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Original Article

Analytical approach to study water infiltration phenomenon in unsaturated soils using reduced differential transform method

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Abstract

In hydrology and soil sciences, infiltration is the process by which water on the ground surface enters the soil, and is described mathematically by Richard's equation. The present paper applies the reduced differential transform method to find the approximate analytical solution of Richard's equation describing infiltration phenomena in porous media. Some standard cases of Richard's equation are discussed to demonstrate the effectiveness and reliability of the method. Comparing approximate analytical solutions obtained by RDTM with exact solutions shows that the proposed method is reliable and accurate and can be applied for solving practical scientific and technological problems. The results obtained are also compared with the analytical solution obtained by some well-known methods available in the literature. The proposed approach does not need any linearization, discretization, or perturbation parameters to obtain the solution for non-linear PDE, and its direct applicability reduces numerical computation. Convergence analysis and error estimation of the approximate solution of Richard's equation is also addressed in this research.

Keywords: richards equation, reduced differential transform method, infiltration phenomena, analytical solution, convergence, error analysis

1. Introduction

Non-linear equations are used to explain most phenomena in the physical world, and these equations have got much attention from scientists, researchers, and engineers. Since a broad class of non-linear equations lacks a precise analytic solution, numerical methods have mainly been used to solve them. The major drawback of numerical methods is that it provides the solution at discontinuous points and therefore it needs an extensive computational resource to have a comprehension of the solution (Asgari, Bagheripour, & Mollazadeh, 2011; Deniz, Bildik, & Sezer, 2017; Ren, Shi, & Vong, 2020). The other issues are stability and convergence criteria (Asgari, Bagheripour, & Mollazadeh, 2011). Non-linear equations can also be solved using analytic methods. The Lyapunov artificial small parameter method, perturbation techniques, and the expansion method are some of the classic analytic approaches. Some new analytic methods for dealing with functional equations have been proposed in the last two decades, including Adomian decomposition method (ADM) (Pamuk,2005), Modified adomain decomposition method (MADM), Differential transform method (Chen & Dai, 2016; Ebrahimi & Mokhtari, 2015; Patel & Dhodiya, 2016), tanh method (Wazwaz,2005), Reduced differential transform method (Keskin & Oturanc, 2009), Variational iteration method (VIM), Homotopy perturbation method(HPM) (Jena & Chakraverty, 2019), Optimal homotopy analysis method (OHAM) (Pathak & Singh, 2015), Sumudu decomposition method (SDM) (Bildik & Deniz, 2016; Ramadan & Alluhaibi, 2016), and Homotopy method (Singh & Sharma,2019), etc. However, each of these methods has one or more drawbacks. While dealing with a complicated non-linear partial

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differential equation, ADM faces complications in calculating Adomain polynomials (Pamuk, 2005). VIM requires to produce the correct function using Lagrange's multipliers and this multiplier can be obtained using variational theory. It also leads to repeated computations and computation of unneeded terms that require time and effort. Laplace transform method can be considered an effective tool to find the exact solution, including the lesser number of non-linear terms, but as the number of non-linear terms increases, it is difficult to find its inverse. HPM requires the identification of parameters and solves the functional equation in each step which requires time and effort. Some of the hybrid methods agree with the exact solution, but it requires computational efforts and resources.

Recently, in the search of the exact solution to non-linear PDE, mathematicians have come up with the combination of two methods namely Elzaki transform and Adomain decomposition method (ETADM) (Varsoliwala & Singh, 2020), Laplace homotopy perturbation method(LHPM) (Johnston *et al*., 2016), Elzaki homotopy transformation perturbation method (Barari *et al*., 2009; Singh & Sharma, 2019), Elzaki projected differential transform method (Khalouta & Kadem, 2018), Homotopy-variational iteration method (Cherif & Ziane, 2018), Double Laplace iterative method (Dhunde & Waghmare, 2019), Elzaki Homotopy perturbation method (Jena & Chakraverty,2019), Laplace Adomain decomposition method (Agbavon, Appadu, & Khumalo, 2019), Laplace variational iteration method (Shah & Singh, 2017), New iterative transform and Homotopy perturbation method, (Shah & Singh, 2019), Homotopy analysis transform method (HATM) (Hoshyar, Ganji, & Abbasi, 2015).In many engineering and science problems, an analytical solution is preferable because it demonstrates the importance of the variables and how important they are concerning the other variables. With good mathematical backing, engineers or scientists who formulated the problem in their model will see the impact of the inputs on the outputs (their influence on the output and the degree of that influence). So, in the current research, we discuss the analytical solution for the non-linear infiltration process described by Richard's equation, leading to better simulation of soil behavior.

Richard's equation is one of the most popular non-linear partial differential equations to study the infiltration process introduced by Richard's. Infiltration is the mechanism of water saturation into the soil, which is considered one of the interesting issues by water and geotechnical engineers, as shown in Figure 1. The process of infiltration is affected by several factors, including soil hydrological properties like porosity, hydraulic conductivity, rainfall intensity, soil moisture content, etc. The water table serves as the boundary between saturated and unsaturated flows. Unsaturated and saturated flow arises above and beneath the water table, respectively. Buckingham introduces the basic concepts that describe the fluid flow through porous media. He suggested that the water content is a significant factor affecting water flow in unsaturated soil. So, he introduces Buckingham law that describes the idea of unsaturated hydraulic conductivity. He also introduces the concept of moisture diffusivity, defined as unsaturated hydraulic conductivity times soil-water characteristic curve. Richard applies the equation of continuity to Buckingham law to obtain the extended version of Darcy's law. He derives a mathematical equation describing the movement of water in non-swelling, unsaturated soil. The mixed-based, h- based, and theta-based formulations are the three primary forms of the Richards equation published in the literature where theta represents the volumetric moisture content, and h represents the pressure potential based on weight.

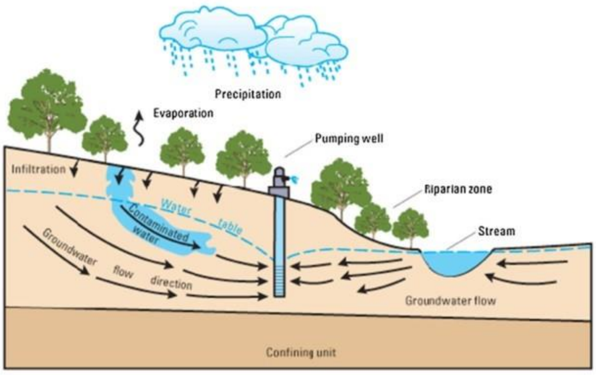

Figure 1. Schematic Diagram for Infiltration Process

2. Formulation of Richard's Equation

Richard's equation (RE) is a PDE that describes the non-saturated flow in soils, which is predicated based on Buckingham's studies done at the start of the 20th century (Asgari, Bagheripour, & Mollazadeh, 2011). It is found by coupling the continuity equation with Darcy's law, so one needs to mention both. The continuity equation and Darcy law are given by (Cherif & Ziane, 2018):

$$\frac{\partial \mu}{\partial \tau} = -\frac{\partial q}{\partial \omega} , \tag{1}$$

$$q = -K\frac{\partial H}{\partial \omega} , \tag{2}$$

where *K* represents hydraulic conductivity. Suction head (H) is defined as energy due to soil suction force in saturated flow. It is assumed that the positive z-axis is heading downward, so the total suction head (H) is given by $h = H - \omega$. The mathematical form of Darcy's law using Equation (1) can be expressed as:

$$q = -K\frac{\partial}{\partial \omega}\left(h+\omega\right) = -K\left(\frac{\partial h}{\partial \omega}+1\right)\cdot \tag{3}$$

Using Equation (3), Equation (2) reduces to $\frac{\partial \mu}{\partial \tau}=\frac{\partial}{\partial \omega}\left[K\left(\frac{\partial h}{\partial \omega}+1\right)\right]$. (4)

Equation (4) is known as mixed form. To solve Equation (4), the relation between hydraulic conductivity, saturation and pressure is required. The idea of differential water capacity is used to eliminate either h or μ, which is the rate of change of the soil water retention curve which is given by (Barari *et al*., 2009):

$$C(h)=\frac{\partial \mu}{\partial h}. \tag{5}$$

The h-based expression of Richard's equation is obtained from Equations (4) and (5) as follows:

$$C(h)\frac{\partial h}{\partial \tau}=\frac{\partial}{\partial \omega}\left(K\frac{\partial h}{\partial \omega}\right)+\frac{\partial K}{\partial \omega}. \tag{6}$$

Equation (6) is useful for the mathematical modeling of water in unsaturated soils. To derive the third form of Richard's equation, pore water diffusivity term D is introduced and is given by

$$D=\frac{K}{C}=\frac{K}{\frac{d\mu}{dh}}=K\frac{dh}{d\mu}. \tag{7}$$

From Equation (3), and Equation (7), we get

$$\frac{\partial \mu}{\partial \tau}=\frac{\partial}{\partial \omega}\left(D\frac{\partial \mu}{\partial \omega}\right)+\frac{\partial K}{\partial \omega}. \tag{8}$$

Here note that hydraulic conductivity and water diffusivity are highly dependent on moisture content, so they are treated as dependent parameters due to which it required proper estimation. In literature, several models have been prompted to determine the parameters, namely hydraulic conductivity and water diffusivity, among that "Brooks and Corey's model" (Brooks & Corey, 1964; Corey, 1994) and also the "Van Genuchten model" (Van Genuchten, 1980) are widely applied. The "Van Genuchten model" relates soil water pressure head, unsaturated hydraulic conductivity, and water content, mathematical which agrees with the experimental data, but the functional form is too intricate to apply in many analytical approaches. On the other hand, "Brooks and Corey's model" provides a more accurate definition. Now we consider the relation between water diffusivity and hydraulic conductivity according to "Brook's and Corey's model, "which is given by (Brooks & Corey, 1964; Corey, 1994):

$$D(\mu)=\frac{K_s}{\alpha\lambda(\mu_s-\mu_r)}\left(\frac{\mu-\mu_r}{\mu_s-\mu_r}\right)^{2+\frac{1}{\lambda}}, \tag{9}$$

$$K(\mu)=K_s\left(\frac{\mu-\mu_r}{\mu_s-\mu_r}\right)^{3+\frac{2}{\lambda}}, \tag{10}$$

where $D(\mu)$, $K(\mu)$ are water diffusivity and hydraulic conductivity. K_s, μ_r, μ_s represents saturated conductivity, residual water content, and saturated water content in the porous medium, α and λ are determined through experimental data. After some consideration in the "Brook's and Corey's model," improvement in soil parameters obtain the following equations (Brooks & Corey,1964; Corey,1994):

$$D(\mu)=D_0(\chi+1)\mu^{\chi},\gamma\geq 0, \tag{11}$$

$$K(\mu)=K_0\mu^{k},k\geq 1. \tag{12}$$

where particle size, pore size distribution, etc., are given by K_0, D_0, m, k, $\int_0^1 D(\theta)d\theta=1$ for all values of x. Due to normalization of diffusivity and $0<\mu<1$. The conductivity varies linearly, parabolically, and cubically with water content, when k takes the value 1, 2, 3, respectively, clear from Equation (12). Using Equations (8), (11), and (12), we get:

$$\frac{\partial \mu}{\partial \tau}=D_0(\chi+1)\frac{\partial}{\partial \omega}\left(\mu^{\chi}\frac{\partial \mu}{\partial \omega}\right)+K_0\frac{\partial}{\partial \omega}\mu^{k}. \tag{13}$$

The time and depth variables are independent variables in Equation (13). A new variable is introduced, a linear combination of time and depth using the travelling wave technique. To evaluate these transform equations, the Tangent-hyperbolic function is generally used. Therefore, the theta-based Richard's equation in order of (x, 1) is as follows:

$$\frac{\partial \mu}{\partial \tau}+\mu^{\chi}\frac{\partial \mu}{\partial \omega}-\frac{\partial^2 \mu}{\partial \omega^2}=0. \tag{14}$$

and the exact solution of Equation (14) is given by

$$\mu(\omega,\tau)=\left(\frac{\gamma}{2}\right)\left(1+\tanh\left(A-(z-Bt)\right)^{\frac{1}{n}}\right), \tag{15}$$

where $A=-\frac{\chi\beta+\beta|\chi|}{4(1+\chi)}(\chi\neq 0), B=\frac{\beta\gamma}{(1+\chi)}$ an arbitrary constant γ is considered as 1. The initial condition can be obtained by considering $t=0$ in the exact solution for Equation (14). The solution of Equation (14) is obtained for different values x using RDTM.

In literature, many case studies, as well as analytical and numerical approaches, have been utilized to study the infiltration phenomena (Chavez-Negrete, Santana-Quinteros & Dominguez-Mota, 2021; Khan *et al*., 2021; Nasseri *et al*., 2008; Nualtong *et al*., 2021a, 2021b, Shah & Singh, 2017, 2019) but some of the solutions apply elementary initial and geometrical conditions (Ghotbi, Omidvar & Barari, 2011; Pamuk, 2005; Witelsiki, 1997). To overcome these difficulties, many numerical approaches, namely finite element solutions, the finite difference has been applied. Nevertheless, to study the effect of input parameters on the output and degree of influence by numerical approach leads to multiple simulations is often time-consuming and requires extensive resources.HPM requires a perturbation parameter to reduce non-linear PDE's complexity, which requires effort. To convert a given non-linear PDE into a non-linear ODE, ADM does not need any transformations or perturbations; the main difficulty is calculating Adomain polynomials. However, an iterative method like variational iteration and Homotopy methods can resolve the drawbacks of the Adomain approach, but still difficult to apply them to highly non-linear infiltration equations. In addition to that, they are sensitive to the initial condition. Some authors applied hybrid methods such as Elzaki transform homotopy perturbation method, Elzaki transforms Adomain decomposition Method (Varsoliwala & Singh, 2020), Modified homotopy method (Shah & Singh, 2019) to solve Richard's equation in search of the analytical solution. This method requires extensive computational efforts, and the solution obtained by these hybrid methods is not satisfactory.

Moreover, no single best technique deals with the highly non-linear PDE describing infiltration phenomena efficiently. On some models/problems, some methods converge faster to a solution, while others do not. That is why non-linear models of infiltration phenomena continue to require a wide range of analytical, semi-analytical, and numerical methods to increase the convergence of the solutions and was the motivation behind the research.

In the current research, the reduced differential transform method using MATLAB is applied to non-linear PDE arising in infiltration phenomena. The main merit of the method is that it converts the non-linear differential equation into an algebraic system and provides the solution in the form of convergent series. After selecting the suitable values of the parameter, this approach leads to a more accurate and realistic solution that describes linear and non-linear behavior in the infiltration process. Furthermore, using MATLAB, it can be easily extended to all classes of non-linear equations.

3. Reduced Differential Transform Method

This section deals with some essential mathematical preliminaries and definitions of the proposed method required for better understanding. RDTM concept has been derived from the two-dimensional Taylor's series expansion w.r.t to specific variable *x* or *t*. Consider a function $\xi(\omega,\tau)$ and assume that $\xi(\omega,\tau)=h(\omega)\,l(\tau)$. According to the definition of the one-dimensional differential transform method (ODDTM), the given function $\xi(\omega,\tau)$ can be expressed as:

$$\xi(\omega,\tau)=\sum_{\kappa=0}^{\infty}H_{\kappa}\omega^{\kappa}\sum_{\eta=0}^{\infty}L_{\eta}\tau^{k}=\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}W_{\kappa,\eta}(\kappa,\eta)\,\omega^{\kappa}\tau^{\eta} \quad , \tag{16}$$

where $W_{\kappa,\eta}=H_{\kappa}L_{\eta}$ is called the spectrum of $\xi(\omega,\tau)$. The basic concept of RDTM is as follows (Keskin & Oturanc, 2009):

Definition. In the domain of interest, if $\xi(\omega,\tau)$ is a continuously differentiable function, then the spectrum or transform the form of $\xi(\omega,\tau)$ w.r.t τ at τ_0 is defined as

$$\xi_{\kappa}(\omega)=\frac{1}{\kappa!}\left[\frac{\partial^{\kappa}}{\partial\tau^{\kappa}}\xi(\omega,\tau)\right]_{\tau=\tau_0}, \tag{17}$$

where $\xi(\omega,\tau)$ is the original function and $\xi_{\kappa}(\omega)$ is the reduced differential transform of $\xi(\omega,\tau)$. The inverse reduced differential transform of $\xi_{\kappa}(\omega)$ is defined as:

$$\xi(\omega,\tau)=\sum_{\kappa=0}^{\infty}\xi_{\kappa}(\omega)(\tau-\tau_0)^{\kappa} \quad . \tag{18}$$

From Equation (17) and Equation (18), we get

$$\xi(\omega,\tau)=\sum_{\kappa=0}^{\infty}\left[\frac{1}{\kappa!}\left[\frac{\partial^{\kappa}}{\partial\tau^{\kappa}}\xi(\omega,\tau)\right]_{\tau=\tau_0}\right](\tau-\tau_0)^{\kappa}. \tag{19}$$

The fundamental operations of RDTM that can be derived using Equations (17) and (18) are mentioned below in Table 1.

Table 1. Fundamental operation of RDTM w.r.t to variable *t*

Original function	Transformed function
$\theta_1(x,y)\pm\theta_2(x,y)$	$\Theta_1(\omega,\ell)\pm\Theta_2(\omega,\ell)$
$\lambda\theta_1(x,y)$	$\lambda\Theta_1(\omega,\ell)$
$\frac{\partial\xi(\omega,\tau)}{\partial\omega}$	$\frac{\partial\xi(\omega,\ell)}{\partial\omega}$
$\frac{\partial\xi(\omega,\tau)}{\partial\tau}$	$(\vartheta+1)\xi(\omega,\vartheta+1)$
$\frac{\partial\xi(\omega,\tau)}{\partial\omega\partial\tau}$	$(\vartheta+1)\frac{\partial\xi(\omega,\vartheta+1)}{\partial\omega}$
$\xi_1(\omega,\tau)\xi_2(\omega,\tau)$	$\sum_{\eta=0}^{\ell}\xi_1(\omega,\eta)\xi_2(\omega,\ell-\eta)$
$\omega^{\mathrm{A}}\tau^{\mathrm{B}}$	$\omega^{\mathrm{A}}\delta(h-\mathrm{B})$ where $\delta(h-\mathrm{B})=\begin{cases}1 & ,h=\mathrm{B}\\ 0 & ,h\neq\mathrm{B}\end{cases}$

3.1 Implementation of RDTM

This section discusses the convergence of the solution obtained by RDTM. First of all, we show that the solution obtained by the proposed method exists as a power series in terms of *t* or *x*. Consider the following non-linear PDE:

$$v_\tau = w(\omega,\tau,\mu,\mu_\omega,\mu_{\omega\omega},.............), \tag{20}$$

with initial condition $v(\omega,0)=v_0(\omega)$. (21)

Applying fundamental operation of RDTM from Table 1 to Equation (20), transformed recursive formula is given by:

$$(l+1)V_{l+1}(\omega)=W\left(\xi,v_l,\frac{dV_l(\omega)}{d\omega},\frac{d^2V_l(\omega)}{d\omega^2},............\right), \tag{22}$$

and the transformed initial condition is $v(\omega)=v_0(\omega)$, (23)

where $V_{l+1}(\omega)$ and $W\left(\omega,v_l,\frac{dV_l(\omega)}{d\omega},\frac{d^2V_l(\omega)}{d\omega^2},............\right)$ are transformed form obtained by applying RDTM to the original function $v(\omega,\tau)$ and $w(x,\tau,\mu,\mu_\omega,\mu_{\omega\omega},.............)$ in the kth iteration. Substituting the value of $V_l(\omega)$ for $l=0,1,2,3,4,......n$ in Equation (18), the approximate analytical series solution of Equation (20) with initial condition Equation (21) is given by

$$v(\omega,\tau)=\sum_{l=0}^{\infty}V_l(\omega)(\tau-\tau_0)^l\ . \tag{24}$$

3.2 Convergence and error analysis of RDTM

The convergence of series solution obtained by RDTM has been discussed by Seyyedeh Roodabeh Moosavi Noori and Nasir. We recall the theorems from (Moosavi Noori & Taghizadeh, 2021) that guarantee the convergence of the series solution Equation (24).

Theorem 1. If $\zeta_k(\omega,\tau)=V_l(\omega)(\tau-\tau_0)^l$, then the series solution $\sum_{i=0}^{n}\zeta_l(\omega,\tau)$ for Equation (20), $\forall\, l\in N\cup\{0\}$ (i) is convergent if there exist $0<\eta<1$ such that $\|\zeta_{l+1}\|\le\eta\|\zeta_l\|$, (ii) is divergent if there exist $\eta>1$ such that $\|\zeta_{l+1}\|\ge\eta\|\zeta_l\|$.

The truncation error of the series Equation (24), which is a specific case of Banach's fixed point theorem, is investigated in Theorem 1.

Theorem 2. Suppose $\sum_{i=0}^{n}\zeta_i(\omega,\tau)$ is required series solution, where $\zeta_l(\omega,\tau)=V_k(\omega)(\tau-\tau_0)^l$, converges to $\varepsilon(\omega,t)$.If $\sum_{i=0}^{n}\zeta_i(\omega,\tau)$ is the truncated series used to approximate the solution and then estimated maximum absolute truncated error is as $\left\|\varepsilon(\omega,t)-\sum_{i=0}^{m}\zeta_i(\omega,\tau)\right\|\leq\frac{1}{1-\eta}\eta^{m+1}\|\zeta_0\|$.

One can refer to the proof of Theorem 1 & 2 (Moosavi Noori & Taghizadeh, 2021).

From Theorem 1 and 2, it is concluded that series solution obtained using RDTM for non-linear Equation (20) converges to an exact solution when there exists $0<\eta<1$ such that $\|\zeta_{l+1}\|\leq\eta\|\zeta_l\|$, for $\forall l\in N\cup\{0\}$. In addition, $\left\|\varepsilon(\omega,t)-\sum_{i=0}^{m}\zeta_i(\omega,\tau)\right\|\leq\frac{1}{1-\eta}\eta^{m+1}\|\zeta_0\|$ represents the maximum estimated absolute truncated error.

4. Application of RDTM to Richard's Equation

The series solution of Richard's equation based on "Brooks and Corey's model" is discussed in this section using RDTM. For simplicity, we have discussed it for two different values of *x*, although this method can solve the equation for any value of *x* which introduces more non-linear terms in Richard's equation.

Case I. Considering *x*=1, Equation (14) reduces to
$$\frac{\partial\mu}{\partial\tau}+\mu\frac{\partial\mu}{\partial\omega}-\frac{\partial^2\mu}{\partial\omega^2}=0, \tag{25}$$

with initial condition
$$\mu(\omega,0)=\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right). \tag{26}$$

The exact solution of Equation (25) is given in reference (Witelski, 1997, 2005).

Applying RDTM to Equation (25), we obtain the following recursive formula:
$$(\iota+1)\mu_{(\iota+1)}(\omega)+\sum_{\eta=0}^{\iota}\mu_\eta(\omega)\frac{\partial\mu_{\iota-\eta}(\omega)}{\partial\omega}-\frac{\partial^2\mu_\iota(\omega)}{\partial\omega^2}=0, \tag{27}$$

with transformed initial condition
$$\mu_0(\omega)=\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right). \tag{28}$$

For $\iota=0$, Equation (27) reduces to
$$\mu_1(x)=\frac{\partial^2\mu_0(\omega)}{\partial\omega^2}-\mu_0(\omega)\frac{\partial\mu_0(\omega)}{\partial\omega}. \tag{29}$$

Substituting values from Equation (28) into Equation (29), we get
$$\mu_1(x)=\frac{\partial^2}{\partial\omega^2}\left(\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right)\right)-\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right)\frac{\partial}{\partial\omega}\left(\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right)\right),$$
$$\mu_1(\omega)=-\frac{1}{8}\left(2\sec h\left(-\frac{\omega}{4}\right)\sec h\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right)\left(-\frac{1}{4}\right)\right)-\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right)\left(-\frac{1}{8}\sec^2 h\left(-\frac{\omega}{4}\right)\right)$$
$$\mu_1(\omega)=-\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right)+\left(\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)+\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)\tanh\left(-\frac{\omega}{4}\right)\right),$$
$$\mu_1(\omega)=\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right), \tag{30}$$

For $\iota=1$, Equation (27) reduces to
$$2\mu_2(\omega)=\frac{\partial^2\mu_1(\omega)}{\partial\omega^2}-\left(\mu_0(\omega)\frac{\partial}{\partial\omega}\mu_1(\omega)+\mu_1(\omega)\frac{\partial}{\partial\omega}\mu_0(\omega)\right). \tag{31}$$

Substituting values from Equations (28), and (30) into Equation (31), we get

$$2\mu_2(\omega)=\frac{\partial^2}{\partial\omega^2}\left(\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)\right)-\left(\begin{array}{l}\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right)\frac{\partial}{\partial x}\left(\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)\right)\\+\left(\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)\right)\frac{\partial}{\partial x}\left(\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right)\right)\end{array}\right),$$

$$2\mu_2(\omega)=-\frac{3}{128}\sec h^4\left(-\frac{\omega}{4}\right)+\frac{1}{64}\sec h^2\left(-\frac{\omega}{4}\right)$$
$$-\frac{1}{32}\sec h^2\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right)\left[\left(\frac{1}{2}+\frac{1}{2}\tan h\left(-\frac{\omega}{4}\right)\right)-\frac{1}{128}\sec h^4\left(-\frac{\omega}{4}\right)\right],$$

$$2\mu_2(\omega)=-\frac{3}{128}\sec h^4\left(-\frac{\omega}{4}\right)+\frac{1}{64}\sec h^2\left(-\frac{\omega}{4}\right)-\frac{1}{64}\sec h^2\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right)$$
$$-\frac{1}{64}\tan^2 h\left(-\frac{\omega}{4}\right)\sec h^2\left(-\frac{\omega}{4}\right)+\frac{1}{128}\sec h^4\left(-\frac{\omega}{4}\right),$$

$$\mu_2(\omega)=-\frac{1}{128}\sec h^2\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right). \tag{32}$$

For $\iota=2$, Equation (27) reduces to

$$3\mu_3(\omega)=\frac{\partial^2\mu_2(\omega)}{\partial\omega^2}-\left[\mu_0(\omega)\frac{\partial}{\partial\omega}\mu_2(\omega)+\mu_1(\omega)\frac{\partial}{\partial\omega}\mu_1(\omega)+\mu_2(\omega)\frac{\partial}{\partial\omega}\mu_0(\omega)\right]. \tag{33}$$

Substituting values from Equations (30) and (31) into Equation (33), we get

$$3\mu_3(\omega)=\frac{\partial^2}{\partial\omega^2}\left(-\frac{1}{128}\sec h^2\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right)\right)$$
$$-\left[\begin{array}{l}\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right)\frac{\partial}{\partial\omega}\left(-\frac{1}{128}\sec h^2\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right)\right)\\+\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)\frac{\partial}{\partial\omega}\left(\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)\right)\\+\left(-\frac{1}{128}\sec h^2\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right)\right)\frac{\partial}{\partial\omega}\frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right)\end{array}\right],$$

$$3\mu_3(\omega)=-\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)\tan \mathrm{h}^3\left(\frac{-\omega}{4}\right)-$$
$$\left[\begin{array}{l}\frac{1}{1024}\sec h^4\left(-\frac{\omega}{4}\right)-\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)+\frac{1}{512}\sec h^4\left(-\frac{\omega}{4}\right)\tanh\left(-\frac{\omega}{4}\right)\\+\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)\tanh\left(\frac{-\omega}{4}\right)\left(\sec h^2\left(-\frac{\omega}{4}\right)-1\right)\end{array}\right],$$

$$3\mu_3(\omega)=-\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)\tan \mathrm{h}^3\left(\frac{-\omega}{4}\right)-$$
$$\left[\begin{array}{l}\frac{1}{1024}\sec h^4\left(-\frac{\omega}{4}\right)-\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)+\frac{1}{512}\sec h^4\left(-\frac{\omega}{4}\right)\tanh\left(-\frac{\omega}{4}\right)\\+\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)\tanh^3\left(\frac{-\omega}{4}\right)\end{array}\right],$$

$$\mu_3(\omega)=\frac{1}{3}\left\{\begin{array}{l}-\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)\tan \mathrm{h}^3\left(\frac{-\omega}{4}\right)-\\\qquad\left[\begin{array}{l}\frac{1}{1024}\sec h^4\left(-\frac{\omega}{4}\right)-\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)+\frac{1}{512}\sec h^4\left(-\frac{\omega}{4}\right)\tanh\left(-\frac{\omega}{4}\right)\\+\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)\tanh^3\left(\frac{-\omega}{4}\right)\end{array}\right]\end{array}\right\}.$$

Similarly, the rest of the coefficient of the series solution can be obtained using MATLAB software. Using the definition of inverse transform and coefficient values $\mu_{\iota}(\omega)$ for $\iota = 0,1,2,3....$, the approximate analytical solution of Equation (24) for x =1 is given by

$$\mu(\omega,\tau) = \sum_{\iota=0}^{n} \mu_{\iota}(\omega) t^{\iota} = \mu_0(\omega) + \mu_1(\omega).\tau + \mu_2(\omega).\tau^2 + \mu_3(\omega).\tau^3 + ...,$$

$$\mu(\omega,t) = \frac{1}{2}\left(1+\tan h\left(-\frac{\omega}{4}\right)\right) + \left(\frac{1}{16}\sec h^2\left(-\frac{\omega}{4}\right)\right)\tau - \frac{1}{128}\sec h^2\left(-\frac{\omega}{4}\right)\tan h\left(-\frac{\omega}{4}\right)\tau^2$$
$$+\frac{1}{3}\left\{\begin{bmatrix} -\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)\tan \mathrm{h}^3\left(\frac{-\omega}{4}\right) - \\ \left[\frac{1}{1024}\sec h^4\left(-\frac{\omega}{4}\right) - \frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right) + \frac{1}{512}\sec h^4\left(-\frac{\omega}{4}\right)\tanh\left(-\frac{\omega}{4}\right)\right] \\ +\frac{1}{512}\sec h^2\left(-\frac{\omega}{4}\right)\tanh^3\left(\frac{-\omega}{4}\right) \end{bmatrix}\right\}\tau^3 + ...$$

Case II. Considering x =2, Equation (23) reduces to $\dfrac{\partial\mu}{\partial\tau} + \mu^2\dfrac{\partial\mu}{\partial\omega} - \dfrac{\partial^2\mu}{\partial\omega^2} = 0,$ (34)

with initial condition $\mu(\omega,0) = \dfrac{1}{2}\left(1-\tan h\left(\dfrac{\omega}{3}\right)\right)^{\frac{1}{3}}$. (35)

The exact solution of Equation (34) is given by $\mu(\omega,\tau) = \dfrac{1}{2}\left(1+\tan h\left(-\dfrac{\omega}{3}+\dfrac{\tau}{8}\right)\right)^{\frac{1}{2}}$. (36)

Applying RDTM to Equation (34), we obtain the following recursive formula:

$$(l+1)\mu_{(l+1)}(\omega) - \frac{\partial^2\mu_l}{\partial\omega^2} + \sum_{\eta=0}^{l}\sum_{s=0}^{\eta}\mu_{l-\eta}(\omega)\mu_{\eta-s}(\omega)\frac{\partial\mu_s(\omega)}{\partial\omega} = 0, \tag{37}$$

with the transformed initial condition is $\mu_0(\omega) = \dfrac{1}{2}\left(1-\tan h\left(\dfrac{\omega}{3}\right)\right)^{\frac{1}{2}}$. (38)

For $l=0$, Equation (37) reduces to $\mu_1(\omega) = \dfrac{\partial^2\mu_0(\omega)}{\partial\tau^2} - \mu_0(\omega)\mu_0(\omega)\dfrac{\partial\mu_0(\omega)}{\partial\omega},$ (39)

Substituting the value from Equation (38) into Equation (39), we get

$$\mu_1(\omega) = -\frac{1}{36}\frac{\sec h^2\left(\frac{\omega}{3}\right)}{\left(1-\tanh\left(\frac{\omega}{3}\right)\right)^{\frac{1}{2}}}\left[\frac{1}{2} - \frac{3}{2}\tanh\left(\frac{\omega}{3}\right)\right] + \frac{1}{48}\left(1-\tan h\left(\frac{\omega}{3}\right)\right)\left(1-\tanh\left(\frac{\omega}{3}\right)\right)^{\frac{1}{2}},$$

$$\mu_1(\omega) = \frac{1}{48}\frac{\sec h^2\left(\frac{\omega}{3}\right)}{\left(1-\tanh\left(\frac{\omega}{3}\right)\right)^{\frac{1}{2}}}\left[\frac{1}{3} + \tanh\left(\frac{\omega}{3}\right)\right].$$

Similarly, the rest of the coefficient of the series solution can be obtained using MATLAB software. Solution of $\mu_2(\omega), \mu_3(\omega)$ were too long to be mentioned here; therefore, they are shown graphically Using the definition of inverse transform and coefficient values $\mu_{\iota}(\omega)$ for $\iota = 0,1,2,3....$ the approximate analytical solution of Equation (24) for x =2 is given by

$$\mu(\omega,\tau) = \sum_{\iota=0}^{n} \mu_{\iota}(\omega) t^{\iota} = \mu_0(\omega) + \mu_1(\omega).\tau + \mu_2(\omega).\tau^2 + \mu_3(\omega).\tau^3 + ...,$$

$$\mu(\omega,\tau) = \frac{1}{2}\left(1-\tan h\left(\frac{\omega}{3}\right)\right)^{\frac{1}{2}} + \frac{1}{48}\frac{\sec h^2\left(\frac{\omega}{3}\right)}{\left(1-\tanh\left(\frac{\omega}{3}\right)\right)^{\frac{1}{2}}}\left[\frac{1}{3} + \tanh\left(\frac{\omega}{3}\right)\right]\tau + ...$$

5. Results and Discussion

Soil moisture content can be traced from the obtained solution and its variation w.r.t to depth and time are determined. These are shown graphically as well as numerically in this section. From Figure 2, it is observed that the content of moisture decreases as depth (ω) increases at the time (τ), and the content of moisture increases as time increases at any depth for x =1 respectively. From Table 1, it is clear that for a fixed value of depth (ω=1) if the time τ =0, 1, 2, 3, 4 is increased, the moisture content will increase, and for a fixed value of time (τ =1) if the depth ω=1, 2, 3 is increased, the moisture content will decrease, which depends on the maximum value of the initial condition. These numerical values of moisture content for depth ω=1, 2, 3 and τ =0, 1, 2, 3, 4 are mentioned in Table 1 for case (I). To examine the accuracy of the RDTM solution, the absolute error of the eight-term approximate analytical solution is mentioned in Table 2. Table 3 presents the error of the third, fifth, and seventh terms for the parameter value x =1 for ω=0, 1, and τ =0.1, 0.4, 0.7. The comparison of obtained solution for ω=1, 2, 3, 4, 5 for τ=1 by ETHPM, DTM, HPM, HAM is presented in Table 4.

From Figure.3, it is observed that the content of moisture decreases as depth increases at any given time, and the content of moisture increases as time increases at any depth for x =2 respectively. These numerical values are presented for depth ω=1, 2, 3 and τ =0, 1, 2, 3, 4 are mentioned in Table 5 for case (II). The effect of the parameter x can be observed in Figures 2 and 3. To examine the accuracy of RDTM solution for x =2, the absolute error of the eight-term approximate analytical solution is mentioned in Table 5. Table 7 presents the error of the third, fifth, and seventh terms for the parameter value x =2 for ω =0, 1,4, and τ =0.1, 0.4, 0.7. The comparison of obtained solution for ω =0, 1, 2, 3, 4, 5 for τ =1 by ETHPM, DTM, HPM, HAM is presented numerically in Table 6.

6. Conclusions

Richard's equation, which describes the behavior of unsaturated infiltration regions in soil, was successfully solved using the RDTM in this paper. The result obtained for two different cases illustrate the effectiveness and preciseness of the method. The comparison of the approximate analytical solution obtained by RDTM is more accurate than the solution

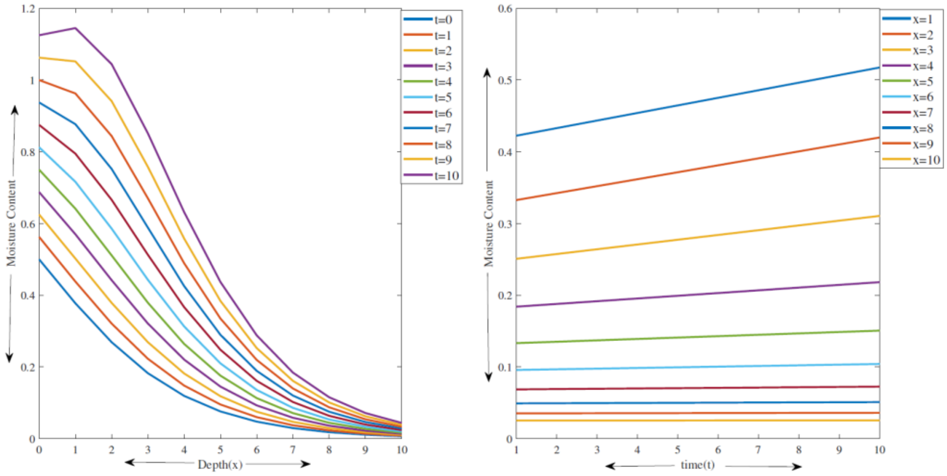

Figure 2. Moisture content versus depth (ω) and time (τ) for case(I)

Table 2. Comparison of exact solution with RDTM at fourth iteration with their absolute error for case(I)

x	τ	Exact	RDTM	RDTM Error
1	0	0.377540668798145	0.377540668798145	0
	1	0.437823499114202	0.437823500013543	8.99341 x 10^{-10}
	2	0.500000000000000	0.500000225965729	2.25966 x 10^{-7}
	3	0.562176500885798	0.562182109933744	5.60905 x 10^{-6}
	4	0.622459331201855	0.622512891320115	5.35601 x 10^{-5}
2	0	0.268941421369995	0.268941421369995	0
	1	0.320821300824607	0.320821301114114	2.89507 x 10^{-10}
	2	0.377540668798145	0.377540755983945	8.71858 x 10^{-8}
	3	0.437823499114202	0.437826057355635	2.5582 x 10^{-6}
	4	0.500000000000000	0.500028545864182	2.85459 x 10^{-5}
3	0	0.182425523806356	0.182425523806356	0
	1	0.222700138825309	0.222700138525035	3.00274 x 10^{-10}
	2	0.268941421369995	0.268941348118777	7.32512 x 10^{-8}
	3	0.320821300824607	0.320819547613199	1.75321 x 10^{-6}
	4	0.377540668798145	0.377524704679392	1.59641 x 10^{-5}

Table 3. The error of the ι^{th} approximate solutions, ι=3, 5, 7 in comparison of exact solution of case (I) at some points

ω	τ	$\lVert\mu(\omega,\tau)-\mu_3(\omega,\tau)\rVert$	$\lVert\mu(\omega,\tau)-\mu_5(\omega,\tau)\rVert$	$\lVert\mu(\omega,\tau)-\mu_7(\omega,\tau)\rVert$
0	0.1	0.0000003255	0.00000000651	0.0000000000651
	0.4	0.0000208125	0.000000416459	0.00000000416
	0.7	0.000111313	0.00000222966	0.000000223
1	0.1	0.000000252623	0.00000000736	0.0000000000736
	0.4	0.0000164841	0.000000474107	0.00000000474
	0.7	0.0000899617	0.00000255709	0.000000256

Table 4. Comparison of result obtained by HAM, HPM, DTM, RDTM, ETHPM and NITHPM for different value ω of and fixed value τ for x=1

	Moisture content						
ω	τ = 1						
	HAM	HPM	DTM	RDTM	ETHPM	NITHPM	Exact
1	0.4378	0.438	0.4382	0.437823	0.4379	0.43809	0.437823
2	0.3208	0.3209	0.3211	0.320821	0.3209	0.320934	0.320821
3	0.2227	0.226	0.2228	0.222700	0.2227	0.22672	0.2227
4	0.148	0.1479	0.148	0.148047	0.148	0.14795	0.148047
5	0.0953	0.0952	0.0953	0.095349	0.0953	0.0953	0.095349

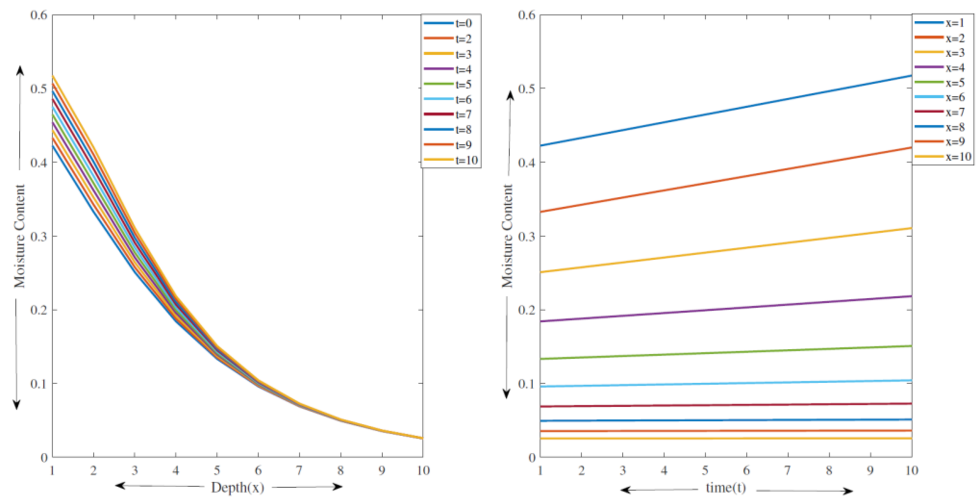

Figure 3. Moisture content versus depth (ω) and time (τ) for case(II)

Table 5. Comparison of exact solution with RDTM at fourth iteration with their absolute error for x=2

x	τ	Exact	RDTM	Absolute error
1	0	0.411851691288371	0.411851691288371	0
	1	0.42695224600747	0.425280071093563	0.001672175
	2	0.441974240288073	0.433484860573789	0.00848938
	3	0.4568532551803070	0.400339475183649	0.05651378
2	0	0.3229617061866970	0.322961706186697	0
	1	0.337302251098934	0.340821613600009	0.003519363
	2	0.351897571133328	0.359266423482335	0.007368852
	3	0.366702482518182	0.410900397197997	0.044197915
3	0	0.244134104563575	0.244134104563575	0
	1	0.256292030241682	0.262709863755740	0.006417834
	2	0.268865905672212	0.278475345776725	0.00960944
	3	0.281841360414786	0.283146676521009	0.001305316

Table 6. Comparison of result obtained by HAM, HPM, DTM, RDTM, ETHPM and NITHPM for x=2

Moisture content							
ω	$\tau = 1$						
	HAM	HPM	DTM	RDTM	ETHPM	NITHPM	Exact
1	0.6252	0.6252	0.6252	0.4253268	0.6252	0.625208	0.426952
2	0.4977	0.4977	0.4979	0.3407757	0.4969	0.496899	0.337302
3	0.3802	0.3803	0.3803	0.262730	0.379	0.379047	0.256292
4	0.2826	0.2826	0.2826	0.196861	0.2814	0.281372	0.189808
5	0.2065	0.2065	0.2065	0.1447589	0.2055	0.205505	0.138452

Table 7. The error of the ι^{th} approximate solutions, $\iota=3, 5, 7$ in comparison of exact solution of case (II) at some points

ω	τ	$\left\|\mu(\omega,\tau)-\mu_3(\omega,\tau)\right\|$	$\left\|\mu(\omega,\tau)-\mu_5(\omega,\tau)\right\|$	$\left\|\mu(\omega,\tau)-\mu_7(\omega,\tau)\right\|$
0	0.1	0.0000001155	0.00000000672	0.0000000000245
	0.4	0.0000308125	0.0000003659	0.00000000503
	0.7	0.0002313	0.00000423	0.000000211
1	0.1	0.0000001626	0.00000000748	0.0000000000287
	0.4	0.00004816	0.0000002516	0.00000000685
	0.7	0.000017899	0.000001803	0.000000573256
4	0.1	0.0000002867	0.000000009873	0.0000000000354
	0.4	0.000003163813	0.00000011194	0.00000000285
	0.7	0.00002287	0.000002462	0.0000002153

obtained using another well-known analytical method which is clear from Tables 4 and 6. The error analysis shows that converges of RDTM is faster as compared to other methods existing in the literature. It is worth mentioning that RDTM is capable of overcoming difficulties like perturbation parameters, discretization, etc., arising to determine the solution of non-linear problems. Therefore, the straight forward applicability and capacity to reduce the computational work make the RDTM method a promising tool for solving Richard's equation and can be applied to different non-linear problems. The proposed method can be utilized to obtain solution of a highly non-linear PDE equation describing the solute transport phenomena. The symbolic computation of RDTM for more than 50 terms is time-consuming is one of the limitations of RDTM.

References

Agbavon, K. M., Appadu, A. R., & Khumalo, M. (2019). On the numerical solution of Fisher's equation with coefficient of diffusion term much smaller than coefficient of reaction term. *Advances in Difference Equations*, *146(1)*, 1-33.

Asgari, A., Bagheripour, M. H., & Mollazadeh, M. (2011). A generalized analytical solution for a non-linear infiltration equation using the exp-function method. *Scientia Iranica*, *18*(1), 28-35.

Barari, A., Omidvar, M., Ghotbi, A. R., & Ganji, D. D. (2009). Numerical analysis of Richards' problem for water penetration in unsaturated soils. *Hydrology and Earth System Sciences Discussions*, *6*(5), 6359-6385.

Bildik, N., & Deniz, S. (2016). The use of Sumudu decomposition method for solving predator- prey systems. *Mathematical Sciences Letters an International Journal*, *5*(3), 285-289.

Brooks, R., & Corey, T. (1964). Hydraulic properties of porous media. *Hydrology Papers, Colorado State University*, *24*, 1-37.

Chavez-Negrete, C., Santana-Quinteros, D., & Domínguez-Mota, F. (2021). A solution of richards' equation by generalized finite differences for stationary flow in a dam. *Mathematics*, *9*(14), 1604.

Chen, X., & Dai, Y. (2016). Differential transform method for solving Richards equation. *Applied Mathematics and Mechanics*, *37*(2), 169-180.

Cherif, M. H., & Ziane, D. (2018). Variational iteration method combined with new transform to solve fractional partial differential equations. *Universal Journal of Mathematics and Applications*, *1*(2), 113-120.

Corey, A. T. (1994). *Mechanics of immiscible fluids in porous media.* Denver, CO: Water Resource Publication.

Deniz, S., Bildik, N., & Sezer, M. (2017). A note on stability analysis of Taylor collocation method. *Celal Bayar Universitesi Fen Bilimleri Dergisi*, *13*(1), 149-153.

Dhunde, R. R., & Waghmare, G. L. (2019). Double Laplace iterative method for solving non-linear partial differential equations. *New Trends Mathematical Sciences*, *7*(2), 138-149.

Ebrahimi, F., & Mokhtari, M. (2015). Transverse vibration analysis of rotating porous beam with functionally graded microstructure using the differential transform method. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, *37*(4), 1435-1444.

Ghotbi, A. R., Omidvar, M., & Barari, A. (2011). Infiltration in unsaturated soils–an analytical approach. *Computers and Geotechnics*, *38*(6), 777-782.

Hoshyar, H., Ganji, D. D., & Abbasi, M. (2015). Determination of temperature distribution for porous fin with temperature-dependent heat generation by homotopy analysis method. *Journal of Applied Mechanical Engineering*, *4*(1),1-5.

Jena, R. M., & Chakraverty, S. (2019). Solving time-fractional Navier–Stokes's equations using homotopy perturbation Elzaki transform. *Springer Nature Applied Sciences*, *1*(1),1-13.

Johnston, S. J., Jafari, H., Moshokoa, S. P., Ariyan, V. M., & Baleanu, D. (2016). Laplace homotopy perturbation method for Burger's equation with space-and time-fractional order. *Open Physics*, *14*(1), 247-252.

Keskin, Y., & Oturanc, G. (2009). Reduced differential transform method for partial differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, *10*(6), 741-750.

Khan, F. Y., Ashraf, A., Akhter, G., Baig, M. A., & Baig, S. A. (2021). Appraisal of groundwater recharge in Neelum watershed (Upper Indus Basin) using geospatial water balance technique. *Science of The Total Environment*, *785(147318)*,1-9.

Khalouta, A., & Kadem, A. (2018). Mixed of elzaki transform and projected differential transform method for a non-linear wave-like equations with variable coefficients. *Preprints*, *2018*. doi:10.20944/preprints 201808.0088.

Loyinmi, A. C., & Akinfe, T. K. (2020). Exact solutions to the family of Fisher's reaction-diffusion equation using Elzaki homotopy transformation perturbation method. *Engineering Reports*, *2*(2), 1-32.

Moosavi Noori, S. R., & Taghizadeh, N. (2021). Study of convergence of reduced differential transform method for different classes of differential equations. *International Journal of Differential Equations*, *2021*(1),1-16.

Nasseri, M., Shaghaghian, M. R., Daneshbod, Y., & Seyyedian, H. (2008). An analytic solution of water transport in unsaturated porous media. *Journal of Porous Media*, *11*(6),591-601.

Nualtong, K., Chinram, R., Khwanmuang, P., Kirtsaeng, S., & Panityakul, T. (2021a). An efficiency dynamic seasonal regression forecasting technique for high variation of water level in Yom River Basin of Thailand. *AIMS Environmental Science*, *8*(4), 283-303.

Nualtong, K., Panityakul, T., Khwanmuang, P., Chinram, R., & Kirtsaeng, S. (2021b). A Hybrid Seasonal Box Jenkins-ANN Approach for Water Level Forecasting in Thailand. *Environment and Ecology Research*, *9*(3),93-106.

Pamuk, S. (2005). Solution of the porous media equation by Adomian's decomposition method. *Physics Letters A*, *344*(24), 184-188.

Patel, Y., & Dhodiya, J. M. (2016). Application of differential transform method to solve linear, non-linear reaction convection diffusion and convection diffusion problem. *International Journal of Pure and Applied Mathematics*, *109*(3), 529-538.

Pathak, S. P., & Singh, T. (2015). Optimal Homotopy Analysis Methods for Solving the Linear and Nonlinear Fokker-Planck Equations. *Journal of Advances in Mathematics and Computer Science, 7(3)*, 209-217.

Ramadan, M. L., & Alluhaibi, M. S. (2016). Application of Sumudu decomposition method for solving non-linear wave-like equations with variable coefficients. *Electronic Journal of Mathematical Analysis and Applications*, *4*(1), 116-124.

Ren, J., Shi, D., & Vong, S. (2020). High accuracy error estimates of a Galerkin finite element method for non-linear time fractional diffusion equation. *Numerical Methods for Partial Differential Equations*, *36*(2), 284-301.

Shah, K., & Singh, T. (2019). Modified approach to solve non-linear equation arising in infiltration phenomenon. *SeMA Journal*, *76*(1), 79-95.

Shah, K., & Singh, T. (2017). An approximate solution of θ-based Richard's equation by combination of new integral transform and homotopy perturbation method. *Journal of the Nigerian Mathematical Society*, *36*(1), 85-100.

Singh, P., & Sharma, D. (2019). Comparative study of homotopy perturbation transformation with homotopy perturbation Elzaki transform method for solving non-linear fractional PDE. *Nonlinear Engineering*, *9*(1), 60-71.

Van Genuchten, M. T. (1980). A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. *Soil science society of America journal, 44(5)*, 892-898.

Varsoliwala, A. C., & Singh, T. R. (2020). An approximate analytical solution of non-linear partial differential equation for water infiltration in unsaturated soils by combined Elzaki Transform and Adomian Decomposition Method. *Journal of Physics: Conference Series*, *1473(1)*,1-11.

Wazwaz, A. M. (2005). The tanh method for generalized forms of non-linear heat conduction and Burgers–Fisher equations. *Applied Mathematics and Computation*, *169*(1), 321-338.

Witelski, T. P. (2005). Motion of wetting fronts moving into partially pre-wet soil. *Advances in Water Resources, 28*(10), 1133-1141.

Witelski, T. P. (1997). Perturbation Analysis for Wetting Fronts in Richard's Equation. *Transport in Porous Media*, *27*(2), 121-134.