Songklanakarin J. Sci. Technol.
44 (1), 201-209, Jan. - Feb. 2022

http://www.sjst.psu.ac.th

Original Article

Mertens' Theorem for some certain intermediate growths

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Received: 22 April 2021; Revised: 24 May 2021; Accepted: 28 June 2021

Abstract

The dynamical Mertens' Theorem gives asymptotics for weighted averages of numbers of closed orbits in *S*-integer dynamical systems, which are constructed from arithmetic data, namely $K = \mathbb{Q}$, $\xi = 2$, and *S* a subset of rational primes. The way of finding such asymptotic expressions is an analogue of Mertens' Theorem in analytic number theory. In this article, we focus on the dynamical Mertens' Theorem of some certain growths in case *S* and its complement are infinite subsets of all rational primes. More precisely, our aim is to find out a leading coefficient which comes from the main term in the asymptotic expression of our interested setting. Moreover, the real interval covering such a coefficient will be provided in some certain examples.

Keywords: prime numbers, closed orbits, periodic points, Mertens' Theorem, *S*-integer dynamical systems

1. Introduction

In 1874, the Polish-Austrian mathematician, Franciszek Mertens (Mark, 2021) studied the sum of the reciprocals of the prime numbers which presents the weighted averages of numbers of prime numbers. He published the famous theorem on the sum as follows.

Theorem 1.1. Let $x \geq 1$ be any real number. Then

$$\sum_{p\leq x}\frac{1}{p} = \log\log\lfloor x\rfloor + \gamma + \sum_{m=2}^{\infty}\mu(m)\frac{\log\{\zeta(m)\}}{m} + \delta, \tag{1}$$

where γ is the *Euler's constant*, μ is the *Mobius function*, ζ is the *Riemann zeta function*, $\lfloor x\rfloor$ is the largest integer less than or equal to x and

$$\delta < \frac{4}{\log(\lfloor x\rfloor + 1)} + \frac{2}{\lfloor x\rfloor\log\lfloor x\rfloor}.$$

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Alternatively, an asymptotic expression of (1) may be illustrated like

$$\sum_{p\leq x}\frac{1}{p} = \log\log x + B + O\left(\frac{1}{\log x}\right)$$

or more simply presented as

$$\sum_{p\leq x}\frac{1}{p} \sim \log\log x + B,$$

where p runs over all primes with $p \leq x$ and

$$B = \gamma + \sum_{p}\left\{\log(1-\frac{1}{p}) + \frac{1}{p}\right\},$$

called the *Mertens' constant*. Such expressions are called the *Mertens' Theorem in analytic number theory* (or it may be called as the *classical Mertens' Theorem*), which is actually an asymptotic expression for weighted averages of the number of prime numbers.

Now, let us give the meaning of the symbols O and $\sim$, which have just appeared previously and the first notation called the *Big Oh* will be mainly used from now on. For any functions f and g on $\mathbb{R}$, we set the meaning of such symbols as follows:

1. $f = O(g)$ means that there exists $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \geq 0$; that is, the ratio $\frac{f(x)}{g(x)}$ stays bounded as $x \to \infty$.
2. $f \sim g$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$; that is, the function $f(x)$ and $g(x)$ are very closed together when $x \to \infty$.

Let α be a continuous map from a compact metric space (X, d) onto itself. A closed orbit τ of length $|\tau| = n$ for such a map is a set of the form

$$\{x, \alpha(x), \alpha^2(x), \alpha^3(x), \ldots, \alpha^n(x) = x\},$$

where n is the smallest non-nagative integer such that $\alpha^n(x) = x$, when x is arbitrarily chosen in X. We write $O_\alpha(n)$ for the number of closed orbits of length n. The *dynamical Mertens' Theorem*, which is an analogue of the classical one giving asymptotics for weighted averages of the number of closed orbits concerns the expression like

$$\mathcal{M}_\alpha(N) = \sum_{|\tau|\leq N} \frac{1}{e^{h(\alpha)|\tau|}}, \tag{2}$$

where $h(\alpha)$ denotes the topological entropy of the map and its more detail may be seen in Bowen (1973); Sawian (2010). This is indeed an analogue of the classical one. Note that this analogy has been arisen because of a prime number and a closed orbit. Another applicable form of the dynamical Mertens' Theorem is written as

$$\mathcal{M}_\alpha(N) = \sum_{1\leq n\leq N} \frac{O_\alpha(n)}{e^{h(\alpha)n}}, \tag{3}$$

which comes from the meaning of $O_\alpha(n)$.

In 1991, results about the asymptotic behaviour of (3) were studied by Richard (1991) and others (William, 1983; William & Mark, 1983) in case α is a hyperbolic diffeomorphism. They showed that

$$\mathcal{M}_\alpha(N) \sim \log N + C \tag{4}$$

for some constant C. The main term in (4) will be significantly changed if we do not consider in the case of hyperbolicity. The simplest non-hyperbolic systems in group automorphisms are those constructed using arithmetic data, namely $K = \mathbb{Q}$, $\xi = 2$, and rings of S-integers (denoted as R_S), where S is a subset of all rational primes. For more detail see Graham, Richard, Shaun and Tom (2007), (2010); Sawian (2010); Vijay, Everest and Ward (1997). It roughly says that, if X is the character group of

$$R_S = \left\{\frac{a}{b} \in \mathbb{Q} \mid \text{prime dividing } b \text{ lie in } S\right\},$$

then we obtain that $\alpha: X \to X$ is the homomorphism dual to the map $x \to 2x$ on the ring R_S. Such a map is called an *S-integer dynamical map*, which is the simplest one-dimensional case. If S is empty, we get the simplest specific map in this line called the *circle doubling map*. We naturally classify an S-integer dynamical map into 3 types depending on the cardinality of S: i) S is finite, ii) S is infinite, but its complement is finite and iii) S and its complement are infinite. From now on, α always means an S-integer dynamical map corresponding to a subset of all rational primes S, and its entropy $h(\alpha)$ is equal to $\log 2$.

In Graham *et al.* (2007), they considered in the case S is finite and illustrated that there exist an explicit rational leading coefficient k_S and a constant C_S for which

$$\mathcal{M}_\alpha(N) \sim k_S \log N + C_S.$$

More refinement can be illustrated in the following theorem.

Theorem 1.2. Let $\alpha: X \to X$ be an S- integer map, where X is connected and S is finite. Then there are constants $k_S \in \mathbb{Q}$, C_S and $\delta > 0$ for which

$$\mathcal{M}_\alpha(N) = k_S \log N + C_S + O(N^{-\delta}).$$

If S is a co-finite set of primes (S is infinite, but its complement is finite), then $O_\alpha(n)$ is polynomial bounded implying that its orbit growth is very slow. This makes (2) or (3) not interesting. However, in 2010, the author has found another suitable form in order to get an asymptotic expression of such a form (Sawian, 2011). In this paper, we study some certain examples of the intermediate growth meaning that the set *S* and also its complement are considered as being infinite. It additionally, says that the number of such a set is huge. It turns out that not only can we provide a leading coefficient arising from the constructed system, but also we are able to restrict such a coefficient to some real interval. The main approach how to obtain this bound is to apply the recipe given the author in Theorem 5.9 (Sawian, 2011). So, the next section will be again given about this approach in order to complete Section 3. later.

2. Background

Regarding (3), the non-negative integer sequence $O_\alpha(n)$ will play a major role in our setting. To find out its formula, we first denote $\mathbb{N}^0$ and α^0 to be the set of all natural numbers including 0 and the identity map on X, respectively. For any dynamical system (X,α), define

$$\mathscr{L}_\alpha(n) = \{x \in X \mid \#\{\alpha^k(x)\}_{k\in\mathbb{N}^0} = n\},$$

$$\mathcal{F}_\alpha(n) = \{x \in X \mid \alpha^n(x) = x\} \text{ and}$$

$$\mathcal{O}_\alpha(n) = \{\tau \mid \tau \text{ is closed orbits of length } n\}$$

to be the set of points of least period n under α, the set of points of period n under α, and the set of orbits of length n under α, respectively. We next write

$$L_\alpha(n) = |\mathcal{L}_\alpha(n)|,\ F_\alpha(n) = |\mathcal{F}_\alpha(n)| \text{ and } O_\alpha(n) = |\mathcal{O}_\alpha(n)|$$

as the number of points of least period n under α, the number of points of period n under α and the number of orbits of length n under α, respectively. Since $\mathcal{F}_\alpha(n)$ is a disjoint union of $\mathcal{L}_\alpha(d)$ for every natural number d dividing n, we immediately obtain that $F_\alpha(n) = \sum_{d|n} L_\alpha(d)$. It's not hard to see that $L_\alpha(n) = nO_\alpha(n)$ for any natural number n. Then the relation between $O_\alpha(n)$ and $F_\alpha(n)$ are subsequently derived in general as

$$O_\alpha(n) = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) F_\alpha(d) \tag{5}$$

by applying the Mobius inversion formula together with the two equations earlier.
In our setting, the equation (5) is consequently illustrated as

$$O_\alpha(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) |2^d - 1||2^d - 1|_S, \tag{6}$$

since we have particularly known that $F_\alpha(n) = |2^n - 1||2^n - 1|_S$ for any natural number n. For more detail, see in Theorem 5.2 (Sawian, 2010). Here let us clarify the notation appearing in (6) as follows: for each natural number t, we set $|t|_S = \prod_{p\in S}|t|_p$, where $|t|_p = p^{-m}$, where m is the highest power of p dividing t. Plugging the formula (6) into (3), we roughly yield the expression of the dynamical Mertens' Theorem in our certain way written as

$$\mathcal{M}_\alpha(N) = \underbrace{\sum_{n\le N} \frac{|2^n - 1|_S}{n}}_{F_S(N)} + R_S(N), \tag{7}$$

where the error term $R_S(N) = B_S + O\left(2^{-\frac{N}{2}}\right)$ for some constant B_S depending on *S* and $F_S(N)$ will be again written in terms of $\log N$ whose coefficient is viewed as the leading coefficient, and some more error terms. The exploration about this can be studied more in Graham *et al.* (2007), (2010); Sawian, (2010); Vijay *et al.* (1997). This leads us to focus only the term $F_S(N)$ in order to get its coefficient.

The tool is to find out the leading coefficient in case *S* is finite can be seen in Proposition 5.3 appearing in Graham *et al.* (2007) when $K = \mathbb{Q}$ particularly. That is, expanding and calculating the following formula

$$F_S(N) = \sum_{T\subseteq S} \sum_{\substack{n\le N, o_T|n, \\ m_p\nmid n\ \forall p\notin T}} \frac{|2^n - 1|_T}{n}, \tag{8}$$

where m_p is the multiplication order 2 modulo p and $o_T = \text{lcm}\{m_p \mid p \in T\}$ for any $T \subseteq S$ lead us to eventually obtain such a coefficient. However, the calculation will become much more complicated and will spend long time when S is getting large. Here, we are going to propose the faster formula in order to obtain such a coefficient in case S is finite based on the recipe in the original formula in (8). More precisely, we write $F_{S_k}(N)$ in terms of $F_{S_{k-1}}(N)$, where S_k is a set of k distinct primes for any natural number k by rearranging the term $F_{S_k}(N) := F_S(N)$ in (8) as illustrated in the following theorem.

Theorem 2.1. Let S_k be a set of k distinct primes, $k \geq 1$ and $S_0 = \emptyset$. Write $S_k = \{p_1, p_2, \dots, p_k\} = S_{k-1} \cup \{p_k\}$, where $p_1 < p_2 < \cdots < p_k$ and for any $T \subseteq S_{k-1}$, set $T' = T \cup \{p_k\}$. Then

$F_{S_k}(N) = F_{S_{k-1}}(N) - E_{S_{k-1}}(N)$, where $E_{S_{k-1}}(N)$ is written as

$$\sum_{T \in G_{S_{k-1}}} \frac{|2^{o_T} - 1|_T}{o_{T'}} \left(\sum_{t \leq N/o_{T'}} \frac{|t|_T}{t} - |o_{T'}|_{p_k} |2^{m_{p_k}} - 1|_{p_k} \sum_{t \leq N/o_{T'}} \frac{|t|_{T'}}{t} \right)$$ and

$$G_{S_{k-1}} = \{T \subseteq S_{k-1} \mid \forall p \in S_k \backslash T',\ m_p \nmid o_{T'}\}.$$

Let $k_S = k_{\alpha,S}$ denote the leading coefficient of $\log N$ in the sum $F_S(N)$ for any finite set of primes S. Then we have $k_{S_k} < k_{S_{k-1}}$ under the same situation appearing in Theorem 2.1. The proofs of this fact and also Theorem 2.1 can be seen in page 102-105 (Sawian, 2010).

In order to clarify why this theorem has arisen, we need to provide some explicit examples. Before doing this, the following facts that will play a crucial role in our work are introduced, and their proofs can be found in (Tom, 1976; Sawian, 2016).

Lemma 2.2. For $N \geq 1$, we have

$$\sum_{n \leq N} \frac{1}{n} = \log N + \gamma + O(1/N).$$

Lemma 2.3. For any finite subset T of rational primes, we have

$$\sum_{n \leq N} \frac{|n|_T}{n} = K_T \log N + C_T + O(1/N),$$
where $K_T = \prod_{p \in T} \frac{p}{p+1}$ and $C_T = K_T \left(\gamma - \sum_{p \in T} \frac{p \log p}{p^2 - 1} \right)$.

In fact, the proofs of the following examples are given in (Sawian, 2010), but for the sake of completeness for understanding more about our doing here, we need to show them again.

Example 2.4. Let $\alpha: X \to X$ be the S-integer dynamical systems dual to $x \to 2x$ with X connected and $S = S_1 = \{3\}$. Then $k_S = \frac{5}{8}$ and moreover, we have

$$F_S(N) = \frac{5}{8} \log N + C_1 + O(1/N)$$
for some constant C_2 depending on S.

Proof. We note that $m_3 = 2$ and $G_\emptyset = \{\emptyset\}$. By Theorem 2.1, we obtain that

$$\begin{aligned} F_{S_1}(N) &= F_\emptyset(N) - E_\emptyset(N) \\ &= \sum_{n \leq N} \frac{1}{n} - \frac{1}{2} \left(\sum_{n \leq N/2} \frac{1}{n} - \frac{1}{3} \sum_{n \leq N/2} \frac{|n|_3}{n} \right) \\ &= \left(1 - \frac{1}{2} + \frac{3}{24} \right) \log N + C + O\left(\frac{1}{N} \right) \end{aligned}$$
by applying Lemma 2.2 and Lemma 2.3, respectively.

Example 2.5. Let $\alpha: X \to X$ be the S-integer dynamical systems dual to $x \to 2x$ with X Connected and $S = S_2 = \{3,5\}$. Then $k_S = \frac{55}{96}$ and moreover, we have

$$F_S(N) = \frac{55}{96}\log N + C_2 + O(1/N).$$

for some constant C_2 depending on S.

Proof. We note that $m_3 = 2, m_5 = 4$ and $G_{S_1}=\{\{3\}\}$. By applying Theorem 2.1 with $S_1 = \{3\}$ and $S_2 = \{3,5\}$, we obtain that

$$\begin{aligned} F_{S_2}(N) &= F_{S_1}(N) - E_{S_1}(N) \\ &= \sum_{n \le N} \frac{|2^n - 1|_3}{n} - \frac{1}{4 \times 3}\left(\sum_{n \le N/4} \frac{|n|_3}{n} - \frac{1}{4 \times 3 \times 5} \sum_{n \le N/4} \frac{|n|_{\{3,5\}}}{n} \right) \\ &= \left(\frac{5}{8} - \frac{1}{12}\left(\frac{3}{4}\right) + \frac{1}{60}\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \right) \log N + C_2 + O\left(\frac{1}{N}\right) \end{aligned}$$

by using the previous example and applying Lemma 2.3.

3. The Certain Intermediate Example

Recall that we are focusing on the S-integer dynamical systems, says α constructed via the data, namely $K = \mathbb{Q}, \xi = 2$, and S is a subset of all rational primes. Indeed, many certain intermediate examples arising from such a map have been studied in Steven, Sawian, Shaun and Tom (2013). Let a_n be an integer sequence. For each positive integer n, a primitive prime divisor of a_n is the prime number p that divides a_n, but for every natural number m<n, a_m can not be divided by p. The well known fact named *Zsigmondy's theorem* appearing in (Karl, 1892) states that the Mersenne sequence $2^n - 1$ always has a primitive prime divisor except $n = 1,6$. It is a consequence that the primitive prime divisor of the sequence $2^{2^n} - 1$ always exists for any natural number n. This leads us to define p_n as the greatest primitive prime divisor of the sequence $2^{2^n} - 1$. Setting

$S = \{ p_n \mid n \in \mathbb{N}\} = \{3, 5, 17, 257, \ldots\}$,

which here is one set of such a certain intermediate example. Note that $m_{p_n} = 2^n$ for any natural number n. Fixed a natural number N, let

$$S(N) = \{p_n \in S \mid m_{p_n} \le N\}$$

$$\begin{aligned} S(2^N) &= \{p_n \in S \mid 2^n \le 2^N\} \\ &= \{p_1, p_2, p_3, \ldots, p_N\}. \end{aligned}$$

Thus, $|S(2^N)| = N$, which implies that $|S(N)| = \lfloor \log_2 N \rfloor$. Consequently, we write $S(N) = \left\{p_1, p_2, p_3, \ldots, p_{\lfloor \log_2 N \rfloor}\right\}$.

For each $i = 0,1,2, \ldots, \lfloor \log_2 N \rfloor - 1$, we generally set

$$S(N-i) = \left\{p_1, p_2, p_3, \ldots, p_{\lfloor \log_2 N \rfloor - i}\right\}.$$

Importantly, for each $T \subseteq S(N-i)$, we have $O_{T'} = 2^{\lfloor \log_2 N \rfloor - i}$, where $T' = T \cup \{p_{\lfloor \log_2 N \rfloor - i}\}$. Then the following lemma is immediately derived.

Lemma 3.1 Given a natural number N, we have

$$G_{S(N-i)} = \{S(N-i)\}$$

for all $i = 0,1,2, \ldots, \lfloor \log_2 N \rfloor - 1$.

Lemma 3.2 Given a natural number N, we have $F_S(N) = F_{S(N)}(N)$.

Proof. Let N and n be natural numbers such that $n \ge \lfloor \log_2 N \rfloor + 1$ and $n \le N$. We intend to show that $|2^n - 1|_S = |2^n - 1|_{S(N)}$. By Lemma 3.22 (Sawian, 2010), we have

$$|2^n-1|_p=\begin{cases}|n|_p|2^{m_p}-1| & \text{if } m_p|n\\ 1 & \text{if } m_p\nmid n\end{cases} \tag{9}$$

for every prime *p*. It follows that

$$2^{p_n}>2^n\geq 2^{\lfloor\log_2 N\rfloor+1}>2^{\log_2 N}=N\geq n,$$

which implies that $2^{p_n}\nmid n$ or $m_{p_n}\nmid n$. By the fact in (9), we get $|2^n-1|_{p_n}=1.$
Consequently, we can conclude that

$$|2^n-1|_S=|2^n-1|_{S(N)}\underbrace{\prod_{i=1}^{\infty}|2^n-1|_{p_{\lfloor\log_2 N\rfloor+i}}}_{=1}$$

which finally leads to the completion of our intention. Hence, the proof of this lemma is finished.

To be conveniently from here on, we shall again give some notations as follows: for each natural number *i*, let $T_i=\{p_j\mid j=1,2,\dots,i\}$, $\quad K_{T_i}=\prod_{p\in T_i}\frac{p}{p+1}$

and $\epsilon_i\geq 1$ is the highest power of p_i dividing $2^{O_{T_i}}-1$. Additionally set $T_0=\emptyset$, and notice that $|T_i|=i$ and $O_{T_i}=2^i$. To be shortly, we moreover set $\mathcal{P}_i=p_1^{\epsilon_1}p_2^{\epsilon_2}\dots p_i^{\epsilon_i}$ and $\mathcal{P}_0=1$, and fix a natural number N, setting

$$\mathcal{K}_N=\sum_{i=\lfloor\log_2 N\rfloor}^{\infty}\frac{K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)$$

$$\mathcal{J}_N=\sum_{i=\lfloor\log_2 N\rfloor}^{\infty}\frac{iK_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)$$

and

$$\mathcal{L}_N=\sum_{i=\lfloor\log_2 N\rfloor}^{\infty}\frac{(i+1)K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right).$$

Lemma 3.3 For each natural number N, we have $\mathcal{K}_N=O(1/N)$, $\mathcal{J}_N=O(\log N/N)$ and $\mathcal{L}_N=O(\log N/N)$.

Proof. Since the series $\sum_{i=0}^{\infty}\frac{1}{2^i}$ and $\sum_{i=0}^{\infty}\frac{i}{2^i}$ converge, there exist constants C_3 and C_4 for which

$$|\mathcal{K}_N|\leq\sum_{i=\lfloor\log_2 N\rfloor}^{\infty}\frac{1}{2^i}=\frac{1}{2^{\lfloor\log_2 N\rfloor}}\sum_{i=0}^{\infty}\frac{1}{2^i}\leq\frac{C_3}{N}$$

and

$$|\mathcal{J}_N|\leq\sum_{i=\lfloor\log_2 N\rfloor}^{\infty}\frac{i}{2^i}=\frac{\lfloor\log_2 N\rfloor}{2^{\lfloor\log_2 N\rfloor}}\sum_{i=0}^{\infty}\frac{i}{2^i}\leq\frac{C_4\log N}{N}.$$

Thus, $\mathcal{K}_N=O(1/N)$ and $\mathcal{J}_N=O(\log N/N)$, which leads us to reach $\mathcal{L}_N=O(\log N/N)$ as $\mathcal{L}_N=\mathcal{J}_N+\mathcal{K}_N$.

Lemma 3.4 The series

$$\sum_{i=0}^{\infty}\frac{K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right) \tag{10}$$

and

$$\sum_{i=0}^{\infty}\frac{(i+1)K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right) \tag{11}$$

are convergent, says that they are equal to some real number C_S^* and B_S^*, respectively. Moreover, we get $0.43\leq C_S^*\leq 0.56.$

Proof. We are able to set the series in (10) and (11) as real numbers C_S^* and B_S^*, respectively as they are bounded by the convergent series $\sum_{i=0}^{\infty}\frac{i}{2^i}$. Let us consider

$$C_S^* = \frac{1}{2}\left(1-\frac{3}{3\times 4}\right)+\frac{3}{2^2\times 4\times 3}\left(1-\frac{5}{5\times 6}\right)+\frac{3\times 5}{2^3\times 4\times 6\times 3\times 5}\left(1-\frac{17}{17\times 18}\right)$$
$$+\underbrace{\sum_{i=3}^{\infty}\frac{K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)}_{\leq\frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2^i}=\frac{1}{8}}.$$

This implies that $0.43 \leq C_S^* \leq 0.56.$
Now, we are ready to show the main result.

Theorem 3.5 For each natural number N, we have

$$F_S(N) = k_S\log N + k_S\gamma - B_S^*\log 2 + O(\log N/N),$$

where $k_S = 1 - C_S^*$. Moreover, we get $0.44 \leq k_S \leq 0.57.$

Proof. Fix $N \geq 1$, and notice that $F_S(N) = F_{S(N)}(N)$ by referring Lemma 3.2. Applying the recipe appearing in Theorem 2.1 N times to the finite sets $S(N), S(N-1), S(N-2), \ldots, S(1)$, in order together with the fact in Lemma 3.1, we eventually reach the rearranged formula as

$$F_S(N) = F_\emptyset(N) - \sum_{i=0}^{\lfloor\log_2 N\rfloor-1}\frac{1}{2^{i+1}\mathcal{P}_i}\left(\underbrace{\sum_{t\leq N/2^{i+1}}\frac{|t|_{T_i}}{t}}_{A} - \frac{1}{p_{i+1}^{\epsilon_{i+1}}}\underbrace{\sum_{t\leq N/2^{i+1}}\frac{|t|_{T_{i+1}}}{t}}_{B}\right)$$

For each $i = 0,1,2,\ldots,\lfloor\log_2 N\rfloor - 1$, we can obtain due to Lemma 2.3 that

$$A = K_{T_i}\log N - (i+1)K_{T_i}\log 2 + K_{T_i}\left(\gamma - \sum_{p\in T_i}\frac{p\ \log p}{p^2-1}\right) + O(2^{i+1}/N)$$

and

$$B = K_{T_{i+1}}\log N - (i+1)K_{T_{i+1}}\log 2 + K_{T_{i+1}}\left(\gamma - \sum_{p\in T_{i+1}}\frac{p\ \log p}{p^2-1}\right) + O(2^{i+1}/N).$$

Consequently,

$$A - \frac{1}{p_{i+1}^{\epsilon_{i+1}}}B = \left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)K_{T_i}\log N - \left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)(i+1)K_{T_i}\log 2$$
$$+\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)K_{T_i}\gamma - K_{T_i}\sum_{p\in T_i}\frac{p\ \log p}{p^2-1}$$
$$+\frac{K_{T_{i+1}}}{p_{i+1}^{\epsilon_{i+1}}}\sum_{p\in T_{i+1}}\frac{p\ \log p}{p^2-1} + \left(1-\frac{1}{p_{i+1}^{\epsilon_{i+1}}}\right)O\left(\frac{2^{i+1}}{N}\right).$$

By plugging such information above into the main formula $F_S(N)$ and applying Lemma 2.2 to the term $F_\emptyset(N)$, we eventually obtain that

$$F_S(N) = \left(1 - \sum_{i=0}^{\lfloor\log_2 N\rfloor-1}\frac{K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)\right)\log N$$
$$-\left(\sum_{i=0}^{\lfloor\log_2 N\rfloor-1}\frac{(i+1)K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)\right)\log 2 + \left(1 - \sum_{i=0}^{\lfloor\log_2 N\rfloor-1}\frac{K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(1-\frac{p_{i+1}}{p_{i+1}^{\epsilon_{i+1}}(p_{i+1}+1)}\right)\right)\gamma$$

$$+\sum_{i=0}^{\lfloor \log_2 N\rfloor-1}\frac{K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(\sum_{p\in T_i}\frac{p\ \log p}{p^2-1}\right)+\sum_{i=0}^{\lfloor \log_2 N\rfloor-1}\frac{K_{T_{i+1}}}{2^{i+1}\mathcal{P}_{i+1}}\left(\sum_{p\in T_{i+1}}\frac{p\ \log p}{p^2-1}\right)$$
$$+\sum_{i=0}^{\lfloor \log_2 N\rfloor-1}\left(\frac{K_{T_i}(p_{i+1}^{\epsilon_{i+1}}-1)O(2^{i+1}/N)}{2^{i+1}\mathcal{P}_{i+1}}\right)+O\left(\frac{1}{N}\right).$$

It follows by Lemma 3.3 and Lemma 3.4 that

$$|F_S(N)-k_S\log N-k_S\gamma+\ B_S^*\log 2|=|\mathcal{K}_N\log N|+|\mathcal{K}_N\gamma|+|\mathcal{L}_N\log 2|$$
$$+\left|\sum_{i=0}^{\lfloor \log_2 N\rfloor-1}\frac{K_{T_i}}{2^{i+1}\mathcal{P}_i}\left(\sum_{p\in T_i}\frac{p\ \log p}{p^2-1}\right)\right|+\left|\sum_{i=0}^{\lfloor \log_2 N\rfloor-1}\frac{K_{T_{i+1}}}{2^{i+1}\mathcal{P}_{i+1}}\left(\sum_{p\in T_{i+1}}\frac{p\ \log p}{p^2-1}\right)\right|$$
$$+\left|\sum_{i=0}^{\lfloor \log_2 N\rfloor-1}\left(\frac{K_{T_i}(p_{i+1}^{\epsilon_{i+1}}-1)O\left(\frac{2^{i+1}}{N}\right)}{2^{i+1}\mathcal{P}_{i+1}}\right)+O\left(\frac{1}{N}\right)\right|\le\frac{C_5\log N}{N},$$

for some constant C_5. Hence, the refined asymptotic expression in the theorem is proven. Furthermore, the real number k_S belonging to the interval $[0.44, 0.57]$ can be shown directly by using Lemma 3.4.

4. Discussion

In the same setting, for each natural number l and n, let p_n^l be the greatest primitive prime divisor of the sequence $2^{l^n}-1$, and let

$$S_l\ =\{\ p_n^l\ |\ n\in\mathbb{N}\},$$

which implies that $m_{p_n^l}=l^n$ for any natural number l and n. This is of course a generalization of our set $S=S_2$, and also we are able to obtain $k_{S_l}=1-C_l^*$ for which

$$C_l^*:=C_{S_l}^*=\sum_{i=0}^{\infty}\frac{K_{T_i}}{l^{i+1}\mathcal{P}_i^l}\left(1-\frac{p_{i+1}^l}{p_{i+1}^{\epsilon_{i+1}^l}(p_{i+1}^l+1)}\right),$$

where $\epsilon_{i+1}^l\ge 1$ is the highest power of p_i^l dividing $2^{o_{T_i^l}}-1$ such that

$T_i^l=\{p_1^l,p_2^l,p_3^l,\dots,p_i^l\}\subseteq S_l$ and $\mathcal{P}_i^l=\epsilon_1^l\epsilon_2^l\epsilon_3^l\dots\epsilon_i^l$

together with $T_i^l=\emptyset$ and $\mathcal{P}_0^l=1$, the index i is running over all natural numbers. The approach to get the real number C_l^* is to follow the same step directly as we have just done in the case $S=S_2$, and we note that the real interval covering the corresponding number k_{S_l} can be computed easily for every natural number $l\ge 2$. Now, we would like to conclude by referring (7) that

$$\mathcal{M}_{\alpha^l}(N)=k_{S_l}\log N+k_{S_l}\gamma-\ B_{S_l}^*\log l+B_{S_l}+O\left(2^{-\frac{N}{2}}\right)+O(\log N/N) \tag{12}$$

for any natural number $l\ge 2$, and note that α^l is the S_l-integer dynamical systems in our setting. In 2013, the partial sum $\mathcal{M}_{\alpha^l}(N)$, presented as *Merten's Theorem* has roughly illustrated as

$$\mathcal{M}_{\alpha^l}(N)\sim k_{S_l}\log N,$$

which may be found on pages 189-191 in Steven *et al.* (2013). In this paper, its greater refinement has been expressed as shown in (12) by using a not strongly different approach. Being motivated by such the result mentioned above, we moreover conjecture that k_{S_l} is a transcendental number for every natural number $l\ge 2$.

Acknowledgements

This research was fully supported by the Development and Promotion of Science and Technology Talent Project (DPST), Thailand grant 006/2555.

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