วารสารคณิตศาสตร์ โดยสมาคมคณิตศาสตร์แห่งประเทศไทย ในพระบรมราชูปถัมภ์
ปริมา 67 เล่มที่ 706 มกราคม – เมษายน 2565
http://www.mathassociation.net Email: MathThaiOrg@gmail.com

ค่าพื้นของค่าเฉลี่ยของรากที่สี่ของจำนวนเต็ม

The Floor of The Average for The Fourth Roots of Integers

DOI: 10.14456/mj-math.xxxx.xx

สมคิด อินเทพ[1] และ บุญยงค์ ศรีพลแผ้ว[2,*]
[1, 2] ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยบูรพา ชลบุรี 20131

Somkid Intep[1] and Boonyong Sriponpaew[2,*]
[1, 2] Department of Mathematics, Faculty of Science, Burapha University, Chonburi 20131

Email: [1] intep@buu.ac.th [2]boonyong@buu.ac.th

วันที่รับบทความ : 2 พฤษภาคม 2564 วันที่แก้ไขบทความ : 16 กรกฎาคม 2564 วันที่ตอบรับบทความ : 15 เมษายน 2565

บทคัดย่อ

จุดมุ่งหมายของบทความนี้คือการสร้างลำดับโดยประมาณสำหรับลำดับของค่าเฉลี่ยของรากที่สี่ของจำนวนเต็ม n พจน์แรก และเพื่อที่จะพิสูจน์ว่าลำดับทั้งสองมีค่าพื้นเดียวกัน ในอีกทางหนึ่งได้แสดงว่าการสร้างลำดับในรูปแบบเดียวกันนี้ไม่สามารถใช้ได้ในกรณีของรากที่ห้าและรากที่หก

คำสำคัญ: ค่าพื้น ค่าเฉลี่ย รากที่สี่

---

* ผู้เขียนหลัก

ABSTRACT

The aim of this paper is to construct an estimated sequence for the sequence of the mean of the fourth roots of the first n integers and to prove that both sequences share the same floor. On the other hand, we show that the same pattern of sequence construction is not applicable for the fifth-root and the sixth-root cases.

Keywords: Floor, Average, Fourth roots

1. Introduction

It has been a long time that many researchers have studied for sum of square roots of non-negative integers. For instance, a mathematical formula of this sum is proposed by Ramanujan [2] as

$$\sum_{k=1}^{n}\sqrt{k} = \frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n}-\frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k}}+\frac{1}{6}\sum_{k=0}^{\infty}\frac{1}{\left(\sqrt{n+k}+\sqrt{n+k+1}\right)^{3}}.$$

Later, Merca [1] focused on the arithmetic mean of the square roots of the first n integers and provided bounds of this mean as follows:

$$\left(\frac{2}{3}+\frac{1}{8n}-\frac{1}{8n\sqrt{n+1}}\right)\sqrt{n+1} < \frac{1}{n}\sum_{k=1}^{n}\sqrt{k} < \left(\frac{2}{3}+\frac{1}{6n}-\frac{1}{6n\sqrt{n+1}}\right)\sqrt{n+1}.$$

Also he conjectured that the sequence has the same floor as a certain approximated sequence, that is,

$$\left\lfloor \frac{1}{n}\sum_{k=1}^{n}\sqrt{k} \right\rfloor = \left\lfloor \left(\frac{2}{3}+\frac{1}{6n}\right)\sqrt{n+1} \right\rfloor.$$

In Zacharias [5], this conjecture was proved by Zacharias. The key of the proof is to find the partition sets in which both sequences have the same floor. Zacharias presented the partition sets in the modulo 2 pattern as, for $k \in \mathbb{N}\cup\{0\}$,

$$B_{1,k} = \left[\frac{9(2k+1)^2-5}{4},\frac{9(2k+2)^2-8}{4}\right]\cap\mathbb{N}$$

and

$$B_{2,k} = \left[\frac{9(2k+2)^2-4}{4}, \frac{9(2k+3)^2-9}{4}\right] \cap \mathbb{N},$$

where $\bigcup_{k=0}^{\infty} B_{1,k} \cup B_{2,k} = \mathbb{N}$ forms a partition of $\mathbb{N}$.

Afterwards Wihler [4] derived a formula for the arithmetic mean of the $r-$th roots of the first n integers where $r \geq 1$ as

$$\sum_{k=v}^{n} \sqrt[r]{k} = \frac{r}{r+1}\sqrt[r]{n+1}\left(n+\frac{1-\frac{1}{r}}{2}\right) - \frac{r}{r+1}\sqrt[r]{v}\left(v-\frac{1+\frac{1}{r}}{2}\right) - \frac{\delta_{v,n,r}}{12r} \qquad (*)$$

with $\delta_{v,n,1} = 0$ and $\sigma_r(v+2, n+2) < \delta_{v,n,r} < \sigma_r(v,n)$ for $r > 1,$ where

$$\sigma_r(v,n) = \begin{cases} 2-\frac{1}{r}-n^{-1+\frac{1}{r}} & \text{if } v=1, \\ (v-1)^{-1+\frac{1}{r}} - n^{-1+\frac{1}{r}} & \text{if } v \geq 2. \end{cases}$$

Recently, Sriponpaew [3] defined the estimated sequence for average of the cube roots of integers by picking up the first two terms from the Wihler's formula as

$$\left(\frac{3}{4}+\frac{1}{4n}\right)\sqrt[3]{n+1} - \frac{1}{4n}.$$

Furthermore, we generated the partition sets in the modulo 9 pattern as follows: for $k \in \mathbb{N} \cup \{0\},$

$$B_{1,k} = \begin{cases} \left[\frac{64(9k+1)^3-37}{27}, \frac{64(9k+2)^3-53}{27}\right] \cap \mathbb{N} & \text{if } k=0,1,2, \\ \left[\frac{64(9k+1)^3-37}{27}, \frac{64(9k+2)^3-80}{27}\right] \cap \mathbb{N} & \text{if } k \geq 3, \end{cases}$$

$$B_{2,k} = \begin{cases} \left[\dfrac{64(9k+2)^3-26}{27}, \dfrac{64(9k+3)^3-54}{27}\right] \cap \mathbb{N} & \text{if } k = 0,1,2, \\ \left[\dfrac{64(9k+2)^3-53}{27}, \dfrac{64(9k+3)^3-54}{27}\right] \cap \mathbb{N} & \text{if } k \geq 3, \end{cases}$$

and for $j = 3,4,\ldots,9,$ define $B_{j,k}$ as

$$B_{j,k} = \left[\frac{64(9k+j)^3-b_j}{27}, \frac{64(9k+j+1)^3-b_{j+1}-27}{27}\right] \cap \mathbb{N},$$

where

$$b_j = \begin{cases} 27 & \text{if } j \equiv 0 \text{ (mod3)}, \\ 2(25-i) & \text{if } j \not\equiv 0 \text{ (mod3) and } j = 2^i, \\ j(5+(-1)^i i) & \text{if } j \text{ is prime and } j \equiv i \text{ (mod3)}, 0 < i < 3, \end{cases}$$

and $b_{10} = 37.$ Consequently, $\bigcup_{k=0}^{\infty}\bigcup_{j=1}^{9} B_{j,k} = \mathbb{N}$ forms a partition of $\mathbb{N}.$ The result shown in [3] is that both sequences share the same floor in each set.

In this article, we concentrate on the floor of arithmetic mean value S_n of the fourth roots of the first n integers. In the process of proof, we construct the sequence $A(n)$ from the formula of Wihler which share the same floor with S_n. Also, we define the highly sophisticated partition sets which are applicative for the fourth roots in order to prove the following main theorem.

Theorem 1.1 (Main Theorem) For any positive integer $n,$

$$\left\lfloor \frac{1}{n}\sum_{k=1}^{n}\sqrt[4]{k} \right\rfloor = \left\lfloor \left(\frac{4}{5}+\frac{3}{10n}\right)\sqrt[4]{n+1}-\frac{3}{10n} \right\rfloor.$$

Moreover, based on some numerical results, we can show that the same technique of construction for the approximated sequence is not suitable for the cases of the fifth roots and the sixth roots.

2. Preliminaries

The goal of this article is to derive the floor of mean value of the fourth roots of the first n integers into the floor of a simpler sequence. In particular, we define the sequence

$$S_n = \frac{1}{n}\sum_{k=1}^{n}\sqrt[4]{k}$$

and prove that

$$\lfloor S_n \rfloor = \left\lfloor \left(\frac{4}{5}+\frac{3}{10n}\right)\sqrt[4]{n+1}-\frac{3}{10n} \right\rfloor.$$

To simplify the process, we define the functions

$$A(x) = \left(\frac{4}{5}+\frac{3}{10x}\right)\sqrt[4]{x+1}-\frac{3}{10x}$$

and

$$L(x) = \left(\frac{4}{5}+\frac{3}{10x}\right)\sqrt[4]{x+1}-\frac{323}{960x},$$

for all real $x \geq 1.$ Firstly, we verify that both functions are strictly increasing.

Lemma 2.1 For $x \geq 1,$ $A(x)$ and $L(x)$ are strictly increasing functions.

Proof. To show the strict monotonicity of $A(x),$ we consider its derivative as

$$A'(x) = \frac{8x^2-9x-12+12(x+1)^{\frac{3}{4}}}{40x^2(x+1)^{\frac{3}{4}}}.$$

After we simplify its numerator, we have

$$8x^2-9x-12+12(x+1)^{\frac{3}{4}} = 8\left(x-\frac{9}{16}\right)^2+12\left((x+1)^{\frac{3}{4}}-\frac{155}{128}\right) > 0$$

for all $x \geq 1.$ This implies that $A(x)$ is a strictly increasing function for all $x \geq 1.$ Similarly, the numerator of derivative of $L(x)$ are algebraically simplified to be

$$192\left(x-\frac{9}{16}\right)^2+323\left((x+1)^{\frac{3}{4}}-\frac{1395}{1292}\right) > 0 \text{ for all } x \geq 1.$$

Hence, $L(x)$ is also a strictly increasing function for all $x \geq 1.$ □

We need to prove that S_n is increasing in order to consider only boundaries of the partition sets when we deal with the integer part of S_n.

Lemma 2.2 $S_n = \dfrac{1}{n}\sum_{k=1}^{n}\sqrt[4]{k}$ is a strictly increasing sequence.

Proof. It is well-known that $f(x) = x^{\frac{1}{4}}$ for $x \geq 1$ is a strictly increasing function. Therefore, we have

$$\sum_{k=1}^{n}\sqrt[4]{k} < \sum_{k=1}^{n}\sqrt[4]{n+1} = n\ \sqrt[4]{n+1}.$$

After adding $n\sum_{k=1}^{n}\sqrt[4]{k}$ on the both sides of the inequality, we obtain that

$$(n+1)\sum_{k=1}^{n}\sqrt[4]{k} < n\sum_{k=1}^{n+1}\sqrt[4]{k}.$$

That is $S_n = \dfrac{1}{n}\sum_{k=1}^{n}\sqrt[4]{k} < \dfrac{1}{n+1}\sum_{k=1}^{n+1}\sqrt[4]{k} = S_{n+1}$, which implies that S_n is a strictly increasing sequence. □

We observe the values of both (S_n) and $(A(n))$ calculated by using MATLAB and define the partition sets where both sequences share the same floor. Let $B_{1,0} = [1,37] \cap \mathbb{N}$ and $B_{2,0} = [38,195] \cap \mathbb{N}$. For $k \in \mathbb{N} \cup \{0\}$ and $j = 1, 2, \ldots, 64$ with $(j,k) \neq (1,0),(2,0)$, we define R_j as the remainder of division of $|32 - j|$ by 32, i.e. $R_j \equiv |32 - j| \pmod{32}$ where $0 \leq R_j < 32$ and define the set $B_{j,k}$ as follows:

$$B_{j,k} = \left[\frac{625(64k+j)^4 - b_j}{256}, \frac{625(64k+j+1)^4 - b_{j+1} - 256}{256}\right] \cap \mathbb{N},$$

where

$$b_j = \begin{cases} 512 & \text{if } R_j \equiv 0 \pmod 4, \\ 528 & \text{if } R_j \equiv 2 \pmod 4, \\ 497 & \text{if } R_j = 1, \\ 577 & \text{if } R_j = 3, \\ 609 & \text{if } R_j = 5, \\ 593 & \text{if } R_j = 7, \\ 401 & \text{if } R_j = 9, \\ 545 & \text{if } R_j = 11, \\ 385 & \text{if } R_j = 13, \\ 433 & \text{if } R_j = 15, \\ 561 & \text{if } R_j = 17, \\ 513 & \text{if } R_j = 19, \\ 417 & \text{if } R_j = 21, \\ 529 & \text{if } R_j = 23, \\ 465 & \text{if } R_j = 25, \\ 481 & \text{if } R_j = 27, \\ 449 & \text{if } R_j = 29, \\ 625 & \text{if } R_j = 31, \end{cases}$$

and $b_{65} = 625.$

Notice that boundary points of $B_{j,k}$ are all integers satisfying $1+\max B_{j,k} = \min B_{j+1,k}$ and $1+\max B_{64,k} = \min B_{1,k+1}$. In conclusion, we have that $\{B_{j,k}\}$ is the partition of $\mathbb{N}$.

From (*), we obtain that

$$\frac{1}{n}\sum_{k=1}^{n}\sqrt[4]{k} = \left(\frac{4}{5}+\frac{3}{10n}\right)\sqrt[4]{n+1}-\frac{3}{10n}-\frac{\delta_{1,n,4}}{48n}$$

where $2^{-\frac{3}{4}}-n^{-\frac{3}{4}} < \delta_{1,n,4} < \frac{7}{4}-n^{-\frac{3}{4}}$. Note that for any $n>2$, $0<\delta_{1,n,4} < \frac{7}{4}.$

Consequently,

$$A(n) > \frac{1}{n}\sum_{k=1}^{n}\sqrt[4]{k} > L(n) \quad \text{for} \quad n>2.$$

The aim of the main theorem is to show that for any $n \in B_{j,k}$ the integer part of $A(n)$ and S_n is $64k+j.$ More precisely, $64k+j \le S_n < A(n) < 64k+j+1.$ Notice that $L(n)$ and $A(n)$ are strictly increasing and the value of S_n is between $L(n)$ and $A(n).$ It is enough to show that $L(r) > 64k + j$ and $A(s) < 64k + j + 1$ for any $r = \min B_{j,k} = \dfrac{625(64k+j)^4 - b_j}{256}$ and $s = \max B_{j,k} = \dfrac{625(64k+j+1)^4 - b_{j+1}}{256} - 1.$

To verify these inequalities, we investigate signs of coefficients of some Taylor expansions. Nevertheless, for some basic numbers $n,$ we estimate the floor of S_n and $A(n)$ by simple calculation.

3. Proof of Main Theorem

Proof. Let $n \in \mathbb{N}.$

<u>Case 1</u> $n \in B_{1,k}$

Subcase 1.1 $k = 0$

For $n \in B_{1,0}$ that is $1 \le n \le 37,$ the bounds of S_n and $A(n)$ are the following:

$$1 = S_1 < S_n < S_{37} \approx 1.9978$$

and

$$1.0081 \approx A(1) < A(n) < A(37) \approx 1.9983.$$

Hence $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 1$ for $1 \le n \le 37.$

Subcase 1.2 $k \ge 1$

Let $r = \dfrac{625(64k+1)^4 - b_1}{256} = \dfrac{625(64k+1)^4 - 625}{256}.$

We will prove that $L(r) > 64k + 1.$ By using MATLAB, we expand the following expression to be the Taylor series about 1 as

$$
\begin{aligned}
&(768r+288)^4(r+1)-(960r(64k+1)+323)^4 \\
&\quad = 4754754611942748176203200082619973196689 5 \\
&\qquad + 77464020311942565872078500723969913520000 0(k-1) \\
&\qquad + 590934443509973123783202240639766272000000 0(k-1)^2 \\
&\qquad + 2802076512741853123493641218758585548800000 0(k-1)^3 \\
&\qquad + 9244432412831239650387840937925156536320000 0(k-1)^4 \\
&\qquad + 2250173899249117309341107277493822291968000 00(k-1)^5 \\
&\qquad + 4180347241963833282966383460404428800000000 00(k-1)^6 \\
&\qquad + 6046723082828228669527351424031129600000000 00(k-1)^7 \\
&\qquad + 6882532487035574340988103685801246720000000 00(k-1)^8 \\
&\qquad + 6185271925602071353881401449512960000000000 00(k-1)^9 \\
&\qquad + 4374449368210858330089188037859737600000000 00(k-1)^{10} \\
&\qquad + 2409167142925335246539178639360000000000000 00(k-1)^{11} \\
&\qquad + 1012914734961035836478598414336000000000000 00(k-1)^{12} \\
&\qquad + 3143053504207546991876505600000000000000000 0(k-1)^{13} \\
&\qquad + 6788349707073106765204684800000000000000000(k-1)^{14} \\
&\qquad + 9117803534130224889247825920000000000000000(k-1)^{15} \\
&\qquad + 573767527668661893464064000000000000000000(k-1)^{16} \ > \ 0.
\end{aligned}
$$

Then we derive the above inequality into $(768r+288)\sqrt[4]{r+1} \ > \ 960r(64k+1)+323.$

We have $L(r) \ = \ \left(\frac{4}{5}+\frac{3}{10r}\right)\sqrt[4]{r+1}-\frac{323}{960r} \ > \ 64k+1$ by direct calculation.

Let $s=\frac{625(64k+2)^4-b_2-256}{256}=\frac{625(64k+2)^4-784}{256}$. We will show that $A(s)<64k+2$. Similarly, the following expression are written as the Taylor expansion about 1 as

$$
\begin{aligned}
&(10s(64k+2)+3)^4-(8s+3)^4(s+1)\\
&\quad = 1095370904641882775710909107248840 4\\
\end{aligned}
$$

$$
\begin{aligned}
(10s(64k+2)+3)^4-(8s+3)^4(s+1) = \; & 10953709046418827757109091072488404\\
&+1696158901587250583996872310637505 20(k-1)\\
&+123115143130545341740002883223538240 0(k-1)^2\\
&+5560325905943677952822429251120128000(k-1)^3\\
&+17488957096089532314859379642818560000(k-1)^4\\
&+40621294930019226943217203622707200000(k-1)^5\\
&+99643568429174106784851547914240000000(k-1)^7\\
&+108486397100314238227268920934400000000(k-1)^8\\
&+93324256201346188695935385600000000000(k-1)^9\\
&+63221184162282908838568919040000000000(k-1)^{10}\\
&+33372558966296612654442086400000000000(k-1)^{11}\\
&+13456894494578024210497536000000000000(k-1)^{12}\\
&+4007064398391505059840000000000000000(k-1)^{13}\\
&+830964796745664430080000000000000000(k-1)^{14}\\
&+107221699928436768768000000000000000(k-1)^{15}\\
&+6485183463413514240000000000000000(k-1)^{16} \; > 0.
\end{aligned}
$$

By the same calculation, we have

$$64k+2 > \left(\frac{4}{5}+\frac{3}{10s}\right)\sqrt[4]{s+1}-\frac{3}{10s}=A(s).$$

Since $A(n)$ and $L(n)$ are increasing, for $r \le n \le s$, we have

$$64k+1 < L(r) < L(n) < S_n < A(n) < A(s) < 64k+2.$$

Hence $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 64k+1$ for $k \ge 1$.

<u>Case 2</u> $n \in B_{2,k}$

Subcase 2.1 $k=0$

We consider $n \in B_{2,0}$ so that $38 \le n \le 195$. Since S_n and $A(n)$ are strictly increasing sequences,

$$2.0105 \approx S_{38} < S_n < S_{195} \approx 2.9974$$

and

$$2.0110 \approx A(38) < A(n) < A(195) \approx 2.9975.$$

Hence $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 2$ for $38 \le n \le 195.$

Subcase 2.2 $k \ge 1$

Let $r = \dfrac{625(64k+2)^4 - b_2}{256} = \dfrac{625(64k+2)^4 - 528}{256}$.

We will show that $L(r) > 64k+2$. Again we consider that the Taylor expansion about 1 of $(768r+288)^4(r+1)-(960r(64k+2)+323)^4$ in terms of k is greater than zero for $k \ge 1$ as the following.

$$\begin{aligned}
&(768r+288)^4(r+1)-(960r(64k+2)+323)^4 \\
&= 66827799487353113154609400406065315541572 7 \\
&\quad +10400114950554878959352664454883733411164160(k-1) \\
&\quad +75867603840583500904679025425153247200870400(k-1)^2 \\
&\quad +344362702240926543282997452509306552057856000(k-1)^3 \\
&\quad +1088550155865703890324832023114330710999040000(k-1)^4 \\
&\quad +2541000229559109600433119988914263477452800000(k-1)^5 \\
&\quad +4530928649775465308350805797614611595264000000(k-1)^6 \\
&\quad +6295443882813605556499487033900075581440000000(k-1)^7 \\
&\quad +6888301792756014127024691114710715596800000000(k-1)^8 \\
&\quad +5955100114930619634516627670066790400000000000(k-1)^9 \\
&\quad +4054270203976961665595455761348034560000000000(k-1)^{10} \\
&\quad +2150765292754342181898829079720755200000000000(k-1)^{11} \\
&\quad +871564931093289201074214436601856000000000000(k-1)^{12} \\
&\quad +260813809742924716184237506560000000000000000(k-1)^{13} \\
&\quad +54354342663695314594163589120000000000000000(k-1)^{14} \\
&\quad +7048224061829361439522947072000000000000000(k-1)^{15} \\
&\quad +428413087325934213786501120000000000000000(k-1)^{16} > 0 .
\end{aligned}$$

In the similar process to Case 1, we have $L(r) > 64k+2$.

Let $s = \dfrac{625(64k+3)^4 - b_3 - 256}{256} = \dfrac{625(64k+3)^4 - 705}{256}$. The same technique of the Taylor expansion about 1 is used for the following algebraic expression

$$
\begin{aligned}
&(10s(64k+3)+3)^4-(8s+3)^4(s+1)\\
&\quad= 6522182691627129366698257394526050\\
&\quad\quad +9927180988873778602317568917446652 5(k-1)\\
&\quad\quad +708260082620689530531688729770735000(k-1)^2\\
&\quad\quad +31441091427552003298834455631771 00000(k-1)^3\\
&\quad\quad +972011680948091716102873915331905 0000(k-1)^4\\
&\quad\quad +221904247750922055303080796524352 00000(k-1)^5\\
&\quad\quad +386974419684278741045495613209600 00000(k-1)^6\\
&\quad\quad +525837649323466682277308948480000 00000(k-1)^7\\
&\quad\quad +562683070201256288811857018880000 00000(k-1)^8\\
&\quad\quad +475732800807017442896248832000000 00000(k-1)^9\\
&\quad\quad +316741494374714911001831014400000 00000(k-1)^{10}\\
&\quad\quad +164323372233843131107246080000000 00000(k-1)^{11}\\
&\quad\quad +651202152984041236725760000000000 0000(k-1)^{12}\\
&\quad\quad +190568908111432646656000000000000 0000(k-1)^{13}\\
&\quad\quad +388379788591874703360000000000000 000(k-1)^{14}\\
&\quad\quad +492491137251100590080000000000000 00(k-1)^{15}\\
&\quad\quad +292733975779082240000000000000000 0(k-1)^{16}>0.
\end{aligned}
$$

By positivity of coefficients of the Taylor series of $(10s(64k+3)+3)^4-(8s+3)^4\ (s+1)$ in term of k about 1, we get $A(s)<64k+3.$

Thus, for $r\le n\le s,$ we obtain that

$$64k+2<L(n)<S_n<A(n)<64k+3.$$

Hence $\lfloor S_n\rfloor=\lfloor A(n)\rfloor=64k+2.$

<u>Case 3</u> $n\in B_{3,k}$

Let $r=\dfrac{625(64k+3)^4-449}{256}$. By using the Maclaurin series expansion, we gain

$$
\begin{aligned}
&(768r+288)^4(r+1)-(960r(64k+3)+323)^4\\
&\quad= 1566251642318639505111+5856962749607605262208 0k\\
&\quad\quad +10194329686304320962969600k^2\\
&\quad\quad +1097187308676744111194112000k^3\\
&\quad\quad +81785824949797735225098240000k^4\\
&\quad\quad +4479926087277225077165260800000k^5
\end{aligned}
$$

$$\begin{aligned}
&+1866370868767347099744337920000000k^{6}\\
&+6035290884358828786565775360000000k^{7}\\
&+153160198360576307538532761600000000k^{8}\\
&+3061598521606783995425558720000000000k^{9}\\
&+48062265635436921267117096960000000000k^{10}\\
&+586475431819508949402098073600000000000k^{11}\\
&+5454620023724177977994379264000000000000k^{12}\\
&+37388742873545543453245440000000000000000k^{13}\\
&+178159299899929429341634560000000000000000k^{14}\\
&+527356109875019020303859712000000000000000k^{15}\\
&+730597318564762811010908160000000000000000k^{16} > 0.
\end{aligned}$$

Consequently, we gain $L(r) > 64k+3$.

Let $s = \dfrac{625(64k+4)^4 - 768}{256}$. In the same process, we obtain $A(s) < 64k+4$ and conclude that $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 64k+3.$

For the remaining cases, to show that

$$\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 64k + j, \text{ for } j = 4, 5, \ldots, 64,$$

we use similar arguments to those in case 3, i.e., we expand

$$(768r+288)^4(r+1) - (960r(64k+j)+323)^4$$

and

$$(10s(64k+j+1)+3)^4 - (8s+3)^4(s+1)$$

into the Maclaurin series in the following form

$$c_{j,0} + c_{j,1}k + c_{j,2}k^2 + \cdots + c_{j,16}k^{16}$$

where $c_{j,k} > 0$. □

In an analogous way, we define approximated sequences as

$$A_5(n) = \left(\frac{5}{6} + \frac{1}{3n}\right)\sqrt[5]{n+1} - \frac{1}{3n}$$

and

$$A_6(n) = \left(\frac{6}{7} + \frac{5}{14n}\right)\sqrt[6]{n+1} - \frac{5}{14n}$$

for the fifth roots and the sixth roots respectively. Nonetheless, the numerical results show that

$$\lfloor A_5(146930) \rfloor = 9 \neq 8 = \left\lfloor \frac{1}{146930} \sum_{k=1}^{146930} \sqrt[5]{k} \right\rfloor$$

and

$$\lfloor A_6(661026) \rfloor = 8 \neq 7 = \left\lfloor \frac{1}{661026} \sum_{k=1}^{661026} \sqrt[6]{k} \right\rfloor.$$

These imply that the similar pattern of formulation for the estimated sequences $A_5(n)$ and $A_6(n)$ is not applicable.

Acknowledgements

The authors would like to thank Detchat Samart for the useful suggestions which improved the presentation of the paper.

References

[1] Merca, M. (2017). On the arithmetic mean of the square roots of the first n positive integers. *The College Mathematics Journal*, 48, p. 129 – 133.

[2] Ramanujan, S. (1915). On the sum of the square roots of the first n natural numbers. *Journal of the Indian Mathematical Society*, 7, p. 173 – 175.

[3] Sriponpaew, B. and Intep, S. (2020). The floor of the arithmetic mean of the cube roots of the first n integers. *Bulletin of the Australian Mathematical Society,* 102, p. 261 – 267.

[4] Wihler, T. P. (2018) Rounding the arithmetic mean value of the square roots of the first n integers. Retrieved from https://arxiv.org/pdf/1803.00362.

[5] Zacharias, J. (2018). Proof of a conjecture of Merca on an average of square roots. *The College Mathematics Journal*, 49, p. 342 – 345.