has a non-trivial solution, but the others have only trivial solutions, (iii) if $\sigma(l, m, n) > 1$, then (1.1) has either no solutions or infinitely many solutions which come from its possible signature shown as

$$(2,2,n) \text{ for } n \geq 2, \text{ or } (2,3,n) \text{ for } n = 3, 4, 5,$$

for instance, the signature $(2,2,2)$ has infinitely many non-trivial solutions depending on two parameters and it is well-known as the *Pythagorean equation*. Moreover, we normally know that the signatures $(2,2,n)$ has no integer solution when n is an even integer greater than 2 and this fact may be seen in the book of Andreescu *et al.*, (Andreescu, 2010). The three distinguished cases above are respectively called the

*Corresponding author
Email address: jsawia@kku.ac.th

hyperbolic, *parabolic* and *spherical cases*. We will consider throughout this paper only the spherical one.

Now, we will divide our discussion about some results concerning the Fermat-type equations with the signature $(2,m,n)$ and $y = p$ is a fixed rational prime into 2 cases: (note that these equations may be viewed as the *generalized Ramanujan-Nagell equation* and to be in order, we are going to use the unknown y instead of z from now on).

Case A: Let $m = 2k + 1$, where k is a positive integer, p be odd such that $p \not\equiv 7 (\mathrm{mod}\ 8)$, and $n \geq 3$ be an odd integer with $\gcd(n,h) = 1$, where h is the class number of the number field $\mathbb{Q}(\sqrt{-p})$. When $n \geq 5$ is not a multiple of 3, the equation $x^2 + p^{2k+1} = y^n$ with $\gcd(p,x) = 1$ has exactly two families of solutions given by

$$p = 19, n = 5, k = 5M, x = 22434 \times 19^{5M}, y = 55 \times 19^{2M}, \text{ and}$$
$$p = 341, n = 5, k = 5M, x = 2759646 \times 341^{5M}, y = 377 \times 19^{2M},$$

and its proof may be seen in (Arif & Abu Muriefah, 2002). The particular equation $x^2 + p^{2k+1} = y^3, p \neq 3$ has exactly one solution $(p,k,x,y) = (11,1,9324,443)$ due to (Lin Zhu, 2011) and moreover, Arif and Abu Muriefah illustrated in 1998 that the equation $x^2 + 3^m = y^n$ has the unique solution given by

$$m = 5 + 6N, x = 10 \times 33^N, y = 7 \times 32^N, \text{ and } n = 3.$$

More detail can be seen in (Arif & Abu Muriefah, 1998).

Case B: Let $m = 2k$, where k is a positive integer. In 2008, Be´rczes and Pink showed that all integer solutions of the equation $x^2 + p^{2k} = y^n$ are

$$(x,y,p,n,k) = (11,5,2,3,1), (46,13,3,3,2), (524,65,7,3,1), (2,5,11,3,1),$$
$$(278,5,29,7,1), (38,5,41,5,1), (52,17,47,3,1), (1405096,12545,97,3,1)$$

under the following conditions:

(1) x, y, n, k are integer unknowns satisfying $x \geq 1, y > 1, n \geq 3$ is prime and $k \geq 0$ with $\gcd(x,y) = 1$, and

(2) $2 \leq p \leq 100$.

Moreover, they gave the particular case $n = p$ that this equation has no integer solution when the conditions (1) holds and $p \geq 5$. See more detail in (B´erczes & Pink, 2008).

Inspired by all mentioned information previously, we will introduce a new look that enables us to provide more suitable primes p for being an existence of an integer solution to the certain equation $x^2 + p^2 = y^5$. However, providing their all solutions will not be studied yet in this paper. The key idea of this line is to follow the property of the unique factorization in the ring of Gaussian integers and use MAGMA program at some points. It turns out that we can eventually describe the solvability of the certain equation $x^2 + p^2 = y^5$, where p is a fixed prime number considered in two cases as $p \equiv 1 (\mathrm{mod}\ 4)$ or $p \equiv 3 (\mathrm{mod}\ 4)$, and one more particular case $p = 2$ with $\gcd(p,x) = 1$ will be illustrated that it always has no any integer solution. Regarding the Bunyakovsky conjecture, the magnitude of such a prime arising in the first case and also its analogous result in the case $n = 3$ will be discussed. We would like to refer this implication to the last section and here are the statements of the mentioned results.

Theorem 1.1 Let p be a prime number such that $p \equiv 1 (\mathrm{mod}\ 4)$. Then the Diophantine equation $x^2 + p^2 = y^5$ has an integral solution (x,y) with $\gcd(p,x) = 1$ if and only if $p = 5n^2 - 10n + 1$ for some non-zero square integer n.

Theorem 1.2 Let p be a prime number such that $p \equiv 3 (\mathrm{mod}\ 4)$. Then the Diophantine equation $x^2 + p^2 = y^5$ has no integral solution.

Corollary 1.1 The Diophantine equation $x^2 + 4 = y^5$ with $\gcd(2,x) = 1$ has no integral solution.

2. Some Important Properties in Gaussian Integers

In order to complete our purpose, let us give the necessary and sufficient condition for having an integer solution of the Fermat-type equation with signatures $(2,2,n)$ as follows:

Lemma 2.1 Let n be any natural number and α, β, γ be non-zero and non-unit Gaussian integers such that β and γ are coprime. If $\alpha^n = \beta\gamma$, then there exist β_1, γ_1 and unit elements u, v in Gaussian integers for which

$$\beta = u\beta_1 \text{ and } \gamma = v\gamma_1,$$

where β_1 and γ_1 are coprime.

Applying the unique factorization in the ring of Gaussian integers, the proof of this lemma is done easily. Indeed, the above lemma can be true for any unique factorization domain, and its analogue in the sense of using the unique prime ideal factorization has been illustrated in (Alaca & Kenneth, 2004). The following two lemmas taken from (Andreescu, Andrica & Cucurezeanu, 2010) and (Jaidee &Wannalookkhee, 2020), respectively are shown below.

Lemma 2.2 Let n be a natural number such that $n \geq 3$. If the equation $x^2 + y^2 = z^n$ has an integer solution (x,y) with $\gcd(x,y) = 1$, then $x + yi$ and $x - yi$ are coprime.

Lemma 2.3 Let n be a natural number such that $n \geq 3$, and p be a prime number with $p \equiv 3(\text{mod } 4)$. If the equation $x^2 + p^2 = y^n$ has an integer solution (x, y), then $\gcd(x, p) = 1$.
Note that the first two lemmas are going to play an important role in proving the following theorem, and its proof may be provided in (Andreescu *et al.*, 2010).

Theorem 2.1 Let n be an integer greater than 1. Then the equation $x^2 + y^2 = z^n$ with $\gcd(x, y) = 1$ has an integer solution if and only if the equation

$$x + yi = u(a + bi)^n$$

has an integer solution (x, y, a, b) for some $u \in \{\pm i, \pm 1\}$.

Now, we are ready to show all solutions depending on two integer parameters of Diophantine equation $x^2 + y^2 = z^5$, and the main tool used to obtain such solutions is the unique factorization in the ring of Gaussian integers and some facts in elementary number theory.

Theorem 2.2 The Diophantine equation $x^2 + y^2 = z^5$ has an integral solution (x, y, z) with $\gcd(x, y) = 1$ if and only if there exist integers a and b for which

$$x = a^5 - 10a^3b^2 + 5ab^4,\ y = 5a^4b - 10a^2b^3 + b^4 \text{ and } z = a^2 + b^2,$$

where $\gcd(a, b) = 1$ and $a \not\equiv b(\text{mod } 2)$.

Proof. Assume that the equation $x^2 + y^2 = z^5$ has an integral solution (x, y, z) with $\gcd(x, y) = 1$. The term on the left-hand side of this equation may be written in terms of two Gaussian integers like

$$(x + yi)(x - yi) = z^5. \tag{2.2}$$

Note that every unit in Gaussian integer can be written as the fifth power of some unit in Gaussian integer. Then we can obtain by Theorem 2.1 that $(x + yi) = (a + bi)^5$ for some integers a and b. Expanding the fifth power and equating the real and imaginary parts, we eventually reach

$$x = a^5 - 10a^3b^2 + 5ab^4,\ y = 5a^4b - 10a^2b^3 + b^5.$$

Clearly, $\gcd(a, b) = 1$, and this leads us to get $a \not\equiv b(\text{mod } 2)$. Otherwise, we can reach a contradiction. Observe that $(x - yi) = (a - bi)^5$. Following (2.2), we finally have $z = a^2 + b^2$. Conversely, let

$$x = a^5 - 10a^3b^2 + 5ab^4,\ y = 5a^4b - 10a^2b^3 + b^5 \text{ and } z = a^2 + b^2$$

for some integers a and b with $\gcd(a, b) = 1$ and $a \not\equiv b(\text{mod } 2)$. It is not hard to see that the triple (x, y, z) is the solution to the considered equation. It remains to show only $\gcd(x, y) = 1$. Suppose that $\gcd(x, y) = d \geq 2$. Then there exists a prime number p such that $p|d$. Thus $p|x$ and $p|y$. Thus, we divide our consideration into 4 cases as follows:

Case 1: $p|a$ and $p|b$. This leads to a contradiction.

Case 2: $p|a$ and $p|(5a^4b - 10a^2b^3 + b^5)$. Then $p|b$, leading to a contradiction.

Case 3: $p|(a^5 - 10a^3b^2 + 5ab^4)$ and $p|b$. Then $p|a$, leading to a contradiction.

Case 4: $p|(a^4 - 10a^2b^2 + 5b^4)$ and $p|(5a^4 - 10a^2b^2 + b^4)$. If $p = 2$, then $2|(a^4 + 5b^4)$, which is a contradiction. Now, let us suppose that $p > 2$. We first observe that $2|(a^4 - b^4)$ and then combine it with each original condition in the current case. Lastly, we can reach the facts that

$$p|b^2(5a^2 - 3b^2) \text{ and } p|a^2(3a^2 - 5b^2),$$

respectively. Again, let us consider all possible subcases coming from these facts:

Subcase 4.1: $p|a^2$ and $p|b^2$. Then $p|\gcd(a, b)$, a contradiction.

Subcase 4.2: $p|b^2$ and $p|(3a^2 - 5b^2)$. Then $p|3a^2$. If $p > 3$, then $p|a$. This leads to a contradiction. Thus, $p = 3$ and we have $3|(a^4 - b^4)$. This immediately contradicts $\gcd(a, b) = 1$, as we have known that $3|b$.

Subcase 4.3: $p|a^2$ and $p|(5a^2 - 3b^2)$. Similar to the subcase 4.2.

Subcase 4.4: $p|(5a^2 - 3b^2)$ and $p|(3a^2 - 5b^2)$. We first observe that $p|(a^2 - b^2)$ and combine it with each original condition in our current subcase above. We can eventually get $p|a$ and $p|b$, respectively. Here is a contradiction. Hence the proof of this theorem is completed.

In the paper of Conrad appearing in (Conrad, 2020), he has given the concept how to prove the analogue of Theorem 2.2 in case $n = 3$ shown below.

Theorem 2.3 The Diophantine equation $x^2 + y^2 = z^3$ has an integral solution (x, y, z) with $\gcd(x, y) = 1$ if and only if there exist integers a and b for which

$$x = a^3 - 3ab^2 + 5ab^4,\ y = 3a^2b - b^3 \text{ and } z = a^2 + b^2,$$

where $\gcd(a, b) = 1$ and $a \not\equiv b(mod\ 2)$.

3. Proofs of Theorem 1.1, Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.1 Suppose that the equation $x^2+p^2=y^5$ has an integer solution with $\gcd(p,x)=1$ such that $p\equiv 1(\mathrm{mod}\ 4)$. By Theorem 2.2, we obtain that

$$p=5a^4b-10a^2b^3+b^5=b(5a^4-10a^2b^2+b^4)$$

for some integers a,b such that $\gcd(a,b)=1$ and $a\not\equiv b(\mathrm{mod}\ 2)$. If $b=\pm1$, then $\pm p=5a^4-10a^2+1$. Since $a^2\geq 4$, so we eventually get $p=5n^2-10n+1$ for some non-zero square integer n. If $b=\pm p$, then we must only have $5a^4-10a^2p^2+p^4=1$because the other possibility contradicts the value of $5a^4-10a^2p^2+p^4$ in modulo 4. By applying the ThueSolve function in MAGMA (Bosma, Cannon & Playoust, 1997), we know that $(x,y)=(\pm1,0)$ are only integer solutions of the Thue equation $x^4-10x^2y^2+5y^4=1$. This leads to a contradiction. Conversely, let $p=5n^2-10n+1$ such that $n=m^2$ for some non-zero integer m. Notice that m is even. Choose $a=m$ and $b=1$ and then apply Theorem 2.2, we can conclude that

$$(x,y)=(m(m^4-10m^2+5),m^2+1)$$

is the solution to the equation $x^2+p^2=z^5$ with $\gcd(p,x)=1$. Here we can complete the proof.

Proof of Theorem 1.2 Suppose that the equation $x^2+p^2=y^5$ has an integer solution such that $p\equiv 3(\mathrm{mod}\ 4)$. By Lemma 2.3 and Theorem 2.2, we obtain that

$$p=5a^4b-10a^2b^3+b^5=b(5a^4-10a^2b^2+b^4)$$

for some integers a,b such that $\gcd(a,b)=1$ and $a\not\equiv b(\mathrm{mod}\ 2)$. If $b=\pm1$, then $\pm p=5a^4-10a^2+1$. Since $a^2\geq 4$ and a is even, it follows that $p=5a^4-10a^2+1\equiv 1(\mathrm{mod}\ 4)$, leading to a contradiction. If $b=\pm p$, then we must have $5a^4-10a^2p^2+p^4=1$ because a is even and $p\equiv 3(\mathrm{mod}\ 4)$. By applying the ThueSolve function in MAGMA (Bosma *et al.*, 1997), we know that $(x,y)=(\pm1,0)$ are only integer solutions of the Thue equation $x^4-10x^2y^2+5y^4=1$. This is a contradiction.

Proof of Corollary 1.3 Following the proof appearing in Theorem 1.2, we can finish the proof of this corollary. Note that each case divided to consider here leads to a contradiction easily without applying MAGMA.

Following the same recipe as we just did in the proof of the above theorems, we are able to obtain the smaller results below without using MAGMA and in fact, their proofs can be found in (Jaidee &Wannalookkhee, 2020).

Theorem 3.1 Let p be a prime number such that $p\equiv 1(\mathrm{mod}\ 4)$. Then the Diophantine Equation $x^2+p^2=y^3$ has an integral solution (x,y) with $\gcd(x,p)=1$ if and only if $p=\sqrt{12n^2+1}$ for some non-zero integer n.

Theorem 3.2 Let p be a prime number such that $p\equiv 3(\mathrm{mod}\ 4)$. Then the Diophantine equation $x^2+p^2=y^3$ has an integral solution (x,y) if and only if $p=\sqrt{12n^2+1}$ or $p=12n^2-1$ for some non-zero integer n.

Theorem 3.3 The Diophantine equation $x^2+4=y^3$ has an integral solution.

Corollary 3.1 Let p be a prime number. Then the Diophantine equation $x^2+p^2=y^3$ has an integral solution (x,y) with $\gcd(x,y)=1$ if and only if $p=2, p=\sqrt{12n^2+1}$ or $p=12n^2-1$ for some non-zero integer n.

4. The Link to Bunyakovsky Conjecture

Victor Y. Bunyakovsky, the Ukrainian mathematician conjectured in (Bunyakovsky,1857) that a non-constant polynomial $f(x)$ over rational integers produces infinitely many rational primes if the polynomial $f(x)$ satisfies the following statements.

(1) Its leading coefficient is a positive integer.
(2) The greatest common divisor of all coefficients of such a polynomial is 1.
(3) It must be an irreducible polynomial over $\mathbb{Z}$.
(4) For each prime p, there exists $n\in\mathbb{Z}_p$ for which $f(n)\not\equiv 0(\mathrm{mod}\ p)$ and $\mathbb{Z}_p$ is the set of integers modulo p.

Indeed, if f is linear, that is, $f(x)=ax+b$, where a and b are integers with $\gcd(a,b)=1$, it produces rational prime pattern written as the infinite sequence $an+b$ for an integer n. This fact is well-known as Dirichlet's Theorem (Borevich,1966). However, the Bunyakovsky conjecture hasn't been proven in case f is not linear, and many number theorists have been in vain attempting to solve it in general or even to provide some certain examples in particular. See more detail in (Conrad, 2021).

The possible value of primes p satisfying the equation $x^2+p^2=y^3$ and $x^2+p^2=y^5$ are respectively illustrated in the Table 1. and Table 2.

Note that the direct approaches used to obtain such data can be found in (Jaidee &Wannalookkhee, 2020). Consequently, the results of B´erczes and Pink as mentioned earlier in the case B have been generalized in the case $(n,k)=(3,1)$ and $(5,1)$.

Remark 4.1 To be continued doing further research about this, let us leave the following conjectures:

Conjecture 1: There are infinitely many primes $p\equiv 3(\mathrm{mod}\ 4)$ satisfying the Diophantine equation $x^2+p^2=y^3$.

Conjecture 2: There are infinitely many primes $p\equiv 1(\mathrm{mod}\ 4)$ with $\gcd(p,x)=1$ satisfying the Diophantine equation $x^2+p^2=y^5$.

We firmly believe that the Conjectures 1 and 2 might be true as the Fermat- type equations with signature (2,2,3) and (2,2,5) always have infinitely many integral solutions together with being motivated by our information shown before. So, if the Conjecture 1 and Conjecture 2 hold, then the polynomials $12x^2-1$ and $5x^4-10x^2+1$ will produce infinitely many primes, respectively. These would be some explicit examples supporting the Bunyakovsky conjecture.

Table 1. The possible values of primes p satisfying the equation $x^2+p^2=y^3$.

Conditions of p	Possible values p
$p=\sqrt{12n^2+1}$, $p\equiv 1(\mathrm{mod}\ 4)$, and $p<9863382151$	97, 708158977
$p=\sqrt{12n^2+1}$, $p\equiv 3(\mathrm{mod}\ 4)$, and $p<9863382151$	7
$p=12n^2-1$ $p\equiv 3(\mathrm{mod}\ 4)$, and $p<46146251$	11, 47, 107, 191, 431, 587, 971, 1451, 2027, 2351, 2699, 3467,4799,5807, 6911, 7499, 8111, 8747, 10091, 10799, 14699, 15551, 16427, 17327, 18251, 25391, 27647, 36299, 41771, 44651, 55487, 57131,62207, 67499, 71147, 67499, 71147, 74891, 80687, 92927, 99371,103787,106031, 124847, 132299, 137387, 139967, 158699, 161471, 164267, 167087, 175691, 184511, 202799, 215471, 221951, 235199,266411, 277247, 284591, 295787, 299567, 303371, 314927, 334667, 338687, 350891, 355007, 359147, 380207, 401867, 406271, 437771, 442367, 460991, 470447, 499391, 504299, 514187, 524171, 549551, 554699, 591407, 602111, 618347, 657071, 691199, 720299, 726191, 756011, 768107, 786431, 811199, 823727, 836351, 855467, 868331, 874799, 954287, 961067, 995327, 1009199, 1023167, 1044299, 1058507, 1072811, 1079999, 1123631, 1138367, 1153199, 1175627, 1183151, 1190699, 1221131, 1236491, 1251947, 1291007, 1338671, 1420031, 1486847,1512299, 1529387, 1537967, 1555199, 1563851, 1598699, 1607471, 1651691, 1696511, 1732799, 1760267, 1797227, 1806527, 1843967, 1853387, 1929611, 1939247, 1948907, 1958591, 2046827, 2056751, 2066699, 2096687, 2116799, 2208491, 2249867, 2281151, 2365631, 2473391, 2561327, 2605871, 2617067, 2650799, 2707499, 2730347, 2753291, 2846027, 2892971, 2904767, 2916587, 2999999, 3072431, 3121199, 3257291, 3269807, 3383531, 3447551, 3460427, 3525167, 3538187, 3564299, 3577391, 3629999, 3656447, 3709631, 3736367, 3749771, 3776651, 3967499, 4009007, 4022891, 4050731, 4078667, 4106699, 4148927, 4163051, 4305611, 4465199, 4479851, 4494527, 4523951, 4583087, 4627691, 4642607, 4657547, 4687499, 4717547, 4762799, 4808267, 4823471, 4838699, 5007791, 5023307, 5085611, 5116907, 5132591, 5274827, 5322671, 5338667, 5370731, 5435147, 5451311, 5467499, 5483711, 5532491, 5696651, 5812991, 5846447, 5863211, 5896811, 5947391, 6032171, 6134699, 6238091, 6324911, 6359807, 6394799, 6429887, 6482699, 6553451, 6571199, 6642431, 6660299, 6714047, 6840299, 6876587, 7041071, 7059467, 7226111, 7263407, 7319531, 7357067, 7375871, 7413551, 7470251, 7584299, 7660811, 7834367, 8108207, 8167499, 8227007, 8286731, 8306687, 8487371, 8608907, 8649611, 8690411, 8710847, 8772299, 8895851, 8978699, 9166511, 9187499, 9250607, 9377471, 9398699, 9419951, 9441227, 9676847, 9937199, 9959051, 10090667, 10156799, 10178891, 10201007, 10245311, 10289711, 10423487, 10490699, 10535627, 10580651, 10603199, 10807211, 10829999, 10898507, 10944299, 11128427, 11174699, 11244287, 11360747, 11430911, 11666351, 11690027, 11928107, 12338351, 12387071,12509291, 12681407, 12929327, 13204811, 13229999, 13483199, 13585151, 13610699, 13790207, 13945007, 14022731, 14048687, 14572847, 14625791, 14652299, 14891951,14918699, 14945471, 14999087, 15025931, 15052799, 15160511, 15187499, 15513227, 15704831, 15787307, 15814847, 15897611, 15952907, 16119371, 16314671, 16398731,16426799, 16793867, 16850699, 16993199, 17164991, 17193707, 17453231, 17482187,17627327, 17656427, 17743871, 17802287, 17948747, 18007499, 18213887, 18273071, 18391727, 18480971, 18600299, 18690047, 18780011, 18870191, 18990767, 19142027, 19415807, 19660799, 19783871, 20000171, 20093231, 20311211, 20342447, 20404991, 20561771, 20687627, 21067499, 21131147, 21386699, 21418751, 21450827, 21482927, 21547199, 21611567, 21708299, 21869999, 21902411, 22129967, 22162571, 22391471, 22753547, 22919087, 23151851, 23620907, 23654591, 23688299, 23722031, 23891051, 23992751, 24026699, 24060671, 24094667, 24162731, 24470207, 24607487, 24641867, 24676271, 25125707, 25195211, 25229999, 25299647, 25439231, 25474187, 25509167, 25579199, 25684427, 25895531, 25930799, 25966091, 26142911, 26249291, 26284799, 26320331, 26462699, 26569727, 26784431, 27144191, 27397451, 27433727, 27542699, 27724799, 27761291, 27907499, 28127531, 28274699, 28348427, 28459199, 28718507, 28867211, 28978991, 29090987, 29203199, 29278127, 30108671, 30222827, 30260927, 30299051, 30337199, 30604907, 30681611, 31221227, 31453931, 31609547, 31648511, 31961087, 32157227, 32314571, 32590847, 32868299, 32947787, 32987567, 33266699, 33346667, 33386687, 33466799, 33747947, 33828491, 34151627, 34192127, 34354367, 34516991, 34639211, 34679999, 34761647, 35335871, 35956331, 36039467, 36247727, 36414767, 36582191, 36792011, 37340351, 37382699, 37764911, 37807499, 37978091, 38320427, 38492171, 38578187, 38707391, 38879999, 39139631, 39182987, 39226367, 39443627, 39617867, 39748799, 39923711, 40011311, 40274687, 40318667, 40583051, 40627199, 40804031, 41203307, 41292299, 41470571, 42097547, 42142511, 42367691, 42729227, 42819851, 43046831, 43183307, 43274411, 43823051, 44098667, 44144687, 44236799, 44282891, 44698799, 44884271, 45209771, 45817391, 45911231, 46146251

Table 2. The possible values of primes p satisfying the equation $x^2 + p^2 = y^5$.

Conditions of p	Possible values p
$p \equiv 3 (\mathrm{mod}\ 4)$	None
$p = 5n^4 - 10n^2 + 1$, $p \equiv 1 (\mathrm{mod}\ 4)$, and $p < 97825078122$	41, 6121, 19841, 102241, 190121, 521641, 596001, 1166441, 2278121, 49141121, 22366121, 56548841, 64764001, 73843241, 83845121, 106860641, 120001001, 299770241, 424581121, 905185121, 1710325121, 2271646121, 3616705121, 3982688641, 4375769441, 4582878121, 9723609001, 10485910121, 11712316001, 12144070441, 13043278121, 13511161441, 18310326121, 25792716641, 28181278121, 32526145121, 33452110121, 34397706241, 36348895841, 46175089001, 49855105121, 78519246121, 87774625121, 95749273441, 97825078121

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