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Original Article

Conceptual interpretation of interval valued $\bar{T}$- normed fuzzy β-subalgebra

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Abstract

Triangular norm is a sort of binary operation often used in the fields such as fuzzy logic, probabilistic metric spaces and so on. In this paper, the concept of interval valued $\bar{T}$-normed fuzzy β-subalgebra is proposed and its associated outcomes investigated. Furthermore, the intersection between two $\bar{T}$-interval valued fuzzy β-subalgebra is presented. Moreover, the characteristics of homomorphism and endomorphism on $\bar{T}$- interval valued fuzzy β-subalgebras have been studied.

Keywords: $\bar{T}$-normed, normed fuzzy, β-algebra, β-subalgebra, interval valued fuzzy β-algebra, T-fuzzy

1. Introduction

T-norms is a generalization of the conjunction of two-valued logic that fuzzy logic is usually studied by classical logic. In fact, classical Boolean conjunctions are both commutative and associative. Monotonicity ensures that if the conjunctive truth value increases, the conjunctive truth degree will not decrease. In 1965, Zadeh (1965) proposed fuzzy sets and he further extended the idea of linguistic variable and its applications. The fuzzy sets have been connected in algebraic structures beginning from Rosenfeld (1971). Neggers and Kim (2002) initiated β-algebras in which two binary operations have been used. Zadeh (1975) provided the idea of interval valued fuzzy subsets in which the membership functions are evaluation an intervals of numbers rather than the numbers. Biswas (1994) presented the thought of fuzzy subgroups with interval valued membership. The author expressed a vital and adequate condition for an i–v fuzzy subset to be an i–v fuzzy subgroup. Moreover, he had concluded that the Intersection of two i–v fuzzy subgroups is again an i–v fuzzy subgroup. Cagman and Deli (2012a, 2012b, 2015) presented t-norm and t-conorm products of fuzzy parameterized soft sets (FP-soft sets) and constructed AND-FP-soft decision making and OR-FP-soft decision making methods. Further, relations on FP-soft sets applied to decision making problems and also provided the applications of decision making on FP-soft set theory.

Menger (1942) has initiated the idea of probabilistic metric spaces which prompts extra contribution into the decision making concepts and speculations of corporative recreations. Specifically, in the system of hypotheses of fuzzy sets, the T-norms have been comprehensively utilized for fuzzy operations, fuzzy logics and fuzzy connections. An explicit

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study regarding the properties and also the connected components of t-norms are reported by Klement, Mesiar and Pap (2000, 2004). Intuitionistic (S, T)-fuzzy lie ideals of lie algebras were protracted by Muhammed Akram (2007). The concept of bipolar valued Q-fuzzy application in building sciences was established by Muthumeenakshi, Muralikrishna and Sabarinathan (2018). The notion of bipolar valued fuzzy sets and their applications has been initiated by Lee (2000). Kim (2007) projected intuitionistic (T, S)-normed fuzzy subalgebras of BCK-algebras. Tapan senapati (2015) examined the concept of T- fuzzy KU-subalgebras of KU-algebra.

Jun and Kim (2012) proposed the concept of β-subalgebras and related topics in which some interesting results were studied. Singh and Kumar (2012) originated the idea of interval-valued fuzzy graph representation of concept lattice. Anasri and Chandramouleeswaran (2014) introduced the inception of T-fuzzy β-subalgebras of β-algebras. Hemavathi, Muralikrishna and Palanivel (2015) introduced an interval valued fuzzy β-subalgebras of β-algebra. Muralikrishna, Sujatha and Chandramouleeswaran (2017) illustrated the notion of (S,T)-Normed intuitionstic fuzzy β-subalgebras. Recently, Borumand Saeid, Muralikrishna and Hemavathi (2019) initiated the concept of binormed intuitionistic β-ideals of β-algebras and some appealing results were explored. This article focused the conviction of interval valued $\overline{T}$-Normed fuzzy β-subalgebra and related discussions.

2. Preliminaries

This section recalls some basic definitions needed for this work.

Definition 2.1 A β-algebra is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:
1. $x - 0 = x$
2. $(0 - x) + x = 0$
3. $(x - y) - z = x - (z - y)\ \forall\ x, y, z\ \in\ X.$

Example 2.2 Let $X = \{0,1,2,3\}$ be a β-algebra with constant 0. The binary operations $+$ and $-$are defined on X by the following Cayley's table

Table 1. Cayley's table

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1

-	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	0	1
3	3	2	1	0

It shows $(X, +, -, 0)$ is a β-algebra.

Example 2.3 Consider Set of all integers Z. $(Z, +, -, 0)$ is an infinite β $-$algebra where 0 , $+$ and $-$ have usual meanings.

Definition 2.4 A non-empty subset A of a β $-$algebra $(X, +, -, 0)$ is called a β-subalgebra of X, if
1. $x + y \in A$
2. $x - y \in A, \quad \forall\ x, y \in A.$

Example 2.5 In the above example of the $\beta -$ algebra, the subset $A = \{0,3\}$ is a β-subalgebra of X.

Definition 2.6 Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be two $\beta -$ algebras. A mapping $f: X \to Y$ is said to be a $\beta -$ homomorphism, if
(i) $f(x + y) = f(x) + f(y)$
(ii) $f(x - y) = f(x) - f(y).$

Definition 2.7 A fuzzy set in X is defined as a function $\sigma: X \to [0,1]$. For each element x in X, $\sigma(x)$ is called the membership value of $x \in X$ and X is a universal set.

Definition 2.8 An interval valued fuzzy set (briefly i-v fuzzy set) A defined on X is given by A=$\{(x, [\sigma_A^L(x), \sigma_A^U(x)])\}\ \forall\ x \in X$ (briefly denoted by $A = [\sigma_A^L, \sigma_A^U]$), where σ_A^L and σ_A^U are two fuzzy sets in X such that $\sigma_A^L(x) \leq \sigma_A^U(x)\ \ \forall\ x \in X$. Let $\overline{\sigma}_A(x) = [\sigma_A^L(x), \sigma_A^U(x)]\ \ \forall\ x \in X$ and let $D[0,1]$ denotes the family of all closed sub intervals of [0,1]. If $\sigma_A^L(x) = \sigma_A^U(x) = c$, say, where $0 \leq c \leq 1$, then we have $\overline{\sigma}_A(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0,1]$.

Thus $\overline{\sigma}_A(x) \in D[0,1]\ \ \forall\ x \in X$, and therefore the i-v fuzzy set A is given by $A = \{(x, \overline{\sigma}_A(x))\}\ \forall\ x \in X$, where $\overline{\sigma}_A: X \to D[0,1]$. Now let us define what is known as $refined\ mimimum$ (briefly $rmin$) of two elements in $D[0,1]$. We also define the symbols "$\geq$" , "$\leq$", and "=" in case of two elements in $D[0,1]$. Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0,1]$.

Then we have $rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}]$; $D_1 \geq D_2$ if and only if $a_1 \geq a_2,\ b_1 \geq b_2$; Similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$.

Definition 2.9 Let $\overline{\sigma}_A$ be an interval valued fuzzy subset in X. $\overline{\sigma}_A$ is said to be interval valued fuzzy(i_v_ fuzzy) β −sub algebra of X if
(i) $\overline{\sigma}_A(x+y) \geq rmin\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\}$.
(ii) $\overline{\sigma}_A(x-y) \geq rmin\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\}$ $\forall\, x, y \in X$.

Example 2.10 Consider the β −algebra X defined in the example 2.2. Define an i_v_ fuzzy set on X as follows,

$$\overline{\sigma}_A(x) = \begin{cases} [0.4,0.6]: & x = 0 \\ [0.3,0.5]: & x = 1 \\ [0.2,0.4]: & x = 2,3 \end{cases}$$

$\therefore A = \{x, \overline{\sigma}_A(x) : x \in X\}$ is an i_v_ fuzzy β −sub algebra of X.

Definition 2.11 A triangular norm that is a t −norm is a function function $T: [0,1] \times [0,1] \to [0,1]$ is called a T −norm, if it satisfies the following conditions,

1. $T(x, 1) = x$ (boundary condition)
2. $T(x, y) = T(y, x)$(commutativity)
3. $T(T(x, y), z) = T(x, T(y, z))$(associativity)
4. $T(x, y) \leq T(x, z)$ if $y \leq z$ $\forall x, y, z \in [0,1]$(monotonicity)

The following are some t −norms used in general.

1. standard t −norm (T_M): $T(x, y) = min(x, y)$
2. Bounded difference t −norm (T_L): $T(x, y) = max(0, x + y - 1)$
3. Algebraic product t −norm (T_P): $T(x, y) = xy$
4. Drastic intersection:

$$T_D: T(x, y) = \begin{cases} x \quad when & y = 1 \\ y \quad when & x = 1 \\ 0 & otherwise. \end{cases}$$

Definition 2.12 Let σ be a fuzzy set in a β −algebra X. For a given t −norm T, σ is called a T − fuzzy β −subalgebra of X if,

1. $\sigma(x + y) \geq T\{\sigma(x), \sigma(y)\}$
2. $\sigma(x - y) \geq T\{\sigma(x), \sigma(y)\}$ $\forall\, x, y \in X$.

3. Interval Valued $\overline{T}$ −normed Fuzzy β −subalgebras

This section is dedicated to the notion of interval valued $\overline{T}$ −normed fuzzy β −subalgebra of a β −algebra and proved some related results. Also in the rest of the paper, X is a β −algebra unless otherwise specified.

Definition 3.1 An interval valued triangular norm (i_v_ t-norm) denoted by $\overline{T}$ −norm is a function $\overline{T}: D[0,1] \times D[0,1] \to D[0,1]$ if it satisfies the following conditions,

1. $\overline{T}(\overline{x}, \overline{1}) = \overline{x}$ (boundary condition)
2. $\overline{T}(\overline{x}, \overline{y}) = \overline{T}(\overline{y}, \overline{x})$(commutativity)
3. $\overline{T}(\overline{T}(\overline{x}, \overline{y}), \overline{z}) = \overline{T}(\overline{x}, \overline{T}(\overline{y}, \overline{z}))$(associativity)
4. $\overline{T}(\overline{x}, \overline{y}) \leq \overline{T}(\overline{x}, \overline{z})$ if $\overline{y} \leq \overline{z}$ (monotonicity) $\forall$ $\overline{x}, \overline{y}, \overline{z} \in D[0,1]$.

The following are some $\overline{t}$ −norms used in general,

1. Standard $\overline{t}$ −norm $(\overline{T}_M)$: $\overline{T}(\overline{x}, \overline{y}) = rmin(\overline{x}, \overline{y})$
2. Bounded difference $\overline{t}$ −norm $(\overline{T}_L)$: $\overline{T}(\overline{x}, \overline{y}) = rmax(\overline{0}, \overline{x} + \overline{y} - \overline{1})$
3. Algebraic product $\overline{t}$ −norm $(\overline{T}_P)$: $\overline{T}(\overline{x}, \overline{y}) = \overline{x}\ \overline{y}$
4. Drastic intersection:

$$\overline{T}_D: \overline{T}(\overline{x}, \overline{y}) = \begin{cases} \overline{x} \quad when & \overline{y} = \overline{1} \\ \overline{y} \quad when & \overline{x} = \overline{1} \\ \overline{0} & otherwise. \end{cases}$$

Definition 3.2 Let $(X, +, -, 0)$ be a β −algebra. An i_v_ fuzzy set $A = \{\langle x, \overline{\sigma}_A(x)\rangle / x \in X\}$ is called $\overline{T}$ − i_v_ fuzzy β −subalgebra X, if it satisfies

i. $\overline{\sigma}_A(x + y) \geq \overline{T}\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\}$
ii. $\overline{\sigma}_A(x - y) \geq \overline{T}\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\}$ $\forall x, y \in X$.

Example 3.3 Let $X = \{0,1,2,3\}$ be a β-algebra with constant 0 and two binary operation $+$ and $-$are defined on X by the following Cayley's table

Table 2. Cayley's table

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

The $\overline{T}$ −i_v_ fuzzy set $A = \{\langle x, \overline{\sigma}_A(x)\rangle / x \in X\}$ such that

$$\overline{\sigma}_A(x) = \begin{cases} [0.3,0.6]: & x = 0 \\ [0.2,0.5]: & x = 2 \\ [0.1,0.4]: & x = 3,1. \end{cases}$$

Hence $\overline{\sigma}_A(x)$ is a $\overline{T}$ −i_v_ fuzzy β −subalgebra in X with respect to the $\overline{t}$ −norms T_L, T_P, T_M. Verification:

For $2,3 \in X$,

$$\begin{aligned} \overline{\sigma}_A(x+y) &\geq \overline{T}\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\} \\ \Rightarrow \overline{\sigma}_A(2+3) &\geq \overline{T}\{\overline{\sigma}_A(2), \overline{\sigma}_A(3)\} \\ \Rightarrow [0.1,0.4] &\geq \overline{T}\{[0.2,0.5],[0.1,0.4]\} \end{aligned}$$

$$(i)\overline{T}_M : \overline{\sigma}_A(x+y) = [0.1,0.4] \geq \overline{T}(\overline{x}, \overline{y}) \geq rmin\{[0.2,0.5],[0.1,0.4]\} \geq [min(0.2,0.1), min(0.5,0.4)] \geq [0.1,0.4]$$

$$(ii)\overline{T}_L : \overline{\sigma}_A(x+y) = [0.1,0.4] \geq rmax[\overline{0}, (\overline{x} + \overline{y} - \overline{1})] \geq rmax\{\overline{0}, [x^L + y^L - 1, x^U + y^U - 1]\} = rmax\{\overline{0}, [0.2 + 0.1 - 1, 0.5 + 0.4 - 1]\} = rmax[\overline{0}, \{-0.7, -0.1\}] = [max\{0, -0.7\}, max\{0, -0.1\}] \geq [0,0]$$

$$(iii)\overline{T}_P : \overline{\sigma}_A(x+y) = [0.1,0.4] \geq \overline{T}(\overline{x}, \overline{y}) \geq \overline{x}\,\overline{y} = [0.2,0.5][0.1,0.4] \geq [0.02,0.2]$$

(iv)$\overline{T}_D : \overline{\sigma}_A(x+y) = [0.3,0.7] \geq \overline{T}(\overline{x}, \overline{y}) =$

$$\begin{cases} \overline{\sigma}_A(2) & when\ \overline{\sigma}_A(3) = \overline{1} \\ \overline{\sigma}_A(3) & when\ \overline{\sigma}_A(2) = \overline{1} \\ \overline{0} & otherwise. \end{cases}$$

Similarly, It can be verified for any elements in X.

Example 3.4 Consider the β −algebra X in example 2.2, the $\overline{T}$ −i_v_ fuzzy β −subalgebra $A = \{\langle x, \overline{\sigma}_A(x)\rangle / x \in X\}$ of X such that $\overline{\sigma}_A$ is not a $\overline{T}$ −i_v_ fuzzy β −subalgebra of X with respect to the $\overline{t}$ −norms T_L, T_P & T_M if,

$$\overline{\sigma}_A(x) = \begin{cases} [0.2,0.4]: & x = 0 \\ [0.3,0.6]: & x = 3 \\ [0.1,0.7]: & x = 2 \\ [0.4,0.5]: & x = 1. \end{cases}$$

For, $\overline{\sigma}_A(3+1) = \overline{\sigma}_A(0) = [0.2,0.4] \ngeq \overline{T}\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\} = [0.3,0.5](in T_M)$.

Lemma 3.5 If $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are two $\overline{T}$ − i_v_ fuzzy β −sub algebra of X, then $\overline{\sigma}_1 \cap \overline{\sigma}_2$ is also an $\overline{T}$ − i_v_ fuzzy β −subalgebra of X.

Proof. For $x, y \in X$,

$$\begin{aligned} (\overline{\sigma}_1 \cap \overline{\sigma}_2)(x+y) &= rmin\{\overline{\sigma}_1(x+y), \overline{\sigma}_2(x+y)\} \\ &\geq rmin\{\overline{T}\{\overline{\sigma}_1(x), \overline{\sigma}_1(y)\}, \overline{T}\{\overline{\sigma}_2(x), \overline{\sigma}_2(y)\}\} \\ &\geq \overline{T}\{min\{\overline{\sigma}_1(x), \overline{\sigma}_2(x)\} min\{\overline{\sigma}_1(y), \overline{\sigma}_2(y)\}\} \\ &= \overline{T}\{(\overline{\sigma}_1 \cap \overline{\sigma}_2)(x), (\overline{\sigma}_1 \cap \overline{\sigma}_2)(y)\}. \end{aligned}$$

Similarly, $(\overline{\sigma}_1 \cap \overline{\sigma}_2)(x-y) \geq \overline{T}\{(\overline{\sigma}_1 \cap \overline{\sigma}_2)(x), (\overline{\sigma}_1 \cap \overline{\sigma}_2)(y)\}$.

Hence $(\overline{\sigma}_1 \cap \overline{\sigma}_2)$ is a $\overline{T}$ − i_v_ fuzzy β −sub algebra of X. The above theorem can be generalised as follows:

Corollary 3.6 If $\{\overline{\sigma}_i : i = 1,2,3\ldots\}$ be a family of $\overline{T}$ −i_v_ fuzzy β −subalgebras of X, then $\cap\ \overline{\sigma}_i$ is also a $\overline{T}$ −i_v_ fuzzy β −subalgebras of X, where $(\cap\ \overline{\sigma}_i)(x) = inf \overline{\sigma}_i(x)\ \ \forall x \in X$.

Lemma 3.7 Let X be a β −algebra and σ be a $\overline{T}$ − i_v_ fuzzy β − subalgebra of X with the property $\overline{T}(\overline{a}, \overline{a}) = \overline{a}, \forall\ \overline{a} \in D[0,1]$. Then

1. $\overline{\sigma}(x) \leq \overline{\sigma}(0)\ \ \forall x \in X$

2. $\overline{\sigma}(x) \leq \overline{\sigma}(x^*) \leq \overline{\sigma}(0)$ $\forall x \in X$ where $x^* = 0 - x$.

Proof. Let $x \in X$.

1. $\overline{\sigma}(0) = \overline{\sigma}(x - x) \geq \overline{T}\{\overline{\sigma}(x), \overline{\sigma}(x)\} = \overline{\sigma}(x)$,
2. $\overline{\sigma}(x^*) = \overline{\sigma}(0 - x) \geq \overline{T}\{\overline{\sigma}(0), \overline{\sigma}(x)\} = \overline{\sigma}(x)$.
   Also, $\overline{\sigma}(0) = \overline{\sigma}(x^* - x^*) \geq \overline{T}\{\overline{\sigma}(x^*), \overline{\sigma}(x^*)\} = \overline{\sigma}(x^*)$.

Theorem 3.8 For the $\overline{T} -$ i_v_ fuzzy $\beta -$subalgebra $A = \{\langle x, \overline{\sigma}_A(x)\rangle / x \in X\}$ of X, the set $\chi_{:\overline{\sigma}_A} = \{x \in X: \overline{\sigma}_A(x) = \overline{\sigma}_A(0)\}$ is a $\beta -$subalgebra of X.

Proof. For any $x, y \in \chi_{\overline{\sigma}} \Rightarrow \overline{\sigma}_A(x) = \overline{\sigma}_A(0) = \overline{\sigma}_A(y)$.

$$\begin{aligned} \overline{\sigma}_A(x + y) &\geq \overline{T}\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\} \\ &= \overline{T}\{\overline{\sigma}_A(0), \overline{\sigma}_A(0)\} \\ &= \overline{\sigma}_A(0) ------------(1) \end{aligned}$$

Now,

$$\begin{aligned} \overline{\sigma}_A(x - y) &\geq \overline{T}\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\} \\ &= \overline{T}\{\overline{\sigma}_A(0), \overline{\sigma}_A(0)\} \\ &= \overline{\sigma}_A(0) ------------(2) \\ \overline{\sigma}_A(0) &= \overline{\sigma}_A(0 + 0) \\ &\geq \overline{T}\{\overline{\sigma}_A(0), \overline{\sigma}_A(0)\} \\ &= \overline{T}\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\} \\ &= \overline{\sigma}_A(x + y) ------------(3) \\ \overline{\sigma}_A(0) &= \overline{\sigma}_A(0 - 0) \\ &\geq \overline{T}\{\overline{\sigma}_A(0), \overline{\sigma}_A(0)\} \\ &= \overline{T}\{\overline{\sigma}_A(x), \overline{\sigma}_A(y)\} \\ &= \overline{\sigma}_A(x - y) ------------(4) \end{aligned}$$

(1) & (3) implies $\overline{\sigma}_A(x + y) = \overline{\sigma}_A(0)$ & (2) & (4) implies $\overline{\sigma}_A(x - y) = \overline{\sigma}_A(0)$.
Hence $\overline{\sigma}_A(x - y) = \overline{\sigma}_A(0) = \overline{\sigma}_A(x + y)$.
Thus $x + y$ & $x - y \in \chi_{\overline{\sigma}_A}$ proving that $\chi_{\overline{\sigma}_A}$ is a $\beta -$sub algebra of X.

Definition 3.9 Let $f: X \rightarrow Y$ be a $\beta -$homomorphism with A and B be two $\overline{T} -$i_v_ fuzzy $\beta -$subalgebras in X and Y respectively. The inverse image of B under f is defined by $f^{-1} = \{f^{-1}(\overline{\sigma}_B(x)) : x \in X\}$ such that $f^{-1}(\overline{\sigma}_B(x)) = \overline{\sigma}_B(f(x))$.

Theorem 3.10 Let $f: X \rightarrow Y$ be a $\beta -$ homomorphism. If B is a $\overline{T} -$i_v_ fuzzy $\beta -$subalgebra of Y, then $f^{-1}(B)$ is a $\overline{T} -$ i_v_ fuzzy $\beta -$subalgebra of X.

Proof. Let B be an i_v_ fuzzy $\beta -$subalgebra of Y.
For any $x, y \in Y$,

$$\begin{aligned} f^{-1}(\overline{\sigma}_B(x + y)) &= \overline{\sigma}_B(f(x + y)) \\ &= \overline{\sigma}_B(f(x) + f(y)) \\ &\geq \overline{T}\{\overline{\sigma}_B(f(x)), \overline{\sigma}_B(f(y))\} \\ &\geq \overline{T}\{f^{-1}(\overline{\sigma}_B(x)), f^{-1}(\overline{\sigma}_B(y))\} \end{aligned}$$

and $f^{-1}(\overline{\sigma}_B(x - y)) \geq \overline{T}\{f^{-1}(\overline{\sigma}_B(x)), f^{-1}(\overline{\sigma}_B(y))\}$.
Hence $f^{-1}(B)$ is a $\overline{T} -$ i_v_fuzzy $\beta -$subalgebra of X.

Theorem 3.11 Let X be a $\beta -$algebras. Let $f: X \rightarrow X$ be an endomorphism of $\beta -$ algebra. If A is $\overline{T} -$ i_v_ fuzzy $\beta -$subalgebra of X, then $f(A)$ is a $\overline{T} -$ i_v_ fuzzy $\beta -$subalgebra of X.

Proof. Let A be a $\overline{T} -$ i_v_ fuzzy $\beta -$subalgebra of X, $\forall\ x, y \in X$.

$$\begin{aligned} \overline{\sigma}_f(x + y) &= \overline{\sigma}(f(x + y)) \\ &= \overline{\sigma}(f(x) + f(y)) \\ &= \overline{\sigma}(f(x)) + \overline{\sigma}(f(y)) \\ &\geq \overline{T}\{\overline{\sigma}(f(x)), \overline{\sigma}(f(y))\} \\ &= \overline{T}\{\overline{\sigma}_f(x), \overline{\sigma}_f(y)\} \end{aligned}$$

and $\overline{\sigma}_f(x-y) \geq \overline{T}\left\{\overline{\sigma}_f(x), \overline{\mu}_f(y)\right\}$.
Hence $f(A)$ is an $\overline{T}-$ i_v_ fuzzy $\beta-$subalgebras.

Theorem 3.12 Let A be any $\beta-$subalgebra of X and $\overline{\sigma}: X \to D[0,1]$ be a i_v_ fuzzy set defined

$$\overline{\sigma}(x) = \begin{cases} [t_0, t_1]: & x \in A \\ [s_0, s_1]: & x \notin A \end{cases}$$

with $[t_0, t_1] > [s_0, s_1]$. Then $\overline{\sigma}$ is a $\overline{T}-$i_v_ fuzzy $\beta-$subalgebra of X with respect to the $\overline{t}-$ norm $\overline{T}_M, \overline{T}_P$ & $\overline{T}_L$.

Proof. Consider the $\overline{t}-$norm $\overline{T}_M$. Let $x, y \in X$.

Case (i) If $x, y \in A \Rightarrow x+y$ & $x-y \in A$.
Hence $\overline{\sigma}(x+y) = [t_0, t_1]$ & $\overline{\sigma}(x-y) = [t_0, t_1]$

$$\begin{aligned} \Rightarrow \overline{\sigma}(x+y) &= [t_0, t_1] = rmin\{[t_0, t_1], [t_0, t_1]\} \\ &= \overline{T}_M\{[t_0, t_1], [t_0, t_1]\} \\ &\geq \overline{T}_M\{[t_0, t_1], [t_0, t_1]\} \\ &\therefore \overline{\sigma}(x+y) \geq \overline{T}_M\{\overline{\sigma}(x), \overline{\sigma}(y)\}. \end{aligned}$$

Similarly, $\overline{\sigma}(x-y) \geq \overline{T}_M\{\overline{\sigma}(x), \overline{\sigma}(y)\}$.

Case (ii) If $x, y \notin A$, then

$$\begin{aligned} \overline{\sigma}(x) = \overline{\sigma}(y) &= [s_0, s_1] = rmin\{[s_0, s_1], [s_0, s_1]\} \\ &= \overline{T}_M\{[s_0, s_1], [s_0, s_1]\} \\ &\geq \overline{T}_M\{[s_0, s_1], [s_0, s_1]\}. \end{aligned}$$

If $x+y \in A \Rightarrow \overline{\sigma}(x+y) = [t_0, t_1] > [s_0, s_1]$ and
if $x-y \notin A \Rightarrow \overline{\sigma}(x+y) = [s_0, s_1]$.
Hence

$$\begin{aligned} \overline{\sigma}(x+y) \geq [s_0, s_1] &= rmin\{[s_0, s_1], [s_0, s_1]\} \\ &\geq \overline{T}_M\{[s_0, s_1], [s_0, s_1]\} \\ &= \overline{T}_M\{\overline{\sigma}(x), \overline{\sigma}(y)\}. \end{aligned}$$

Similarly, $\overline{\sigma}(x-y) \geq \overline{T}_M\{\overline{\sigma}(x), \overline{\sigma}(y)\}$.

Case (iii) If $x \in A, y \notin A$, then $\overline{\sigma}(x) = [t_0, t_1]$ & $\overline{\sigma}(y) = [s_0, s_1]$.
Hence

$$\begin{aligned} \overline{T}_M\{\overline{\sigma}(x), \overline{\sigma}(y)\} &\leq \overline{T}_M\{[t_0, t_1][s_0, s_1]\} \\ &= rmin\{[t_0, t_1], [s_0, s_1]\} \\ &= [s_0, s_1], \ \ Since [t_0, t_1] > [s_0, s_1]. \end{aligned}$$

If $x+y \in A$, then $\overline{\sigma}(x+y) = [t_0, t_1] > [s_0, s_1]$ and
if $x+y \notin A$ then $\overline{\sigma}(x+y) = [s_0, s_1]$.
In both the cases

$$\overline{\sigma}(x+y) \geq [s_0, s_1] \geq \overline{T}_M\{\overline{\sigma}(x), \overline{\sigma}(y)\}.$$

Similarly, $\Rightarrow \overline{\sigma}(x-y) \geq \overline{T}_M\{\overline{\sigma}(x), \overline{\sigma}(y)\}$.
Hence $\overline{\sigma}$ is a $\overline{T}_M$i_v_ fuzzy $\beta-$subalgebra of X.

Case (iv) Interchanging the roles of x & y in case (iii) prove $\overline{\sigma}$ is a $\overline{T}_M$ i_v_ fuzzy $\beta-$ subalgebra of X when $x \notin A, y \in A$.
For the $\overline{t}-$norm $\overline{T}_P$. Let $x, y \in X$.

Case (i) Since A is a $\beta-$subalgebra of X, if $x, y \in A$, then $x+y$ & $x-y \in A$.
Hence $\overline{\sigma}(x) = \overline{\sigma}(y) = \overline{\sigma}(x+y) = [t_0, t_1]$ and hence

$$\begin{aligned} \overline{T}_P\{\overline{\sigma}(x), \overline{\sigma}(y)\} &= \overline{\sigma}(x)\overline{\sigma}(y) \\ &= [t_0, t_1][t_0, t_1] \\ &\leq [t_0, t_1] \\ &= \overline{\sigma}(x+y). \end{aligned}$$

Similarly, $\overline{\sigma}(x-y) \geq \overline{T}_P\{\overline{\sigma}(x), \overline{\sigma}(y)\}$.

Case (ii) If $x, y \notin A$, then $\overline{\sigma}(x) = \overline{\sigma}(y) = [s_0, s_1]$
if $(x+y) \in A$ then $\overline{\sigma}(x+y) = [t_0, t_1] > [s_0, s_1]$ and if $(x+y) \notin A$ then $\overline{\sigma}(x+y) = [s_0, s_1]$

In both the cases

$$\begin{aligned}\overline{\sigma}(x+y) \geq [s_0, s_1] &\geq [s_0, s_1][s_0, s_1] \\ &= \{\overline{\sigma}(x), \overline{\sigma}(y)\} \\ &= \overline{T}_P\{\overline{\sigma}(x), \overline{\sigma}(y)\}.\end{aligned}$$

Similarly, $\overline{\sigma}(x-y) \geq \overline{T}_P\{\overline{\sigma}(x), \overline{\sigma}(y)\}$.
Hence $\overline{\sigma}$ is a $\overline{T}$ i_v_ fuzzy β −subalgebra of X.

Case (iii) If $x \in A, y \notin A$, then $\overline{\sigma}(x) = [t_0, t_1]$ and $\overline{\sigma}(y) = [s_0, s_1]$.
Hence

$$\begin{aligned}\overline{\sigma}(x+y) &\geq [t_0, t_1][s_0, s_1] \\ &= \{\overline{\sigma}(x), \overline{\sigma}(y)\} \\ &= \overline{T}_P\{\overline{\sigma}(x), \overline{\sigma}(y)\}.\end{aligned}$$

Similarly, $\overline{\sigma}(x-y) \geq \overline{T}_P\{\overline{\sigma}(x), \overline{\sigma}(y)\}$
Hence $\overline{\sigma}$ is a $\overline{T}$ i_v_ fuzzy β −subalgebra of X.

Case (iv) Interchanging the roles of x & y in case (iii) prove $\overline{\sigma}$ is a $\overline{T}-$ i_v_ fuzzy $\beta-$ subalgebra of X when $x \notin A, y \in A$.
For the $\overline{t}$ −norm $\overline{T}_L$ Let $x, y \in X$.

Case (i) Since A is a β −subalgebra of X, if $x, y \in A$, then $x+y$ & $x-y \in A$.
Hence $\overline{\sigma}(x) = \overline{\sigma}(y) = \overline{\sigma}(x+y) = [t_0, t_1]$ and hence

$$\begin{aligned}\overline{T}_L\{\overline{\sigma}(x), \overline{\sigma}(y)\} &= rmax\{\overline{\sigma}(x) + \overline{\sigma}(y) - \overline{1}, \overline{0}\} \\ &= rmax\{[t_0, t_1] + [t_0, t_1] - \overline{1}, \overline{0}\} \\ &= rmax\{[t_0 + t_0 - 1, t_1 + t_1 - 1], \overline{0}\} \\ &= rmax\{[2t_0 - 1, 2t_1 - 1], \overline{0}\} \\ &= \{max[2t_0 - 1, 0], max[2t_1 - 1, 0]\} \\ &= \begin{cases}[2t_0 - 1, 2t_1 - 1] & : if [t_0, t_1] \\ \overline{0} & : if [t_0, t_1]\end{cases} \\ &= [t_0, t_1] \\ &= \overline{\sigma}(x+y).\end{aligned}$$

Similarly, $\overline{\sigma}(x-y) \geq \overline{T}_L\{\overline{\sigma}(x), \overline{\sigma}(y)\}$.

Case (ii) If $x, y \notin A$, then

$$\begin{aligned}&\overline{\sigma}(x) = \overline{\sigma}(y) = [s_0, s_1] \\ \overline{T}_L\{\overline{\sigma}(x), \overline{\sigma}(y)\} &= rmax\{\overline{\sigma}(x) + \overline{\sigma}(y) - \overline{1}, \overline{0}\} \\ &= rmax\{[s_0, s_1] + [s_0, s_1] - \overline{1}, \overline{0}\} \\ &= rmax\{[s_0 + s_0 - 1, s_1 + s_1 - 1], \overline{0}\} \\ &= rmax\{[2s_0 - 1, 2s_1 - 1], \overline{0}\} \\ &= \{max[2s_0 - 1, 0], max[2s_1 - 1, 0]\} \\ &= \begin{cases}[2s_0 - 1, 2s_1 - 1] & : if [s_0, s_1] \\ 0 & : if [s_0, s_1]\end{cases} \\ &= [s_0, s_1] \\ &= \overline{\sigma}(x+y).\end{aligned}$$

Similarly, $\overline{\sigma}(x-y) \geq \overline{T}_L\{\overline{\sigma}(x), \overline{\sigma}(y)\}$.

Case (iii) If $x \in A, y \notin A$, then

$$\begin{aligned}&\overline{\sigma}(x) = [t_0, t_1] \,\&\, \overline{\sigma}(y) = [s_0, s_1] \\ \overline{T}_L\{\overline{\sigma}(x), \overline{\sigma}(y)\} &= rmax\{\overline{\sigma}(x) + \overline{\sigma}(y) - \overline{1}, \overline{0}\} \\ &= rmax\{[t_0, t_1] + [s_0, s_1] - \overline{1}, \overline{0}\} \\ &= rmax\{[t_0 + s_0 - 1, t_1 + s_1 - 1], \overline{0}\} \\ &= \{max[t_0 + s_0 - 1, 0], max[t_1 + s_1 - 1, 0]\} \\ &= \begin{cases}[t_0 + s_0 - 1] & : if [s_0, s_1] \\ 0 & : if [s_0, s_1]\end{cases} \\ &= [t_0, t_1] \\ &= \overline{\sigma}(x+y).\end{aligned}$$

Similarly, $\overline{\sigma}(x-y) \geq \overline{T}_L\{\overline{\sigma}(x), \overline{\sigma}(y)\}$.
Hence $\overline{\sigma}$ is a $\overline{T}$-i_v_ fuzzy β −subalgebra of X.

Case (iv) Interchanging the roles of x & y in case (iii) prove $\overline{\sigma}$ is a $\overline{T}$ − i_v_ fuzzy β − subalgebra of X when $x \notin A, y \in A$.

Definition 3.13 Let $\overline{\sigma}_1$ & $\overline{\sigma}_2$ be two $\overline{T}$ − i_v_ fuzzy subset of any X. Then the $\overline{T}$ − product $\overline{\sigma}_1 \times_{\overline{T}} \overline{\sigma}_2$ of $\overline{\sigma}_1$ & $\overline{\sigma}_2$ is defined by $(\overline{\sigma}_1 \times_{\overline{T}} \overline{\sigma}_2) = \overline{T}\{\overline{\sigma}_1(x), \overline{\sigma}_2(x)\}\ \forall x \in X$.

Definition 3.14 A $\overline{t}$ −norm $\overline{T'}$ is said to dominate a $\overline{t}$ −norm $\overline{T}$ if

$$\overline{T'}(\overline{T}(\overline{a},\overline{b}),\overline{T}(\overline{c},\overline{d}) \geq \overline{T'}(\overline{T}(\overline{a},\overline{c}),\overline{T}(\overline{b},\overline{d}) \qquad \forall \overline{a},\overline{b},\overline{c},\overline{d} \in D[0,1].$$

Theorem 3.15 Let $\overline{\sigma}_1$ & $\overline{\sigma}_2$ be two $\overline{T}$ −i_v_ fuzzy β −subalgebras of X, if a $\overline{t}$ −norm $\overline{T_2}$ dominates $\overline{t}$ −norm $\overline{T_1}$,then the $\overline{T_2}$ −product $(\overline{\sigma}_1 \times_{\overline{T_1}} \overline{\sigma}_2)$ of $\overline{\sigma}_1$ & $\overline{\sigma}_2$ is a $\overline{T}$ −i_v_ fuzzy β −subalgebra of X.

Proof. For any $x, y \in X$,

$$\begin{aligned}(\overline{\sigma}_1 \underset{\overline{T_2}}{\times} \overline{\sigma}_2)(x+y) &= \overline{T_2}(\overline{\sigma}_1(x+y), \overline{\sigma}_2(x+y)) \\ &\geq \overline{T_2}(\overline{T_1}(\overline{\sigma}_1(x), \overline{\sigma}_1(y)), \overline{T_1}(\overline{\sigma}_2(x), \overline{\sigma}_2(y))) \\ &\geq \overline{T_1}(\overline{T_2}(\overline{\sigma}_1(x), \overline{\sigma}_1(y)), \overline{T_2}(\overline{\sigma}_2(x), \overline{\sigma}_2(y))) \\ &= \overline{T_1}((\overline{\sigma}_1 \underset{\overline{T_2}}{\times} \overline{\sigma}_2)(x), (\overline{\sigma}_1 \underset{\overline{T_2}}{\times} \overline{\sigma}_2)(y)) \\ (\overline{\sigma}_1 \underset{\overline{T_2}}{\times} \overline{\sigma}_2)(x-y) &= \overline{T_2}(\overline{\sigma}_1(x-y), \overline{\sigma}_2(x-y)) \\ &\geq \overline{T_2}(\overline{T_1}(\overline{\sigma}_1(x), \overline{\sigma}_1(y)), \overline{T_1}(\overline{\sigma}_2(x), \overline{\sigma}_2(y))) \\ &\geq \overline{T_1}(\overline{T_2}(\overline{\sigma}_1(x), \overline{\sigma}_1(y)), \overline{T_2}(\overline{\sigma}_2(x), \overline{\sigma}_2(y))) \\ &= \overline{T_1}((\overline{\sigma}_1 \underset{\overline{T_2}}{\times} \overline{\sigma}_2)(x), (\overline{\sigma}_1 \underset{\overline{T_2}}{\times} \overline{\sigma}_2)(y)).\end{aligned}$$

Hence $(\overline{\sigma}_1 \times_{\overline{T_2}} \overline{\sigma}_2)$ is a $\overline{T_1}$ −i_v_ fuzzy β −subalgebra of X.

Theorem 3.16 Let $f: X \to Y$ be an homomorphism of β −algebras. Let $\overline{\sigma}_1$ & $\overline{\sigma}_2$ be two $\overline{T}$ −i_v_ fuzzy β −subalgebras of Y. If a $\overline{t}$ −norms $\overline{T'}$ −i_v_ fuzzy dominates $\overline{T}$, then the inverse image of $\overline{T'}$ −i_v_ fuzzy product of $\overline{\sigma}_1$ & $\overline{\sigma}_2$ is same as the $\overline{T'}$ −i_v_ fuzzy product of inverse image of $\overline{\sigma}_1$ & $\overline{\sigma}_2$.

Proof. Let $\overline{\sigma}_1$ & $\overline{\sigma}_2$ be two $\overline{T}$ −i_v_ fuzzy β −subalgebras of Y. By theorem 3.15, the inverse image of $\overline{\sigma}_1$ & $\overline{\sigma}_2$,inverse image of $f^{-1}(\overline{\sigma}_1)$ & $f^{-1}(\overline{\sigma}_2)$ are $\overline{T}$ −i_v_ fuzzy β −subalgebra of X. Since $\overline{T'}$ dominates $\overline{T}$, by theorem 3.15
(1) $f^{-1}(\overline{\sigma}_1) \times_{\overline{T'}} f^{-1}(\overline{\sigma}_2)$ is a $\overline{T}$ −i_v_ fuzzy β −subalgebras of X.
(2) $\overline{\sigma}_1 \times_{\overline{T'}} \overline{\sigma}_2$ is a $\overline{T}$ −i_v_ fuzzy β −subalgebras of Y.
Hence by theorem 3.15, the inverse image of $f^{-1}(\overline{\sigma}_1 \times_{\overline{T'}} \overline{\sigma}_2)$is a $\overline{T}$ −i_v_ fuzzy β −subalgebras of X. Now for any $x \in X$,

$$\begin{aligned}f^{-1}(\overline{\sigma}_1 \underset{\overline{T'}}{\times} \overline{\sigma}_2)(x) &= (\overline{\sigma}_1 \underset{\overline{T'}}{\times} \overline{\sigma}_2)(f(x)) \\ &= \overline{T'}(\overline{\sigma}_1(f(x)), \overline{\sigma}_2(f(x))) \\ &= \overline{T'}(f^{-1}(\overline{\sigma}_1)(x), f^{-1}(\overline{\sigma}_2)(x)) \\ &= \left(f^{-1}(\overline{\sigma}_1) \underset{\overline{T'}}{\times} f^{-1}(\overline{\sigma}_2)\right)(x).\end{aligned}$$

Hence the inverse images of $\overline{T'}$ −i_v_ fuzzy product of $\overline{\sigma}_1$ & $\overline{\sigma}_2$ is same as the $\overline{T'}$ −i_v_ fuzzy product of inverse image of $\overline{\sigma}_1$ & $\overline{\sigma}_2$.

4. Conclusions

An investigation on interval valued $\overline{T}$ −normed fuzzy β −subalgebra is done and various captivating properties are observed. Further, the intersection between two $\overline{T}$-interval valued fuzzy β-subalgebra is studied. Moreover, some entrancing results on Cartesian product and the inverse image have been explored. In future, this can be extended in various algebraic structures.

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