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ดีเทอร์มิแนนต์ของเมทริกซ์ที่สร้างจากค่าของพหุนาม

On Determinants of Matrices Generated From Values of Polynomials

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บทคัดย่อ

บทความนี้ศึกษาดีเทอร์มิแนนต์ของเมทริกซ์จัตุรัสขนาด $n \times n$ ที่มีสมาชิกเป็นค่าของพหุนาม ผู้เขียนแสดงว่าดีเทอร์มิแนนต์มีค่าเป็นศูนย์ เมื่อดีกรีของพหุนามไม่เกิน $n-2$ และพิสูจน์สูตรทั่วไปสำหรับดีเทอร์มิแนนต์ในกรณีที่พหุนามมีดีกรีเป็น $n-1$ ซึ่งผลที่ได้สามารถนำไปอธิบายการเป็น

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อิสระเชิงเส้นต่อกันของเซตของพหุนามที่เกิดจากการเลื่อนและการสเกลตัวแปรภายใต้เงื่อนไขที่เหมาะสมบางประการ

คำสำคัญ: เมทริกซ์แวนเดอร์มอนด์ ดีเทอร์มิแนนต์ การประมาณพหุนาม

ABSTRACT

We consider a certain class of $n \times n$ matrices whose elements are given by values of polynomials. We show that their determinants vanish if the degree of the corresponding polynomial does not exceed $n-2$ and give a general formula for their determinants when the corresponding polynomial has degree $n-1$. As an immediate consequence, we deduce linear independence of sets of translations and scalings of a polynomial under suitable assumptions.

Keywords: Vandermonde matrix, Determinant, Polynomial interpolation

1. Introduction and The Main Results

Students learn from an elementary linear algebra class that, for any positive integer n, a system of n linear equations over the complex numbers with n unknowns has either a unique solution, infinitely many solutions, or no solutions. The first case occurs if and only if the matrix of coefficients of the system is nonsingular; i.e., it has nonzero determinant. Therefore, it is an interesting problem to determine whether the determinant of a given matrix vanishes. There are various of classes of matrices with zero determinant. Simple examples include those whose entries, enumerated in a zig-zag direction, form an arithmetic sequence, a geometric sequence, or a homogeneous linear recursive sequence, as illustrated below:

$$\begin{pmatrix} 2 & 5 & 8 \\ 11 & 14 & 17 \\ 20 & 23 & 26 \end{pmatrix}, \quad \begin{pmatrix} 2 & -6 & 18 \\ -54 & 162 & -486 \\ 1458 & -4374 & 13122 \end{pmatrix}, \quad \begin{pmatrix} 2 & 7 & 9 \\ 16 & 25 & 41 \\ 66 & 107 & 173 \end{pmatrix}$$

It is easily seen from their construction that the row (or column) vectors of these matrices are linearly dependent, so they all have zero determinant. In fact, the rank (i.e., the dimension of the row space) or an upper bound of the rank of each matrix in these classes is explicitly known [2]. A less obvious example is the following matrix

$$\begin{pmatrix} 3^2 & 4^2 & 6^2 & 8^2 \\ 4^2 & 5^2 & 7^2 & 9^2 \\ 5^2 & 6^2 & 8^2 & 10^2 \\ 6^2 & 7^2 & 9^2 & 11^2 \end{pmatrix}$$

The fact that its determinant is zero is an immediate consequence of a result due to Yandl and Swenson [5].

Proposition 1.1 [3, Prop.1, Prop.2] Let k and $\mathbf{n}$ be positive integers. For real numbers $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ define

$$\mathrm{C} = \begin{pmatrix} (a_1+b_1)^k & (a_1+b_2)^k & (a_1+b_3)^k & \cdots & (a_1+b_n)^k \\ (a_2+b_1)^k & (a_2+b_2)^k & (a_2+b_3)^k & \cdots & (a_2+b_n)^k \\ (a_3+b_1)^k & (a_3+b_2)^k & (a_3+b_3)^k & \cdots & (a_3+b_n)^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (a_n+b_1)^k & (a_n+b_2)^k & (a_n+b_3)^k & \cdots & (a_n+b_n)^k \end{pmatrix}$$

Then

$$\det(\mathrm{C}) = \begin{cases} 0 & \text{if } k \le n-2, \\ (-1)^{\lfloor \frac{n}{2} \rfloor} \prod_{j=1}^{k} \binom{k}{j} \prod_{1 \le i < j \le n} (a_j - a_i) \prod_{1 \le i < j \le n} (b_j - b_i) & \text{if } k = n-1, \end{cases} \quad (1)$$

where $\lfloor \cdot \rfloor$ is the floor function.

Yandl and Swenson [5] gave a simple proof of Proposition 1.1 by rewriting the matrix C as a product of two matrices involving Vandermonde matrices, whose determinants are well-known. It turns out that one can use the same approach to extend Proposition 1.1 to a much larger class of matrices. In particular, we have the following result.

Theorem 1.1 Let $\mathrm{n} \geq 2$ be an integer. For each $l \in \{1,2,\dots,n\}$, let $\mathrm{f_l(x)} \in \mathbb{C}[\mathrm{x}]$ be a polynomial of degree not exceeding $\mathrm{n}-2$. For any $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{C}$, let $\mathrm{A} = \left[\mathrm{f_j}\left(\mathrm{a_i}+\mathrm{b_j}\right)\right]_{1\leq \mathrm{i,j} \leq \mathrm{n}}$ and $\mathrm{B} = \left[\mathrm{f_j}\left(\mathrm{a_i}\cdot\mathrm{b_j}\right)\right]_{1\leq \mathrm{i,j} \leq \mathrm{n}}$. Then $\det(\mathrm{A}) = \det(\mathrm{B}) = 0$.

To illustrate Theorem 1.1, we randomly choose $n = 4$, $f_1(x) = -x^2+4$, $f_2(x) = x^2 - x + 1, f_3(x) = x + 3, f_4(x) = 4x^2 - 5x, a_1 = 1, a_2 = -3, a_3 = 4$, $a_4 = -8$, $b_1 = -7$, $b_2 = 5$, $b_3 = 4$, and $b_4 = 2$. Then the matrices

$$\mathrm{A} = \begin{pmatrix} f_1(a_1+b_1) & f_2(a_1+b_2) & f_3(a_1+b_3) & f_4(a_1+b_4) \\ f_1(a_2+b_1) & f_2(a_2+b_2) & f_3(a_2+b_3) & f_4(a_2+b_4) \\ f_1(a_3+b_1) & f_2(a_3+b_2) & f_3(a_3+b_3) & f_4(a_3+b_4) \\ f_1(a_4+b_1) & f_2(a_4+b_2) & f_3(a_4+b_3) & f_4(a_4+b_4) \end{pmatrix}$$

$$= \begin{pmatrix} -32 & 31 & 8 & 21 \\ -96 & 3 & 4 & 9 \\ -5 & 73 & 11 & 114 \\ -221 & 13 & -1 & 174 \end{pmatrix}$$

$$\mathrm{B} = \begin{pmatrix} f_1(a_1b_1) & f_2(a_1b_2) & f_3(a_1b_3) & f_4(a_1b_4) \\ f_1(a_2b_1) & f_2(a_2b_2) & f_3(a_2b_3) & f_4(a_2b_4) \\ f_1(a_3b_1) & f_2(a_3b_2) & f_3(a_3b_3) & f_4(a_3b_4) \\ f_1(a_4b_1) & f_2(a_4b_2) & f_3(a_4b_3) & f_4(a_4b_4) \end{pmatrix}$$

$$= \begin{pmatrix} -45 & 21 & 7 & 6 \\ -437 & 241 & -9 & 174 \\ -780 & 381 & 19 & 216 \\ -3132 & 1641 & -29 & 1104 \end{pmatrix}$$

both have zero determinant. It is clear that one can deduce [5, Prop. 1] from Theorem 1.1 by choosing $f_j(x) = x^k$ for all $1 \leq \mathrm{j} \leq \mathrm{n}$.

Proof of Theorem 1.1 For each $\mathrm{l} \in \{1,2,\dots,n\}$, let

$$f_l(x) = c_{l,n-2}x^{n-2} + c_{l,n-1}x^{n-1} + \cdots + c_{l,1}x + c_{l,0}.$$

Then by expanding and rearranging terms we have

$$f_j(a_i+b_j)=a_i^{n-2}c_{j,n-2}\binom{n-2}{0}+a_i^{n-3}\left(c_{j,n-2}\binom{n-2}{1}b_j+c_{j,n-3}\binom{n-3}{0}\right)+\cdots$$
$$+\left(c_{j,n-2}\binom{n-2}{n-2}b_j^{n-2}+\cdots+c_{j,1}\binom{1}{1}b_j+c_{j,0}\binom{0}{0}\right).$$

Therefore, we can write the matrix A as $\mathrm{A}=\mathrm{A}_1\mathrm{A}_2$, where A_1 and A_2 are $n\times n$ matrices given by

$$\mathrm{A}_1=\begin{pmatrix} a_1^{n-2} & a_1^{n-3} & \cdots & 1 & 0\\ a_2^{n-2} & a_2^{n-3} & \cdots & 1 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a_n^{n-2} & a_n^{n-3} & \cdots & 1 & 0\end{pmatrix}$$

and the j^{th} column of A_2 is

$$\begin{pmatrix} c_{j,n-2}\binom{n-2}{0}\\ c_{j,n-2}\binom{n-2}{1}+c_{j,n-3}\binom{n-3}{0}\\ \vdots\\ c_{j,n-2}\binom{n-2}{n-2}b_j^{n-2}+\cdots+c_{j,1}\binom{1}{1}b_j+c_{j,0}\binom{0}{0}\\ 0\end{pmatrix}.$$

Similarly, the matrix B can be decomposed into the product of A_1 and another matrix whose j^{th}column is

$$\begin{pmatrix} c_{j,n-2}b_j^{n-2}\\ c_{j,n-3}b_j^{n-3}\\ \vdots\\ c_{j,0}\\ 0\end{pmatrix}.$$

Since $\det(\mathrm{A}_1)=0$, it follows from the multiplicativity of the determinant that $\det(\mathrm{A})=\det(\mathrm{B})=0$. □

We also generalize [5, Prop. 2], which corresponds to the case $k=n-1$ in (1), as follows.

Theorem 1.2 Let $\mathrm{n} \geq 2$ be an integer and let $\mathrm{f(x)} = \mathrm{c_{n-1}x^{n-1}} + \mathrm{c_{n-2}x^{n-2}} + \cdots + \mathrm{c_1x} + \mathrm{c_0} \in \mathbb{C}[\mathrm{x}]$. For any $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{C}$, let $\mathrm{D} = \left[\mathrm{f}\left(\mathrm{a_i} + \mathrm{b_j}\right)\right]_{1\leq \mathrm{i,j} \leq \mathrm{n}}$ and $\mathrm{E} = \left[\mathrm{f}\left(\mathrm{a_i} \cdot \mathrm{b_j}\right)\right]_{1\leq \mathrm{i,j} \leq \mathrm{n}}$. Then we have

$$\det(\mathrm{D}) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} c_{n-1}^{n} \prod_{k=0}^{n-1} \binom{n-1}{k} \prod_{1\leq i<j\leq n} \left(a_j - a_i\right) \prod_{1\leq i<j\leq n} \left(b_j - b_i\right), \tag{2}$$

$$\det(\mathrm{E}) = \prod_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{c_k} \prod_{1\leq i<j\leq n} \left(a_j - a_i\right) \prod_{1\leq i<j\leq n} \left(b_j - b_i\right). \tag{3}$$

Remark 1.1 It should be noted that we require a *single polynomial* to generate all elements of each matrix in Theorem 1.2, while the polynomials involved in Theorem 1.1 could vary from columns to columns. We attempted to generalize formulas (2) and (3) to matrices whose elements are generated from polynomials which vary from columns to columns but never succeeded since the calculations involved became too complicated to be handled systematically.

Recall that a *Vandermonde matrix* is a matrix of the form

$$\mathrm{V} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix}$$

If V is a square matrix (i.e., $\mathrm{m} = \mathrm{n}$), we have the following formula for its determinant:

$$\det(\mathrm{V}) = \prod_{1\leq i<j\leq n} (x_j - x_i)$$

which is famously known as the *Vandermonde determinant*. Before proving Theorem 1.2, we shall prove the following lemma, which can be seen as a modest generalization of the Vandermonde determinant.

Lemma 1.1 Let $n \geq 2$ be an integer. For $b_1, \ldots, \mathrm{b_n}, \mathrm{r_{11}}, \mathrm{r_{21}}, \mathrm{r_{22}}, \mathrm{r_{31}}, \mathrm{r_{32}}, \mathrm{r_{33}}, \ldots, \mathrm{r_{n1}}, \mathrm{r_{n2}}, \ldots, \mathrm{r_{nn}} \in \mathbb{C}$ define an $n \times n$ matrix M by

$$\mathrm{M} = \begin{pmatrix} r_{11} & r_{21}b_1 + r_{22} & r_{31}b_1^2 + r_{32}b_1 + r_{33} & \cdots & \sum_{j=1}^{n} r_{nj}b_1^{n-j} \\ r_{11} & r_{21}b_2 + r_{22} & r_{31}b_2^2 + r_{32}b_1 + r_{33} & \cdots & \sum_{j=1}^{n} r_{nj}b_2^{n-j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{11} & r_{21}b_n + r_{22} & r_{31}b_n^2 + r_{32}b_1 + r_{33} & \cdots & \sum_{j=1}^{n} r_{nj}b_n^{n-j} \end{pmatrix}.$$

Then

$$\det(\mathrm{M}) = r_{11}r_{21}\ldots r_{n1} \prod_{1 \leq i < j \leq n} \left(b_j - b_i\right).$$

Proof Assume first that $\mathrm{r_{j1}} \neq 0$ for all $1 \leq j \leq n$. Then we can perform column operations to get rid of the small powers of b_i in each column as follows:

$$\mathrm{M} \xrightarrow{c_2 - \frac{r_{22}}{r_{11}}c_1} \begin{pmatrix} r_{11} & r_{21}b_1 & r_{31}b_1^2 + r_{32}b_1 + r_{33} & \cdots & \sum_{j=1}^{n} r_{nj}b_1^{n-j} \\ r_{11} & r_{21}b_2 & r_{31}b_2^2 + r_{32}b_1 + r_{33} & \cdots & \sum_{j=1}^{n} r_{nj}b_2^{n-j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{11} & r_{21}b_n & r_{31}b_n^2 + r_{32}b_1 + r_{33} & \cdots & \sum_{j=1}^{n} r_{nj}b_n^{n-j} \end{pmatrix} \xrightarrow{c_3 - \frac{r_{32}}{r_{21}}c_2 - \frac{r_{33}}{r_{11}}c_1} \cdots$$

$$\longrightarrow \begin{pmatrix} r_{11} & r_{21}b_1 & r_{31}b_1^2 & \cdots & r_{n1}b_1^{n-1} \\ r_{11} & r_{21}b_2 & r_{31}b_2^2 & \cdots & r_{n1}b_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{11} & r_{21}b_n & r_{31}b_n^2 & \cdots & r_{n1}b_n^{n-1} \end{pmatrix} := \widetilde{\mathrm{M}}.$$

Observe that the matrix $\widetilde{\mathrm{M}}$ is a Vandermonde matrix with the j^{th} column multiplied by the constant r_{j1}. Since the determinant is invariant under column operations, we have

$$\det(\mathrm{M}) = \det(\widetilde{\mathrm{M}}) = r_{11}r_{21}\cdots r_{n1}\prod_{1\le i<j\le n}(b_j - b_i).$$

If $r_{j1} = 0$ for some $1 \le \mathrm{j} \le \mathrm{n}$, then either the j^{th} column of M is zero or it is a linear combination of the previous columns. Therefore, we have $\det(\mathrm{M}) = 0$ in this case and the proof is complete. □

Proof of Theorem 1.2 Following the argument in the proof of Theorem 1.1, we express the matrix D as a product of $\mathrm{n} \times \mathrm{n}$ matrices $\mathrm{D_1}$ and $\mathrm{D_2}$, where

$$\mathrm{D_1} = \begin{pmatrix} \mathrm{a_1^{n-1}} & \mathrm{a_1^{n-2}} & \cdots & \mathrm{a_1} & 1 \\ \mathrm{a_2^{n-1}} & \mathrm{a_2^{n-2}} & \cdots & \mathrm{a_2} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{a_n^{n-1}} & \mathrm{a_n^{n-2}} & \cdots & \mathrm{a_n} & 1 \end{pmatrix}$$

and the j^{th} column of $\mathrm{D_2}$ is

$$\begin{pmatrix} c_{n-1}\binom{n-1}{0} \\ c_{n-1}\binom{n-1}{1}b_j + c_{n-2}\binom{n-2}{0} \\ \vdots \\ c_{n-1}\binom{n-1}{n-1}b_j^{n-1} + \cdots + c_1\binom{1}{1}b_j + c_0\binom{0}{0} \end{pmatrix}.$$

By switching $\left\lfloor \frac{n}{2} \right\rfloor$ pairs of columns in $\mathrm{D_1}$, we transform it into a Vandermonde matrix. Hence we have

$$\det(\mathrm{D_1}) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}\prod_{1\le i<j\le n}(a_j - a_i). \tag{4}$$

On the other hand, we apply Lemma 1.1 to the transpose of $\mathrm{D_2}$ to obtain

$$\det(\mathrm{D_2}) = c_{n-1}^{n}\prod_{k=0}^{n-1}\binom{n-1}{k}\prod_{1\le i<j\le n}(b_j - b_i). \tag{5}$$

Then (2) follows from (4) and (5). Next, we write the matrix E as $\mathrm{E} = \mathrm{D_1}\mathrm{E_2}$, where

$$E_2 = \begin{pmatrix} c_{n-1}b_1^{n-1} & c_{n-1}b_2^{n-1} & \cdots & c_{n-1}b_{n-1}^{n-1} & c_{n-1}b_n^{n-1} \\ c_{n-2}b_1^{n-2} & c_{n-2}b_2^{n-2} & \cdots & c_{n-2}b_{n-1}^{n-2} & c_{n-2}b_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_0 & c_0 & \cdots & c_0 & c_0 \end{pmatrix}.$$

Consider

$$E_2^t = \begin{pmatrix} c_{n-1}b_1^{n-1} & c_{n-2}b_1^{n-2} & \cdots & c_0 \\ c_{n-1}b_2^{n-1} & c_{n-2}b_2^{n-2} & \cdots & c_0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1}b_n^{n-1} & c_{n-2}b_n^{n-2} & \cdots & c_0 \end{pmatrix}.$$

We again switch columns of E_2^t and take the determinant to obtain

$$\det(E_2) = \det(E_2^t) = (-1)^{\lfloor \frac{n}{2} \rfloor} \prod_{k=0}^{n-1} c_k \prod_{1 \le i < j \le n} (b_j - b_i).$$

Therefore, we have

$$\det(E) = \det(D_1)\det(E_2) = \prod_{k=0}^{n-1} c_k \prod_{1 \le i < j \le n} (a_j - a_i) \prod_{1 \le i < j \le n} (b_j - b_i)$$

as desired. □

2. Application and Further Discussions

As an easy application of Theorem 1.2, we deduce the linear independence of sets of functions constructed from a single polynomial by translation and scaling.

Corollary 2.1 Let k be a positive integer, $f(x) \in \mathbb{C}[x]$ a polynomial of degree k, and let $b_0, b_1, \ldots, b_k$ be complex numbers which are all distinct.

(i) The set $B_1 = \{ f(x + b_0), f(x + b_1), \ldots, f(x + b_k)\}$ is linearly independent.

(ii) If the coefficients of $f(x)$ are all nonzero, then the set $B_2 = \{ f(b_0 x), f(b_1 x), \ldots, f(b_k x)\}$ is linearly independent.

In particular, under the assumptions above, the sets B_1 and B_2 form bases for the subspace P_k of $\mathbb{C}[x]$ consisting of the polynomials of degree not exceeding k.

Proof Let $c_0, c_1, \dots, c_k \in \mathbb{C}$ such that

$$c_0 f(x+b_0) + c_1 f(x+b_1) + \cdots + c_k f(x+b_k) = 0.$$

Let $a_0, a_1, \dots, a_k$ be distinct complex numbers. Then for each integer i such that $0 \le i \le k$ we have

$$c_0 f(a_i+b_0) + c_1 f(a_i+b_1) + \cdots + c_k f(a_i+b_k) = 0,$$

which can be written in terms of matrices as follows:

$$\begin{pmatrix} f(a_0+b_0) & f(a_0+b_1) & \cdots & f(a_0+b_k) \\ f(a_1+b_0) & f(a_1+b_1) & \cdots & f(a_1+b_k) \\ \vdots & \vdots & \ddots & \vdots \\ f(a_k+b_0) & f(a_k+b_1) & \cdots & f(a_k+b_k) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By Theorem 1.2, we have that the coefficient matrix has nonzero determinant, so it is nonsingular. This implies that $c_0 = c_1 = \cdots = c_k = 0$. Hence B_1 must be linearly independent.

One can use the same argument to show that B_2 is linearly independent. (The assumption that the coefficients of $f(x)$ do not vanish is required in this case since the determinant of the corresponding coefficient matrix involves the product of these quantities.) □

As closing remarks, we propose a few questions which might be of interest to the reader.

Question 2.1 There are several results in the literature concerning ranks of matrices whose determinants are zero. What can we say about the ranks of matrices A and B in Theorem 1.1 ?

Using our results and some basic facts in linear algebra, we can partially answer this question. For example, let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $n-k$ where $2 \le k \le n$ and, for $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{C}$, let $A = \left[f\left(a_i + b_j\right)\right]_{1 \le i,j \le n}$. Then it is clear from Theorem 1.1 that all $(n-k+2) \times (n-k+2)$ minors of A vanish. It is a well-known fact that the rank of a matrix $\mathbf{M}$ is the largest order of any nonvanishing minor in $\mathbf{M}$ (see, for example, [3, Sec. 5.2]). Therefore, we can conclude that $\mathrm{rank}(\mathrm{A}) \leq n - k + 1$.

Similarly, if $\mathrm{B} = \left[f\left(a_i \cdot b_j\right)\right]_{1 \leq i,j \leq n}$, then $\mathrm{rank}(\mathrm{B}) \leq n - k + 1$.

Question 2.2 It is natural to ask whether there exist formulas similar to (2) and (3) for $n \times n$ matrices whose elements are values of polynomials of degree n or higher.

The reader should notice immediately that the strategy employed in the proof of Theorem 1.2 is not applicable in these cases due to limitation on sizes of matrices in the decomposition. A possible approach to tackle this problem, which is kindly suggested by one of the referees, is to apply the *Cauchy-Binet formula* [4, Sect. 3.2]. To demonstrate this formula, consider *the* following matrix:

$$\mathrm{A} = \begin{pmatrix} (a_1 + b_1)^2 & (a_1 + b_2)^2 \\ (a_2 + b_1)^2 & (a_2 + b_2)^2 \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2b_1 & 2b_2 \\ b_1^2 & b_2^2 \end{pmatrix}.$$

Using the Cauchy-Binet formula, we have

$$\det(\mathrm{A}) = \begin{vmatrix} a_1^2 & a_1 \\ a_2^2 & a_2 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 2b_1 & 2b_2 \end{vmatrix} + \begin{vmatrix} a_1 & 1 \\ a_2 & 1 \end{vmatrix} \begin{vmatrix} 2b_1 & 2b_2 \\ b_1^2 & b_2^2 \end{vmatrix} + \begin{vmatrix} a_1^2 & 1 \\ a_2^2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ b_1^2 & b_2^2 \end{vmatrix}.$$

Then by some manipulations and the Vandermonde determinant one can deduce

$$\det(\mathrm{A}) = (a_1 - a_2)(b_2 - b_1)\big(2(a_1 a_2 + b_1 b_2) + (a_1 + a_2)(b_1 + b_2)\big).$$

It would definitely be desirable to extend this formula to more general cases.

Question 2.3 Corollary 2.1 gives an affirmative answer to a problem in interpolation theory (see, for example, [1, Problem 2, p. 75]). It implies that given a degree k polynomial $f(x) \in \mathbb{R}[x]$ and distinct real numbers $x_0, x_1, \ldots, x_k$, any polynomial $p(x) \in P_k$ can be written uniquely as

$$p(x) = \sum_{i=0}^{k} \alpha_i f(x - x_i),$$

where $\alpha_i \in \mathbb{R}, 0 \leq i \leq k$. For a given set of k points $(x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k)$ in $\mathbb{R}^2$, can one develop a method for constructing a polynomial which interpolates this data set using k distinct shifts of a single polynomial $f(x)$?

This approach is quite different from the classical methods such as Newton polynomial interpolation and Lagrange's interpolation formula [1, Chapter 9 – 10].

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References

[1] Cheney, E. W., and Light, W. A. (2009). *A Course in Approximation Theory*. Providence, R. I.: American Mathematical Society.

[2] Lee, C., and Peterson, V. (2014). The Rank of Recurrence Matrices. *College Mathematics Journal,* 45 (3), p. 207 – 215.

[3] Mirsky, L. (1990). *An Introduction to Linear Algebra*. Reprint of the 1972 edition. New York, N. Y.: Dover Publications, Inc.

[4] Tao, T. (2012). *Topics in Random Matrix Theory*. Providence, R. I.: American Mathematical Society.

[5] Yandl, A. L., and Swenson, C. (2012). A Class of Matrices with Zero Determinant. *Mathematics Magazine,* 85 (2), p. 126 – 130.