Songklanakarin J. Sci. Technol.
44 (2), 421-432, Mar. - Apr. 2022

http://www.sjst.psu.ac.th

Original Article

Confidence intervals for the coefficient of variation and the difference between coefficients of variation of inverse-gamma distributions

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Received: 5 June 2021; Revised: 2 September 2021; Accepted: 26 September 2021

Abstract

The aim of this study is to establish new confidence intervals for the single coefficient of variation of an inverse-gamma distribution using Bayesian methods based on the Jeffreys, reference, and uniform priors and compare them with the Wald method. The Bayesian methods are constructed with either the credible confidence interval or the highest posterior density (HPD) interval. These concepts were extended to find the difference between the coefficients of variation for two independent inverse-gamma populations. The performances of the proposed confidence intervals were evaluated using coverage probabilities and expected lengths via Monte Carlo simulations. The results indicate that the Bayesian HPD interval based on the reference prior can be recommended for constructing confidence intervals for the coefficient of variation of a single inverse-gamma distribution and the Bayesian HPD interval based on the Jeffreys prior can be recommended for constructing confidence intervals for the difference between the coefficients of variation of two inverse-gamma distributions. Rainfall data from northern Thailand were used to illustrate the efficacies of the proposed methods.

Keywords: inverse gamma distribution, coefficient of variation, Bayesian method, the highest posterior density, coverage probability

1. Introduction

The inverse-gamma distribution is right-skewed and a member of the two-parameter family of continuous probability distributions on the positive real line. Also called the inverted gamma distribution or the reciprocal gamma distribution, it is most often used as a conjugate prior distribution in Bayesian statistics (Glen & Leemis, 2017). In general, the inverse-gamma distribution is applied to point estimation. For example, Abid and Al-Hassany (2016) estimated the maximum likelihood, moments, percentile, least-squares, and weighted least-squares estimators for an inverted gamma distribution. Llera and Beckmann (2016) introduced five algorithms based on the moment, maximum likelihood, and full Bayesian estimation of the parameters of an inverse-gamma distribution. However, interval estimation of the parameters of an inverse-gamma distribution has not yet been reported. Interval estimation is defined as the estimation of a population parameter by specifying a range of values bounded by upper and lower limits within which the true value is asserted to lie. It is distinct from point estimation in which a single value is assigned as the true value of the parameter. However, interval estimation provides more information on a population than point estimation (Casella & Berger, 2002).

The coefficient of variation is one of the parameters of interest for studying interval estimation because it is a statistical measure of the dispersion of data points around the mean. It is a helpful quantity to describe the variation when evaluating results from different populations (Liu, 2012), which is commonly used to compare the data dispersion between distinct series of data. Therefore, many scholars have investigated confidence intervals for parameter functions of the coefficient of variation. For example, Pang, Leung, Huang, and Liu (2005) proposed confidence intervals for the coefficient of variation of a three-parameter Weibull distribution by using a simulation-based Bayesian approach. Mahmoudvand and Hassani (2009) proposed confidence

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intervals for the coefficient of variation of a normal distribution. Buntao and Niwitpong (2013) provided confidence intervals for the difference between the coefficients of variation of lognormal distributions. Sangnawakij and Niwitpong (2016) examined confidence intervals for the single coefficient of variation and the difference between the coefficients of variation of two-parameter exponential distributions using the method of variance of estimates recovery (MOVER), the generalized confidence interval (GCI), and the asymptotic confidence interval. Thangjai and Niwitpong (2017) proposed confidence intervals based on adjusted MOVER, GCI, and a large sample method for weighted coefficients of variation. Chankham, Niwitpong, and Niwitpong (2019) proposed new confidence intervals for the coefficient of variation and the difference between the coefficients of variation of inverse Gaussian distributions using GCI and the percentile bootstrap (PB) confidence interval. Yosboonruang, Niwitpong, and Niwitpong (2019) established new confidence intervals for the single coefficient of variation of a delta-lognormal distribution using Bayesian methods and compared them with the fiducial GCI (FGCI). Recently, Kaewprasert, Niwitpong, and Niwitpong (2020) proposed the score, Wald, and PB methods to establish confidence intervals for the single coefficient of variation of an inverse-gamma distribution, and subsequently recommended the Wald method for constructing confidence intervals in this scenario.

The goal of this study is to propose new confidence intervals for the single coefficient of variation of an inverse-gamma distribution using Bayesian methods and comparing them with the Wald method proposed by Kaewprasert, Niwitpong, and Niwitpong (2020). Moreover, the approach is extended to the difference between the coefficients of variation of two independent inverse-gamma distributions.

2. Materials and Methods

Let $Y \sim \text{Gamma}(a,b)$ with shape parameter a and rate parameter b. The transformation $X = g(Y) = \dfrac{1}{Y}$ is defined as an inverse-gamma distribution with probability density function (pdf) X defined as

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}(x)^{-\alpha-1}\exp\left(-\frac{\beta}{x}\right) \quad , x > 0,\ \alpha > 0,\ \beta > 0. \tag{1}$$

with the shape parameter α and scale parameter β, denoted as $X \sim \text{IG}(\alpha,\beta)$.

The population mean and variance of the inverse-gamma distribution are respectively defined as

$$E(X) = \frac{\beta}{\alpha-1} \qquad for \quad \alpha > 1 \tag{2}$$

and

$$Var(X) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \qquad for \quad \alpha > 2. \tag{3}$$

The coefficient of variation of an inverse-gamma distribution can be modified as follows:

$$\tau = CV(X) = \frac{\sqrt{Var(X)}}{E(X)} = \frac{1}{\sqrt{\alpha-2}} . \tag{4}$$

Since α is an unknown parameter, it must be estimated.

The log-likelihood function of α and β is respectively given by

$$\ln L(\alpha,\beta) = -\sum\frac{\beta}{X_i} - (\alpha+1)\sum\ln X_i - n\ln\Gamma(\alpha) + n\alpha\ln\beta . \tag{5}$$

Thus, the respective maximum likelihood estimators for α and β, are

$$\hat{\alpha} = \frac{1}{2\left[\dfrac{\sum\ln X_i}{n} + \ln\dfrac{-\sum X_i^{-1}}{n}\right]} \quad \text{and} \quad \hat{\beta} = \frac{n\alpha}{-\sum X_i^{-1}} . \tag{6}$$

Therefore, the sample coefficient of variation for τ is given by $\hat{\tau} = \dfrac{1}{\sqrt{\hat{\alpha}-2}}$.

2.1 Confidence intervals for a single of coefficient of variation of inverse gamma distribution

We approximate fiducial quantities based on cube-root transformed samples. Let $W_i = Y_i^{1/3}$, for i = 1, 2, …. n (Krishnamoorthy & Wang, 2016). Therefore, using the Wilson and Hilferty (1931) approximation, $W_i = Y_i^{1/3} = (1/X_i)^{1/3} = X_i^{-1/3}$ is approximately normal with mean μ and variance σ^2 respectively given by

$$\mu = \left(\frac{\alpha}{\beta}\right)^{1/3}\left(1-\frac{1}{9\alpha}\right) \quad \text{and} \quad \sigma^2 = \frac{1}{9\alpha^{1/3}\beta^{2/3}}. \tag{7}$$

For $W_i \sim N(\mu,\sigma^2)$, the pdf of a normal distribution is as follows:

$$f(w_i;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(w_i-\mu)^2\right). \tag{8}$$

To find the confidence interval for the coefficient of variation, we use the following approach that requires the expressions in Equation (7). Define

$$\overline{W} = \frac{1}{n}\sum_{i=1}^{n} W_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(W_i-\overline{W})^2. \tag{9}$$

Recall that

$$\frac{(n-1)S^2}{\sigma^2} = U \sim \chi^2_{n-1} \quad \text{and} \quad \overline{W} = \mu + Z\frac{\sigma}{\sqrt{n}} \tag{10}$$

where U is a chi-squared distribution with *n*-1 degrees of freedom and *Z* is a standard normal random variable.

Solving Equation (10) for μ and σ^2, let $\overline{w}$ and s be the observed values of $\overline{W}$ and S, respectively. The fiducial quantities for the parameter are given by

$$\mu = \overline{w} + \frac{Z}{\sqrt{\chi^2_{n-1}}}\sqrt{\frac{(n-1)s^2}{n}} \quad \text{and} \quad \sigma^2 = \frac{(n-1)s^2}{\chi^2_{n-1}}. \tag{11}$$

Solving the fiducial quantities for α and β can be obtained in terms of μ and σ^2, respectively, as follows:

$$\alpha = \frac{1}{9}\left[\left(1+\frac{\mu^2}{2\sigma^2}\right)+\left(\left(1+\frac{\mu^2}{2\sigma^2}\right)^2-1\right)^{1/2}\right] \tag{12}$$

and

$$\beta = \frac{1}{27\alpha^{1/2}(\sigma^2)^{3/2}}. \tag{13}$$

Let (μ,σ^2) be an unknown parameter. The likelihood of W is

$$p(w_i;|\mu,\sigma^2) \propto (\sigma^2)^{-n/2}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(w_i-\mu)^2\right). \tag{14}$$

Therefore, the Fisher information matrix is defined as

$$I(\mu,\sigma^2) = \begin{bmatrix} n/\sigma^2 & 0 \\ 0 & n/2\sigma^4 \end{bmatrix}. \tag{15}$$

In the following section, the Bayesian methods based on the Jeffreys, reference, and uniform priors are presented.

2.1.1 The Jeffreys Prior

The Jeffreys (1961) prior is defined as $p(\theta) = \sqrt{\det(I(\theta))}$, where $I(\theta)$ is the Fisher information matrix. From the Fisher information matrix $I(\mu,\sigma^2)$, the Jeffreys prior is obtained as

$$p(\mu,\sigma^2) \propto 1/\sigma^2. \tag{16}$$

The joint posterior density function is defined as

$$p(\mu,\sigma^2|w) \propto \frac{1}{\sigma^2}\prod_{i=1}^{n}(\sigma^2)^{-1/2}\exp\left(-\frac{1}{2\sigma^2}(w_i-\mu)^2\right). \tag{17}$$

The respective marginal posteriors of μ and σ^2 for the Jeffreys prior (Dongchu & Keying, 1996) are

$$\mu|\sigma^2,w \sim N(\hat{\mu},\sigma^2/n) \tag{18}$$

and

$$\sigma^2|w \sim IG(n/2, W_n/2) \tag{19}$$

where $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} w_i$ and $W_n = \sum_{i=1}^{n}(W_i - \overline{W})^2$.

The Bayesian confidence interval for the coefficient of variation is constructed by substitution by $\mu|\sigma^2, w$ and $\sigma^2|w$ in Equations (12) and (3), respectively.

Therefore, the $100(1-\gamma)\%$ two-sided confidence interval for τ based on the Jeffreys prior is defined as

$$CI_\tau^J = \left(L_\tau^J, U_\tau^J\right) \quad (20)$$

where L_τ^J and U_τ^J are the lower and upper bounds of the $100(1-\gamma)\%$ credible confidence interval and the HPD interval of τ, respectively.

Let $p(\theta|x)$ be a posterior density function. When $p(\theta|x)$ is not symmetric, the HPD interval is as defined by Box and Tiao (1992), and the $100(1-\gamma)\%$ HPD interval for θ is simply given by

$$R(p_\gamma) = \{\theta : p(\theta|x) \geq p_\gamma\} \quad (21)$$

where *R* is a region in the parameter space of θ and p_γ is the largest constant value such that $P(\theta \in R(p_\gamma)) \geq 1-\gamma$ (Chen & Shao, 1998).

2.1.2 The reference prior

The reference prior approach was developed by Bernardo (1979) and modified for multiparameter problems by Berger and Bernardo (1992). Although the approach cannot be simply described, it can be roughly thought of as trying to modify the Jeffreys prior by reducing the dependence among the parameters (Yang & Berger, 1998).

The reference prior is given by

$$p(\mu, \sigma^2) \propto 1/\sigma \ . \quad (22)$$

The joint posterior density function of the reference prior is defined as

$$p(\mu, \sigma^2|w) \propto \frac{1}{\sigma}\prod_{i=1}^{n}(\sigma^2)^{-1/2}\exp\left(-\frac{1}{2\sigma^2}(w_i-\mu)^2\right). \quad (23)$$

The respective marginal posteriors of μ and σ^2 for the reference prior (Dongchu & Keying, 1996) are

$$\mu|\sigma^2, w \sim \mathrm{N}(\hat{\mu}, \sigma^2/\mathrm{n}) \quad (24)$$

and

$$\sigma^2|w \sim \mathrm{IG}((n-1)/2, \mathrm{W_n}/2) \quad (25)$$

where $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} w_i$ and $W_n = \sum_{i=1}^{n}(W_i - \overline{W})^2$.

Equations (24) and (25) can be substituted into Equations (12) and (3), respectively.
Therefore, the $100(1-\gamma)\%$ two-sided confidence interval for τ based on the reference prior is defined as

$$CI_\tau^R = \left(L_\tau^R, U_\tau^R\right) \quad (26)$$

where L_τ^R and U_τ^R are the lower and upper bounds of the $100(1-\gamma)\%$ credible confidence interval of the coefficient of variation.

2.1.3 The uniform prior

A prior just means a constant density with the value of the constant typically chosen as 1, as popularized by Laplace (1812). The uniform prior is defined as

$$p(\mu, \sigma^2) \propto 1 . \quad (27)$$

The joint posterior density function for the uniform prior is defined as

$$p(\mu, \sigma^2|w) \propto 1\prod_{i=1}^{n}(\sigma^2)^{-1/2}\exp\left(-\frac{1}{2\sigma^2}(w_i-\mu)^2\right). \quad (28)$$

The respective marginal posteriors of μ and σ^2 for the uniform prior are given by

$$\mu|\sigma^2, w \sim \mathrm{N}(\hat{\mu}, \sigma^2/\mathrm{n}) \quad (29)$$

and

$$\sigma^2 | w \sim \mathrm{IG}((n-3)/2, \mathrm{W_n}/2) \quad (30)$$

where $\hat{\mu} = \frac{1}{\mathrm{n}} \sum_{i=1}^{n} w_i$ and $W_n = \sum_{i=1}^{n} (W_i - \overline{W})^2$.

Equations (29) and (30) can be substituted into Equations (12) and (3), respectively.
Therefore, the $100(1-\gamma)\%$ two-sided confidence interval for τ based on the uniform prior is defined as

$$CI_\tau^U = \left(L_\tau^U, U_\tau^U\right) \quad (31)$$

where L_τ^U and U_τ^U are the lower and upper bounds of the $100(1-\gamma)\%$ credible confidence interval of the coefficient of variation.

Algorithm 1

Step 1 Generate x_i from IG distribution.
Step 2 Compute $x^{-1/3}$.
Step 3 Compute $\mu | \sigma^2, w$ from Equation (18), Equation (24), and Equation (29).
Step 4 Compute $\sigma^2 | w$ from Equation (19), Equation (25), and Equation (30).
Step 5 Compute α and τ from Equation (12), and Equation (3) by substituting $\mu | \sigma^2, w$ and $\sigma^2 | w$.
Step 6 Repeat Step 3-5 5,000 times.
Step 7 Compute the 95% credible and HPD interval for τ from Equation (20), Equation (26), and Equation (31).
Step 8 Repeat Step 1-7 15,000 times to compute the CP and the EL.

2.2 Confidence intervals for the difference of coefficients of variation of inverse gamma distribution

In this section, we explain the methods for constructing the confidence intervals for the difference between the coefficients of variation of two independent inverse-gamma distributions. Suppose that $X_1 \sim \mathrm{IG}(\alpha_1, \beta_1)$ and $X_2 \sim \mathrm{IG}(\alpha_2, \beta_2)$ are independent, then the difference between their coefficients of variation is simply

$$\psi = \tau_1 - \tau_2 = \frac{1}{\sqrt{\alpha_1 - 2}} - \frac{1}{\sqrt{\alpha_2 - 2}} . \quad (32)$$

The confidence intervals for the parameter ψ can be constructed as follows.

2.2.1 The Bayesian method based on the Jeffreys prior

The marginal posteriors of μ_i and σ_i^2 are defined in Equations (18) and (19), respectively. To construct the Bayesian methods based on the Jeffreys prior, ψ is substituted by μ_i and σ_i^2 , for i = 1,2, in Equations (12) and (32), respectively.

Therefore, the $100(1-\gamma)\%$ two-sided confidence interval for ψ based on the Jeffreys prior is defined as

$$CI_\psi^J = \left(L_\psi^J, U_\psi^J\right) \quad (33)$$

where L_ψ^J and U_ψ^J are the lower and upper bounds of the $100(1-\gamma)\%$ credible confidence intervals for the difference between the coefficients of variation of two inverse-gamma distributions.

2.2.2 The Bayesian method based on the reference prior

The marginal posteriors of μ_i and σ_i^2 are defined in Equations (24) and (25), respectively. Moreover, μ_i and σ_i^2 can be substituted into Equations (12) and (32), respectively.
Therefore, the $100(1-\gamma)\%$ two-sided confidence interval for ψ based on the reference prior is defined as

$$CI_\psi^R = \left(L_\psi^R, U_\psi^R\right) . \quad (34)$$

2.2.3 The Bayesian method based on the uniform prior

The marginal posterior of μ_i and σ_i^2, for i = 1, 2, are defined in Equations (29) and (30), respectively. Subsequently, μ_i and can be substituted into Equation (12). By the difference between the coefficients of variation of two inverse-gamma distributions in Equation (32), the $100(1-\gamma)\%$ two-sided confidence interval for ψ based on the uniform prior is given by

$$CI_{\psi}^{U} = \left(L_{\psi}^{U}, U_{\psi}^{U}\right). \quad (35)$$

Algorithm 2

Step 1 Generate x_i; i = 1, 2 from IG distribution.

Step 2 Compute $x_1^{-1/3}$ and $x_2^{-1/3}$.

Step 3 Compute $\mu_i \left| \sigma^2, w \right.$; i = 1, 2 from Equation (18), Equation (24), and Equation (29).

Step 4 Compute $\sigma_i^2 \left| w \right.$; i = 1, 2 from Equation (19), Equation (25), and Equation (30).

Step 5 Compute α_i for i = 1, 2 from Equation (12).

Step 6 Compute ψ from Equation (36).

Step 7 Repeat Step 3-6 5,000 times.

Step 8 Compute the 95% credible and HPD interval for ψ from Equation (33), Equation (34), and Equation (35).

Step 9 Repeat Step 1-8 15,000 times to compute the CP and the EL.

3. Simulation Studies

To compare the performances of the proposed methods, their coverage probabilities and expected lengths were estimated via Monte Carlo simulation with the R statistical program (R Core Team, 2020). In each scenario, the best-performing confidence interval was chosen with a coverage probability greater than or close to the nominal confidence level and the shortest expected length.

For one inverse-gamma population, the data were generated for an inverse-gamma distribution with $\beta = 1$ and varying α to obtain the required coefficient of variation $\tau =$ 0.1, 0.2, 0.3, 0.4, and 0.5. The sample sizes were $n =$ 10, 30, 50, and 100. Subsequently, the performances of the confidence intervals at the nominal confidence interval of 95% for τ were computed.

For two inverse-gamma populations, the data were generated for two independent inverse-gamma distributions with $X_i \sim \mathrm{IG}(\alpha_i, \beta_i)$, for i = 1, 2, where β_i was fixed at 1 and $\alpha_i = \frac{1}{\tau_i^2} + 2$. The coefficients of variation were set as (τ_1, τ_2) = (0.1, 0.2), (0.2, 0.3), (0.2, 0.4), (0.3, 0.4), (0.3, 0.5), (0.4, 0.5), and (0.5, 0.5). For equal sample sizes ($n_1 = n_2$), we used (10, 10), (30, 30), (50, 50), and (100, 100), and for unequal sample sizes ($n_1 \neq n_2$), we used (10, 30), (30, 50), and (50, 100). Next, the coverage probabilities and expected lengths for the 95% confidence intervals for the difference between the coefficients of variation n ψ were evaluated. For all of the simulations, the number of replications was set as 15,000, with 5,000 repetitions used for the Bayesian methods.

4. Results and Discussion

The coverage probabilities and expected lengths of the 95% confidence intervals for τ are reported in Table 1.It can be seen that the Bayesian HPD intervals based on the reference and uniform priors were greater than or close to the nominal confidence level of 0.95 in almost all cases. Although the coverage probabilities of the Bayesian credible confidence intervals using all three priors were less than the nominal confidence level 0.95 for almost all cases, their expected lengths were shorter than those obtained with the Wald method. Moreover, for sample size n = 50 and 100 with $\tau =$ 0.1 and 0.2, the Bayesian HPD interval based on the uniform prior performed better than the other methods. Therefore, the Bayesian HPD interval based on the reference prior is recommended for constructing the confidence interval for the coefficient of variation of a single inverse-gamma distribution.

The coverage probabilities and expected lengths of the 95% two-sided confidence interval for ψ with equal and unequal sample sizes are listed in Tables 2 and 3, respectively. The results show that the Bayesian HPD intervals provided coverage probabilities that were greater than or close to the nominal confidence level 0.95 in almost all cases, with the expected lengths of the Bayesian HPD interval based on the Jeffreys prior being the shortest. For equal and unequal sample sizes, the

Bayesian HPD interval based on the Jeffreys prior performed better than the other methods in almost all cases, except for equal sample size with (τ_1, τ_2) = (0.1, 0.2) when the Bayesian HPD interval based on the reference prior performed better than the others. Therefore, the Bayesian HPD interval based on the Jeffreys prior is recommended for constructing the confidence interval for the difference between the coefficients of variation of two inverse-gamma distributions with equal and unequal sample sizes.

Table 1. The coverage probabilities and expected lengths of the 95% confidence intervals for the coefficient of variation of a single inverse-gamma distribution.

n	τ	Coverage probability (Expected length)						
		CI_J	CI_R	CI_U	$CI_{HPD.J}$	$CI_{HPD.R}$	$CI_{HPD.U}$	CI_W
10	0.10	0.9376 (0.1009)	0.9498 (0.1153)	0.9277 (0.1630)	0.9198 (0.0931)	**0.9501** **(0.1053)**	0.9724 (0.1445)	0.9722 (0.2172)
	0.20	0.9394 (0.2259)	0.9477 (0.2667)	0.9282 (0.4105)	0.9228 (0.2034)	**0.9528** **(0.2357)**	0.9822 (0.3419)	0.9991 (0.7585)
	0.30	0.9348 (0.4267)	0.9510 (0.5244)	0.9281 (0.8701)	0.9192 (0.3627)	**0.9534** **(0.4321)**	0.9857 (0.6666)	0.9971 (1.5023)
	0.40	0.9380 (0.7703)	0.9506 (0.9569)	0.9381 (1.5349)	0.9214 (0.6081)	**0.9530** **(0.7340)**	0.9873 (1.1196)	0.9912 (1.8331)
	0.50	0.9382 (1.1665)	0.9542 (1.4136)	0.9492 (2.1593)	0.9202 (0.8798)	**0.9518** **(1.0453)**	0.9846 (1.5450)	0.9813 (2.0490)
30	0.10	0.9440 (0.0534)	0.9514 (0.0556)	0.9408 (0.0601)	0.9356 (0.0518)	**0.9516** **(0.0539)**	0.9532 (0.0582)	0.9605 (0.0609)
	0.20	0.9471 (0.1146)	0.9489 (0.1193)	0.9464 (0.1301)	0.9398 (0.1106)	**0.9500** **(0.1150)**	0.9614 (0.1250)	0.9560 (0.1332)
	0.30	0.9470 (0.1940)	0.9498 (0.2029)	0.9435 (0.2261)	0.9438 (0.1850)	**0.9520** **(0.1930)**	0.9642 (0.2138)	0.9591 (0.2359)
	0.40	0.9431 (0.3161)	0.9457 (0.3358)	0.9404 (0.3879)	0.9410 (0.2935)	**0.9536** **(0.3100)**	0.9692 (0.3522)	0.9562 (0.4315)
	0.50	0.9473 (0.4827)	0.9522 (0.5255)	0.9456 (0.6419)	0.9450 (0.4286)	**0.9585** **(0.4607)**	0.9754 (0.5463)	0.9864 (0.8125)
50	0.10	0.9497 (0.0408)	0.9472 (0.0417)	0.9451 (0.0436)	0.9443 (0.0400)	0.9478 (0.0409)	**0.9530** **(0.0427)**	0.9556 (0.0439)
	0.20	0.9466 (0.0868)	0.9508 (0.0887)	0.9480 (0.0931)	0.9423 (0.0848)	0.9490 (0.0867)	**0.9540** **(0.0908)**	0.9549 (0.0947)
	0.30	0.9469 (0.1444)	0.9506 (0.1484)	0.9443 (0.1574)	0.9426 (0.1402)	**0.9530** **(0.1440)**	0.9574 (0.1524)	0.9531 (0.1622)
	0.40	0.9462 (0.2260)	0.9484 (0.2341)	0.9467 (0.2512)	0.9442 (0.2167)	**0.9523** **(0.2241)**	0.9614 (0.2395)	0.9509 (0.2673)
	0.50	0.9462 (0.3271)	0.9480 (0.3391)	0.9463 (0.3762)	0.9439 (0.3074)	**0.9535** **(0.3177)**	0.9681 (0.3490)	0.9455 (0.4219)
100	0.10	0.9534 (0.0295)	0.9492 (0.0288)	0.9470 (0.0295)	0.9452 (0.0281)	0.9492 (0.0285)	**0.9510** **(0.0291)**	0.9514 (0.0296)
	0.20	0.9478 (0.0603)	0.9490 (0.0610)	0.9480 (0.0624)	0.9444 (0.0595)	0.9480 (0.0601)	**0.9521** **(0.0615)**	0.9512 (0.0633)
	0.30	0.9470 (0.0995)	0.9493 (0.1008)	0.9478 (0.1035)	0.9451 (0.0978)	**0.9500** **(0.0990)**	0.9519 (0.1016)	0.9477 (0.1065)
	0.40	0.9453 (0.1527)	0.9472 (0.1555)	0.9453 (0.1601)	0.9440 (0.1493)	**0.9504** **(0.1520)**	0.9540 (0.1563)	0.9451 (0.1689)
	0.50	0.9508 (0.2138)	0.9474 (0.2173)	0.9491 (0.2263)	0.9458 (0.2074)	**0.9522** **(0.2108)**	0.9600 (0.2192)	0.9419 (0.2454)

Table 2. The coverage probabilities and expected lengths of the 95% confidence intervals for the difference between the coefficients of variation of two inverse-gamma distributions ($n_1 = n_2$)

(n_1, n_2)	(τ_1, τ_2)	Coverage probability (Expected length)					
		CI_J	CI_R	CI_U	$CI_{HPD.J}$	$CI_{HPD.R}$	$CI_{HPD.U}$
(10, 10)	(0.1, 0.2)	0.9354 (0.2585)	0.9546 (0.3045)	0.9717 (0.4682)	0.9492 (0.2473)	**0.9688** **(0.2890)**	0.9908 (0.4330)
	(0.2, 0.3)	0.9391 (0.5182)	0.9525 (0.6346)	0.9728 (1.0651)	**0.9637** **(0.4876)**	0.9772 (0.5911)	0.9948 (0.9731)
	(0.2, 0.4)	0.9406 (0.8366)	0.9510 (1.0453)	0.9729 (1.6959)	**0.9535** **(0.7262)**	0.9735 (0.8950)	0.9924 (1.4458)

Table 2. Continued.

(n_1, n_2)	(τ_1, τ_2)	Coverage probability (Expected length)					
		CI_J	CI_R	CI_U	$CI_{HPD.J}$	$CI_{HPD.R}$	$CI_{HPD.U}$
(10, 10)	(0.3, 0.4)	0.9362 (0.9756)	0.9534 (1.2227)	0.9774 (2.0139)	**0.9709** **(0.8964)**	0.9852 (1.1200)	0.9965 (1.8622)
	(0.3, 0.5)	0.9426 (1.3343)	0.9548 (1.6487)	0.9783 (2.5868)	**0.9663** **(1.1686)**	0.9793 (1.4477)	0.9966 (2.3251)
	(0.4, 0.5)	0.9384 (1.5972)	0.9537 (1.9873)	0.9811 (3.1383)	**0.9814** **(1.4645)**	0.9888 (1.8309)	0.9973 (2.9446)
	(0.5, 0.5)	0.9402 (1.9336)	0.9508 (2.3932)	0.9807 (3.6612)	**0.9873** **(1.7913)**	0.9912 (2.2251)	0.9978 (3.4632)
(30, 30)	(0.1, 0.2)	0.9450 (0.1281)	0.9504 (0.1336)	0.9557 (0.1459)	0.9476 (0.1259)	**0.9565** **(0.1312)**	0.9654 (0.1431)
	(0.2, 0.3)	0.9442 (0.2304)	0.9475 (0.2420)	0.9535 (0.2690)	**0.9541** **(0.2265)**	0.9582 (0.2377)	0.9691 (0.2636)
	(0.2, 0.4)	0.9442 (0.3402)	0.9469 (0.3618)	0.9536 (0.4177)	**0.9520** **(0.3247)**	0.9583 (0.3438)	0.9730 (0.3918)
	(0.3, 0.4)	0.9446 (0.3831)	0.9456 (0.4097)	0.9541 (0.4689)	**0.9615** **(0.3734)**	0.9664 (0.3979)	0.9757 (0.4523)
	(0.3, 0.5)	0.9475 (0.5336)	0.9467 (0.5847)	0.9520 (0.7004)	**0.9631** **(0.4984)**	0.9659 (0.5403)	0.9773 (0.6343)
	(0.4, 0.5)	0.9432 (0.6095)	0.9480 (0.6647)	0.9538 (0.8014)	**0.9712** **(0.5841)**	0.9756 (0.6335)	0.9832 (0.7534)
	(0.5, 0.5)	0.9457 (0.7417)	0.9488 (0.8159)	0.9598 (1.0065)	**0.9794** **(0.7122)**	0.9820 (0.7796)	0.9894 (0.9529)
(50, 50)	(0.1, 0.2)	0.9460 (0.0967)	0.9478 (0.0989)	0.9531 (0.1040)	0.9465 (0.0955)	**0.9505** **(0.0976)**	0.9582 (0.1026)
	(0.2, 0.3)	0.9450 (0.1712)	0.9481 (0.1757)	0.9528 (0.1858)	**0.9520** **(0.1692)**	0.9554 (0.1736)	0.9628 (0.1835)
	(0.2, 0.4)	0.9480 (0.2446)	0.9492 (0.2527)	0.9487 (0.2713)	**0.9520** **(0.2381)**	0.9557 (0.2457)	0.9629 (0.2631)
	(0.3, 0.4)	0.9481 (0.2747)	0.9496 (0.2836)	0.9517 (0.3035)	**0.9583** **(0.2707)**	0.9612 (0.2793)	0.9661 (0.2985)
	(0.3, 0.5)	0.9458 (0.3622)	0.9504 (0.3786)	0.9512 (0.4154)	**0.9574** **(0.3502)**	0.9641 (0.3650)	0.9693 (0.3978)
	(0.4, 0.5)	0.9483 (0.4109)	0.9515 (0.4296)	0.9545 (0.4685)	**0.9675** **(0.4029)**	0.9698 (0.4206)	0.9741 (0.4570)
	(0.5, 0.5)	0.9484 (0.4851)	0.9489 (0.5087)	0.9524 (0.5588)	**0.9704** **(0.4771)**	0.9717 (0.4994)	0.9763 (0.5471)
(100, 100)	(0.1, 0.2)	0.9486 (0.0669)	0.9521 (0.0677)	0.9501 (0.0694)	0.9480 (0.0663)	**0.9530** **(0.0671)**	0.9533 (0.0688)
	(0.2, 0.3)	0.9478 (0.1172)	0.9486 (0.1186)	0.9503 (0.1219)	**0.9514** **(0.1162)**	0.9524 (0.1176)	0.9535 (0.1209)
	(0.2, 0.4)	0.9477 (0.1650)	0.9498 (0.1680)	0.9503 (0.1731)	**0.9585** **(0.1626)**	0.9531 (0.1654)	0.9571 (0.1704)
	(0.3, 0.4)	0.9494 (0.1846)	0.9498 (0.1867)	0.9518 (0.1929)	**0.9535** **(0.1829)**	0.9540 (0.1850)	0.9562 (0.1911)
	(0.3, 0.5)	0.9462 (0.2372)	0.9472 (0.2422)	0.9491 (0.2507)	**0.9508** **(0.2332)**	0.9540 (0.2380)	0.9557 (0.2462)
	(0.4, 0.5)	0.9488 (0.2667)	0.9427 (0.2714)	0.9504 (0.2816)	**0.9581** **(0.2641)**	0.9534 (0.2686)	0.9592 (0.2786)
	(0.5, 0.5)	0.9444 (0.3085)	0.9472 (0.3143)	0.9509 (0.3279)	**0.9567** **(0.3061)**	0.9606 (0.3118)	0.9636 (0.3252)

Table 3. The coverage probabilities and expected lengths of the 95% confidence intervals for the difference between the coefficients of variation of two inverse-gamma distributions ($n_1 \neq n_2$)

(n_1, n_2)	(τ_1, τ_2)	Coverage probability (Expected length)					
		CI_J	CI_R	CI_U	$CI_{HPD.J}$	$CI_{HPD.R}$	$CI_{HPD.U}$
(10, 30)	(0.1, 0.2)	0.9448 (0.1590)	0.9502 (0.1742)	0.9614 (0.2202)	**0.9585** **(0.1574)**	0.9662 (0.1722)	0.9793 (0.2150)

Table 3. Continued.

(n_1, n_2)	(τ_1, τ_2)	Coverage probability (Expected length)					
		CI_J	CI_R	CI_U	$CI_{HPD.J}$	$CI_{HPD.R}$	$CI_{HPD.U}$
(10, 30)	(0.2, 0.3)	0.9422 (0.3156)	0.9492 (0.3552)	0.9606 (0.5002)	**0.9624** **(0.3090)**	0.9690 (0.3448)	0.9864 (0.4674)
	(0.2, 0.4)	0.9419 (0.4117)	0.9480 (0.4588)	0.9658 (0.6206)	**0.9689** **(0.4031)**	0.9742 (0.4484)	0.9893 (0.5984)
	(0.3, 0.4)	0.9404 (0.5728)	0.9528 (0.6792)	0.9592 (1.0329)	**0.9696** **(0.5465)**	0.9797 (0.6365)	0.9910 (0.9207)
	(0.3, 0.5)	0.9394 (0.7054)	0.9502 (0.8219)	0.9616 (1.2186)	**0.9779** **(0.6757)**	0.9834 (0.7822)	0.9917 (1.1334)
	(0.4, 0.5)	0.9363 (1.0038)	0.9546 (1.2000)	0.9632 (1.8331)	**0.9720** **(0.9230)**	0.9856 (1.0884)	0.9940 (1.6139)
	(0.5, 0.5)	0.9375 (1.3649)	0.9502 (1.6447)	0.9656 (2.4283)	**0.9651** **(1.1958)**	0.9786 (1.4214)	0.9938 (2.0561)
(30, 50)	(0.1, 0.2)	0.9473 (0.1032)	0.9478 (0.1064)	0.9546 (0.1128)	**0.9520** **(0.1023)**	0.9522 (0.1055)	0.9603 (0.1119)
	(0.2, 0.3)	0.9462 (0.1888)	0.9500 (0.1948)	0.9542 (0.2095)	**0.9558** **(0.1873)**	0.9601 (0.1933)	0.9638 (0.2079)
	(0.2, 0.4)	0.9472 (0.2580)	0.9475 (0.2682)	0.9537 (0.2897)	**0.9580** **(0.2532)**	0.9584 (0.2630)	0.9655 (0.2838)
	(0.3, 0.4)	0.9455 (0.3080)	0.9469 (0.3214)	0.9565 (0.3515)	**0.9618** **(0.3048)**	0.9628 (0.3181)	0.9722 (0.3475)
	(0.3, 0.5)	0.9462 (0.3927)	0.9492 (0.4108)	0.9533 (0.4549)	**0.9622** **(0.3836)**	0.9659 (0.4011)	0.9718 (0.4427)
	(0.4, 0.5)	0.9457 (0.4782)	0.9485 (0.5064)	0.9586 (0.5711)	**0.9713** **(0.4698)**	0.9728 (0.4965)	0.9802 (0.5577)
	(0.5, 0.5)	0.9461 (0.6178)	0.9496 (0.6653)	0.9548 (0.7918)	**0.9718** **(0.5921)**	0.9767 (0.6338)	0.9844 (0.7414)
(50, 100)	(0.1, 0.2)	0.9500 (0.0735)	0.9512 (0.0746)	0.9578 (0.0769)	**0.9525** **(0.0731)**	0.9534 (0.0741)	0.9600 (0.0764)
	(0.2, 0.3)	0.9473 (0.1339)	0.9493 (0.1361)	0.9520 (0.1411)	**0.9521** **(0.1331)**	0.9543 (0.1353)	0.9565 (0.1403)
	(0.2, 0.4)	0.9461 (0.1777)	0.9497 (0.1809)	0.9537 (0.1874)	**0.9508** **(0.1759)**	0.9552 (0.1790)	0.9597 (0.1855)
	(0.3, 0.4)	0.9453 (0.2142)	0.9499 (0.2191)	0.9514 (0.2288)	**0.9555** **(0.2128)**	0.9580 (0.2176)	0.9607 (0.2272)
	(0.3, 0.5)	0.9477 (0.2622)	0.9480 (0.2684)	0.9517 (0.2810)	**0.9582** **(0.2595)**	0.9561 (0.2657)	0.9626 (0.2782)
	(0.4, 0.5)	0.9474 (0.3203)	0.9464 (0.3282)	0.9507 (0.3478)	**0.9588** **(0.3173)**	0.9620 (0.3250)	0.9657 (0.3441)
	(0.5, 0.5)	0.9409 (0.4033)	0.9444 (0.4171)	0.9514 (0.4506)	**0.9586** **(0.3942)**	0.9634 (0.4069)	0.9706 (0.4374)

5. An Empirical Application

Kaewprasert, Niwitpong, and Niwitpong (2020) provided the data of yearly rainfall amounts (mm) that we used to compute the confidence intervals for the coefficient of variation of a single inverse-gamma distribution. To illustrate the efficacies of the confidence intervals proposed in this paper, we used monthly rainfall data (mm) from the Mae Taeng district, Chiang Mai province, Thailand (Upper Northern Region Irrigation Hydrology Center, 2021). There were 27 observations from July, 1994 to 2020. The density and Q-Q plots for the rainfall data showing that they follow an inverse-gamma distribution are shown in Figure.1. For the two inverse-gamma populations, the yearly rainfall data (mm) from the Chae Hom and Mae Tha districts in Lampang province with the same sample sizes were used to compute the confidence intervals for the difference between the coefficients of variation of two inverse-gamma distributions. There were 23 observations from 1998 to 2020. The density and Q-Q plots of the two sets of rainfall data showing that they follow inverse-gamma distributions are exhibited in Figures 2 and 3, respectively. We tested the distributions of these datasets using the minimum Akaike information criterion (AIC) and the Bayesian information criterion (BIC), which are respectively defined as

$$AIC = -2\ln L + 2k \quad (36)$$

and

$$BIC = -2\ln L + k\ln(n) \quad (37)$$

where L is the likelihood function, k is the number of parameters, and n is the number of recorded measurements. It was found that both of the rainfall datasets fit an inverse-gamma distribution, as confirmed by the AIC and BIC value in Table 4 because the AIC and BIC values for this distribution were the smallest.

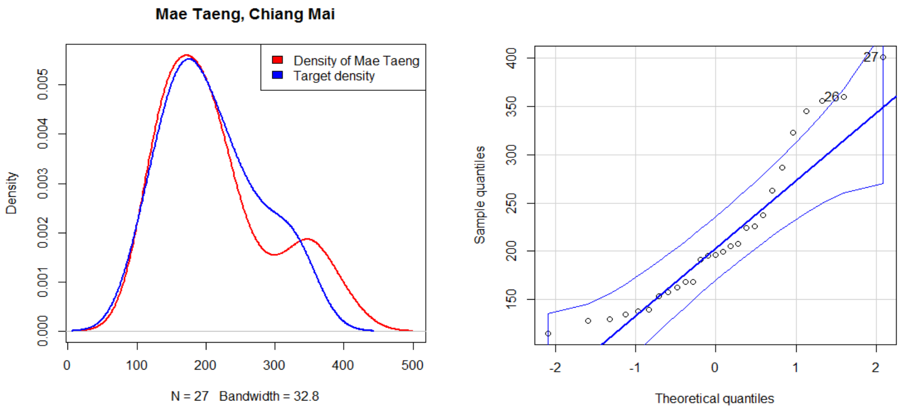

Figure 1. Density and Q-Q plots to check that the rainfall dataset from Mae Taeng, Chiang Mai, fits an inverse-gamma distribution

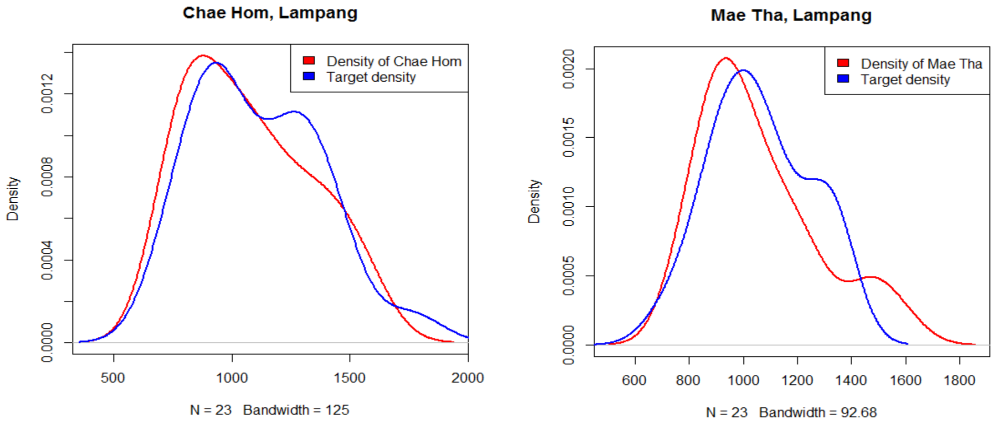

Figure 2. The densities of the rainfall data from the Chae Hom and Mae Tha districts, Lampang province

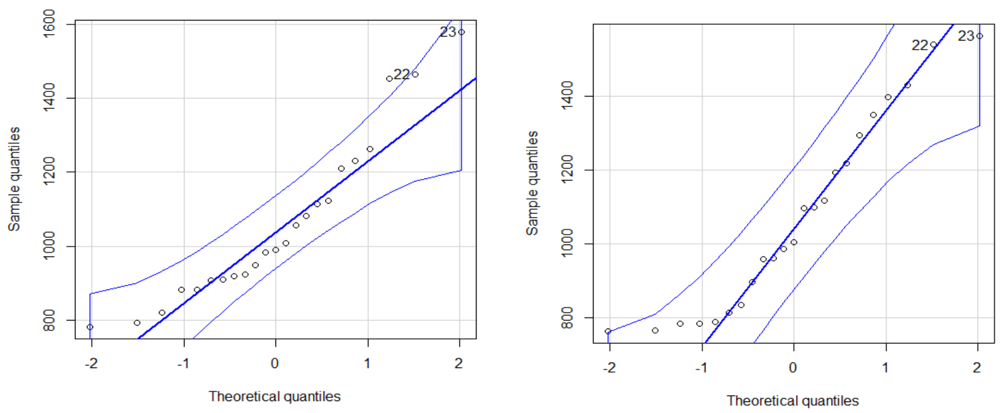

Figure 3. Q-Q plots for fitting the rainfall data from the Chae Hom and Mae Tha districts, Lampang, to inverse-gamma distributions

Table 4. AIC and BIC results to check the distributions of the rainfall datasets

Rainfall data	Densities	Normal	Cauchy	Exponential	Log normal	Gamma	IG
Mae Taeng, Chiang Mai	AIC	316.95	323.38	346.08	310.53	311.95	309.65
	BIC	319.54	325.98	347.38	313.12	314.54	312.25
Chae Hom, Lampang	AIC	324.03	335.78	368.93	322.00	322.43	321.71
	BIC	326.30	338.05	370.07	324.27	324.70	323.98
Mae Tha, Lampang	AIC	316.33	321.41	368.34	312.83	313.80	312.04
	BIC	318.61	323.68	369.47	315.10	316.07	314.31

The summary statistics were computed for the rainfall dataset from Mae Taeng district as n = 27, $\hat{\alpha}$ = 8.62, and the maximum likelihood estimator for τ as $\hat{\tau}$ = 0.39. The 95% confidence intervals for τ were calculated, as reported in Table 5. In accordance with the simulation results in the previous section, the expected length of the Bayesian HPD intervals based on the reference and uniform priors were shorter than that using the Wald method but longer than the Bayesian HPD interval based on the Jeffreys prior. Thus, the Bayesian HPD intervals using the three priors performed better than the Wald method in terms of length.

We used our proposed confidence intervals to estimate the difference between the coefficients of variation for the rainfall datasets from the Chae Hom and Mae Tha districts, Lampang province. Here, we present the summary statistical values: the number of observations was equal (n_1 = n_2 = 23), $\hat{\alpha}_1$ = 18.76, $\hat{\alpha}_2$ = 28.22, $\hat{\tau}_1$ = 0.24, $\hat{\tau}_2$ = 0.19, and $\hat{\psi}$ = 0.05. Finally, the 95% confidence intervals for ψ were calculated, as reported in Table 6. It was found that the length of the Bayesian HPD interval based on the Jeffreys prior was the shortest and performed better than the respective Bayesian credible confidence interval, which corresponds to the results of the simulation study.

6. Conclusions

We proposed new confidence intervals for the single and difference between the coefficients of variation of inverse-gamma distributions. The performance of the confidence intervals was evaluated using the coverage probability and expected length through Monte Carlo simulations. For the single coefficient of variation of an inverse-gamma distribution, the simulation study results show that the coverage probabilities of the Bayesian HPD intervals based on the reference and uniform priors were greater than or close to the nominal confidence level of 0.95 in almost all cases. Moreover, the Bayesian credible confidence intervals and the Bayesian HPD intervals performed better than the Wald method in terms of expected length.

For the difference between the coefficients of variation of two inverse-gamma populations, the Bayesian HPD interval based on the Jeffreys prior performed well in terms of coverage probability and its expected length was shorter than the other methods. Therefore, the Bayesian HPD interval based on the Jeffreys prior is recommended for constructing the confidence interval for the difference between the coefficients of variation of inverse-gamma distributions.

Acknowledgements

The authors would to thank the referees whose valuable comments and suggestions. The first author is grateful for receiving the financial support from Science Achievement Scholarship of Thailand. This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-65-BASIC-35.

Table 5. The 95% confidence intervals for the single coefficient of variation for the rainfall dataset from Mae Taeng district, Chiang Mai province

Methods	Confidence intervals for τ		Length of intervals
	Lower bound	Upper bound	
CI_J	0.2970	0.5557	0.2587
CI_R	0.3024	0.5781	0.2757
CI_U	0.3142	0.6321	0.3179
$CI_{HPD.J}$	0.2968	0.5134	0.2166
$CI_{HPD.R}$	0.3022	0.5320	0.2298
$CI_{HPD.U}$	0.3140	0.5758	0.2617
CI_W	0.2985	0.7032	0.4047

Table 6. The 95% confidence intervals for the difference between the coefficients of variation of the rainfall datasets from the Chae Hom and Mae Tha districts, Lampang province

Methods	Confidence intervals for ψ		Length of intervals
	Lower bound	Upper bound	
CI_J	-0.0447	0.1895	0.2342
CI_R	-0.0488	0.1999	0.2487
CI_U	-0.0589	0.2251	0.2840
$CI_{HPD.J}$	-0.0379	0.1895	0.2275
$CI_{HPD.R}$	-0.0412	0.1999	0.2411
$CI_{HPD.U}$	-0.0491	0.2251	0.2742

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