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*Original Article*

# Characterizing some regularities of ordered semigroups by their anti-hybrid ideals

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**Abstract**

We focus on mappings called hybrid structures. A hybrid structure can be regarded as a combination of a soft set and a fuzzy set. Therefore, with the help of the products of union softs sets and fuzzy sets, we can define a new product of hybrid structure. This product defined the concepts of anti-hybrid left (resp., right, bi-) ideals in ordered semigroups. This paper considers a relationship between ideals and anti-hybrid ideals in ordered semigroups. We characterize some classes of ordered semigroups by anti-hybrid left (resp., right, bi-) ideals of ordered semigroups.

**Keywords**: ordered semigroup, hybrid structure, anti-hybrid left ideal, anti-hybrid right ideal, anti-hybrid bi-ideal

## 1. Introduction

Zadeh (1965) introduced the concept of fuzzy sets as a generalization of crisp sets. Since it deals with uncertainties, fuzzy set theory can be applied in many mathematics and computer sciences branches. This concept was also used to investigate some properties of algebraic structures. It was applied to groups, so-called fuzzy groups, by Rosenfeld (1971). Later on, Kuroki (1979, 1981) studied semigroups properties through fuzzy sets.

Properties of many generalizations of semigroups were considered using fuzzy sets. The concept of fuzzy ordered semigroups was initialed by Kehayopulu and Tsingelis (2002). A fuzzy ordered semigroup is a structure consisting of the set of all fuzzy sets defined on an ordered semigroup, a binary operation defined on it. Many results in ordered semigroups were studied in terms of fuzzy ordered semigroups. Several regularities of ordered semigroups can be characterized using fuzzy sets.

Zeb and Khan (2011) defined a new binary operation on all fuzzy sets defined on an ordered semigroup. This structure is called an anti-fuzzy ordered semigroup. They defined the notion of anti-fuzzy quasi-ideals and characterized some classes of ordered semigroups by this new algebra.

Several generalized concepts of fuzzy sets can be used to investigate some properties of algebras, for example, the concepts of cubic sets and neutrosophic cubic sets. Khan Jun, Gulistan, and Yaqoob (2015) studied a generalized version of cubic sets, a generalization of fuzzy sets. They defined various ideals in semigroups and their related properties are investigated. Khan, Gulistan, Yaqoob, and Shabir (2018) generalized the concept of fuzzy points to neutrosophic cubic points. Some important ideals in terms of neutrosophic cubic were defined and studied by this generalization.

The concept of soft sets is a new mathematical concept that Molodtsov (1999) introduced to generalize fuzzy sets. It deals with some problematic uncertainties. Some real-world problems can be studied through this concept, also some properties of algebraic structures. By the definition of soft sets, there are two possibilities to define the product of soft sets. That is, we obtain two algebras with the same universe set. We call these two algebras uni-soft sets and int-soft sets.

Uni-soft sets were first used to characterize ordered semigroups by Khan, Jun, Ali Shah, and Khan (2016). They defined the notion of uni-soft quasi-ideals and used it to characterize left (resp., right) simple ordered semigroups and completely ordered semigroups. Khan, Khan, and Jun (2017) defined uni-soft bi-ideals and uni-soft interior ideals in ordered semigroups. They used these uni-soft ideals to characterize weakly regular, intra-regular, left weakly regular, and semisimple ordered semigroups. Khan, Khan, Uzair Khan, and

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Khan (2020) considered the properties of uni-soft bi-ideals in ordered semigroups. They provided two types of prime uni-soft bi-ideals in ordered semigroups. Left and right simple of ordered semigroups were considered.

Int-soft sets were applied to study semigroups in 2014 by Song, Kim, and Jun (2014). They introduced the notions of int-soft left (resp., right) ideals and int-soft quasi-ideals in semigroups. These notions were used to characterize regular semigroups. Later, some further properties of int-soft left (resp., right) ideals in semigroups are considered by Jun and Song (2015). Moreover, the authors defined int-soft (generalized) bi-ideals, and their characterizations were provided. Int-soft sets were applied to ordered semigroups by Muhiuddin and Mahboob (2020). They introduced the notions of int-soft left (resp., right) ideals, int-soft interior ideals, and int-soft bi-ideals in ordered semigroups. Furthermore, characterizations of ordered semigroups were provided. In 2021, Muhiuddin, Alenzea, Mahboob, and Al-Masarwah (2021) used the notions of int-soft left (resp., right) ideals in ordered semigroups to characterize convex soft sets.

A hybrid structure is a mapping combining a soft set and a fuzzy set together. Hybrid structures were applied to consider properties of an algebraic structure by Jun, Song, and Muhiuddin (2018). They applied hybrid structure to BCK/BCI-algebras and linear spaces. Since a hybrid structure is defined in terms of a soft set and a fuzzy set, there are at least four possibilities to define a product of any given two hybrid structures. The set of all hybrid structures together with an operation of int-soft sets and an operation of anti-fuzzy sets, one obtains a new algebra. This algebra is also called the hybrid structure, and some properties of algebras can be studied through hybrid structures. Anis, Khan, and Jun (2017) introduced the notions of hybrid subsemigroups and left (resp., right) ideals in semigroups. They characterized these notions by the hybrid product. Elavarasan and Jun (2022) characterized regular and intra-regular semigroups in terms of hybrid ideals and hybrid bi-ideals. Ordered semigroups can also be considered in terms of hybrid structures. Mekwian and Lekkoksung (2021) characterized regular and intra-regular ordered semigroups via hybrid left (resp., right) ideals in ordered semigroups.

Uni soft sets and fuzzy sets can also define a binary operation on the set of all hybrid structures. This algebra is called an anti-hybrid structure. Anti-hybrid structures were first applied to investigate and characterize some algebraic structures in 2021. Linesawat *et al.* (2021) introduced anti-hybrid left (resp., right) ideals in ordered semigroups, and their related properties were provided. Mekwian *et al.* (2021) introduced anti-hybrid quasi-ideals in ordered semigroups. This notion was characterized in terms of the anti-hybrid product. Sarasit, Linesawat, Lekkoksung, and Lekkoksung (2021) characterized subsemigroups of ordered semigroups in terms of anti-hybrid subsemigroups.

In this paper, we discuss a connection between ideals and ant-hybrid ideals in ordered semigroups. We characterize some classes of ordered semigroups in the context of anti-hybrid left (resp., right) ideals and anti-hybrid bi-ideals.

## 2. Preliminaries

This section recalls the basic terms and definitions from the ordered semigroup theory and the hybrid structure theory that we will use.

A groupoid $(S;\cdot)$ is an algebra consisting of a nonempty set $S$ and a (binary) operation $\cdot$ on $S$.

**Definition 2.1** A *groupoid* $(S;\cdot)$, which is associative, that is

$$x\cdot(y\cdot z)=(x\cdot y)\cdot z$$

for all $x, y, z \in S$, is called a *semigroup*.

**Definition 2.2** (Fuchs, 1963) A structure $(S;\cdot,\leq)$ is called an *ordered semigroup* if the following conditions are satisfied:

(1) $(S;\cdot)$ is a semigroup.

(2) $(S;\leq)$ is a partially ordered set.

(3) For every $a,b,c\in S$ if $a\leq b$, then $a\cdot c\leq b\cdot c$ and $c\cdot a\leq c\cdot b$.

For simplicity, we denote an ordered semigroup $(S;\cdot,\leq)$ by its carrier set as a bold letter **S**.

For $K\subseteq S$, we denote

$$(K] := \{a\in S \mid a\leq k \text{ for some } k\in K\}.$$

Let $A$ and $B$ be two nonempty subsets of $S$. Then we define

$$AB := \{ab \mid a\in A \text{ and } b\in B\}.$$

Let **S** be an ordered semigroup. A nonempty subset $A$ of S is called a *subsemigroup* of **S** if $AA\subseteq A$.

**Definition 2.3** (Kehayopulu, 1990) Let **S** be an ordered semigroup. A nonempty subset $A$ of $S$ is called a *left (resp., right) ideal* of **S** if

(1) $SA\subseteq A$ (resp., $AS\subseteq A$).

(2) For $x,y\in S$, if $x\leq y$ and $y\in A$, then $x\in A$.

A nonempty subset $I$ of S is called an *(two-sided) ideal* of **S** if it is both a left and a right ideal of **S**.

**Definition 2.4** (Kehayopulu, 1992a) Let **S** be an ordered semigroup. A subsemigroup $B$ of S is called a *bi-ideal* of **S** if

(1) $BSB\subseteq B$.

(2) For $x,y\in S$, if $x\leq y$ and $y\in B$, then $x\in B$.

Let $a \in S$, we denoted $L(a), R(a), I(a)$ and $B(a)$ the smallest left ideal, smallest right ideal, smallest ideal and smallest bi-ideal of **S** containing $a$, repectively. It is easy to verify that

$$L(a) = (a \cup Sa], R(a) = (a \cup Sa], I(a) = (a \cup Sa \cup aS \cup SaS],$$

and

$$B(a) = (a \cup aa \cup aSa].$$

In what follows, let $I$ be the unit interval, $S$ is a set of parameters and $\mathcal{P}(U)$ denotes the power set of an initial universe set $U$.

**Definition 2.5** (Anis *et al.*, 2017) A hybrid structure in $S$ over $U$ is defined to be a mapping

$$\tilde{f}_\lambda := (\tilde{f}, \lambda) : S \to \mathcal{P}(U) \times I, x \mapsto \left(\tilde{f}(x), \lambda(x)\right),$$

where

$$\tilde{f} : S \to \mathcal{P}(U) \text{ and } \lambda : S \to I$$

are mappings.

Let us denote $H(S)$ the set of all hybrid structures in $S$ over $U$. We define an order $\ll$ on $H(S)$ as follows. For any $\tilde{f}_\lambda, \tilde{g}_\gamma \in H(S)$,

$$\tilde{f}_\lambda \ll \tilde{g}_\gamma \Leftrightarrow \tilde{f} \sqsubseteq \tilde{g}, \lambda \succcurlyeq \gamma,$$

where $\tilde{f} \sqsubseteq \tilde{g}$ means that $\tilde{f}(x) \subseteq \tilde{g}(x)$ and $\lambda \succcurlyeq \gamma$ means that $\lambda(x) \geq \gamma(x)$ for all $x \in S$ and $\tilde{f}_\lambda = \tilde{g}_\gamma$ if $\tilde{f}_\lambda \ll \tilde{g}_\gamma$ and $\tilde{g}_\gamma \ll \tilde{f}_\lambda$.

**Definition 2.6** (Anis *et al.*, 2017) Let $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ be hybrid structures in $S$ over $U$. Then the hybrid union of $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ is denoted by $\tilde{f}_\lambda \Cup \tilde{g}_\gamma$ and is defined to be a hybrid structure

$$\tilde{f}_\lambda \Cup \tilde{g}_\gamma : S \to \mathcal{P}(U) \times I, x \mapsto \left((\tilde{f} \cup \tilde{g})(x), (\lambda \wedge \gamma)(x)\right),$$

where

$$(\tilde{f} \cup \tilde{g})(x) := \tilde{f}(x) \cup \tilde{g}(x) \text{ and } (\lambda \wedge \gamma)(x) := \min\{\lambda(x), \gamma(x)\}.$$

We denote $\tilde{S}_S$ the hybrid structure in $S$ over $U$ and is defined as follows:

$$\tilde{S}_S : S \to \mathcal{P}(U) \times I : x \mapsto \left(\tilde{S}(x), S(x)\right),$$

where

$$\tilde{S}(x) := \emptyset \text{ and } S(x) := 1.$$

It is not difficult to see that $\tilde{f}_\lambda \gg \tilde{S}_S$ for any hybrid structure $\tilde{f}_\lambda$ in $S$ over $U$.

Let $a \in S$. Then, we set

$$\mathbf{S}_a := \{(x, y) \in S \times S \mid a \leq xy\}.$$

**Definition 2.7** (Sarasit *et al.*, 2021) Let $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ be hybrid structures in $S$ over $U$. Then the hybrid products of $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ is denoted by $\tilde{f}_\lambda \otimes \tilde{g}_\gamma$ and is defined to be a hybrid structure

$$\tilde{f}_\lambda \otimes \tilde{g}_\gamma : S \to \mathcal{P}(U) \times I, x \mapsto \left((\tilde{f} \odot \tilde{g})(x), (\lambda \circ \gamma)(x)\right),$$

where

$$(\tilde{f} \odot \tilde{g})(x) := \begin{cases} \bigcap_{(a,b) \in \mathbf{S}_x} \left(\tilde{f}(a) \cup \tilde{g}(b)\right) & \text{if } \mathbf{S}_x \neq \emptyset, \\ U & \text{otherwise,} \end{cases}$$

and

$$(\lambda \circ \gamma)(x) := \begin{cases} \bigvee_{(a,b) \in \mathbf{S}_x} \{\min\{\lambda(x), \gamma(x)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A \subseteq S$. We denote by $\chi_{A^c}(\tilde{S}_S)$ the characteristic hybrid structure of complement of $A$ in $S$ over $U$ and is defined to be a hybrid structure

$$\chi_{A^c}(\tilde{S}_S) : S \to \mathcal{P}(U) \times I, x \mapsto \left(\chi_{A^c}(\tilde{S})(x), \chi_{A^c}(S)(x)\right),$$

where

$$\chi_{A^c}(\tilde{S})(x) := \begin{cases} \emptyset & \text{if } x \in A, \\ U & \text{otherwise,} \end{cases}$$

and

$$\chi_{A^c}(S)(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

We set $\chi_{A^c}(\tilde{S}_S) := \tilde{S}_S$ if $A = S$.

## 3. Results

This section introduces the concept of anti-hybrid bi-ideals and studies its properties. Finally, we characterize regular ordered semigroups and intra-regular ordered semigroups using anti-hybrid left ideals, anti-hybrid right ideals, and anti-hybrid bi-ideals.

**Definition 3.1** (Sarasit *et al*., 2021) Let **S** be an ordered semigroup. A hybrid structure $\tilde{f}_\lambda$ in $S$ over $U$ is called an *anti-hybrid subsemigroup* in $S$ over $U$ if the following statements are satisfied. For every $x, y \in S$,

(1) $\tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$,
(2) $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$.

**Example 3.2** Let $S = \{a, b, c, d\}$. Define a binary operation * on $S$ by the following table:

| * | a | b | c | d |
|---|---|---|---|---|
| a | a | a | a | a |
| b | a | a | a | a |
| c | a | a | b | b |
| d | a | a | b | b |

We define an order relation $\leq$ on $S$ as follows:

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(b,c)\}.$$

Then, $(S;*,\leq)$ is an ordered semigroup. Let $U = \{1, 2, 3\}$. We define a hybrid structure $\tilde{f}_\lambda$ in $S$ over $U$ as follows:

| S | $\tilde{f}$ | λ |
|---|---|---|
| a | {1, 2} | 0 |
| b | {1} | 1 |
| c | U | 0.8 |
| d | ∅ | 0.5 |

By carefully calculation, we see that $\tilde{f}_\lambda$ is an anti-hybrid subsemigroup in $S$ over $U$. Comparing to the definition of hybrid subsemigroups by Mekwian and Lekkoksung (2021), it is not difficult to see that $\tilde{f}_\lambda$ is not a hybrid subsemigroup in $S$ over $U$ since $\tilde{f}(cc) \subseteq \tilde{f}(c) \cap \tilde{f}(c)$.

**Definition 3.3** (Linesawat *et al.*, 2021) Let **S** be an ordered semigroup. A hybrid structure $\tilde{f}_\lambda$ in $S$ over $U$ is called an *anti-hybrid left (resp., right) ideal* in $S$ over $U$ if the following statements are satisfied. For every $x, y \in S$,

(1) $\tilde{f}(xy) \subseteq \tilde{f}(y)$ (resp., $\tilde{f}(xy) \subseteq \tilde{f}(x)$),
(2) $\lambda(xy) \geq \lambda(y)$ (resp., $\lambda(xy) \geq \lambda(x)$),
(3) if $x \leq y$, then $\tilde{f}(x) \subseteq \tilde{f}(y)$ and $\lambda(x) \geq \lambda(y)$.

A hybrid structure $\tilde{f}_\lambda$ in $S$ over $U$ is called an *anti-hybrid (two-sided) ideal* if it is both an anti-hybrid left and an anti-hybrid right ideal in $S$ over $U$.

**Example 3.4** Let $S = \{a, b, c, d\}$. Define a binary operation * on $S$ by the following table:

| * | a | b | c | d |
|---|---|---|---|---|
| a | a | a | a | a |
| b | a | a | a | a |
| c | a | a | b | a |
| d | a | a | b | b |

We define an order relation $\leq$ on $S$ as follows:

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b)\}.$$

Then, $(S;*,\leq)$ is an ordered semigroup. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$. We define a hybrid structure $\tilde{f}_\lambda$ in $S$ over $U$ as follows:

| $S$ | $\tilde{f}$ | λ |
|---|---|---|
| *a* | $\{u_5\}$ | *0.8* |
| *b* | $\{u_1, u_5\}$ | *0.5* |
| *c* | $\{u_1, u_2, u_5\}$ | *0.1* |
| *d* | $\{u_1, u_2, u_4, u_5\}$ | *0.3* |

By carefully calculation, we see that $\tilde{f}_\lambda$ is an anti-hybrid ideal in *S* over *U*. Comparing to the definition of hybrid ideals by Mekwian and Lekkoksung (2021), it is not difficult to see that $\tilde{f}_\lambda$ is not a hybrid ideal in *S* over *U* since $\tilde{f}(bb) \subseteq \tilde{f}(b)$.

**Definition 3.5** (Linesawat *et al*., 2021) Let **S** be an ordered semigroup. An anti-hybrid subsemigroup $\tilde{f}_\lambda$ in $S$ over $U$ is called an *anti-hybrid bi-ideal* in $S$ over $U$ if the following statements are satisfied. For every $x, y, z \in S$,

(1) $\tilde{f}(xyz) \subseteq \tilde{f}(x) \cup \tilde{f}(z)$,

(2) $\lambda(xyz) \geq \min\{\lambda(x), \lambda(z)\}$,

(3) if $x \leq y$, then $\tilde{f}(x) \subseteq \tilde{f}(y)$ and $\lambda(x) \geq \lambda(y)$.

**Example 3.6** Let $S = \{a, b, c, d\}$. Define a binary operation * on *S* by following table:

| ∗ | *0* | *a* | *b* | *c* |
|---|---|---|---|---|
| *0* | *0* | *0* | *0* | *b* |
| *a* | *0* | *0* | *0* | *b* |
| *b* | *0* | *0* | *0* | *b* |
| *c* | *b* | *b* | *b* | *c* |

We define an order relation $\leq$ on *S* as follows:

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(b,c)\}.$$

Then, $(S; *, \leq)$ is an ordered semigroup. Let *U* = [0, 1]. We define the hybrid structure $\tilde{f}_\lambda$ in *S* over *U* as follows:

| $S$ | $\tilde{f}$ | λ |
|---|---|---|
| *0* | [0, 0.2] | *0.8* |
| *a* | [0, 0.4] | *0.5* |
| *b* | [0, 0.5] | *0.4* |
| *c* | [0, 0.6] | *0.2* |

By carefully calculation, we see that $\tilde{f}_\lambda$ is an anti-hybrid bi-ideal in *S* over *U*.

**Lemma 3.7** Let **S** be an ordered semigroup and $\tilde{f}_{1_\lambda}, \tilde{f}_{2_\gamma}, \tilde{g}_{1_\alpha}, \tilde{g}_{2_\delta}$ be hybrid structures in $S$ over $U$ such that $\tilde{f}_{1_\lambda} \ll \tilde{g}_{1_\alpha}$ and $\tilde{f}_{2_\gamma} \ll \tilde{g}_{2_\delta}$. Then $\tilde{f}_{1_\lambda} \otimes \tilde{f}_{2_\gamma} \ll \tilde{g}_{1_\alpha} \otimes \tilde{g}_{2_\delta}$.

**Proof**. Let $a \in S$. If $\mathbf{S}_a = \emptyset$, then $(\tilde{f}_1 \odot \tilde{f}_2)(a) = U, (\tilde{g}_1 \odot \tilde{g}_2)(a) = U, (\lambda \circ \gamma)(a) = 0$ and $(\alpha \circ \delta)(a) = 0$. Thus, $(\tilde{f}_1 \odot \tilde{f}_2)(a) \subseteq (\tilde{g}_1 \odot \tilde{g}_2)(a)$ and $(\lambda \circ \gamma)(a) \geq (\alpha \circ \delta)(a)$. If $\mathbf{S}_a \neq \emptyset$, then

$$\begin{aligned}(\tilde{f}_1 \odot \tilde{f}_2)(a) &= \bigcap_{(x,y)\in \mathbf{S}_a} \left(\tilde{f}_1(x) \cup \tilde{f}_2(y)\right)\\ &\subseteq \bigcap_{(x,y)\in \mathbf{S}_a} \left(\tilde{g}_1(x) \cup \tilde{g}_2(y)\right)\\ &= (\tilde{g}_1 \odot \tilde{g}_2)(a),\end{aligned}$$

and

$$\begin{aligned}(\lambda \circ \gamma)(a) &= \bigvee_{(x,y)\in \mathbf{S}_a} \{\min\{\lambda(x), \gamma(y)\}\}\\ &\geq \bigvee_{(x,y)\in \mathbf{S}_a} \{\min\{\alpha(x), \delta(y)\}\}\\ &= (\alpha \circ \delta)(a).\end{aligned}$$

Therefore, $\tilde{f}_{1_\lambda} \otimes \tilde{f}_{2_\gamma} \ll \tilde{g}_{1_\alpha} \otimes \tilde{g}_{2_\delta}$.

**Lemma 3.8** Let **S** be an ordered semigroup and $A, B$ be subsets of $S$. Then

(1) $A^c \subseteq B^c$ if and only if $\chi_{A^c}(\tilde{S}_S) \ll \chi_{B^c}(\tilde{S}_S)$,
(2) $\chi_{A^c}(\tilde{S}_S) \Cup \chi_{B^c}(\tilde{S}_S) = \chi_{A^c \cup B^c}(\tilde{S}_S) = \chi_{(A\cap B)^c}(\tilde{S}_S)$,
(3) $\chi_{A^c}(\tilde{S}_S) \otimes \chi_{B^c}(\tilde{S}_S) = \chi_{(AB]^c}(\tilde{S}_S)$.

**Proof**. We only give proof of (3). Let $x \in (AB]^c$. Then $x \notin (AB]$. We obtain that $\chi_{(AB]^c}(\tilde{S})(x) = U$ and $\chi_{(AB]^c}(S)(x) = 0$. Since $x \notin (AB]$, we have $\mathbf{S}_x = \emptyset$. Thus, $(\chi_{A^c}(\tilde{S}) \odot \chi_{B^c}(\tilde{S}))(x) = U$, and $(\chi_{A^c}(S) \circ \chi_{B^c}(S))(x) = 0$. Therefore, $\chi_{A^c}(\tilde{S}_S) \otimes \chi_{B^c}(\tilde{S}_S) = \chi_{(AB]^c}(\tilde{S}_S)$. If $x \notin (AB]^c$, then $\chi_{(AB]^c}(\tilde{S})(x) = \emptyset$ and $\chi_{(AB]^c}(S)(x) = 1$. Since $x \in (AB]$, there exists $a \in A, b \in B$ such that $x \leq ab$ which means that $\mathbf{S}_x \neq \emptyset$. Then,

$$\begin{aligned}(\chi_{A^c}(\tilde{S}) \odot \chi_{B^c}(\tilde{S}))(x) &= \bigcap_{(p,q)\in \mathbf{S}_x} (\chi_{A^c}(\tilde{S})(p) \cup \chi_{B^c}(\tilde{S})(q)) \\ &\subseteq \chi_{A^c}(\tilde{S})(a) \cup \chi_{B^c}(\tilde{S})(b) \\ &= \emptyset.\end{aligned}$$

Thus, $\left(\chi_{A^c}(\tilde{S}) \odot \chi_{B^c}(\tilde{S})\right)(x) = \emptyset$ and then $\chi_{(AB]^c}(\tilde{S})(x) = \left(\chi_{A^c}(\tilde{S}) \odot \chi_{B^c}(\tilde{S})\right)(x)$.
We also obtain that

$$\begin{aligned}(\chi_{A^c}(S) \circ \chi_{B^c}(S))(x) &= \bigvee_{(p,q)\in \mathbf{S}_x} \{\min\{\chi_{A^c}(S)(p), \chi_{B^c}(S)(q)\}\} \\ &\geq \min\{\chi_{A^c}(S)(a), \chi_{B^c}(S)(b)\} \\ &= 1.\end{aligned}$$

Thus, $(\chi_{A^c}(S) \circ \chi_{B^c}(S))(x) = 1$ and then $\chi_{(AB]^c}(S)(x) = (\chi_{A^c}(S) \circ \chi_{B^c}(S))(x)$. Therefore, $\chi_{A^c}(\tilde{S}_S) \otimes \chi_{B^c}(\tilde{S}_S) = \chi_{(AB]^c}(\tilde{S}_S)$.

**Lemma 3.9** Let **S** be an ordered semigroup, $A$ a nonempty subset of $S$. Then the following statements are equivalent.
(1) $A$ is a left (resp., right, bi-, two-sided) ideal of **S**.
(2) $\chi_{A^c}(\tilde{S}_S)$ is an anti-hybrid left (resp., right, bi-, two-sided) ideal in $S$ over $U$.

**Proof**. (1) ⇒ (2). Assume that $A$ is a left ideal of **S**. Let $x, y \in S$. If $y \in A$, then $xy \in A$. Thus, $\chi_{A^c}(\tilde{S})(xy) = \emptyset = \chi_{A^c}(\tilde{S})(y)$ and $\chi_{A^c}(S)(xy) = 1 = \chi_{A^c}(S)(y)$. If $y \notin A$, then $\chi_{A^c}(\tilde{S})(xy) \subseteq U = \chi_{A^c}(\tilde{S})(y)$ and $\chi_{A^c}(S)(xy) \geq 0 = \chi_{A^c}(S)(y)$. Let $x, y \in S$ such that $x \leq y$. If $y \in A$, then $x \in A$. Thus, $\chi_{A^c}(\tilde{S})(x) = \emptyset \subseteq \chi_A(\tilde{S})(y)$ and $\chi_{A^c}(S)(x) = 1 \geq \chi_{A^c}(S)(y)$. Therefore, $\chi_A(\tilde{S}_S)$ is an anti-hybrid left ideal in $S$ over $U$. For other kinds of ideals can be proved similarly.

(2) ⇒ (1). Let $x, y \in S$ such that $y \in A$. Since $\chi_A(\tilde{S}_S)$ is an anti-hybrid left ideal in $S$ over $U$, $\chi_{A^c}(\tilde{S})(xy) \subseteq \chi_{A^c}(\tilde{S})(y) = \emptyset$. This implies that $\chi_{A^c}(\tilde{S})(xy) = \emptyset$. Hence, $xy \in A$. Now, suppose that $x \leq y$ such that $y \in A$. Since $\chi_A(\tilde{S}_S)$ is an anti-hybrid left ideal in $S$ over $U$, $\chi_{A^c}(\tilde{S})(x) \subseteq \chi_{A^c}(\tilde{S})(y) = \emptyset$. This implies that $\chi_{A^c}(\tilde{S})(x) = \emptyset$. Hence, $x \in A$. This shows that $A$ is a left ideal of **S**. For other kinds of anti-hybrid ideals of $S$ over $U$ can be done similarly.

An ordered semigroup **S** is *intra-regular* (Kehayopulu, 1993) if for each element $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$.

**Lemma 3.10** (Xie & Tang, 2010) Let **S** be an ordered semigroup. Then the following statements are equivalent:
(1) **S** is intra-regular.
(2) $R \cap L \subseteq (LR]$ for every right ideal $R$ and every left ideal $L$ of **S**.
(3) $R(a) \cap L(a) \subseteq (L(a)R(a)]$ for every $a \in S$.

Now we give a characterization of an intra-regular ordered semigroup by anti-hybrid left ideals and anti-hybrid right ideals.

**Theorem 3.11** Let **S** be an ordered semigroup. Then the following statements are equivalent.
(1) **S** is intra-regular.
(2) $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg \tilde{g}_\alpha \otimes \tilde{f}_\lambda$ for every anti-hybrid left ideal $\tilde{g}_\alpha$ and every anti-hybrid right ideal $\tilde{f}_\lambda$ in $S$ over $U$.

**Proof**. (1)⇒(2). Let $\tilde{f}_\lambda$ and $\tilde{g}_\alpha$ be an anti-hybrid right ideal and an anti-hybrid left ideal in $S$ over $U$, respectively. Let $a \in S$. Since **S** is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y = (xa)(ay)$, that is, $\mathbf{S}_a \neq \emptyset$. Thus, we obtain that

$$\begin{aligned}(\tilde{g} \odot \tilde{f})(a) &= \bigcap_{(y,z)\in \mathbf{S}_a} [\tilde{g}(y) \cup \tilde{f}(z)] \\ &\subseteq \tilde{g}(xa) \cup \tilde{f}(ay) \\ &\subseteq \tilde{g}(a) \cup \tilde{f}(a) \\ &= (\tilde{g} \cup \tilde{f})(a) \\ &= (\tilde{f} \cup \tilde{g})(a),\end{aligned}$$

and

$$
\begin{aligned}
(\alpha \circ \lambda)(a) &= \bigvee_{(y,z)\in \boldsymbol{S}_a} \{\min\{\alpha(y), \lambda(z)\}\} \\
&\geq \min\{\alpha(xa), \lambda(ay)\} \\
&\geq \min\{\alpha(a), \lambda(a)\} \\
&= (\alpha \wedge \lambda)(a) \\
&= (\lambda \wedge \alpha)(a).
\end{aligned}
$$

This means that $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg \tilde{g}_\alpha \otimes \tilde{f}_\lambda$.
(2) ⇒ (1). Let $L$ and $R$ be a left ideal and a right ideal of **S**, respectively. Then, by Lemma 3.9, $\chi_{L^c}(\tilde{S}_S)$ and $\chi_{R^c}(\tilde{S}_S)$ is an anti-hybrid left ideal and an anti-hybrid right ideal in $S$ over $U$, respectively. By hypothesis and Lemma 3.8, we obtain that

$$\chi_{(L\cap R)^c}(\tilde{S}_S) = \chi_{L^c}(\tilde{S}_S) \Cup \chi_{R^c}(\tilde{S}_S) \gg \chi_{L^c}(\tilde{S}_S) \otimes \chi_{R^c}(\tilde{S}_S) = \chi_{(LR]^c}(\tilde{S}_S).$$

This implies that $(LR]^c \subseteq (L \cap R)^c$ and then $L \cap R \subseteq (LR]$. Thus, by Lemma 3.10, we obtain that **S** is intra-regular.

**Theorem 3.12** Let **S** be an ordered semigroup. Then the following statements are equivalent.

(1) **S** is intra-regular.
(2) For each element $a \in S$ we have that $\tilde{f}(a) = \tilde{f}(a^2)$ and $\lambda(a) = \lambda(a^2)$ for every anti-hybrid ideal $\tilde{f}_\lambda$ in $S$ over $U$.

**Proof**. (1) ⇒ (2). Let $\tilde{f}_\lambda$ be an anti-hybrid ideal in $S$ over $U$ and $a \in S$. Since **S** is inter-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Then

$$\tilde{f}(a) \subseteq \tilde{f}(xa^2y) \subseteq \tilde{f}(a^2y) \subseteq \tilde{f}(a^2) \subseteq \tilde{f}(a),$$

which means that $\tilde{f}(a) = \tilde{f}(a^2)$ and

$$\lambda(a) \geq \lambda(xa^2y) \geq \lambda(a^2y) \geq \lambda(a^2) \geq \lambda(a).$$

This means that $\lambda(a) = \lambda(a^2)$.
(2) ⇒ (1). Let $a \in S$. Then we have that $\chi_{I^c(a^2)}(\tilde{S}_S)$ is an anti-hybrid ideal in $S$ over $U$. Then, we obtain that $\chi_{I^c(a^2)}(\tilde{S})(a) = \chi_{I^c(a^2)}(\tilde{S})(a^2) = \emptyset$ and $\chi_{I^c(a^2)}(S)(a) = \chi_{I^c(a^2)}(S)(a^2) = 1$. This implies that $a \in I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]$. That is, $a \leq t$ for some $t \in a^2 \cup Sa^2 \cup a^2S \cup Sa^2S$. It is easy to verify that for any case of $t$, we obtain that $a \leq pa^2q$ for some $p, q \in S$. This means that **S** is intra-regular.

**Theorem 3.13** Let **S** be an intra-regular ordered semigroup. Then for any anti-hybrid ideal $\tilde{f}_\lambda$ in $S$ over $U$, we have that

$$\tilde{f}(ab) = \tilde{f}(ba) \text{ and } \lambda(ab) = \lambda(ba) \text{ for all } a, b \in S.$$

**Proof**. Let $\tilde{f}_\lambda$ be an anti-hybrid ideal in $S$ over $U$ and $a, b \in S$. Since **S** is intra-regular, by Theorem 3.12, we have that

$$\tilde{f}(ab) = \tilde{f}((ab)^2) = \tilde{f}(a(ba)b) \subseteq \tilde{f}(ba) = \tilde{f}((ba)^2) = \tilde{f}(b(ab)a) \subseteq \tilde{f}(ab).$$

This implies that $\tilde{f}(ab) = \tilde{f}(ba)$. Moreover,

$$\lambda(ab) = \lambda((ab)^2) = \lambda(a(ba)b) \geq \lambda(ba) = \lambda((ba)^2) = \lambda(b(ab)a) \geq \lambda(ab).$$

This implies that $\lambda(ab) = \lambda(ba)$.

**Lemma 3.14** Let **S** be an ordered semigroup and $\tilde{f}_\lambda$ be an anti-hybrid bi-ideal in $S$ over $U$. Then $\tilde{f}_\lambda \ll \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda$.

**Proof**. Let $x \in S$. If $\mathbf{S}_x = \emptyset$, then

$$\tilde{f}(x) \subseteq U = (\tilde{f} \odot \tilde{S} \odot \tilde{f})(x) \text{ and } \lambda(x) \geq 0 = (\lambda \circ S \circ \lambda)(x).$$

In this case we have $\tilde{f}_\lambda \ll \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda$. If $\mathbf{S}_x \neq \emptyset$, then

$$
\begin{aligned}
(\tilde{f} \odot \tilde{S} \odot \tilde{f})(x) &= \bigcap_{(a,b)\in \mathbf{S}_x} \left[(\tilde{f} \odot \tilde{S})(a) \cup \tilde{f}(b)\right] \\
&= \bigcap_{(a,b)\in \mathbf{S}_x} \left[\bigcap_{(p,q)\in \mathbf{S}_a} \left[\tilde{f}(p) \cup \tilde{S}(q)\right] \cup \tilde{f}(b)\right] \\
&= \bigcap_{(a,b)\in \mathbf{S}_x} \bigcap_{(p,q)\in \mathbf{S}_a} \left[\tilde{f}(p) \cup \tilde{S}(q) \cup \tilde{f}(b)\right] \\
&= \bigcap_{(a,b)\in \mathbf{S}_x} \bigcap_{(p,q)\in \mathbf{S}_a} \left[\tilde{f}(p) \cup \tilde{f}(b)\right] \\
&\supseteq \bigcap_{(a,b)\in \mathbf{S}_x} \bigcap_{(p,q)\in \mathbf{S}_a} \tilde{f}(a) \\
&= \tilde{f}(a),
\end{aligned}
$$

and

$$(\lambda \circ S \circ \lambda)(x) = \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{(\lambda \circ S)(a), \lambda(b)\}\}$$

$$\begin{aligned}
&= \bigvee_{(a,b)\in \mathbf{S}_x} \left\{ \min\left\{ \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\lambda(p), S(q)\}\}, \lambda(b) \right\} \right\} \\
&= \bigvee_{(a,b)\in \mathbf{S}_x} \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\lambda(p), S(q), \lambda(b)\}\} \\
&= \bigvee_{(a,b)\in \mathbf{S}_x} \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\lambda(p), \lambda(b)\}\} \\
&\leq \bigvee_{(a,b)\in \mathbf{S}_x} \bigvee_{(p,q)\in \mathbf{S}_a} \lambda(a) \\
&= \lambda(a).
\end{aligned}$$

In this case we also have that $\tilde{f}_\lambda \ll \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda$. Altogether, we obtain $\tilde{f}_\lambda \ll \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda$.

An ordered semigroup **S** is *regular* (Kehayopulu, 1992b) if for each $a \in S$ there exists $x \in S$ such that $a \leq axa$.

**Lemma 3.15** (Xie & Tang, 2010) Let **S** be an ordered semigroup. Then the following statements are equivalent.

(1) **S** is regular.
(2) $B = (BSB]$ for any bi-ideal $B$ of **S**.

**Lemma 3.16** (Sarasit *et al*, 2021) Let **S** be an ordered semigroup and $\tilde{f}_\lambda$ a hybrid structure in $S$ over $U$. Then the following statements are equivalent.

(1) $\tilde{f}_\lambda$ is an anti-hybrid ideal in $S$ over $U$.
(2) $\tilde{f}_\lambda$ satisfies the following conditions.
 (2.1) $\tilde{f}_\lambda \ll \tilde{f}_\lambda \otimes \tilde{S}_S$ and $\tilde{f}_\lambda \ll \tilde{S}_S \otimes \tilde{f}_\lambda$.
 (2.2) For $x, y \in S$, if $x \leq y$, then $\tilde{f}(x) \subseteq \tilde{f}(y)$ and $\lambda(x) \geq \lambda(y)$.

**Lemma 3.17** Let **S** be an ordered semigroup. Then the following statements are equivalent.

(1) **S** is regular.
(2) $\tilde{f}_\lambda = \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda$ for any anti-hybrid bi-ideal $\tilde{f}_\lambda$ in $S$ over $U$.

**Proof**. **(1) ⇒ (2)**. Let $\tilde{f}_\lambda$ be an anti-hybrid bi-ideal in $S$ over $U$. By Lemma 3.14, we have that $\tilde{f}_\lambda \ll \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda$. On the other hand, we let $a \in S$. Then there exists $x \in S$ such that $a \leq axa$. This means that $\mathbf{S}_a \neq \emptyset$. Then, we obtain that

$$\begin{aligned}
(\tilde{f} \odot \tilde{S} \odot \tilde{f})(a) &= \bigcap_{(b,c)\in \mathbf{S}_a} \left(\tilde{f}(b) \cup (\tilde{S} \odot \tilde{f})(c)\right) \\
&\subseteq \tilde{f}(a) \cup (\tilde{S} \odot \tilde{f})(xa) \\
&= \tilde{f}(a) \cup \left[ \bigcap_{(p,q)\in \mathbf{S}_{xa}} \left(\tilde{S}(p) \cup \tilde{f}(q)\right) \right] \\
&\subseteq \tilde{f}(a) \cup \tilde{S}(x) \cup \tilde{f}(a) \\
&= \tilde{f}(a),
\end{aligned}$$

and

$$\begin{aligned}
(\lambda \circ S \circ \lambda)(a) &= \bigvee_{(b,c)\in \mathbf{S}_a} \{\min\{\lambda(b), (S \circ \lambda)(c)\}\} \\
&\geq \min\{\lambda(a), (S \circ \lambda)(xa)\} \\
&= \min\left\{ \lambda(a), \left[ \bigvee_{(p,q)\in \mathbf{S}_{xa}} \{\min\{S(p), \lambda(q)\}\} \right] \right\} \\
&\geq \min\{\lambda(a), \min\{S(x), \lambda(a)\}\} \\
&= \min\{\lambda(a), S(x), \lambda(a)\} \\
&= \lambda(a).
\end{aligned}$$

Hence, $\tilde{f}_\lambda \gg \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda$. Altogether, we obtain that $\tilde{f}_\lambda = \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda$.

**(2) ⇒ (1)**. Let $B$ be a bi-ideal of **S**. By Lemma 3.9, we have that $\chi_{B^c}(\tilde{S}_S)$ is an anti-hybrid bi-ideal in $S$ over $U$. By hypothesis and Lemma 3.8, we obtain that

$$\chi_{B^c}(\tilde{S}_S) = \chi_{B^c}(\tilde{S}_S) \otimes \tilde{S}_S \otimes \chi_{B^c}(\tilde{S}_S) = \chi_{(BSB]^c}(\tilde{S}_S).$$

This implies that $B^c = (BSB]^c$ and then $B = (BSB]$. By Lemma 3.15, we have that **S** is regular.

Now, we characterize regular ordered semigroups through anti-hybrid bi-ideals and anti-hybrid ideals.

**Theorem 3.18** Let **S** be an ordered semigroup. Then the following statements are equivalent.

(1) **S** is regular.
(2) $\tilde{f}_\lambda \otimes \tilde{g}_\alpha \otimes \tilde{f}_\lambda = \tilde{f}_\lambda \Cup \tilde{g}_\alpha$ for every anti-hybrid bi-ideal $\tilde{f}_\lambda$ and every anti-hybrid ideal $\tilde{g}_\alpha$ in $S$ over $U$.

**Proof.** (1) ⇒ (2). Let $\tilde{f}_\lambda$ and $\tilde{g}_\alpha$ be an anti-hybrid bi-ideal and an anti-hybrid ideal in $S$ over $U$, respectively. Then, by Lemma 3.17, we have

$$\tilde{f}_\lambda \otimes \tilde{g}_\alpha \otimes \tilde{f}_\lambda \gg \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda = \tilde{f}_\lambda.$$

Since $\tilde{g}_\alpha$ is an anti-hybrid ideal in $S$ over $U$, by Lemma 3.16, we have

$$\tilde{f}_\lambda \otimes \tilde{g}_\alpha \otimes \tilde{f}_\lambda \gg \tilde{S}_S \otimes \tilde{g}_\alpha \otimes \tilde{S}_S \gg \tilde{S}_S \otimes \tilde{g}_\alpha \gg \tilde{g}_\alpha.$$

Thus, $\tilde{f}_\lambda \otimes \tilde{g}_\alpha \otimes \tilde{f}_\lambda \gg \tilde{f}_\lambda \Cup \tilde{g}_\alpha$. On the other hand, we let $a \in S$. Since **S** is regular, there exists $x \in S$ such that $a \leq axa \leq ax(axa) = a(xaxa)$ and then $\mathbf{S}_a \neq \emptyset$. Since $\tilde{g}_\alpha$ is an anti-hybrid ideal in $S$ over $U$, so $\tilde{g}(xax) \subseteq \tilde{g}(ax) \subseteq \tilde{g}(a)$ and $\alpha(xax) \geq \alpha(ax) \geq \alpha(a)$, we obtain that

$$\begin{aligned}
(\tilde{f} \odot \tilde{g} \odot \tilde{f})(a) &= \bigcap_{(y,z)\in \mathbf{S}_a} \left[\tilde{f}(y) \cup (\tilde{g} \odot \tilde{f})(z)\right] \\
&\subseteq \left[\tilde{f}(a) \cup (\tilde{g} \odot \tilde{f})(xaxa)\right] \\
&= \left[\tilde{f}(a) \cup \bigcap_{(p,q)\in \mathbf{S}_{xaxa}} \left[\tilde{g}(p) \cup \tilde{f}(q)\right]\right] \\
&\subseteq \left[\tilde{f}(a) \cup \tilde{g}(xax) \cup \tilde{f}(a)\right] \\
&\subseteq \left[\tilde{f}(a) \cup \tilde{g}(a) \cup \tilde{f}(a)\right] \\
&= \tilde{f}(a) \cup \tilde{g}(a) \\
&= (\tilde{f} \cup \tilde{g})(a),
\end{aligned}$$

and

$$\begin{aligned}
(\lambda \circ \alpha \circ \lambda)(a) &= \bigvee_{(y,z)\in \mathbf{S}_a} \{\min\{\lambda(y), (\alpha \circ \lambda)(z)\}\} \\
&\geq \min\{\lambda(a), (\alpha \circ \lambda)(xaxa)\} \\
&= \min\{\lambda(a), \bigvee_{(p,q)\in \mathbf{S}_{xaxa}} \{\min\{\alpha(p), \lambda(q)\}\}\} \\
&\geq \min\{\lambda(a), \alpha(xax), \lambda(a)\} \\
&\geq \min\{\lambda(a), \alpha(a), \lambda(a)\} \\
&= \min\{\lambda(a), \alpha(a)\} \\
&= (\lambda \wedge \alpha)(a).
\end{aligned}$$

This means that $\tilde{f}_\lambda \otimes \tilde{g}_\alpha \otimes \tilde{f}_\lambda \ll \tilde{f}_\lambda \Cup \tilde{g}_\alpha$. Altogether, we have $\tilde{f}_\lambda \otimes \tilde{g}_\alpha \otimes \tilde{f}_\lambda = \tilde{f}_\lambda \Cup \tilde{g}_\alpha$.
(2) ⇒ (1). Let $\tilde{f}_\lambda$ be an anti-hybrid bi-ideal in $S$ over $U$. Since $\tilde{S}_S$ is an anti-hybrid ideal in $S$ over $U$, we have that

$$\tilde{f}_\lambda = \tilde{f}_\lambda \Cup \tilde{S}_S = \tilde{f}_\lambda \otimes \tilde{S}_S \otimes \tilde{f}_\lambda.$$

By Lemma 3.17, we have that **S** is regular.

**Lemma 3.19** (Xie & Tang, 2010) Let **S** be an ordered semigroup. Then the following statements are equivalent.
(1) **S** is regular.
(2) $B \cap L \subseteq (BL]$ for every bi-ideal $B$ and every left ideal $L$ of **S**.
(3) $R \cap B \cap L \subseteq (RBL]$ for every right ideal $R$, every bi-ideal $B$ and every left ideal $L$ of **S**.

We now characterize regular ordered semigroup by using anti-hybrid left ideals, anti-hybrid right ideals, and anti-hybrid bi-ideals.

**Theorem 3.20** Let **S** be an ordered semigroup. Then the following statements are equivalent.
(1) **S** is regular.
(2) $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg \tilde{f}_\lambda \otimes \tilde{g}_\alpha$ for every anti-hybrid bi-ideal $\tilde{f}_\lambda$ and every anti-hybrid left ideal $\tilde{g}_\alpha$ in $S$ over $U$.
(3) $\tilde{h}_\beta \Cup \tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg \tilde{g}_\beta \otimes \tilde{f}_\lambda \otimes \tilde{g}_\alpha$ for every anti-hybrid bi-ideal $\tilde{f}_\lambda$, every anti-hybrid left ideal $\tilde{g}_\alpha$ and every anti-hybrid right ideal $\tilde{h}_\beta$ in $S$ over $U$.

**Proof.** (1) ⇒ (2). Suppose that $\tilde{f}_\lambda$ and $\tilde{g}_\alpha$ is an anti-hybrid bi-ideal and an anti-hybrid left ideal in $S$ over $U$, respectively. Let $a \in S$. Since **S** is regular, there exists $x \in S$ such that $a \leq axa$ and then $\mathbf{S}_a \neq \emptyset$, we obtain that

$$\begin{aligned}
(\tilde{f} \odot \tilde{g})(a) &= \bigcap_{(y,z)\in \mathbf{S}_a} \left[\tilde{f}(y) \cup \tilde{g}(z)\right] \\
&\subseteq \tilde{f}(a) \cup \tilde{g}(xa) \\
&\subseteq \tilde{f}(a) \cup \tilde{g}(a) \\
&= (\tilde{f} \cup \tilde{g})(a),
\end{aligned}$$

and

$$(\lambda \circ \alpha)(a) = \bigvee_{(y,z)\in \mathbf{S}_a} \{\min\{\lambda(y), \alpha(z)\}\} \geq \min\{\lambda(a), \alpha(xa)\} \geq \min\{\lambda(a), \alpha(a)\} = (\lambda \wedge \alpha)(a).$$

Therefore $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg \tilde{f}_\lambda \otimes \tilde{g}_\alpha$.

(2) ⇒ (1). Let $B$ and $L$ be a bi-ideal and a left ideal of **S** , respectively. Then, by Lemma 3.3, $\chi_{B^c}(\tilde{S}_S)$ and $\chi_{L^c}(\tilde{S}_S)$ is an anti-hybrid bi-ideal and an anti-hybrid left ideal in $S$ over $U$, respectively. By hypothesis and Lemma 3.8, we obtain that

$$\chi_{(B\cap L)^c}(\tilde{S}_S) = \chi_{B^c}(\tilde{S}_S) \Cup \chi_{L^c}(\tilde{S}_S) \gg \chi_{B^c}(\tilde{S}_S) \otimes \chi_{L^c}(\tilde{S}_S) = \chi_{(BL]^c}(\tilde{S}_S).$$

This implies means that $(BL]^c \subseteq (B\cap L)^c$ and then $B \cap L \in (BL]$. By Lemma 3.19, we obtain that **S** is regular.

(1) ⇒ (3). Let $\tilde{f}_\lambda, \tilde{g}_\alpha$ and $\tilde{h}_\beta$ be an anti-hybrid bi-ideal, an anti-hybrid left ideal and an anti-hybrid right ideal in $S$ over $U$, respectively. Let $a \in S$. Since **S** is regular, there exists $x \in S$ such that $a \leq axa$ and then $\mathbf{S}_a \neq \emptyset$, we obtain that

$$(\tilde{h} \odot \tilde{f} \odot \tilde{g})(a) = \bigcap_{(y,z)\in \mathbf{S}_a} [\tilde{h}(y) \cup (\tilde{f} \odot \tilde{g})(z)] \subseteq \tilde{h}(ax) \cup (\tilde{f} \odot \tilde{g})(a) \subseteq \tilde{h}(a) \cup \bigcap_{(p,q)\in \mathbf{S}_a} [\tilde{f}(p) \cup \tilde{g}(q)] \subseteq \tilde{h}(a) \cup \tilde{f}(a) \cup \tilde{g}(xa) \subseteq \tilde{h}(a) \cup \tilde{f}(a) \cup \tilde{g}(a) = (\tilde{h} \cup \tilde{f} \cup \tilde{g})(a),$$

and

$$(\beta \circ \lambda \circ \alpha)(a) = \bigvee_{(y,z)\in \mathbf{S}_a} \{\min\{\beta(y), (\lambda \circ \alpha)(z)\}\} \geq \min\{\beta(ax), (\lambda \circ \alpha)(a)\} \geq \min\left\{\beta(a), \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\lambda(p), \alpha(q)\}\}\right\} \geq \min\{\beta(a), \lambda(a), \alpha(xa)\} \geq \min\{\beta(a), \lambda(a), \alpha(a)\} = (\beta \wedge \lambda \wedge \alpha)(a).$$

Therefore, $\tilde{h}_\beta \Cup \tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg \tilde{g}_\beta \otimes \tilde{f}_\lambda \otimes \tilde{g}_\alpha$.

(3) ⇒ (1). Let $R, L$ and $B$ be a right ideal, a left ideal and a bi-ideal of **S**, respectively. Then, by Lemma 3.9, $\chi_{R^c}(\tilde{S}_S)$, $\chi_{L^c}(\tilde{S}_S)$ and $\chi_{(B^c)}(\tilde{S}_S)$ is an anti-hybrid right ideal, an anti-hybrid left ideal and an anti-hybrid bi-ideal in $S$ over $U$, respectively. By hypothesis and Lemma 3.8, we obtain that

$$\chi_{(R\cap B\cap L)^c}(\tilde{S}_S) = \chi_{R^c}(\tilde{S}_S) \Cup \chi_{B^c}(\tilde{S}_S) \Cup \chi_{L^c}(\tilde{S}_S) \gg \chi_{R^c}(\tilde{S}_S) \otimes \chi_{(B^c)}(\tilde{S}_S) \otimes \chi_{L^c}(\tilde{S}_S) = \chi_{(RBL]^c}(\tilde{S}_S).$$

This implies that $(RBL]^c \subseteq (R \cap B \cap L)^c$ and then $R \cap B \cap L \subseteq (RBL]$. By Lemma 3.19, we obtain that **S** is regular.

**Lemma 3.21** (Cao, 2002) Let **S** be an ordered semigroup. Then the following statements are equivalent.

(1) **S** is regular.
(2) $A \cap B = (AB]$ for every right ideal $A$ and every left ideal $B$ of $S$.

**Lemma 3.22** Let **S** be an ordered semigroup. Then the following statements are equivalent.

(1) **S** is regular.
(2) $\tilde{f}_\lambda \Cup \tilde{g}_\alpha = \tilde{f}_\lambda \otimes \tilde{g}_\alpha$ for every anti-hybrid right ideal $\tilde{f}_\lambda$ and every anti-hybrid left ideal $\tilde{g}_\alpha$ in $S$ over $U$.

**Proof.** (1) ⇒ (2). Let $\tilde{f}_\lambda$ and $\tilde{g}_\alpha$ be an anti-hybrid right ideal and an anti-hybrid left ideal in $S$ over $U$, respectively. Let $a \in S$. Then, there exists $x \in S$ such that $a \leq axa$. It follows that $\mathbf{S}_a \neq \emptyset$. We obtain that

$$(\tilde{f} \odot \tilde{g})(a) = \bigcap_{(y,z)\in \mathbf{S}_a} [\tilde{f}(y) \cup \tilde{g}(z)] \subseteq \tilde{f}(a) \cup \tilde{g}(xa) \subseteq \tilde{f}(a) \cup \tilde{g}(a) = (\tilde{f} \cup \tilde{g})(a),$$

and

$$\begin{aligned}(\lambda \circ \alpha)(a) &= \bigvee_{(y,z)\in \mathbf{S}_a} \{\min\{\lambda(y), \alpha(z)\}\} \\ &\geq \min\{\lambda(a), \alpha(xa)\} \\ &\geq \min\{\lambda(a), \alpha(a)\} \\ &= (\lambda \wedge \alpha)(a).\end{aligned}$$

Therefore, $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg \tilde{f}_\lambda \otimes \tilde{g}_\alpha$. Inverse inclusion is obvious. Thus, $\tilde{f}_\lambda \Cup \tilde{g}_\alpha = \tilde{f}_\lambda \otimes \tilde{g}_\alpha$.
(2) ⇒ (1). Let $A$ and $B$ be a right ideal and a left ideal of $\boldsymbol{S}$, respectively. Then, $\chi_{A^c}(\tilde{S}_S)$ and $\chi_{(B^c)}(\tilde{S}_S)$ is an anti-hybrid right ideal and an anti-hybrid left ideal in $S$ over $U$, respectively. By our assumption and Lemma 3.8, we have that

$$\chi_{(A\cap B)^c}(\tilde{S}_S) = \chi_{A^c}(\tilde{S}_S) \Cup \chi_{(B^c)}(\tilde{S}_S) = \chi_{A^c}(\tilde{S}_S) \otimes \chi_{(B^c)}(\tilde{S}_S) = \chi_{(AB]^c}(\tilde{S}_S).$$

This implies that $A \cap B = (AB]$. By Lemma 3.21, **S** is regular.

Finally, we characterize ordered semigroups that are both intra-regular and regular by using anti-hybrid left ideals, anti-hybrid right ideals and anti-hybrid bi-ideals.

**Theorem 3.24**. Let **S** be an ordered semigroup. Then the following statements are equivalent.

(1) **S** is regular and intra-regular.
(2) $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg (\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda)$ for every anti-hybrid bi-ideals $\tilde{f}_\lambda$ and $\tilde{g}_\alpha$ in $S$ over $U$.
(3) $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg (\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda)$ for every anti-hybrid bi-ideal $\tilde{f}_\lambda$ and every anti-hybrid left ideal $\tilde{g}_\alpha$ in $S$ over $U$.
(4) $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg (\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda)$ for every anti-hybrid right ideal $\tilde{f}_\lambda$ and every anti-hybrid bi-ideal $\tilde{g}_\alpha$ in $S$ over $U$.
(5) $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg (\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda)$ for every anti-hybrid right ideal $\tilde{f}_\lambda$ and every anti-hybrid left ideal $\tilde{g}_\alpha$ in $S$ over $U$.

**Proof.** (1) ⇒ (2). Let $\tilde{f}_\lambda$ and $\tilde{g}_\alpha$ be anti-hybrid bi-ideals in $S$ over $U$ and $a \in S$. Since **S** is both regular and intra-regular, there exists $x \in S$ such that $a \leq axa$ and there exist $y, z \in S$ such that $a \leq ya^2z$. Thus

$$a \leq axa \leq ax(axa) \leq ax(ya^2z)xa = (axya)(azxa).$$

This implies that $\mathbf{S}_a \neq \emptyset$. Since

$$\tilde{f}(axya) \subseteq \left(\tilde{f}(a) \cup \tilde{f}(a)\right) = \tilde{f}(a) \text{ and } \tilde{g}(azxa) \subseteq \left(\tilde{g}(a) \cup \tilde{g}(a)\right) = \tilde{g}(a)$$

and

$$\lambda(axya) \geq \min\{\lambda(a), \lambda(a)\} = \lambda(a) \text{ and } \alpha(azxa) \geq \min\{\alpha(a), \alpha(a)\} = \alpha(a),$$

we have that

$$\begin{aligned}(\tilde{f} \odot \tilde{g})(a) &= \bigcap_{(p,q)\in \mathbf{S}_a} [\tilde{f}(p) \cup \tilde{g}(q)] \\ &\subseteq \tilde{f}(axya) \cup \tilde{g}(azxa) \\ &\subseteq \tilde{f}(a) \cup \tilde{g}(a) \\ &= (\tilde{f} \cup \tilde{g})(a),\end{aligned}$$

and

$$\begin{aligned}(\lambda \circ \alpha)(a) &= \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\lambda(p), \alpha(q)\}\} \\ &\geq \min\{\lambda(axya), \alpha(azxa)\} \\ &\geq \min\{\lambda(a), \alpha(a)\} \\ &= (\lambda \wedge \alpha)(a).\end{aligned}$$

This mean that $\tilde{f}_\lambda \otimes \tilde{g}_\alpha \ll \tilde{f}_\lambda \Cup \tilde{g}_\alpha$. In the same way, we can also show that $\tilde{g}_\alpha \otimes \tilde{f}_\lambda \ll \tilde{g}_\alpha \Cup \tilde{f}_\lambda$. Therefore, $(\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda) \ll \tilde{f}_\lambda \Cup \tilde{g}_\alpha$. Since every anti-hybrid left (resp., right) ideal in $S$ over $U$ is an anti-hybrid bi-ideal in $S$ over $U$, the implications (2) ⇒ (3), (2) ⇒ (4), (3) ⇒ (5) and (4) ⇒ (5) are clear.
(5) ⇒ (1). Let $\tilde{f}_\lambda$ and $\tilde{g}_\alpha$ be an anti-hybrid right ideal and an anti-hybrid left ideal in $S$ over $U$, respectively. By hypothesis, we have that $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg (\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda) \gg \tilde{g}_\alpha \otimes \tilde{f}_\lambda$. By Theorem 3.11, we obtain that **S** is intra-regular. Since

$$\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg (\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda) \gg \tilde{f}_\lambda \otimes \tilde{g}_\alpha \gg \tilde{f}_\lambda \otimes \tilde{S}_S \gg \tilde{f}_\lambda,$$

and

$$\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg (\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda) \gg \tilde{f}_\lambda \otimes \tilde{g}_\alpha \gg \tilde{S}_S \otimes \tilde{g}_\alpha \gg \tilde{g}_\alpha,$$

we obtain that $\tilde{f}_\lambda \Cup \tilde{g}_\alpha \gg (\tilde{f}_\lambda \otimes \tilde{g}_\alpha) \Cup (\tilde{g}_\alpha \otimes \tilde{f}_\lambda) \gg \tilde{f}_\lambda \otimes \tilde{g}_\alpha \gg \tilde{f}_\lambda \Cup \tilde{g}_\alpha$. This implies that $\tilde{f}_\lambda \Cup \tilde{g}_\alpha = \tilde{f}_\lambda \otimes \tilde{g}_\alpha$. By Lemma 3.22, we have that **S** is regular.

## 4. Conclusions

In this paper, hybrid structures and operations with the help of union soft sets and fuzzy sets are considered. The properties of anti-hybrid bi-ideals are studied. We characterize regular ordered semigroups and intra-regular ordered semigroups by anti-hybrid left ideals, anti-hybrid right ideals and anti-hybrid bi-ideals. In the future work, we will apply hybrid structures to hyperstructures.

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