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On the Diophantine Equation $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=\frac{1}{2}$

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Abstract

In this paper, we find all solutions of the Diophantine equation $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=\frac{1}{2}$ where x, y and z are positive integers.

Keywords: Diophantine equation

1. Introduction

A Diophantine equation is an equation in which only integer solutions are allowed. The researchers about Diophantine equations are ancient, various and no general method exists. The one well-known problem $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{4}{n}$ has been studied by many researchers, see (1-3, 6).

In 2013, Rabago and Tagle (3) studied some elementary problems involving surface area and volume of a certain regular solid. This problem leads them to solving the specific case of Diophantine problem $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{4}{n}$, i.e. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$. Later, Sandor (5) offered an elementary approach to the solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$.

In 2019, Pakapongpun (4) found positive integral solutions of the Diophantine equation $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{2}$.

In this paper, we find all solutions of the Diophantine equation $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=\frac{1}{2}$ where x, y and z are positive integers.

2. Solutions of the Diophantine equation $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=\frac{1}{2}$

From the Diophantine equation $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=\frac{1}{2}$, where x, y and z are positive integers, we get $\frac{2}{x}<\frac{1}{2}$, $\frac{3}{y}<\frac{1}{2}$ and $\frac{4}{z}<\frac{1}{2}$. This implies that $x \geq 5, y \geq 7$ and $z \geq 9$.

We consider the following 3 cases.

Case 1 : $x \leq y \leq z$ or $x \leq z \leq y$.

If $x \leq y \leq z$, then $\frac{2}{x}+\frac{3}{y}+\frac{4}{z} \leq \frac{9}{x}$.

If $x \leq z \leq y$, then $\frac{2}{x}+\frac{3}{y}+\frac{4}{z} \leq \frac{9}{x}$.

Therefore, $\frac{1}{2}=\frac{2}{x}+\frac{3}{y}+\frac{4}{z} \leq \frac{9}{x}$ which implies that $x \leq 18$. From $x \geq 5$ and $x \leq 18$, we get $5 \leq x \leq 18$. If $x=5$, then $\frac{3}{y}+\frac{4}{z}=\frac{1}{10}$. We have

$$(y-30)(z-40)=1200.$$

We use the function τ to ensure that the total number of divisors of 1200 is 30. Using factors of 1200, we have

$$y-30=1, z-40=1200;$$
$$y-30=2, z-40=600;$$
$$y-30=3, z-40=400;$$
$$y-30=4, z-40=300;$$
$$y-30=5, z-40=240;$$

$y-30=6, z-40=200;$
$y-30=8, z-40=150;$
$y-30=10, z-40=120$
$y-30=12, z-40=100;$
$y-30=15, z-40=80;$
$y-30=16, z-40=75;$
$y-30=20, z-40=60;$
$y-30=24, z-40=50;$
$y-30=25, z-40=48;$
$y-30=30, z-40=40;$
$y-30=40, z-40=30;$
$y-30=48, z-40=25;$
$y-30=50, z-40=24;$
$y-30=60, z-40=20;$
$y-30=75, z-40=16;$
$y-30=80, z-40=15;$
$y-30=100, z-40=12;$
$y-30=120, z-40=10;$
$y-30=150, z-40=8;$
$y-30=200, z-40=6;$
$y-30=240, z-40=5;$
$y-30=300, z-40=4;$
$y-30=400, z-40=3;$
$y-30=600, z-40=2;$
$y-30=1200, z-40=1.$

The solutions (x, y, z) are $(5,31,1240)$, $(5,32,640)$, $(5,33,440)$, $(5,34,340)$, $(5,35,280)$, $(5,36,240)$, $(5,38,190)$, $(5,40,160)$, $(5,42,140)$, $(5,45,120)$, $(5,46,115)$, $(5,50,100)$, $(5,54,90)$, $(5,55,88)$, $(5,60,80)$, $(5,70,70)$, $(5,78,65)$, $(5,80,64)$, $(5,90,60)$, $(5,105,56)$, $(5,110,55)$, $(5,130,52)$, $(5,150,50)$, $(5,180,48)$, $(5,230,46)$, $(5,270,45)$, $(5,330,44)$, $(5,430,43)$, $(5,630,42)$, $(5,1230,41)$.

In the same manner,
if $x=6$, then $\frac{3}{y}+\frac{4}{z}=\frac{1}{6}$. We have

$$(y-18)(z-24)=432.$$

The solutions (x, y, z) are $(6,19,456)$, $(6,20,240)$, $(6,21,168)$, $(6,22,132)$, $(6,24,96)$, $(6,26,78)$, $(6,27,72)$, $(6,30,60)$, $(6,34,51)$, $(6,36,48)$, $(6,42,42)$, $(6,45,40)$, $(6,54,36)$, $(6,66,33)$, $(6,72,32)$, $(6,90,30)$, $(6,126,28)$, $(6,162,27)$, $(6,234,26)$, $(6,450,25)$.

If $x=7$, then $\frac{3}{y}+\frac{4}{z}=\frac{3}{14}$. We have

$$(y-14)(3z-56)=784.$$

The solutions (x, y, z) are $(7,15,280)$, $(7,18,84)$, $(7,21,56)$, $(7,30,35)$, $(7,42,28)$, $(7,63,24)$, $(7,126,21)$, $(7,210,20)$, $(7,798,19)$.

If $x=8$, then $\frac{3}{y}+\frac{4}{z}=\frac{1}{4}$. We have

$$(y-12)(z-16)=192.$$

The solutions (x, y, z) are $(8,13,208)$, $(8,14,112)$, $(8,15,80)$, $(8,16,64)$, $(8,18,48)$, $(8,20,40)$, $(8,24,32)$, $(8,28,28)$, $(8,36,24)$, $(8,44,22)$, $(8,60,20)$, $(8,76,19)$, $(8,108,18)$, $(8,204,17)$.

If $x=10$, then $\frac{3}{y}+\frac{4}{z}=\frac{3}{10}$. We have

$$(y-10)(3z-40)=400.$$

The solutions (x, y, z) are $(10,12,80)$, $(10,15,40)$, $(10,18,30)$, $(10,30,20)$, $(10,60,16)$, $(10,90,15)$, $(10,210,14)$.

If $x=12$, then $\frac{3}{y}+\frac{4}{z}=\frac{1}{3}$. We have

$$(y-9)(z-12)=108.$$

Since $x \leq y \leq z$ or $x \leq z \leq y$, the solutions (x, y, z) are $(12,12,48)$, $(12,13,39)$, $(12,15,30)$, $(12,18,24)$, $(12,21,21)$, $(12,27,18)$, $(12,36,16)$, $(12,45,15)$, $(12,63,14)$, $(12,117,13)$.

If $x=16$, then $\frac{3}{y}+\frac{4}{z}=\frac{3}{8}$. We have

$$(y-8)(3z-32)=256.$$

Since $x \leq y \leq z$ or $x \leq z \leq y$, the solution (x, y, z) is $(16,24,16)$.

If $x=9$, then $\frac{3}{y}+\frac{4}{z}=\frac{5}{18}$.

Case $z \leq y$. We get $\frac{5}{18}=\frac{3}{y}+\frac{4}{z} \leq \frac{7}{z}$, so $9 \leq z \leq 25$.

Since y is a positive integer, the solutions (x, y, z) are $(9,270,15)$, $(9,108,16)$, $(9,54,18)$, $(9,27,24)$.

Case $y \leq z$. We get $\frac{5}{18}=\frac{3}{y}+\frac{4}{z} \leq \frac{7}{y}$, so $9 \leq y \leq 25$.

Since z is a positive integer, the solutions (x, y, z) are $(9,11,792)$, $(9,12,144)$, $(9,14,63)$, $(9,18,36)$.

If $x=11$, then $\frac{3}{y}+\frac{4}{z}=\frac{7}{22}$.

Case $z \leq y$. We get $\frac{7}{22}=\frac{3}{y}+\frac{4}{z} \leq \frac{7}{z}$, so $11 \leq z \leq 22$.

Since y is a positive integer, the solutions (x, y, z) are $(11,286,13)$, $(11,44,16)$, $(11,22,22)$.

Case $y \leq z$. We get $\frac{7}{22}=\frac{3}{y}+\frac{4}{z} \leq \frac{7}{y}$, that is $11 \leq y \leq 22$.

Since z is a positive integer, the solutions (x, y, z) are $(11,11,88)$, $(11,22,22)$.

If $x = 13$, then $\frac{3}{y} + \frac{4}{z} = \frac{9}{26}$.

Case $z \leq y$. We get $\frac{9}{26} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $13 \leq z \leq 20$.

Since y is a positive integer, the solution (x, y, z) is $(13,78,13)$.

Case $y \leq z$. We get $\frac{9}{26} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $13 \leq y \leq 20$.

Since z is a positive integer, there is no solution.

If $x = 14$, then $\frac{3}{y} + \frac{4}{z} = \frac{5}{14}$.

Case $z \leq y$. We get $\frac{5}{14} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $14 \leq z \leq 19$.

Since y is a positive integer, the solutions (x, y, z) are $(14,42,14), (14,28,16)$.

Case $y \leq z$. We get $\frac{5}{14} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $14 \leq y \leq 19$.

Since z is a positive integer, the solutions (x, y, z) are $(14,14,28), (14,18,21)$.

If $x = 15$, then $\frac{3}{y} + \frac{4}{z} = \frac{11}{30}$.

Case $z \leq y$. We get $\frac{11}{30} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $15 \leq z \leq 19$.

Since y is a positive integer, the solution (x, y, z) is $(15,30,15)$.

Case $y \leq z$. We get $\frac{11}{30} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $15 \leq y \leq 19$.

Since z is a positive integer, the solutions (x, y, z) are $(15,15,24), (15,18,20)$.

If $x = 17$, then $\frac{3}{y} + \frac{4}{z} = \frac{13}{34}$.

Case $z \leq y$. We get $\frac{13}{34} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $17 \leq z \leq 18$.

Since y is a positive integer, there is no solution.

Case $y \leq z$. We get $\frac{13}{34} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $17 \leq y \leq 18$.

Since z is a positive integer, there is no solution.

If $x = 18$, then $\frac{3}{y} + \frac{4}{z} = \frac{7}{18}$.

Case $z \leq y$. We get $\frac{7}{18} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $z = 18$.

The solution (x, y, z) is $(18,18,18)$.

Case $y \leq z$. We get $\frac{7}{18} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $y = 18$.

The solution (x, y, z) is $(18,18,18)$.

Case 2 : $y < x \leq z$ or $y \leq z < x$.

If $y < x \leq z$, then $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{y}$.

If $y \leq z < x$, then $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{y}$.

Therefore, $\frac{1}{2} = \frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{y}$ which implies that $y < 18$. From $y$is a positive integer, $y \geq 7$and $y < 18$, we get $7 \leq y \leq 17$.

If $y = 7$, then $\frac{2}{x} + \frac{4}{z} = \frac{1}{14}$. We have

$$(x - 28)(z - 56) = 1568.$$

We use the function $\tau$to ensure that the total number of divisors of 1568 is 18. Using factors of 1568, we have

$x - 28 = 1, z - 56 = 1568;$
$x - 28 = 2, z - 56 = 784;$
$x - 28 = 4, z - 56 = 392;$
$x - 28 = 7, z - 56 = 224;$
$x - 28 = 8, z - 56 = 196;$
$x - 28 = 14, z - 56 = 112;$
$x - 28 = 16, z - 56 = 98;$
$x - 28 = 28, z - 56 = 56;$
$x - 28 = 32, z - 56 = 49;$
$x - 28 = 49, z - 56 = 32;$
$x - 28 = 56, z - 56 = 28;$
$x - 28 = 98, z - 56 = 16;$
$x - 28 = 112, z - 56 = 14;$
$x - 28 = 196, z - 56 = 8;$
$x - 28 = 224, z - 56 = 7;$
$x - 28 = 392, z - 56 = 4;$
$x - 28 = 784, z - 56 = 2;$
$x - 28 = 1568, z - 56 = 1.$

The solutions (x, y, z) are $(29,7,1624)$, $(30,7,840), (32,7,448), (35,7,280), (36,7,252)$, $(42,7,168), (44,7,154), (56,7,112), (60,7,105)$, $(77,7,88), (84,7,84), (126,7,72), (140,7,70)$, $(224,7,64), (252,7,63), (420,7,60), (812,7,58)$, $(1596,7,57)$.

If $y = 8$, then $\frac{2}{x} + \frac{4}{z} = \frac{1}{8}$. We have

$$(x - 16)(z - 32) = 512.$$

The solutions (x, y, z) are $(17,8,544)$, $(18,8,288), (20,8,160), (24,8,96), (32,8,64)$, $(48,8,48), (80,8,40), (144,8,36), (272,8,34)$, $(528,8,33)$.

If $y = 9$, then $\frac{2}{x} + \frac{4}{z} = \frac{1}{6}$. We have

$$(x - 12)(z - 24) = 288.$$

The solutions (x, y, z) are $(13,9,312)$, $(14,9,168), (15,9,120), (16,9,96), (18,9,72)$, $(20,9,60), (21,9,56), (24,9,48), (28,9,42)$, $(30,9,40), (36,9,36), (44,9,33), (48,9,32)$, $(60,9,30), (84,9,28), (108,9,27), (156,9,26)$, $(300,9,25)$.

If $y=10$, then $\frac{2}{x}+\frac{4}{z}=\frac{1}{5}$. We have
$$(x-10)(z-20)=200.$$
The solutions (x,y,z) are $(11,10,220)$, $(12,10,120)$, $(14,10,70)$, $(15,10,60)$, $(18,10,45)$, $(20,10,40)$, $(30,10,30)$, $(35,10,28)$, $(50,10,25)$, $(60,10,24)$, $(110,10,22)$, $(210,10,21)$.

If $y=12$, then $\frac{2}{x}+\frac{4}{z}=\frac{1}{4}$. We have
$$(x-8)(z-16)=128.$$
Since $y<x\le z$ or $y\le z<x$, the solutions (x,y,z) are $(16,12,32)$, $(24,12,24)$, $(40,12,20)$, $(72,12,18)$, $(136,12,17)$.

If $y=14$, then $\frac{2}{x}+\frac{4}{z}=\frac{2}{7}$. We have
$$(x-7)(2z-28)=196.$$
Since $y<x\le z$ or $y\le z<x$, and z is a positive integer, the solutions (x,y,z) are $(21,14,21)$, $(56,14,16)$, $(105,14,15)$.

If $y=11$, then $\frac{2}{x}+\frac{4}{z}=\frac{5}{22}$.
Case $x\le z$. We get $\frac{5}{22}=\frac{2}{x}+\frac{4}{z}\le\frac{6}{x}$, so $12\le x\le 26$. Since z is a positive integer, the solution (x,y,z) is $(12,11,66)$.
Case $z<x$. We get $\frac{5}{22}=\frac{2}{x}+\frac{4}{z}<\frac{6}{z}$, so $11\le z\le 26$. Since x is a positive integer, the solutions (x,y,z) are $(396,11,18)$, $(44,11,22)$, $(33,11,24)$.

If $y=13$, then $\frac{2}{x}+\frac{4}{z}=\frac{7}{26}$.
Case $x\le z$. We get $\frac{7}{26}=\frac{2}{x}+\frac{4}{z}\le\frac{6}{x}$, so $14\le x\le 22$. Since z is a positive integer, there is no solution.
Case $z<x$. We get $\frac{7}{26}=\frac{2}{x}+\frac{4}{z}<\frac{6}{z}$, so $13\le z\le 22$. Since x is a positive integer, the solutions (x,y,z) are $(780,13,15)$, $(104,13,16)$.

If $y=15$, then $\frac{2}{x}+\frac{4}{z}=\frac{3}{10}$.
Case $x\le z$. We get $\frac{3}{10}=\frac{2}{x}+\frac{4}{z}\le\frac{6}{x}$, so $16\le x\le 20$. Since z is a positive integer, the solution (x,y,z) is $(20,15,20)$.
Case $z<x$. We get $\frac{3}{10}=\frac{2}{x}+\frac{4}{z}<\frac{6}{z}$, so $15\le z\le 19$. Since x is a positive integer, the solutions (x,y,z) are $(60,15,15)$, $(40,15,16)$.

If $y=16$, then $\frac{2}{x}+\frac{4}{z}=\frac{5}{16}$.
Case $x\le z$. We get $\frac{5}{16}=\frac{2}{x}+\frac{4}{z}\le\frac{6}{x}$, so $17\le x\le 19$. Since z is a positive integer, there is no solution.
Case $z<x$. We get $\frac{5}{16}=\frac{2}{x}+\frac{4}{z}<\frac{6}{z}$, so $16\le z\le 19$. Since x is a positive integer, the solution (x,y,z) is $(32,16,16)$.

If $y=17$, then $\frac{2}{x}+\frac{4}{z}=\frac{11}{34}$.
Case $x\le z$. We get $\frac{11}{34}=\frac{2}{x}+\frac{4}{z}\le\frac{6}{x}$, so $x=18$. Since z is a positive integer, there is no solution.
Case $z<x$. We get $\frac{11}{34}=\frac{2}{x}+\frac{4}{z}<\frac{6}{z}$, so $17\le z\le 18$. Since x is a positive integer, there is no solution.

Case 3: $z<x\le y$ or $z<y<x$.

If $z<x\le y$, then $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}<\frac{9}{z}$.
If $z<y<x$, then $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}<\frac{9}{z}$.
Therefore, $\frac{1}{2}=\frac{2}{x}+\frac{3}{y}+\frac{4}{z}<\frac{9}{z}$ which implies that $z<18$. From z is a positive integer, $z\ge 9$ and $z<18$, we get $9\le z\le 17$.

If $z=9$, then $\frac{2}{x}+\frac{3}{y}=\frac{1}{18}$. We have
$$(x-36)(y-54)=1944.$$
We use the function τ to ensure that the total number of divisors of 1944 is 24. Using factors of 1944, we have

$$x-36=1, y-54=1944;$$
$$x-36=54, y-54=36;$$
$$x-36=2, y-54=972;$$
$$x-36=72, y-54=27;$$
$$x-36=3, y-54=648;$$
$$x-36=81, y-54=24;$$
$$x-36=4, y-54=486;$$
$$x-36=108, y-54=18;$$
$$x-36=6, y-54=324;$$
$$x-36=162, y-54=12;$$
$$x-36=8, y-54=243;$$
$$x-36=216, y-54=9;$$
$$x-36=9, y-54=216;$$
$$x-36=243, y-54=8;$$
$$x-36=12, y-54=162;$$
$$x-36=324, y-54=6;$$
$$x-36=18, y-54=108;$$
$$x-36=486, y-54=4;$$
$$x-36=24, y-54=81;$$
$$x-36=648, y-54=3;$$
$$x-36=27, y-54=72;$$
$$x-36=972, y-54=2;$$
$$x-36=36, y-54=54;$$
$$x-36=1944, y-54=1.$$

The solutions (x, y, z) are $(37{,}1998{,}9)$, $(38{,}1026{,}9)$, $(39{,}702{,}9)$, $(40{,}540{,}9)$, $(42{,}378{,}9)$, $(44{,}297{,}9)$, $(45{,}270{,}9)$, $(48{,}216{,}9)$, $(54{,}162{,}9)$, $(60{,}135{,}9)$, $(63{,}126{,}9)$, $(72{,}108{,}9)$, $(90{,}90{,}9)$, $(108{,}81{,}9)$, $(117{,}78{,}9)$, $(144{,}72{,}9)$, $(198{,}66{,}9)$, $(252{,}63{,}9)$, $(279{,}62{,}9)$, $(360{,}60{,}9)$, $(522{,}58{,}9)$, $(684{,}57{,}9)$, $(1008{,}56{,}9)$, $(1980{,}55{,}9)$.

In the same manner,
if $z = 10$, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{10}$. We have

$$(x - 20)(y - 30) = 600.$$

The solutions (x, y, z) are $(21{,}630{,}10)$, $(22{,}330{,}10)$, $(23{,}230{,}10)$, $(24{,}180{,}10)$, $(25{,}150{,}10)$, $(26{,}130{,}10)$, $(28{,}105{,}10)$, $(30{,}90{,}10)$, $(32{,}80{,}10)$, $(35{,}70{,}10)$, $(40{,}60{,}10)$, $(44{,}55{,}10)$, $(45{,}54{,}10)$, $(50{,}50{,}10)$, $(60{,}45{,}10)$, $(70{,}42{,}10)$, $(80{,}40{,}10)$, $(95{,}38{,}10)$, $(120{,}36{,}10)$, $(140{,}35{,}10)$, $(170{,}34{,}10)$, $(220{,}33{,}10)$, $(320{,}32{,}10)$, $(620{,}31{,}10)$.

If $z = 11$, then $\frac{2}{x} + \frac{3}{y} = \frac{3}{22}$. We have

$$(3x - 44)(y - 22) = 968.$$

Since x is a positive integer, the solutions (x, y, z) are $(15{,}990{,}11)$, $(16{,}264{,}11)$, $(22{,}66{,}11)$, $(44{,}33{,}11)$, $(55{,}30{,}11)$, $(176{,}24{,}11)$.

If $z = 12$, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{6}$. We have

$$(x - 12)(y - 18) = 216.$$

The solutions (x, y, z) are $(13{,}234{,}12)$, $(14{,}126{,}12)$, $(15{,}90{,}12)$, $(16{,}72{,}12)$, $(18{,}54{,}12)$, $(20{,}45{,}12)$, $(21{,}42{,}12)$, $(24{,}36{,}12)$, $(30{,}30{,}12)$, $(36{,}27{,}12)$, $(39{,}26{,}12)$, $(48{,}24{,}12)$, $(66{,}22{,}12)$, $(84{,}21{,}12)$, $(120{,}20{,}12)$, $(228{,}19{,}12)$.

If $z = 14$, then $\frac{2}{x} + \frac{3}{y} = \frac{3}{14}$. We have

$$(3x - 28)(y - 14) = 392.$$

Since $z < x \leq y$ or $z < y < x$, and y is a positive integer, the solutions (x, y, z) are $(28{,}21{,}14)$, $(42{,}18{,}14)$, $(140{,}15{,}14)$.

If $z = 16$, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{4}$. We have

$$(x - 8)(y - 12) = 96.$$

Since $z < x \leq y$ or $z < y < x$, the solutions (x, y, z) are $(20{,}20{,}16)$, $(24{,}18{,}16)$.

If $z = 13$, then $\frac{2}{x} + \frac{3}{y} = \frac{5}{26}$.

Case $x \leq y$. We get $\frac{5}{26} = \frac{2}{x} + \frac{3}{y} \leq \frac{5}{x}$, so $14 \leq x \leq 26$.
Since y is a positive integer, the solution (x, y, z) is $(26{,}26{,}13)$.
Case $y < x$. We get $\frac{5}{26} = \frac{2}{x} + \frac{3}{y} < \frac{5}{y}$, so $14 \leq y \leq 25$.
Since x is a positive integer, the solutions (x, y, z) are $(416{,}16{,}13)$, $(78{,}18{,}13)$.

If $z = 15$, then $\frac{2}{x} + \frac{3}{y} = \frac{7}{30}$.
Case $x \leq y$. We get $\frac{7}{30} = \frac{2}{x} + \frac{3}{y} \leq \frac{5}{x}$, so $16 \leq x \leq 21$.
Since y is a positive integer, there is no solution.
Case $y < x$. We get $\frac{7}{30} = \frac{2}{x} + \frac{3}{y} < \frac{5}{y}$, so $16 \leq y \leq 21$.
Since x is a positive integer, the solutions (x, y, z) are $(30{,}18{,}15)$, $(24{,}20{,}15)$.

If $z = 17$, then $\frac{2}{x} + \frac{3}{y} = \frac{9}{34}$.
Case $x \leq y$. We get $\frac{9}{34} = \frac{2}{x} + \frac{3}{y} \leq \frac{5}{x}$, so $x = 18$.
Since y is a positive integer, there is no solution.
Case $y < x$. We get $\frac{9}{34} = \frac{2}{x} + \frac{3}{y} < \frac{5}{y}$, so $y = 18$.
Since x is a positive integer, there is no solution.

On the other hand, we can compute directly that all above (x, y, z) satisfy the equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$. Hence, we obtain all solutions of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$ where x, y and z are positive integers. □

Moreover, If (x_0, y_0, z_0) is a positive integral solution of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$ such that $4|z_0$, then $\left(\frac{z_0}{4}, x_0, y_0\right)$ is a positive integral solution of the equation $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$, in (4). On the other hand, if (x_0, y_0, z_0) is a positive integral solution of the Diophantine equation $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$, then $(y_0, z_0, 4x_0)$ is also a positive integral solution of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$.

3. Conclusions

In this paper, we get all solutions of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$ where x, y and z are positive integers.

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