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Original Article

A simple proof of generalizations of number-theoretic sums

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Abstract

For positive integers k, m, and n, let $S_k^m(n)$ be the sum of all elements in the finite set $\{x^k\colon 1 \le x \le n/m, (x,n) = 1\}$. The formula for $S_k^m(n)$ is established and simpler formulae for $S_k^m(n)$ under some conditions on m and n are verified. The explicit formulae for $S_1^{2^a}(n)$ and $S_2^{2^a}(n)$, where $2^a|n$ and $a \ge 1$, are also provided.

Keywords: arithmetic function, Euler's phi-function, Möbius function, Möbius inversion formula

1. Introduction

Throughout this article, let (m,n) denote the greatest common divisor of integers m and n, and $|X|$ denote the number of elements in a finite set X. By an *arithmetic function*, we mean a mapping f from the set of positive integers $\mathbb{N}$ into the field of complex numbers $\mathbb{C}$. There are many interesting examples of arithmetic functions. Both of them are the Euler's phi-function,

$$\phi(n) = |\{x : 1 \le x \le n, (x,n) = 1\}|,$$

and the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^2|n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1p_2\cdots p_r, \text{where all } p_i \text{ are distinct primes.} \end{cases}$$

An arithmetic function f is called *multiplicative* if $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$. It is well-known that ϕ and μ are multiplicative and

$$\phi(n) = n\prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right),$$

where $n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ is its prime factorization (Burton, 2011; Rosen, 2005; Niven, Zuckerman, & Montgomery, 1991).

For positive integers k, m, and n, we define the set of positive integers $R_k^m(n)$ by

$$R_k^m(n) = \left\{x^k : 1 \le x \le \frac{n}{m}, (x,n) = 1\right\}.$$

Observe that $R_k^m(m) = \{1\}$ and $R_k^m(n) = \emptyset$ if $n < m$. Let $\sum X$ denote the sum of all elements in a finite set X of positive integers. Then, we let

$$S_k^m(n) = \sum R_k^m(n).$$

It is clear that $S_k^m(m) = 1$ and it suffices to study $S_k^m(n)$ only in the case $n > m$. Note that $|R_1^1(n)| = \phi(n)$ for all $n \ge 1$ and it was proved in (Burton, 2011) that

$$S_1^1(n) = \frac{n\phi(n)}{2} \qquad (n > 1). \tag{1.1}$$

There is an exercise in (Niven, Zuckerman, & Montgomery, 1991) to calculate $S_2^1(n)$ by using the *Möbius inversion formula* which asserts in the following theorem (Burton, 2011; Rosen, 2005; Niven, Zuckerman, & Montgomery, 1991).

Theorem 1.1. (Möbius Inversion Formula). If F and f are arithmetic functions with $F(n) = \sum_{d|n} f(d)$ for $n \ge 1$, then

$$f(n) = \sum_{d|n}\mu(d)F\left(\frac{n}{d}\right) \qquad (n \ge 1),$$

where the sum $\sum_{d|n}$ is over all divisors d of n.

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The formula for $S_2^1(n)$ is as follows:

$$S_2^1(n) = \frac{n^2}{6}\sum_{d|n}\mu(d)\left(\frac{2n}{d}+3+\frac{d}{n}\right) \qquad (n \geq 1). \tag{1.2}$$

By using the following facts (Burton, 2011):

$$\sum_{d|n}\frac{\mu(d)}{d} = \frac{\phi(n)}{n} \qquad (n \geq 1), \tag{1.3}$$

$$\sum_{d|n}\mu(d) = \begin{cases}1 & ; n = 1\\ 0 & ; n > 1\end{cases}, \tag{1.4}$$

$$\sum_{d|n}\mu(d)d = \psi(n) \qquad (n \geq 1), \tag{1.5}$$

where $\psi(1) = 1$ and $\psi(n) = \prod_{p|n}(1-p)$ for $n > 1$, the product is over the prime divisors p of n, the formula (1.2) can be rewritten as

$$S_2^1(n) = \frac{2n^2\phi(n) + n\psi(n)}{6} \qquad (n > 1), \tag{1.6}$$

In another direction, Baum (1982) provided the formula for $S_1^2(n)$ as follows:

$$S_1^2(n) = \frac{1}{8}\big(n\phi(n) - |r|\psi(n)\big) \qquad (n > 2), \tag{1.7}$$

where $n \equiv r \pmod 4$ with $r \in \{-1,0,1,2\}$, and he advised the reader to prove the following

$$S_2^2(n) = \begin{cases}\dfrac{n^2\phi(n) + 2n\psi(n)}{24} & \text{if } n \equiv 0 \pmod 4,\\[2mm] \dfrac{n^2\phi(n) - n\psi(n)}{24} & \text{if } n \equiv \pm 1 \pmod 4,\\[2mm] \dfrac{n^2\phi(n) - 4n\psi(n)}{24} & \text{if } n \equiv 2 \pmod 4,\end{cases} \qquad (n > 2) \tag{1.8}$$

as an exercise.

Recently, Kanasri, Pornsurat, and Tongron (2019), established the formulae for $S_k^1(n)$ and $S_k^2(n)$, which are the generalizations of (1.1), (1.6) and (1.7), (1.8), respectively. Such formulae are as follows: for any positive integer k, we have

$$S_k^1(n) = \sum_{d|n}\mu(d)d^k g_k\left(\frac{n}{d}\right) \qquad (n \geq 1) \tag{1.9}$$

and for $n > 2$,

$$S_k^2(n) = \begin{cases}\displaystyle\sum_{d|(n/2)}\mu(d)d^k g_k\left(\frac{n}{2d}\right) & \text{if } n \equiv 0 \pmod 4,\\ \displaystyle\sum_{d|n}\mu(d)d^k g_k\left(\frac{n/d-1}{2}\right) & \text{if } n \equiv \pm 1 \pmod 4,\\ \displaystyle\sum_{d|(n/2)}\mu(d)d^k\left(g_k\left(\frac{n}{2d}\right) - 2^k g_k\left(\frac{n/2d-1}{2}\right)\right) & \text{if } n \equiv 2 \pmod 4,\end{cases} \tag{1.10}$$

where

$$g_k(t) = 1^k + 2^k + \cdots + t^k$$

for all positive integers k and t. From (1.9) and (1.10), the explicit formulae for $S_3^1(n)$ and $S_3^2(n)$ are provided in (Kanasri, Pornsurat, & Tongron, 2019) as follows:

$$S_3^1(n) = \frac{n^3\phi(n) + n^2\psi(n)}{4} \qquad (n > 1)$$

and

$$S_3^2(n) = \begin{cases}\dfrac{n^3\phi(n) + 4n^2\psi(n)}{64} & \text{if } n \equiv 0 \pmod 4,\\[2mm] \dfrac{n^3\phi(n) - 2n^2\psi(n) + \psi_3(n)}{64} & \text{if } n \equiv \pm 1 \pmod 4,\\[2mm] \dfrac{n^3\phi(n) - 8n^2\psi(n) + 8\psi_3(n)/7}{64} & \text{if } n \equiv 2 \pmod 4,\end{cases} \qquad (n > 2),$$

where $\psi_3(n) = \prod_{p|n}(1-p^3)$. In general, we have given in (Kanasri, Pornsurat, & Tongron, 2019) that for $m \geq 1$,

$$\psi_m(1) = 1 \text{ and } \psi_m(n) = \prod_{p|n}(1-p^m) \text{ for } n > 1,$$

where the product is over the prime divisors p of n. We note that $\psi_1 = \psi$ and we also obtain

$$\sum_{d|n}\mu(d)d^m = \psi_m(n) \qquad (n > 1).$$

However, there is no any general formula for $S_k^m(n)$ for positive integers k, m, and n. Thus, we are interested in establishing such formula. In this work, we establish the general formula for $S_k^m(n)$ by the use of Möbius inversion formula and then verify some simpler formulae for $S_k^m(n)$ under certain conditions on m and n. We also confirm that the known results (1.9) and (1.10) are special cases of our results. Moreover, the explicit formulae for $S_1^{2^a}(n)$ and $S_2^{2^a}(n)$, where $2^a|n$ and $a \geq 1$, are provided.

2. Main Results

We first establish the formula for $S_k^m(n)$ and then show that this formula yields the known results (1.9) and (1.10).

Theorem 2.1 Let k, m, and n be positive integers with $n > m$. Then

$$S_k^m(n) = \sum_{d|n} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right),$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to a real number x. For $n < dm$, let $g_k(0) = 0$.

Proof. For a positive divisor d of n, define

$$B_d^m = \left\{x^k : 1 \leq x \leq \frac{n}{m}, (x,n) = d\right\}.$$

It is clear that

$$\bigcup_{d|n} B_d^m = \left\{1^k, 2^k, \dots, \left\lfloor \frac{n}{m} \right\rfloor^k\right\} \text{ and } B_{d_1}^m \cap B_{d_2}^m = \emptyset \text{ for } d_1 \neq d_2,$$

which implies that

$$g_k\left(\left\lfloor \frac{n}{m} \right\rfloor\right) = \sum_{i=1}^{\lfloor n/m \rfloor} i^k = \sum_{d|n} \sum B_d^m. \quad (2.1)$$

Next, we show that

$$B_d^m = d^k R_k^m\left(\frac{n}{d}\right). \quad (2.2)$$

If $x^k \in B_d^m$, then $1 \leq x \leq n/m$ and $(x,n) = d$, so $1 \leq x/d \leq n/dm$ and $(x/d, n/d) = 1$. Consequently, $(x/d)^k \in R_k^m(n/d)$ and so $x^k \in d^k R_k^m(n/d)$. On the other hand, if $y^k \in R_k^m(n/d)$, then $1 \leq y \leq n/dm$ and $(y, n/d) = 1$. It follows that $d \leq dy \leq n/m$ and $(dy, n) = d$. This shows that $(dy)^k \in B_d^m$ and the desired result follows.

For $d|n$, we obtain by using (2.2) that

$$\sum B_d^m = d^k \sum R_k^m\left(\frac{n}{d}\right) = d^k S_k^m\left(\frac{n}{d}\right).$$

It follows by (2.1) that

$$g_k\left(\left\lfloor \frac{n}{m} \right\rfloor\right) = \sum_{d|n} d^k S_k^m\left(\frac{n}{d}\right) = \sum_{d|n} \left(\frac{n}{d}\right)^k S_k^m(d),$$

because $\{d \in \mathbb{N} : d|n\} = \{n/d : d \in \mathbb{N} \text{ and } d|n\}$. By the Möbius inversion formula with $f(n) = S_k^m(n)/n^k$ and $F(n) = g_k(\lfloor n/m \rfloor)/n^k$, we get

$$\frac{S_k^m(n)}{n^k} = \sum_{d|n} \mu(d) \frac{d^k}{n^k} g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right).$$

This completes the proof. □

We note that (1.9) follows immediately from Theorem 2.1 because $\lfloor n/d \rfloor = n/d$ for $d|n$.

We observe that the formula (1.10) is divided into three cases and some sums are over all divisors d of $n/2$ while the sum in Theorem 2.1 is over all divisors d of n. To show that (1.10) is a special case of the result in Theorem 2.1, we verify some simpler formulae for $S_k^m(n)$ with $m = p^a$, a prime power, under the condition $m|n$ as the following two propositions.

Proposition 2.2 Let k and n be positive integers. If $m = p^a$ is a prime power such that $pm|n$, then

$$S_k^m(n) = \sum_{d|(n/m)} \mu(d) d^k g_k\left(\frac{n}{dm}\right).$$

Proof. Let $m = p^a$ be a prime power such that $pm|n$. Then $m|n$ and from Theorem 2.1, we have

$$\begin{aligned} S_k^m(n) &= \sum_{d|n} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) \\ &= \sum_{\substack{d|n \\ d|(n/m)}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) + \sum_{\substack{d|n \\ d\nmid(n/m)}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) \\ &= \sum_{d|(n/m)} \mu(d) d^k g_k\left(\frac{n}{dm}\right) + \sum_{\substack{d|n \\ d\nmid(n/m)}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right). \end{aligned} \quad (2.3)$$

We next show that the second sum in (2.3) vanishes. Since $\mu(d) = 0$ whenever $q^2|d$ for some prime q, it suffices to show that if $d|n$ and $d \nmid (n/m)$, then $p^2|d$. Write

$$n = p^b q_1^{c_1} \cdots q_s^{c_s}$$

as its prime factorization, where $b \geq a+1$ and $c_j \in \mathbb{N} \cup \{0\}$ $(1 \leq j \leq s)$. Since $pm|n$, we obtain

$$\frac{n}{m} = p^{b-a} q_1^{c_1} \cdots q_s^{c_s},$$

where $b - a \geq 1$. It follows from the assumption that $p^{b-a+1}|d$. Since $b - a + 1 \geq 2$, we now have $p^2|d$. This completes the proof. □

Proposition 2.3 Let k and n be positive integers. If $m = p^a$ is a prime power such that $n > m$, $m|n$, and $pm \nmid n$, then

$$S_k^m(n) = \sum_{d|(n/m)} \mu(d) d^k \left(g_k\left(\frac{n}{dm}\right) - p^k g_k\left(\left\lfloor \frac{n}{pdm} \right\rfloor\right)\right).$$

Proof. Let $m = p^a$ be a prime power such that $m|n$ and $pm \nmid n$. Write

$$n = p^b q_1^{c_1} \cdots q_s^{c_s}$$

as its prime factorization. Note that $m|n$ and $pm \nmid n$ imply $a = b$. Then

$$\frac{n}{m} = q_1^{c_1} q_2^{c_2} \cdots q_s^{c_s}.$$

From Theorem 2.1, we have

$$\begin{aligned} S_k^m(n) &= \sum_{d|n} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) \\ &= \sum_{\substack{d|n \\ d|(n/m)}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) + \sum_{\substack{d|n \\ d\nmid(n/m)}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) \\ &= \sum_{d|(n/m)} \mu(d) d^k g_k\left(\frac{n}{dm}\right) + \sum_{\substack{d|n \\ d\nmid(n/m)}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right). \end{aligned}$$

For $d|n$, it is not difficult to see that $d \nmid (n/m)$ if and only if $p|d$. Then

$$\begin{aligned} \sum_{\substack{d|n \\ d\nmid(n/m)}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) &= \sum_{\substack{d|n \\ p|d}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) \\ &= \sum_{\substack{pe|n \\ p\nmid e}} \mu(pe)(pe)^k g_k\left(\left\lfloor \frac{n}{pem} \right\rfloor\right), \end{aligned}$$

since $\mu(pe) = 0$ if $p|e$. It is clear that $pe|n$ and $p \nmid e$ if and only if $e|(n/m)$. Consequently,

$$\begin{aligned} \sum_{\substack{d|n \\ d\nmid(n/m)}} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{dm} \right\rfloor\right) &= -p^k \sum_{e|(n/m)} \mu(e) e^k g_k\left(\left\lfloor \frac{n}{pem} \right\rfloor\right), \\ &= -p^k \sum_{d|(n/m)} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{pdm} \right\rfloor\right). \end{aligned}$$

This completes the proof. □

We now verify (1.10) by using Theorem 2.1, Proposition 2.2, and Proposition 2.3. We consider three possible cases for $n > 2$ as follows:

Case 1: $n \equiv 0 \pmod 4$. Then $4|n$. From Proposition 2.2 with $m = 2$, we have

$$S_k^2(n) = \sum_{d|(n/2)} \mu(d) d^k g_k\left(\frac{n}{2d}\right).$$

Case 2: $n \equiv \pm 1 \pmod 4$. Then n is odd. From Theorem 2.1, we have

$$S_k^2(n) = \sum_{d|n} \mu(d) d^k g_k\left(\left\lfloor \frac{n}{2d} \right\rfloor\right).$$

The result follows from the fact that

$$\left\lfloor \frac{n}{2d} \right\rfloor = \left\lfloor \frac{n/d}{2} \right\rfloor = \frac{n/d - 1}{2},$$

since n/d is odd for $d|n$.

Case 3: $n \equiv 2 \pmod 4$. Then $2|n$ and $4 \nmid n$. From Proposition 2.3 with $m = 2$, we have

$$S_k^2(n) = \sum_{d|(n/2)} \mu(d) d^k \left(g_k\left(\frac{n}{2d}\right) - 2^k g_k\left(\left\lfloor \frac{n}{4d} \right\rfloor\right)\right).$$

The result follows from the fact that

$$\left\lfloor \frac{n}{4d} \right\rfloor = \left\lfloor \frac{n/2d}{2} \right\rfloor = \frac{n/2d - 1}{2},$$

since $n/2d$ is odd for $d|(n/2)$.

3. Some Explicit Formulae

In this section, we provide the explicit formulae for $S_1^{2^a}(n)$ and $S_2^{2^a}(n)$, where $2^a|n$ and $a \geq 1$. The following lemma is necessary.

Lemma 3.1 Let p^a be a prime power and n be a positive integer such that $p^a|n$ and $n > p^a$. Then the following statements hold.

(i) $$\phi\left(\frac{n}{p^a}\right) = \begin{cases} \phi(n)/p^a & \text{if } n \equiv 0 \pmod{p^{a+1}}, \\ \phi(n)/\phi(p^a) & \text{if } n \not\equiv 0 \pmod{p^{a+1}}. \end{cases}$$

(ii) $$\psi_m\left(\frac{n}{p^a}\right) = \begin{cases} \psi_m(n) & \text{if } n \equiv 0 \pmod{p^{a+1}}, \\ \psi_m(n)/(1-p^m) & \text{if } n \not\equiv 0 \pmod{p^{a+1}}. \end{cases}$$

Proof. We treat two possible cases.

Case 1: $n \equiv 0 \pmod{p^{a+1}}$. Then write $n = p^b t$ for some positive integers b and t such that $b \geq a+1$ and $p \nmid t$. Since ϕ is multiplicative, we obtain

$$\phi\left(\frac{n}{p^a}\right) = \phi(p^{b-a})\phi(t) = \frac{\phi(p^b)\phi(t)}{p^a} = \frac{\phi(n)}{p^a},$$

$$\psi_m\left(\frac{n}{p^a}\right) = \prod_{q|(p^{b-a}t)} (1-q^m) = \prod_{q|(p^b t)} (1-q^m) = \psi_m(n).$$

Case 2: $n \not\equiv 0 \pmod{p^{a+1}}$. Then we can write $n = p^a t$ for some positive integer t such that $p \nmid t$ and so

$$\phi\left(\frac{n}{p^a}\right) = \phi(t) = \frac{\phi(p^a)\phi(t)}{\phi(p^a)} = \frac{\phi(n)}{\phi(p^a)},$$

$$\psi_m\left(\frac{n}{p^a}\right) = \prod_{q|t}(1-q^m) = \frac{\prod_{q|(p^a t)}(1-q^m)}{1-p^m} = \frac{\psi_m(n)}{1-p^m}.$$

This completes the proof. □

Next, we give the formulae for $S_1^{p^a}(n)$ and $S_2^{p^a}(n)$, where p^a is a prime power such that $p^a|n$, by using Lemma 3.1 as the following.

Proposition 3.2 Let p^a be a prime power and n be a positive integer such that $p^a|n$ and $n > p^a$. Then

$$S_1^{p^a}(n) = \begin{cases} \dfrac{n\phi(n)}{2p^{2a}} & \text{if } n \equiv 0 \pmod{p^{a+1}}, \\ \dfrac{1}{2}\left(\dfrac{n\phi(n)}{p^a\phi(p^a)} - p \displaystyle\sum_{d|(n/p^a)} \mu(d) d \left\lfloor \frac{n}{p^{a+1}d} \right\rfloor \left(\left\lfloor \frac{n}{p^{a+1}d} \right\rfloor + 1\right)\right) & \text{if } n \not\equiv 0 \pmod{p^{a+1}}, \end{cases}$$

$$S_2^{p^a}(n) = \begin{cases} \dfrac{2n^2\phi(n) + np^{2a}\psi(n)}{6p^{3a}} & \text{if } n \equiv 0 \pmod{p^{a+1}}, \\ \dfrac{1}{6}\left(\dfrac{2n^2\phi(n)}{p^{2a}\phi(p^a)} + \dfrac{n\psi(n)}{(1-p)p^a} - p^2 \displaystyle\sum_{d|(n/p^a)} \mu(d) d^2 \left\lfloor \frac{n}{p^{a+1}d} \right\rfloor \left(\left\lfloor \frac{n}{p^{a+1}d} \right\rfloor + 1\right)\left(2\left\lfloor \frac{n}{p^{a+1}d} \right\rfloor + 1\right)\right) & \\ & \text{if } n \not\equiv 0 \pmod{p^{a+1}}. \end{cases}$$

Proof. We consider two possible cases.

Case 1: $n \equiv 0 \pmod{p^{a+1}}$. By using Proposition 2.2, we have

$$\begin{aligned} S_1^{p^a}(n) &= \sum_{d|(n/p^a)} \mu(d) d g_1\left(\frac{n}{p^a d}\right) \\ &= \frac{1}{2} \sum_{d|(n/p^a)} \mu(d) d \left(\frac{n}{p^a d}\right)\left(\frac{n}{p^a d} + 1\right) \end{aligned}$$

$$= \frac{1}{2p^{2a}} \sum_{d|(n/p^a)} \mu(d)\left(\frac{n^2}{d} + p^a n\right)$$
$$= \frac{n^2}{2p^{2a}} \frac{\phi(n/p^a)}{n/p^a}, \qquad \text{by (1.3) and (1.4)}$$
$$= \frac{n^2}{2p^{2a}} \frac{\phi(n)/p^a}{n/p^a}, \qquad \text{by Lemma 3.1 (i)}$$
$$= \frac{n\phi(n)}{2p^{2a}}.$$

Again, by using Proposition 2.2, we obtain

$$S_2^{p^a}(n) = \sum_{d|(n/p^a)} \mu(d) d^2 g_2\left(\frac{n}{p^a d}\right)$$
$$= \frac{1}{6} \sum_{d|(n/p^a)} \mu(d) d^2 \left(\frac{n}{p^a d}\right)\left(\frac{n}{p^a d} + 1\right)\left(\frac{2n}{p^a d} + 1\right)$$
$$= \frac{1}{6p^{3a}} \sum_{d|(n/p^a)} \mu(d)\left(\frac{2n^3}{d} + 3p^a n^2 + np^{2a} d\right)$$
$$= \frac{1}{6p^{3a}}\left(2n^3 \frac{\phi(n/p^a)}{n/p^a} + np^{2a}\psi\left(\frac{n}{p^a}\right)\right), \qquad \text{by (1.3), (1.4) and (1.5)}$$
$$= \frac{1}{6p^{3a}}\left(2n^3 \frac{\phi(n)/p^a}{n/p^a} + np^{2a}\psi(n)\right), \qquad \text{by Lemma 3.1 (i) and (ii)}$$
$$= \frac{2n^2\phi(n) + np^{2a}\psi(n)}{6p^{3a}}.$$

Case 2: $n \not\equiv 0 \pmod{p^{a+1}}$. By using Proposition 2.3, we get

$$S_1^{p^a}(n) = \sum_{d|(n/p^a)} \mu(d) d\left(g_1\left(\frac{n}{p^a d}\right) - p g_1\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\right)\right)$$
$$= \frac{1}{2} \sum_{d|(n/p^a)} \mu(d) d\left(\left(\frac{n}{p^a d}\right)\left(\frac{n}{p^a d} + 1\right) - p\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\right)$$
$$= \frac{1}{2}\left(\frac{n^2}{p^{2a}} \sum_{d|(n/p^a)} \frac{\mu(d)}{d} - p \sum_{d|(n/p^a)} \mu(d) d\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\right), \text{by (1.4)}$$
$$= \frac{1}{2}\left(\frac{n^2}{p^{2a}} \frac{\phi(n/p^a)}{n/p^a} - p \sum_{d|(n/p^a)} \mu(d) d\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\right), \text{by (1.3)}$$
$$= \frac{1}{2}\left(\frac{n\phi(n)}{p^a\phi(p^a)} - p \sum_{d|(n/p^a)} \mu(d) d\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\right), \text{by Lemma 3.1 (i)}$$

and

$$S_2^{p^a}(n) = \sum_{d|(n/p^a)} \mu(d) d^2\left(g_2\left(\frac{n}{p^a d}\right) - p^2 g_2\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\right)\right)$$
$$= \frac{1}{6} \sum_{d|(n/p^a)} \mu(d) d^2\left(\left(\frac{n}{p^a d}\right)\left(\frac{n}{p^a d} + 1\right)\left(\frac{2n}{p^a d} + 1\right) - p^2\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\left(2\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\right)$$
$$= \frac{1}{6} \sum_{d|(n/p^a)} \mu(d) d^2\left(\frac{2n^3}{p^{3a} d^3} + \frac{3n^2}{p^{2a} d^2} + \frac{n}{p^a d} - p^2\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\left(2\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\right)$$
$$= \frac{1}{6}\left(\frac{2n^3}{p^{3a}} \frac{\phi(n/p^a)}{n/p^a} + \frac{n}{p^a}\psi\left(\frac{n}{p^a}\right) - p^2 \sum_{d|(n/p^a)} \mu(d) d^2\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor\left(\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\left(2\left\lfloor\frac{n}{p^{a+1} d}\right\rfloor + 1\right)\right),$$

by (1.3), (1.4), and (1.5)

$$= \frac{1}{6}\left(\frac{2n^2\phi(n)}{p^{2a}\phi(p^a)} + \frac{n\psi(n)}{(1-p)p^a} - p^2 \sum_{d|(n/p^a)} \mu(d)d^2 \left\lfloor \frac{n}{p^{a+1}d} \right\rfloor \left(\left\lfloor \frac{n}{p^{a+1}d} \right\rfloor + 1 \right)\left(2\left\lfloor \frac{n}{p^{a+1}d} \right\rfloor + 1 \right)\right),$$

by Lemma 3.1 (i) and (ii). This completes the proof. □

We note that the formulae for $S_k^{p^a}(n)$ with $k > 2$ can be obtained similarly to such formulae for the cases $k = 1, 2$ as in Proposition 3.2.

Example 3.3 This example illustrates how to find $S_1^3(n)$ and $S_2^3(n)$ for $n = 9$, 15, and 16 by the definition and our results. First, we calculate these $S_k^3(n)$ $(k = 1,2)$ by the definition as follows:

$$S_1^3(9) = \sum \left\{x : 1 \le x \le \frac{9}{3}, (x, 9) = 1\right\} = \sum\{1,2\} = 3,$$

$$S_2^3(9) = \sum \left\{x^2 : 1 \le x \le \frac{9}{3}, (x, 9) = 1\right\} = \sum\{1^2, 2^2\} = 5,$$

$$S_1^3(15) = \sum \left\{x : 1 \le x \le \frac{15}{3}, (x, 15) = 1\right\} = \sum\{1,2,4\} = 7,$$

$$S_2^3(15) = \sum \left\{x^2 : 1 \le x \le \frac{15}{3}, (x, 15) = 1\right\} = \sum\{1^2, 2^2, 4^2\} = 21,$$

$$S_1^3(16) = \sum \left\{x : 1 \le x \le \frac{16}{3}, (x, 16) = 1\right\} = \sum\{1,3,5\} = 9,$$

$$S_2^3(16) = \sum \left\{x^2 : 1 \le x \le \frac{16}{3}, (x, 16) = 1\right\} = \sum\{1^2, 3^2, 5^2\} = 35.$$

Next, we calculate $S_k^3(9)$ and $S_k^3(15)$ $(k = 1,2)$ by Proposition 3.2 as follows:

$$S_1^3(9) = \frac{9\phi(9)}{2\cdot 3^2} = \frac{9\cdot 6}{2\cdot 9} = 3,$$

$$S_2^3(9) = \frac{2\cdot 9^2\phi(9) + 9\cdot 3^2\psi(9)}{6\cdot 3^3} = \frac{2\cdot 81\cdot 6 + 9\cdot 9\cdot(-2)}{6\cdot 27} = 5,$$

$$S_1^3(15) = \frac{1}{2}\left(\frac{15\phi(15)}{3\phi(3)} - 3\sum_{d|5}\mu(d)d\left\lfloor\frac{15}{3^2 d}\right\rfloor\left(\left\lfloor\frac{15}{3^2 d}\right\rfloor + 1\right)\right)$$

$$= \frac{1}{2}\left(\frac{15\cdot 8}{3\cdot 2} - 3(1\cdot 1\cdot 1\cdot 2 + (-1)\cdot 5\cdot 0\cdot 1)\right) = 7,$$

$$S_2^3(15) = \frac{1}{6}\left(\frac{2\cdot 15^2\phi(15)}{3^2\phi(3)} + \frac{15\psi(15)}{(1-3)3} - 3^2\sum_{d|5}\mu(d)d^2\left\lfloor\frac{15}{3^2 d}\right\rfloor\left(\left\lfloor\frac{15}{3^2 d}\right\rfloor + 1\right)\left(2\left\lfloor\frac{15}{3^2 d}\right\rfloor + 1\right)\right)$$

$$= \frac{1}{6}\left(\frac{2\cdot 225\cdot 8}{9\cdot 2} - \frac{15\cdot 8}{2\cdot 3} - 9(1\cdot 1^2\cdot 1\cdot 2\cdot 3 + (-1)\cdot 5^2\cdot 0\cdot 1\cdot 1)\right) = 21.$$

Finally, we calculate $S_1^3(16)$ and $S_2^3(16)$ by using Theorem 2.1 as follows:

$$S_1^3(16) = \sum_{d|16}\mu(d)dg_1\left(\left\lfloor\frac{16}{3d}\right\rfloor\right)$$
$$= 1\cdot 1g_1(5) + (-1)\cdot 2g_1(2) + 0\cdot 4g_1(1) + 0\cdot 8g_1(0) + 0\cdot 16g_1(0)$$
$$= 15 - 6 = 9,$$

$$S_2^3(16) = \sum_{d|16}\mu(d)d^2g_2\left(\left\lfloor\frac{16}{3d}\right\rfloor\right)$$
$$= 1\cdot 1^2g_2(5) + (-1)\cdot 2^2g_2(2) + 0\cdot 4^2g_2(1) + 0\cdot 8^2g_2(0) + 0\cdot 16^2g_2(0)$$
$$= 55 - 20 = 35.$$

Taking $p = 2$ in Proposition 3.2, we get the explicit formulae for $S_1^{2^a}(n)$ and $S_2^{2^a}(n)$, where $2^a|n$ and $a \ge 1$, as the following proposition.

Proposition 3.4 Let n and a be positive integers such that $2^a|n$ and $n > 2^a$. Then

$$S_1^{2^a}(n) = \begin{cases} \dfrac{n\phi(n)}{2^{2a+1}} & \text{if } n \equiv 0 \pmod{2^{a+1}}, \\ \dfrac{n\phi(n)}{2^{2a+1}} - \dfrac{\psi(n)}{4} & \text{if } n \not\equiv 0 \pmod{2^{a+1}}, \end{cases}$$

and

$$S_2^{2^a}(n) = \begin{cases} \dfrac{n^2\phi(n)}{3\cdot 2^{3a}} + \dfrac{n\psi(n)}{3\cdot 2^{a+1}} & \text{if } n \equiv 0 \pmod{2^{a+1}}, \\ \dfrac{n^2\phi(n)}{3\cdot 2^{3a}} - \dfrac{n\psi(n)}{3\cdot 2^{a}} & \text{if } n \not\equiv 0 \pmod{2^{a+1}}. \end{cases}$$

Proof. If $n \equiv 0 \pmod{2^{a+1}}$, then the results easily follow from Proposition 3.2 by taking $p = 2$. Assume now that $n \not\equiv 0 \pmod{2^{a+1}}$, yielding $n/2^a$ is odd. Then, for $d|(n/2^a)$, we have $n/2^a d$ is odd and so

$$\left\lfloor \frac{n}{2^{a+1}d} \right\rfloor = \left\lfloor \frac{(n/2^a)/d}{2} \right\rfloor = \frac{(n/2^a)/d - 1}{2} = \frac{n - 2^a d}{2^{a+1}d}.$$

It follows that

$$\begin{aligned}
&\frac{1}{2}\left(\frac{n\phi(n)}{2^a\phi(2^a)} - 2\sum_{d|(n/2^a)} \mu(d)d \left\lfloor \frac{n}{2^{a+1}d} \right\rfloor \left(\left\lfloor \frac{n}{2^{a+1}d} \right\rfloor + 1\right)\right) \\
&= \frac{1}{2}\left(\frac{n\phi(n)}{2^{2a-1}} - 2\sum_{d|(n/2^a)} \mu(d)d \left(\frac{n-2^a d}{2^{a+1}d}\right)\left(\frac{n-2^a d}{2^{a+1}d} + 1\right)\right) \\
&= \frac{n\phi(n)}{2^{2a}} - \frac{1}{2^{2a+2}}\left(\frac{n^2\phi(n/2^a)}{n/2^a} - 2^{2a}\psi\left(\frac{n}{2^a}\right)\right), \qquad \text{by (1.3) and (1.5)} \\
&= \frac{n\phi(n)}{2^{2a}} - \frac{1}{2^{2a+2}}\left(\frac{2^a n\phi(n)}{2^{a-1}} + 2^{2a}\psi(n)\right), \qquad \text{by Lemma 3.1 (i) and (ii)} \\
&= \frac{n\phi(n)}{2^{2a+1}} - \frac{\psi(n)}{4}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{6}\left(\frac{2n^2\phi(n)}{2^{2a}\phi(2^a)} - \frac{n\psi(n)}{2^a} - 2^2\sum_{d|(n/2^a)} \mu(d)d^2 \left\lfloor \frac{n}{2^{a+1}d} \right\rfloor \left(\left\lfloor \frac{n}{2^{a+1}d} \right\rfloor + 1\right)\left(2\left\lfloor \frac{n}{2^{a+1}d} \right\rfloor + 1\right)\right) \\
&= \frac{1}{6}\left(\frac{n^2\phi(n)}{2^{3a-2}} - \frac{n\psi(n)}{2^a} - 2^2\sum_{d|(n/2^a)} \mu(d)d^2 \left(\frac{n-2^a d}{2^{a+1}d}\right)\left(\frac{n-2^a d}{2^{a+1}d} + 1\right)\left(\frac{n-2^a d}{2^a d} + 1\right)\right) \\
&= \frac{1}{6}\left(\frac{n^2\phi(n)}{2^{3a-2}} - \frac{n\psi(n)}{2^a} - \frac{1}{2^{3a}}\sum_{d|(n/2^a)} \mu(d)\left(\frac{n-2^a d}{d}\right)(n + 2^a d)n\right) \\
&= \frac{1}{6}\left(\frac{n^2\phi(n)}{2^{3a-2}} - \frac{n\psi(n)}{2^a} - \frac{1}{2^{3a}}\left(\frac{n^3\phi(n/2^a)}{n/2^a} - 2^{2a}n\psi\left(\frac{n}{2^a}\right)\right)\right), \qquad \text{by (1.3) and (1.5)} \\
&= \frac{1}{6}\left(\frac{n^2\phi(n)}{2^{3a-2}} - \frac{n\psi(n)}{2^a} - \frac{1}{2^{3a}}\left(2n^2\phi(n) + 2^{2a}n\psi(n)\right)\right), \text{by Lemma 3.1 (i) and (ii)} \\
&= \frac{n^2\phi(n)}{3\cdot 2^{3a}} - \frac{n\psi(n)}{3\cdot 2^a}.
\end{aligned}$$

Hence, the results follow from Proposition 3.2. □

We see that (1.7) and (1.8) for an even integer n can be verified by taking $a = 1$ in Proposition 3.4.

Example 3.5 This example illustrates how to find $S_1^4(n)$ and $S_2^4(n)$ for $n = 16$, 28, and 30 by the definition and our results. First, we calculate these $S_k^4(n)$ $(k = 1,2)$ by the definition as follows:

$$\begin{aligned}
S_1^4(16) &= \sum\left\{x : 1 \le x \le \frac{16}{4}, (x,16) = 1\right\} = \sum\{1,3\} = 4, \\
S_2^4(16) &= \sum\left\{x^2 : 1 \le x \le \frac{16}{4}, (x,16) = 1\right\} = \sum\{1^2,3^2\} = 10, \\
S_1^4(28) &= \sum\left\{x : 1 \le x \le \frac{28}{4}, (x,28) = 1\right\} = \sum\{1,3,5\} = 9, \\
S_2^4(28) &= \sum\left\{x^2 : 1 \le x \le \frac{28}{4}, (x,28) = 1\right\} = \sum\{1^2,3^2,5^2\} = 35, \\
S_1^4(30) &= \sum\left\{x : 1 \le x \le \frac{30}{4}, (x,30) = 1\right\} = \sum\{1,7\} = 8, \\
S_2^4(30) &= \sum\left\{x^2 : 1 \le x \le \frac{30}{4}, (x,30) = 1\right\} = \sum\{1^2,7^2\} = 50.
\end{aligned}$$

Next, we calculate $S_k^4(16)$ and $S_k^4(28)$ $(k = 1,2)$ by Proposition 3.4 as follows:

$$S_1^4(16) = \frac{16\phi(16)}{2^{2\cdot 2+1}} = \frac{16\cdot 8}{32} = 4,$$

$$S_2^4(16) = \frac{16^2\phi(16)}{3\cdot 2^{3\cdot 2}} + \frac{16\psi(16)}{3\cdot 2^{2+1}} = \frac{256\cdot 8}{3\cdot 64} + \frac{16\cdot(-1)}{3\cdot 8} = 10,$$

$$S_1^4(28) = \frac{28\phi(28)}{2^{2\cdot 2+1}} - \frac{\psi(28)}{4} = \frac{28\cdot 12}{32} - \frac{6}{4} = 9,$$

$$S_2^4(28) = \frac{28^2\phi(28)}{3\cdot 2^{3\cdot 2}} - \frac{28\psi(28)}{3\cdot 2^2} = \frac{784\cdot 12}{3\cdot 64} - \frac{28\cdot 6}{3\cdot 4} = 35.$$

Finally, we calculate $S_1^4(30)$ and $S_2^4(30)$ by using Theorem 2.1 as follows:

$$\begin{aligned} S_1^4(30) &= \sum_{d|30} \mu(d) d g_1\left(\left\lfloor \frac{30}{4d} \right\rfloor\right) \\ &= 1\cdot 1g_1(7) + (-1)\cdot 2g_1(3) + (-1)\cdot 3g_1(2) + (-1)\cdot 5g_1(1) \\ &\quad +1\cdot 6g_1(1) + 1\cdot 10g_1(0) + 1\cdot 15g_1(0) + (-1)\cdot 30g_1(0) \\ &= 28 - 12 - 9 - 5 + 6 = 8, \end{aligned}$$

$$\begin{aligned} S_2^4(30) &= \sum_{d|30} \mu(d) d^2 g_2\left(\left\lfloor \frac{30}{4d} \right\rfloor\right) \\ &= 1\cdot 1^2g_2(7) + (-1)\cdot 2^2g_2(3) + (-1)\cdot 3^2g_2(2) + (-1)\cdot 5^2g_2(1) \\ &\quad +1\cdot 6^2g_2(1) + 1\cdot 10^2g_2(0) + 1\cdot 15^2g_2(0) + (-1)\cdot 30^2g_2(0) \\ &= 140 - 56 - 45 - 25 + 36 = 50. \end{aligned}$$

4. Conclusions

For positive integers k, m, and n, let

$$S_k^m(n) = \sum \left\{x^k : 1 \le x \le \frac{n}{m}, (x,n) = 1\right\},$$

where $\sum X$ denotes the sum of all elements in a finite set X of positive integers. The formulae for $S_1^1(n)$ and $S_2^1(n)$ appeared in (Burton, 2011) and (Niven, Zuckerman, & Montgomery, 1991), respectively, while the formulae for $S_1^2(n)$ and $S_2^2(n)$ appeared in (Baum, 1982). Recently, the formulae for $S_k^1(n)$ and $S_k^2(n)$, which are the generalizations of the results mentioned above, was provided by Kanasri, Pornsurat, and Tongron (2019). In the present work, we establish the formula for $S_k^m(n)$ as in Theorem 2.1, which is a generalization of all previous results. Some conditions on m and n yield some simpler formulae for $S_k^m(n)$ as in Proposition 2.2 and Proposition 2.3. We also provide the explicit formulae for $S_1^{2^a}(n)$ and $S_2^{2^a}(n)$, where $2^a|n$, $n > 2^a$, and $a \ge 1$, as in Proposition 3.4.

References

Baum, J. D. (1982). A number-theoretic sum. *Mathematics Magazine, 55*(2), 111-113.

Burton, D. M. (2011). *Elementary number theory.* New York, NY: McGraw-Hill.

Kanasri, N. R., Pornsurat, P., & Tongron, Y. (2019). Generalizations of number-theoretic sums. *Communications of the Korean Mathematical Society, 34*(4), 1105-1115.

McCarthy, P. J. (1986). *Introduction to arithmetical functions.* New York, NY: Springer-Verlag.

Niven, I., Zuckerman, H. S., & Montgomery, H. L. (1991). *An introduction to the theory of numbers.* New York, NY: John Wiley & Sons.

Rosen, K. H. (2005). *Elementary Number Theory and Its Applications*. New York, NY: Addison Wesley.