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On an Open Problem in Complex Valued Rectangular b-Metric Spaces with an Application

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ABSTRACT

The purpose of the paper is to solve problem 1. Moreover, we prove fixed point theorems for contraction mappings in complete rectangular b-metrics and give examples as a satisfying the theorems in such spaces and give examples as a satisfying the theorems in rectangular b-metric spaces. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

Keywords: Fixed point; Contraction mapping; Rectangular b-metric spaces; Integral equation; Fredholm type

1. Introduction

In 2015, George et al. [1] established the concept of rectangular b-metric space as a generalization of metric space (MS) [2], rectangular metric space (RMS) [3] and b-metric space (bMS) [4].

In the same year, Ege [6] established the complex valued rectangular b-metric space (CRbMS) as a generalization of a complex valued metric space (CMS) [5] and rectangular b-metric space (RbMS) [1] and proved an analogue of Banach contraction principle. Author also proved a different contraction principle with a new condition and a fixed point theorem in this space. Finally, author gave an application of Banach contraction principle to linear equations.

The complex metric space was initiated by Azam et al. [5]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$.

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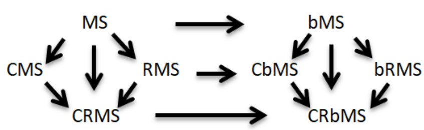
Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
$(C_1) Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$;
$(C_2) Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$;
$(C_3) Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$;
$(C_4) Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$.

Particularly, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C_2), (C_3)$ and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:
(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \precsim bz$ for all $z \in \mathbb{C}$.
(2) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.
(3) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.1 ([6]). Let X be a nonempty set and the mapping $d : X \times X \to \mathbb{C}$ satisfies:
(i) $0 \precsim d(x, y)$ for all $x = y$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iv) there exists a real number $s \geq 1$ such that $d(x, y) \precsim s[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$, and all distinct points $u, v \in X \backslash \{x, y\}$. Then d is called a complex valued rectangular b-metric on X and (X, d) is called a *complex valued rectangular b-metric space (in short CRbMS)* with coefficient .



Note that every complex valued space is a complex valued rectangular b-metric space with coefficient $s = 1$.

Example 1.2 ([6]). Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \mathbb{Z}^+$ and $d : X \times X \to \mathbb{C}$ be defined as follows: $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2t, & \text{if } x, y \in A; \\ \frac{t}{2n}, & \text{if } x \in A \text{ and } y \notin \{2,3\}, \\ t, & \text{otherwise,} \end{cases}$$

where $t > 0$ is a constant. Then (X, d) is a complex valued rectangular b-metric space with coefficient $s = 2$, but (X, d) is not a complex valued rectangular metric space.

Now, we review definition in complex valued rectangular b-metric spaces as follows:

Definition 1.3 ([6]). Let (X, d) be a complex valued rectangular b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) The sequence $\{x_n\}$ is said to be *complex valued convergent* in (X, d) and converges to x, if for every $\epsilon \succ 0$ there exists $n_0 \in N$ such that $d(x_n, x) \prec \epsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

(ii) The sequence $\{x_n\}$ is said to be *complex valued Cauchy sequence* in X if for every $\epsilon \succ 0$ there exists $n_0 \in N$ such that $d(x_n, x_{n+p}) \prec \epsilon$ for all $n > n_0,\ p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(iii) X is said to be a *complete complex valued rectangular b-metric space* if every Cauchy sequence in X converges to some $x \in X$.

Lemma 1.4 ([6]). *Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.5 ([6]). *Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then*

$\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

The main result in paper [6] is the following theorem (The Banach contraction principle theorem in complex valued rectangular b-metric spaces).

Theorem 1.6. *Let (X, d) be a complex valued complete rectangular b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \precsim \alpha d(x, y) \tag{1.1}$$

for all $x, y \in X$, where $\alpha \in [0, \frac{1}{s})$. Then T has a unique fixed point.

In 2017, Mitrović [7], improved $[0, \frac{1}{s})$ to $[0, 1)$ for the Banach contraction principle theorem of George et al.[1] in rectangular b-metric spaces.

The above results naturally bring us to the following open problem.

Open Problem 1. In Theorem 1.6, we can extend the range of α to the case $\alpha \in [\frac{1}{s}, 1)$?

The purpose of this paper is to give some affirmative answers to the questions raised. We prove fixed point theorems for contraction mappings in complete rectangular b-metrics and give examples extend the theorems in such spaces. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

2. Main Results

In this section, we prove a fixed point theorem for contraction mappings in complete rectangular b-metric space and give an example that satisfies main theorem in such spaces.

Theorem 2.1. *Let (X, d) be a complex valued complete rectangular b-metric space. Suppose that $T : X \to X$ is a mapping satisfying: There exists constant α with $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \precsim \alpha d(x, y) \tag{2.1}$$

for all $x, y \in X$. Then T has a unique fixed point. Moreover, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, converges to fixed point for any $x_0 \in X$, where $n = 0, 1, 2, \ldots$.

Proof. Let $\alpha \in [0, 1)$. Since $\lim_{n\to\infty} \alpha^n = 0$, there exists a natural number k_0 such that

$$0 < \alpha^k s < 1, \tag{2.2}$$

for all $k > k_0$. Let x_0 be a arbitrary in X such that $Tx_0 = x_1 \in X$. Define a sequence $\{x_n\}$ by $x_n = Tx_{n-1} \in X$, where $n = 1, 2, \ldots$. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Then $x_n \neq x_{n+k}$ for all $n \geq 0$, $k \geq 1$. Namely, if $x_n = x_{n+k}$ for some $n \geq 0$ and $k \geq 1$ we have that $Tx_n = Tx_{n+k}$ and $x_{n+1} = x_{n+k+1}$. By (2.1), we have

$$\begin{aligned} d(x_{n+q}, x_{m+q}) &= d(Tx_{n+q-1}, Tx_{m+q-1}) \\ &\precsim \alpha d(x_{n+q-1}, x_{m+q-1}) \\ &= \alpha d(Tx_{n+q-2}, Tx_{m+q-2}) \\ &\precsim \alpha^2 d(x_{n+q-2}, x_{m+q-2}) \\ &\precsim \vdots \\ &\precsim \alpha^q d(x_n, x_m), \quad n, m, q \in \mathbb{N}. \end{aligned} \tag{2.3}$$

Similarly, we get for any $n, r \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_{n+r}) &= d(Tx_{n-1}, Tx_{n+r-1}) \\ &\precsim \alpha d(x_{n-1}, x_{n+r-1}) \\ &\precsim \alpha^2 d(x_{n-2}, x_{n+r-2}) \\ &\precsim \vdots \\ &\precsim \alpha^n d(x_0, x_r). \end{aligned} \tag{2.4}$$

Since $0 \leq \alpha < 1$, we get $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. We next prove that $\{x_n\}$ is a Cauchy sequence by letting $m, n \in \mathbb{N}$ with $m, n > k_0$,

$$\begin{aligned} d(x_n, x_m) &\precsim s[d(x_n, x_{n+k_0}) \\ &\quad + d(x_{n+k_0}, x_{m+k_0}) + d(x_{m+k_0}, x_m)] \\ &\precsim s[\alpha^n d(x_0, x_{k_0}) \\ &\quad + \alpha^{k_0} d(x_n, x_m) + \alpha^m d(x_{k_0}, x_0)]. \end{aligned} \tag{2.5}$$

It follows that

$$|d(x_n, x_m)| \leq \frac{s\alpha^n + s\alpha^m}{1 - \alpha^{k_0}} |d(x_0, x_{k_0})|. \tag{2.6}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of a complete rectangular b-metric space (X, d), there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$.
Next, we will show that p is a fixed point of T. Let $n \in \mathbb{N} \bigcup \{0\}$. We have

$$\begin{aligned} d(p, Tp) &\preceq s[d(p, x_n) + d(x_n, x_{n+1}) \\ &\quad + d(x_{n+1}, Tp)] \\ &\preceq s[d(p, x_n) + d(x_n, x_{n+1}) \\ &\quad + d(Tx_n, Tp)] \\ &\preceq s[d(p, x_n) + d(x_n, x_{n+1}) \\ &\quad + \alpha d(x_n, p)]. \end{aligned} \tag{2.7}$$

Taking limit as $n \to \infty$ in (2.7), we get $p = Tp$. Thus p is a fixed point of T. To prove uniqueness, suppose that there exists $p^* \in X$ such that $p^* = Tp^*$. We consider

$$\begin{aligned} d(p, p^*) &= d(Tp, Tp^*) \\ &\preceq \alpha d(p, p^*), \end{aligned} \tag{2.8}$$

which implies

$$|d(p, p^*)| \leq \alpha |d(p, p^*)|, \tag{2.9}$$

and then $p = p^*$. So, the Picard iteration $\{x_n\}_{n=0}$ defined by $x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$ converges to p for any $x_0 \in X$. □

We give examples in order to validate the proved result.

Example 2.2. Let $X = \mathbb{R}$ with $d(x, y) = |x - y|^2 + i|x - y|^2$. Let $T : X \to X$ be given by

$$Tx = \begin{cases} \frac{x^2}{3} & \text{if } x, y \in [-1, 1]; \\ \frac{3x}{4}, & \text{if } x, y \in X \setminus [-1, 1]. \end{cases}$$

Then (X, d) is a complete CRbMS with coefficient $s = 2$. Let $x, y \in X$. Now, we consider, for any $x, y \in X \setminus [-1, 1]$,

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty|^2 + i|Tx - Ty|^2 \\ &= \frac{9}{16}(|x - y|^2 + i|x - y|^2) \\ &\precsim \alpha d(x, y), \end{aligned} \tag{2.10}$$

where $\alpha = \frac{9}{16}$ and $\alpha \in [\frac{1}{s}, 1)$. If $x, y \in [-1, 1]$, $|x| \neq 1$ and $|y| \neq 1$ then

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty|^2 + i|Tx - Ty|^2 \\ &= (|x^2 - y^2|^2 + i|x^2 - y^2|^2) \\ &= \frac{|x + y|^2}{9}(|x - y|^2 + i|x - y|^2) \\ &\precsim \frac{|x + y|^2}{|x + y|^2 + 5}(|x - y|^2 + i|x - y|^2) \\ &= \alpha d(x, y), \end{aligned} \tag{2.11}$$

where $\alpha = \frac{|x+y|^2}{|x+y|^2+5}$ and $\alpha \in [\frac{1}{s}, 1)$, which implies that T has a unique fixed point $0 \in X$.

3. Application

In this section, we endeavor to apply Theorem 2.1 to prove the existence and uniqueness of solution of the following integral equation of Fredholm type:

$$x(t) = \int_a^b G(t, s, x(s))ds + h(t) \tag{3.1}$$

for $t, s \in [a, b]$ where, $G, h \in C([a, b], \mathbb{R})$ (say $X = C([a, b], \mathbb{R})$ Define $d : X \times X \to$

$\mathbb{C}$ by $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2 + i\sup_{t\in[a,b]} |x(t) - y(t)|^2$ for all $x, y \in X$, Then, (X, d) is a complete extended rectangular b-metric space, see example 2.2. Now, we are equipped to state and prove our result as follows:

Theorem 3.1. *For all* $x, y \in X := C([a,b], \mathbb{R})$,

$$G(t,s,x(t),G(t,s,x(y))) \leq \frac{1}{2(b-a)} \tag{3.2}$$

for all $t, s \in [a,b]$. *Then the integral equation (3.1) has a unique solution.*

Proof. Define $T : X \to X$ by $Tx(t) = G(t,s,x(s))ds + h(t)$ for all $t, s \in [a,b]$ It is clear that, x is a fixed point of the operator T if and only if it is a solution of the integral equation For all $x, y \in X$, we get

$$\begin{aligned}
&|fx(t) - fy(t)|^2 \\
&\leq \int_a^b |G(t,s,x(s)) - G(t,s,y(s))|ds \\
&\leq \int_a^b \frac{1}{2(b-a)}|x(s) - y(s)|ds \\
&\leq \frac{1}{4(b-a)^2} \sup_{t\in[a,b]} (\int_a^b ds)^2, \qquad (3.3)
\end{aligned}$$

then $d(fx(t) - fy(t)) \to 0$ as $n \to \infty$ with $\frac{1}{2(b-a)} \in [0,1]$. Hence, the operator T has a unique fixed point, that is, the Fredholm integral Equation (3.1) has a unique solution

□

4. Conclusion

The purpose of this paper is to give some affirmative answers to the questions raised. We proved fixed point theorems for contraction mappings in complete rectangular b-metrics and gave examples that satisfy the theorems in such spaces. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equation:

Let (X, d) be a complex valued complete rectangular b-metric space. Suppose that $T : X \to X$ is a mapping satisfying: There exists constants α with $\alpha \in [0,1)$ such that

$$d(Tx,Ty) \precsim \alpha d(x,y) \tag{4.1}$$

for all $x, y \in X$. Then T has a unique fixed point and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, converges to fixed point for any $x_0 \in X$, where $n = 0, 1, 2, \ldots$.

$$x(t) = \int_a^b G(t,s,x(s))ds + h(t) \tag{4.2}$$

for $t, s \in [a,b]$ where, $G, h \in C([a,b], \mathbb{R})$ (say $X = C([a,b], \mathbb{R})$ For all $x, y \in X := C([a,b], \mathbb{R})$,

$$G(t,s,x(t),G(t,s,x(y))) \leq \frac{1}{2(b-a)} \tag{4.3}$$

for all $t, s \in [a,b]$. Then the integral equation 3.1 has a unique solution.

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